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On a Capacitability Problem Raised in Connection with Linear Programming

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Introduction

Let X and Y be compact Hausdorff spaces, $\Phi(x, y)$ be a universally measurable[1] function on $X \times Y$ which is bounded from below, $g(x)$ be any function on X and $f(y)$ be a universally measurable function which is bounded from above. We denote by $\mathscr{M}$ the set of all non-negative Radon measures satisfying the inequality

$$\int \Phi(x, y) d\mu(y) \leqq g(x) \quad \text{on} \quad X .$$

In the case that $\mathscr{M}$ is not empty, the quantity

$$M = \sup\left\{\int f d\mu ;\ \mu \in \mathscr{M}\right\}$$

was considered by Ohtsuka [4] in connection with a generalization of a theorem in the theory of linear programming. In this paper, we consider the family $\mathscr{M}_K$ of measures in $\mathscr{M}$ supported by a compact subset K of Y and consider a similar quantity

$$M(K) = \sup\left\{\int f d\mu ;\ \mu \in \mathscr{M}_K\right\}$$

in the case that $\mathscr{M}_K$ is not empty. This quantity has a potential theoretic meaning. In fact, Fuglede [2] considered it in case $\Phi \geqq 0$, $g=1$ and $f=1$ and denoted it by cap K. We shall call it Fuglede's capacity in § 11.

For any set $A \subset Y$, we define in § 1 an inner quantity $M_i(A)$ and an outer quantity $M_e(A)$ from $M(K)$ in the same way as the inner capacity $\mathrm{cap}_* A$ and the outer capacity $\mathrm{cap}^* A$ were defined from cap K in [2]. Ohtsuka orally raised the question as to when $M_i(A)$ is equal to $M_e(A)$. We shall give an answer to this question in the present paper.

Kishi [3] examined this problem in the case that $X = Y$, $\Phi(x, y) = \Phi(y, x) > 0$ for all x, $y \in X$, Φ is lower semicontinuous and $g = f = 1$. His main result

1) A function on a compact set is universally measurable if it is measurable with respect to all Radon measures.

is that if Φ satisfies the continuity principle[2)], then $M_i(A)=M_e(A)$ for every analytic set.

Fuglede [2] investigated this problem in case $\Phi\geqq 0$, $g=1$ and $f=1$, and obtained a result which is similar to Kishi's. However, it is impossible, in his case, to retain the continuity principle as its original form, because $X\neq Y$ in general, so that he used conditions (A) and (B) (see §6 and §8) instead of the continuity principle. Fuglede proved $\text{cap}_*A=\text{cap}^*A$ for every analytic set under conditions (A) and (B).

In §2 we shall give the equality $M_i(K)=M_e(K)$ for compact sets K under more general setting than Kishi's or Fuglede's. On account of an interesting result of Ohtsuka stated in §3, we can develop our theory from §5 to §10. There, we consider another quantities $\gamma_i(A)$ and $\gamma_e(A)$, and follow the reasoning in [2]. We shall give in §10 the equality $M_i(A)=M_e(A)$ for every analytic set by applying a useful theorem of Choquet [1]. This contains the results which are mentioned above. In §4 we shall study the properties of $M_i(A)$ and $M_e(A)$ as set functions for later application and in §11 we shall compare our $M_i(A)$ and $M_e(A)$ with Fuglede's capacity.

We always assume that Φ and g are non-negative from §4 to §11 and that f is upper semicontinuous from §5 to §10. In case $g\geqq 0$, we remark that $\mathcal{M}_K$ is not empty for any $K\neq\phi$ and that $M(K)$, $M_i(A)$ and $M_e(A)$ are non-negative. Then we define $M(\phi)=0$ for the empty set ϕ.

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§1. Definitions

Let X and Y be compact Hausdorff spaces, $\Phi(x, y)>-\infty$ be a lower semicontinuous function on $X\times Y$, $g(x)$ ($f(y)$ resp.) be a function on X (Y resp.) which is bounded from below (above resp.) and K be a compact subset of Y. A measure μ will be always a non-negative Radon measure and $S\mu$ will be the support of μ. For a measure μ on Y (ν on X resp.) a potential $\Phi(x, \mu)$ ($\Phi(\nu, y)$ resp.) is defined by $\int\Phi(x, y)d\mu(y)$ ($\int\Phi(x, y)d\nu(x)$ resp.). We shall consider two classes of measures

$$\mathcal{M}_K=\{\mu;\ S\mu\subset K \text{ and } \Phi(x, \mu)\leqq g(x) \text{ on } X\}$$

and

$$\mathcal{M}'_K=\{\nu;\ S\nu\subset X \text{ and } \Phi(\nu, y)\geqq f(y) \text{ on } K\}\ .$$

2) Continuity principle: If a potential $\int\Phi(x,y)d\mu(y)$ of a positive Radon measure μ is finite and continuous as a function on the support of μ, then it is continuous in X.

In case f and g are universally measurable, we define

$$M(K)=\sup\left\{\int f d\mu;\ \mu\in\mathscr{M}_K\right\}\qquad \text{if } \mathscr{M}_K\neq\phi\ ,$$

and

$$M'(K)=\inf\left\{\int g d\nu;\ \nu\in\mathscr{M}'_K\right\}\qquad \text{if } \mathscr{M}'_K\neq\phi\ ,$$

where the empty set is denoted by ϕ. For simplicity, we put

$$M'(K)=\infty \qquad \text{if } \mathscr{M}'_K=\phi\ .$$

But we do not define $M(K)$ for $\mathscr{M}_K=\phi$ except for the special case in §4. By definition, it is easily seen that $M(K)\leqq M'(K)$ if $\mathscr{M}_K\neq\phi$. For any set $A\subset Y$, we define

$$M_i(A)=\sup\{M(K);\ K \text{ is compact and } K\subset A\}$$

and

$$M_e(A)=\inf\{M_i(G);\ G \text{ is open and } G\supset A\}\ .$$

§ 2. Equality $M_i(K)=M_e(K)$

First we observe that $M(K)$, $M_i(A)$ and $M_e(A)$ are increasing set functions, $M(K)=M_i(K)$ for every compact set K and $M_i(A)\leqq M_e(A)$. $M_i(K)$ is not necessarily equal to $M_e(K)$. This is shown by

Example 1. Let $X=Y$ be the interval $\{|x|\leqq 2\}$ in the real line, $K=\{|y|\leqq 1\}$, $\Phi=0$, $g=1$ and $f(y)=0$ if $|y|\leqq 1$ and $=|y|-1$ if $1<|y|\leqq 2$. Then $M_i(K)=0$ and $M_i(G)=\infty$ for any open set $G\supset K$. Hence $M_e(K)=\infty>0=M_i(K)$.

We shall define *conditions* (H_1) *and* (H_2) as follows:

(H_1) *There is a point* $x_0\in X$ *such that* $\Phi(x_0, y)>0$ *for all* $y\in Y$ *and* $g(x_0)<\infty$.

(H_2) $f(y)>0$ *for all* $y\in Y$.

Under condition (H_1), if $\mathscr{M}_Y\neq\phi$, then the set of total masses $\{\mu(Y);\ \mu\in\mathscr{M}_Y\}$ is bounded. Whence $M_i(A)$ and $M_e(A)$ are not equal to $+\infty$ for any set A. As for condition (H_2), we have

Lemma 1.[3] *Assume condition* (H_2). *If* $\mathscr{M}_K\neq\phi$ *and* $M(K)$ *is finite, then the set of total masses* $\{\mu(K);\ \mu\in\mathscr{M}_K\}$ *is bounded.*

3) cf. [4], Lemma 1.

PROOF. If we deny this, then there is a sequence $\{\mu_n\}$ in $\mathscr{M}_K$ such that $\mu_n(K)>n$. We put $\mu_n'=\mu_n/\mu_n(K)$, and choose a vaguely convergent subsequence of $\{\mu_n'\}$. We shall denote it again by $\{\mu_n'\}$ and let μ_0' be the limit. It holds that $S\mu_0'\subset K$, $\mu_0'(K)=1$ and

$$\Phi(x, \mu_0')\leqq\varliminf_{n\to\infty}\Phi(x, \mu_n')=\varliminf_{n\to\infty}[\Phi(x, \mu_n)/\mu_n(K)]\leqq\varliminf_{n\to\infty}[(\max(g(x), 0))/n] .$$

Therefore $\Phi(x, \mu_0')\leqq 0$ if $g(x)$ is finite. Take μ such that $S\mu\subset K$, $\Phi(x, \mu)\leqq g(x)$ on X and $\int f d\mu$ is finite. Then for any positive number t, we have $\Phi(x, \mu+t\mu_0')\leqq g(x)$ on X and hence $\mu+t\mu_0'\in\mathscr{M}_K$. Thus $M(K)\geqq\int f d\mu+t\int f d\mu_0'$. Since $\int f d\mu_0'>0$ by condition (H_2), we have $M(K)=\infty$. This is a contradiction.

REMARK. Consider a class of measures

$$\mathscr{M}_A=\{\mu;\ S\mu\subset A \text{ and } \Phi(x, \mu)\leqq g(x) \text{ on } X\}$$

and a set function $m_i(A)=\sup\left\{\int f d\mu;\ \mu\in\mathscr{M}_A\right\}$ if $\mathscr{M}_A\neq\phi$. It is evident that $m_i(K)=M(K)$ for any compact set K with $\mathscr{M}_K\neq\phi$ and easy to see that, if $\mathscr{M}_A$ is not empty, then we have

$$m_i(A)=\sup\{m_i(K);\ K \text{ is compact and } K\subset A\}=M_i(A) .$$

The equality $M_i(K)=M_e(K)$ for compact sets K is given by

THEOREM 1. *Let f be upper semicontinuous*[4] *and g be any function bounded from below such that $\mathscr{M}_K\neq\phi$. If we assume either condition (H_1) or condition (H_2), then we have $M_i(K)=M_e(K)$.*

PROOF. We may suppose $M_e(K)>-\infty$. In case $-\infty<M_e(K)<\infty$, there is an open set G_0 such that $G_0\supset K$ and $-\infty<M(\bar{G}_0)<\infty$. Let D_0 be the set of all open sets G satisfying $K\subset G\subset G_0$. D_0 is directed by $\subset$. We assume $M_i(K)<M_e(K)$, and take a number α in between. For every $G\in D_0$ there is a measure μ_G of $\mathscr{M}_G$ such that $\int f d\mu_G>\alpha$. The set $\{\mu_G; G\in D_0\}$ is a net and vaguely bounded. In fact, we have on account of condition (H_1) or (H_2)

$$\sup\{\mu_G(Y); G\in D_0\}\leqq\sup\{\mu(Y);\ \mu\in\mathscr{M}_{\bar{G}_0}\}<\infty \qquad \text{(see Lemma 1).}$$

Hence a subnet $\{\mu_\omega;\ \omega\in D_0'\}$ converges vaguely to some measure μ_0. We

4) At the beginning of our paper we assumed that f is bounded from above. If f is upper semicontinuous and does not take the value $+\infty$, then f is bounded from above.

observe that $S\mu_0$ is contained in K because $\bigcap_{G\in D_0} G=K$ and that $\Phi(x, \mu_0)\leqq g(x)$ on X. Therefore $\mu_0\in \mathscr{M}_K$. Since f is upper semicontinuous,

$$\alpha\leqq \overline{\lim_{\omega\in D_0'}}\int f d\mu_\omega\leqq \int f d\mu_0\leqq M_i(K)\ .$$

This contradicts the assumption $M_i(K)<\alpha$. Thus $M_i(K)\geqq M_e(K)$. Next, we shall prove that $M_e(K)=\infty$ implies $M_i(K)=\infty$. Since $M_e(K)=\infty$ does not occur under condition (H_1), our hypothesis is limited to condition (H_2). Let D be the set of all open sets containing K. D is directed in the same way as above. Let n be an arbitrarily fixed positive integer and α be an upper bound of f on Y. For any $G\in D$, there is a measure $\mu_G^{(n)}$ of $\mathscr{M}_G$ such that $\int f d\mu_G^{(n)}>n$. Then $\mu_G^{(n)}(Y)>n/\alpha$. We put $\lambda_G^{(n)}=\mu_G^{(n)}/\mu_G^{(n)}(Y)$, and choose a vaguely convergent subnet $\{\lambda_\omega^{(n)}; \omega\in D'\}$ and let μ_n' be the limit. Then we have $S\mu_n'\subset K$, $\mu_n'(K)=1$ and

$$\Phi(x, \mu_n')\leqq \underline{\lim_{\omega\in D'}}\,\Phi(x, \lambda_\omega^{(n)})\leqq \frac{\alpha}{n}\max(g(x), 0) \qquad \text{on } X\ .$$

We choose a vaguely convergent subsequence of $\{\mu_n'\}$. We shall denote it again by $\{\mu_n'\}$ and let μ_0' be the limit. It follows that $\mu_0'(K)=1$, $S\mu_0'\subset K$ and

$$\Phi(x, \mu_0')\leqq \underline{\lim_{n\to\infty}}\,\Phi(x, \mu_n')\leqq \lim_{n\to\infty}[\alpha(\max(g(x), 0))/n] \qquad \text{on } X\ ,$$

and hence $\Phi(x, \mu_0')\leqq 0$ if $g(x)$ is finite. Take any measure μ of $\mathscr{M}_K$. Then $\mu+t\mu_0'$ belongs to $\mathscr{M}_K$ for any positive number t. Since $\int f d\mu_0'>0$ by condition (H_2) and $M(K)\geqq \int f d\mu+t\int f d\mu_0'$, we conclude $M_i(K)=M(K)=\infty$.

Theorem 1 is not always true if we omit the condition that f is upper semicontinuous. To see this, we give

EXAMPLE 2. Let X, Y, K and g be the same as in Example 1. Take $\Phi=1$ and $f(y)=1$ if $|y|\leqq 1$ and $=2$ if $1<|y|\leqq 2$. Then $M_i(K)=1$ and $M_i(G)=2$ for any open set $G\supset K$. Thus $M_e(K)=2>1=M_i(K)$.

§3. Duality Theorem

If X and Y are discrete spaces, then it is known as duality theorem that $M(K)=M'(K)$. In the general case as ours, Yoshida [5] gave an example such that $M(K)\neq M'(K)$ even if Φ, g and f are non-negative and continuous. Ohtsuka [4] proved

THEOREM 2. *Let f be an upper semicontinuous function and g be a lower semicontinuous function with $\mathscr{M}_K\neq\phi$. If we assume $M(K)>-\infty$ and either* *condition* (H_1) *or condition* (H_2), *then we have* $M(K)=M'(K)$.

Consequently, we need not consider $M_i'(A)=\sup\{M'(K);\ K$ is compact and $K\subset A\}$ and $M_e'(A)=\inf\{M_i'(G);\ G$ is open and $G\supset A\}$ in this case.

Fuglede [2] proved the above theorem in case $g=1$ and $f=1$.

§ 4. Properties of $M_i(A)$ and $M_e(A)$ as set functions

We assume hereafter that Φ *and* g *are non-negative.* On account of this assumption, $\mathscr{M}_K$ is not empty for any compact set $K\neq\phi$. It is evident that $M(K)$, $M_i(A)$ and $M_e(A)$ are non-negative. We define $M(\phi)=0$ for the empty set ϕ. We shall study some properties of $M_i(A)$ and $M_e(A)$ as set functions. We remark that, in case $\Phi\geqq 0$, $g=1$ and $f=1$, we have $M(K)=\mathrm{cap}\,K$, $M_i(A)=\mathrm{cap}_*A$ and $M_e(A)=\mathrm{cap}^*A$ with the notations of Fuglede [2].

LEMMA 2. *Let* f *be universally measurable, and* K_1 *and* K_2 *be compact sets. Then we have*

$$M(K_1\cup K_2)\leqq M(K_1)+M(K_2)\ .$$

PROOF. We may assume that $M(K_1\cup K_2)$ is positive. Let α be a positive number with $M(K_1\cup K_2)>\alpha$. There is a measure μ of $\mathscr{M}_{K_1\cup K_2}$ such that $\int f d\mu>\alpha$. Writing $F=\{y\in S\mu;\ f(y)>0\}$, we see that F is a universally measurable set and $\mu(F)>0$. Let μ_j be the restriction of μ to $F\cap K_j (j=1, 2)$. Then $\Phi(x,\mu_j)\leqq\Phi(x,\mu)\leqq g(x)$ on X because $\Phi\geqq 0$. Since $S\mu_j\subset K_j$, μ_j belongs to $\mathscr{M}_{K_j}$. Therefore

$$\alpha<\int f d\mu\leqq\int_F f d\mu\leqq\int f d\mu_1+\int f d\mu_2\leqq M(K_1)+M(K_2)\ .$$

By the arbitrariness of α, we obtain the inequality.

LEMMA 3. *Let* f *be universally measurable and* B_1 *and* B_2 *be universally measurable sets. Then we have*

$$M_i(B_1\cup B_2)\leqq M_i(B_1)+M_i(B_2)\ .$$

PROOF. We may suppose that $M_i(B_1\cup B_2)$ is positive. For any positive number α smaller than $M_i(B_1\cup B_2)$, there is a compact set K such that $K\subset B_1\cup B_2$ and $M(K)>\alpha$. We can find a measure μ of $\mathscr{M}_K$ such shat $\int f d\mu>\alpha$. Write $F=\{y\in S\mu;\ f(y)>0\}$. Given $\varepsilon>0$, there are compact sets K_1 and K_2 having the following properties:

$$K_1\subset K\cap F\cap B_1,\quad K_2\subset K\cap F\cap B_2,\quad \int_{K\cap F\cap B_1} f d\mu<\int_{K_1} f d\mu+\frac{\varepsilon}{2}$$

and

$$\int_{K\cap F\cap B_2} f d\mu < \int_{K_2} f d\mu + \frac{\varepsilon}{2} .$$

Denoting the restriction of μ to K_j by $\mu_j(\mathrm{j}=1, 2)$, we see $\mu_j \in \mathscr{M}_{K_j}(\mathrm{j}=1, 2)$ and

$$\begin{aligned}\alpha &< \int f d\mu \leqq \int_F f d\mu \leqq \int_{K\cap F\cap B_1} f d\mu + \int_{K\cap F\cap B_2} f d\mu \\ &< \int f d\mu_1 + \int f d\mu_2 + \varepsilon \leqq M(K_1) + M(K_2) + \varepsilon \\ &\leqq M_i(B_1) + M_i(B_2) + \varepsilon .\end{aligned}$$

By the arbitrariness of α and ε, we obtain the desired inequality.

From Lemma 3, we easily deduce

Theorem 3. *Assume that f is universally measurable.*

(α) *For any sequence $\{B_n\}$ of universally measurable sets and for any set A, we have $M_i(\bigcup_{n=1}^{\infty}(B_n \cap A)) \leqq \sum_{n=1}^{\infty} M_i(B_n \cap A)$.*

(β) *For any sequence $\{A_n\}$ of sets, we have $M_e(\bigcup_{n=1}^{\infty} A_n) \leqq \sum_{n=1}^{\infty} M_e(A_n)$.*

Corollary. *$M_e(A-N)=M_e(A)$ if $M_e(N)=0$.*

If we omit the condition $\Phi \geqq 0$, then it is not always valid that $M_e(A_1 \cup A_2) \leqq M_e(A_1) + M_e(A_2)$. In fact, we can construct

Example 3. Let $X=\{x_1, x_2\}$, $Y=\{y_1, y_2\}$, $K_1=\{y_1\}$, $K_2=\{y_2\}$, $g(x_1)=g(x_2)=1$, $f(y_1)=f(y_2)=1$, $\Phi(x_1, y_2)=\Phi(x_2, y_1)=1$, $\Phi(x_1, y_1)=-1$ and $\Phi(x_2, y_2)=1/4$. Then $M(K_1)=M(K_2)=1$ and $M(K_1 \cup K_2)=11/5$. In fact, let ε_y be the unit point measure at y and $\mu = a\varepsilon_{y_1} + b\varepsilon_{y_2}$ with $a, b \geqq 0$. Then $\Phi(x, \mu) \leqq g(x)$ on X means that $-a+b \leqq 1$ and $a+b/4 \leqq 1$. We see $M(K_1)=\sup\{a\}=1$, $M(K_2)=\sup\{b\}=1$ and $M(K_1 \cup K_2)=\sup\{a+b\}=3/5+8/5=11/5$.

Lemma 4.[5] *Let f be universally measurable. Then we have $M_i(A_1 \cup A_2) \leqq M_i(A_1) + M_e(A_2)$ for arbitrary sets A_1 and A_2.*

Proof. We may suppose that $M_i(A_1 \cup A_2)$ is positive. For any positive number α smaller than $M_i(A_1 \cup A_2)$, we can find a compact set $K \subset A_1 \cup A_2$ with $M(K) > \alpha$. We may assume that $M_e(A_2)$ is finite. In this case, given $\varepsilon > 0$, there is an open set G such that $G \supset A_2$ and $M_i(G) < M_e(A_2) + \varepsilon$. Write $F = K - G$. Then F is compact and contained in A_1 and $K = F \cup (K \cap G)$. By Lemma 3, it holds that

5) cf. [2], footnote 3.

$$\alpha < M_i(K) \leqq M_i(F) + M_i(F \cap G) \leqq M_i(A_1) + M_i(G) < M_i(A_1) + M_e(A_2) + \varepsilon \ .$$

By the arbitrariness of α and ε, we obtain the desired inequality.

DEFINITION. We shall say that a property holds *n.e.*[6] (*q.e.* resp.) on A if the M_i-value (M_e-value resp.) of the exceptional set in A is zero.

§5. $\gamma_i(A)$ and $\gamma_e(A)$

We shall define two more classes of measures

$$\Gamma_A^i = \{\nu;\ S\nu \subset X \text{ and } \Phi(\nu, y) \geqq f(y) \text{ n.e. on } A\}$$

and

$$\Gamma_A^e = \{\nu;\ S\nu \subset X \text{ and } \Phi(\nu, y) \geqq f(y) \text{ q.e. on } A\} \ .$$

Fuglede [2] considered these classes in case $g=1$ and $f=1$. From now on, we assume that g is universally measurable. We set

$$\gamma_i(A) = \inf\left\{\int g d\nu;\ \nu \in \Gamma_A^i\right\} \qquad \text{if } \Gamma_A^i \neq \phi$$

and

$$\gamma_i(A) = \infty \qquad \text{if } \Gamma_A^i = \phi \ .$$

For Γ_A^e, we define $\gamma_e(A)$ similarly. In case $\gamma_i(A) = \gamma_e(A)$, we shall simply write $\gamma(A)$ for the common value.

In what follows, except in §11, *we assume that f is upper semicontinuous.* First we have

THEOREM 4. *Assume either condition (H_1) or condition (H_2). Then it is valid that $\gamma_i(A) = \gamma_e(A)$ for any K_σ-set.*

PROOF. Let A be a K_σ-set. It suffices to show that if $\Phi(\nu, y) \geqq f(y)$ n.e. on A, then $\Phi(\nu, y) \geqq f(y)$ q.e. on A. Write $N = \{y \in A;\ \Phi(\nu, y) < f(y)\}$. Since A is a K_σ-set and f is upper semicontinuous, we see by the relation

$$N = \bigcup_{n=1}^{\infty} \{y \in A;\ \Phi(\nu, y) \leqq \left(1 - \frac{1}{n}\right) f(y)\}$$

that N is also a K_σ-set, i.e. N can be expressed as $\bigcup_{n=1}^{\infty} K_n$ with compact sets $\{K_n\}$. By Theorem 1, we have

6) "n.e." ("q.e." resp.) is an abbreviation of "nearly everywhere" ("quasi-everywhere" resp.).

$$0 \leqq M_e(K_n) = M_i(K_n) \leqq M_i(N) = 0 \text{ for each } n \ .$$

It follows from Theorem 3 (β) that $0 \leqq M_e(N) \leqq \sum_{n=1}^{\infty} M_e(K_n) = 0$.

THEOREM 5. *Let g be a lower semicontinuous function. If we assume either condition (H_1) or condition (H_2), then it holds that $M(K) = \gamma(K)$ for every compact set K.*

PROOF. Since $\mathscr{M}'_K \subset \Gamma^i_K$, it follows from Theorem 2 that $0 \leqq \gamma(K) \leqq M'(K) = M(K)$. It suffices to show the converse inequality in case $M(K) > 0$. Let μ be a measure of $\mathscr{M}_K$ with $\int f d\mu > 0$ and ν be any measure of Γ^i_K. Write

$$N = \{ y \in K; \Phi(\nu, y) < f(y) \}$$

and

$$F = \{ y \in K; f(y) > 0 \} \ .$$

Then $N \subset F$, $M_i(N) = 0$ and $\mu(F) > 0$. It is valid that $\mu(N) = 0$. In fact, for any compact set $H \subset N$, the restriction of μ to H belongs to $\mathscr{M}_H$ and we have $\mathscr{M}_H = \{0\}$, because $f(y) > 0$ for all $y \in H$. Thus $\mu(H) = 0$. Since N is a Borel set, we conclude $\mu(N) = 0$. Consequently

$$\int f d\mu \leqq \int_F f d\mu \leqq \int_{(K-N) \cap F} \Phi(\nu, y) d\mu(y) \leqq \int \Phi(\nu, y) d\mu(y)$$

$$= \int \Phi(x, \mu) d\nu(x) \leqq \int g d\nu \ .$$

Thus $M(K) \leqq \gamma(K)$.

COROLLARY. $M_i(A) \leqq \gamma_i(A)$.

We remark that $M(K)$ is not necessarily equal to $\gamma(K)$ without condition (H_1) or (H_2). Yoshida [5] gave an example such that $M'(K) = \gamma(K) > M(K)$.

THEOREM 6. *If we assume either condition (H_1) or condition (H_2), then it holds that $M_e(A) \leqq \gamma_e(A)$.*

PROOF. It suffices to show that $\Phi(\nu, y) \geqq f(y)$ q.e. on A implies $M_e(A) \leqq \int g d\nu$. Let $N = \{ y \in A; \Phi(\nu, y) < f(y) \}$. Then $M_e(N) = 0$. Assume condition (H_1) and let $\nu_t = \nu + t\varepsilon_{x_0}$ with a positive number t. Writing $G_t = \{ y \in Y; \Phi(\nu_t, y) > f(y) \}$, G_t is open and contains $A - N$. By the Corollary of Theorem 5, we see

$$M_e(A-N) \leqq M_i(G_t) \leqq \gamma_i(G_t) \leqq \int g d\nu_t = \int g d\nu + t g(x_0) \ .$$

Letting $t \to 0$, on account of the Corollary of Theorem 3, we obtain $M_e(A) = M_e(A-N) \leqq \int g d\nu$. In case condition (H_2) is assumed, we consider $G_s = \{y \in Y; \Phi(\nu, y) > sf(y)\}$ with $0 < s < 1$ instead of G_t. Then G_s is open and contains $A-N$. The rest of the proof is carried out in the same way as above.

§6. Relation between $\gamma_i(A)$ and $M_i(A)$

In this section, we shall discuss when $M_i(A)$ is equal to $\gamma_i(A)$. We define *condition* (A) as follows:

(A) *For any compact set $K \subset Y$ with $M_i(K) > 0$, there is a nonzero measure μ supported by K such that $\Phi(x, \mu)$ is finite and continuous in X.*

Fuglede [2] defined this condition in case $g=1$ and $f=1$. In the special case that $X=Y$, $g=1$ and $f=1$, it is well-known that condition (A) is verified for any kernel which satisfies the continuity principle. Even if $X=Y$ and $\Phi(x, y) = \Phi(y, x)$ for all $x, y \in X$, our condition (A) is different from Fuglede's. In fact we give

EXAMPLE 4. Let $X = Y$ be the interval $\{|x| \leqq 1\}$ in the real line, $K = \{x = 0\}$, $f=1$, $g(x) = 1/|x|$ and $\Phi(x, y) = 1/(|x| + |y|)$. Then we see $M_i(K) = 1$. But condition (A) is not satisfied. On the other hand, it is easily seen that Fuglede's condition (A) is fulfilled.

Like in the classical case, we have

LEMMA 5.[7] *Let $\{\nu_n\}$ be a sequence of measures on X which converges vaguely to ν_0. Then, under condition (A), we have*

$$\varliminf_{n\to\infty} \Phi(\nu_n, y) \leqq \Phi(\nu_0, y) \qquad n.e. \text{ on } Y \ .$$

PROOF. Write $N = \{y \in Y; \varliminf_{n\to\infty} \Phi(\nu_n, y) > \Phi(\nu_0, y)\}$ and suppose $M_i(N) > 0$. Then there is a compact set $K \subset N$ with $M_i(K) > 0$. By condition (A), we can find a nonzero measure μ supported by K such that $\Phi(x, \mu)$ is finite and continuous in X. It follows from Fatou's lemma that

$$\int \Phi(\nu_0, y) d\mu(y) < \int \varliminf_{n\to\infty} \Phi(\nu_n, y) d\mu(y) \leqq \varliminf_{n\to\infty} \int \Phi(\nu_n, y) d\mu(y)$$

$$= \lim_{n\to\infty} \int \Phi(x, \mu) d\nu_n(x) = \int \Phi(x, \mu) d\nu_0(y) \ .$$

This is a contradiction. Hence $M_i(N) = 0$.

7) cf. [2], Lemme 2.1.

THEOREM 7. *Suppose that g is a positive and lower semicontinuous function and that condition (A) is satisfied. If $\gamma_i(K)$ is finite, then there exists a measure ν_0 such that*

$$\gamma_i(K)=\int g d\nu_0 \qquad and \qquad \nu_0\in\Gamma_K^i \ .$$

(This measure ν_0 is called the optimal measure for $\gamma_i(K)$.)

PROOF. Let t be a positive number larger than $\gamma_i(K)$. We can find a sequence $\{\nu_n\}\subset\Gamma_K^i$ such that $\int g d\nu_n$ tends to $\gamma_i(K)$. Since $s=\inf\{g(x);\ x\in X\}>0$, we see $s\nu_n(X)\leqq\int g d\nu_n<t$ for large n. We can find a subsequence $\{\nu_{n_j}\}$ which converges vaguely to some measure ν_0. By Lemma 5, $\Phi(\nu_0,\ y)\geqq f(y)$ n.e. on K and hence $\nu_0\in\Gamma_K^i$. It follows that

$$\gamma_i(K)=\varliminf_{j\to\infty}\int g d\nu_{n_j}\geqq\int g d\nu_0\geqq\gamma_i(K)\ .$$

We can not omit either condition (A) or the condition $g>0$ in this theorem. In fact, an example in [5] shows that it happens that there is no measure in Γ_K^i which attains $\gamma_i(K)$ if we allow $g(x)=0$. In this example, Φ is finite and continuous. Therefore condition (A) is valid and $M'(K)=\gamma_i(K)$. Fuglede [2] gave an example such that $g>0$ but condition (A) is not valid and there is no optimal measure in Γ_K^i for $\gamma_i(K)$.

THEOREM 8. *Assume that g is positive and lower semicontinuous and that condition (A) is satisfied. If we assume either condition (H_1) or condition (H_2), then $\gamma_i(A)=M_i(A)$. In case $M_i(A)$ is finite, there is a measure ν_0 of Γ_A^i such that $\gamma_i(A)=\int g d\nu_0$.*

PROOF. It is enough to show that $M_i(A)\geqq\gamma_i(A)$ in case $M_i(A)$ is finite. The set D of all compact sets contained in A is directed by $\subset$. For any $K\in D$, there exists a measure ν_K such that

$$\int g d\nu_K=\gamma(K)=M(K)\leqq M_i(A)<\infty$$

and

$$\Phi(\nu_K,\ y)\geqq f(y) \qquad\qquad \text{n.e. on } K$$

because of Theorems 5 and 7. Since $s=\inf\{g(x);\ x\in X\}>0$, $\{\nu_K(X);\ K\in D\}$ is bounded. We can find a subnet $\{\nu_\omega;\ \omega\in D'\}$ which converges vaguely to some ν_0. Let $N=\{y\in A;\ \Phi(\nu_0,\ y)<f(y)\}$ and suppose that $M_i(N)>0$. Then there is a compact set $K_0 \subset N$ with $M_i(K_0) > 0$. By condition (A), there exists a nonzero measure μ_0 supported by K_0 such that $\Phi(x, \mu_0)$ is finite and continuous in X. Consequently $\int \Phi(\nu_0, y) d\mu_0(y) < \int f d\mu_0$. Write $N_K = \{y \in K; \Phi(\nu_K, y) < f(y)\}$ and $F = \{y \in Y; f(y) > 0\}$. Then $N_K \subset F$, N_K is a Borel measurable set and $M_i(N_K) = 0$. We see by the same reasoning as in the proof of Theorem 5 that

$$\mu_0(N_K) = 0 \quad \text{and} \quad \int f d\mu_0 \leqq \int \Phi(\nu_K, y) d\mu_0(y) .$$

We have

$$\int f d\mu_0 \leqq \overline{\lim_{\omega \in D'}} \int \Phi(\nu_\omega, y) d\mu_0(y) = \lim_{\omega \in D'} \int \Phi(x, \mu_0) d\nu_\omega(x)$$

$$= \int \Phi(x, \mu_0) d\nu_0(x) < \int f d\mu_0 .$$

This is absurd. Thus $M_i(N) = 0$. Namely $\Phi(\nu_0, y) \geqq f(y)$ n.e. on A. Therefore $\nu_0 \in \Gamma_A^i$ and

$$M_i(A) \geqq \underline{\lim_{\omega \in D'}} \int g d\nu_\omega \geqq \int g d\nu_0 \geqq \gamma_i(A) .$$

This completes the proof.

§7. Relation between $\gamma_e(A)$ and $M_e(A)$

In this section, we shall discuss when $\gamma_i(G) = \gamma_e(G)$ for every open set G. If we assume either condition (H_1) or condition (H_2) and that any open set in Y is a K_σ-set, then the equality is guaranteed by Theorem 4. On the other hand, even if we do not assume these, by following the method of Fuglede [2], we can obtain the equality under some different conditions.

DEFINITION. *A* real-valued function h on Y is called *quasi-continuous* if, for any $\varepsilon > 0$, there is an open set G_ε such that $M_i(G_\varepsilon) < \varepsilon$ and the restriction of h to $Y - G_\varepsilon$ is finite and continuous.

We define *condition* (B) as follows:

(B) *$\Phi(\nu, y)$ is quasi-continuous on Y for every measure ν on X.*

Fuglede [2] defined condition (B) in case $g = 1$ and $f = 1$. First we shall prove

LEMMA 6.[8]) *Let A be any set. If we assume that, for any $\varepsilon > 0$, there is a set B_ε such that $M_i(B_\varepsilon) = M_e(B_\varepsilon)$, $M_e(A - B_\varepsilon) < \varepsilon$ and $M_e(B_\varepsilon - A) < \varepsilon$, then it is*

8) cf. [2], Lemme 4.3.

valid that $M_i(A)=M_e(A)$.

PROOF. On account of Theorem 3 and Lemma 4, we have

$$M_e(A)\leqq M_e(B_\varepsilon)+M_e(A-B_\varepsilon)<M_i(B_\varepsilon)+\varepsilon$$

and

$$M_i(B_\varepsilon)\leqq M_i(A)+M_e(B_\varepsilon-A)<M_i(A)+\varepsilon\ .$$

Therefore $M_e(A)<M_i(A)+2\varepsilon$. By the arbitrariness of ε, we have $M_e(A)\leqq M_i(A)$. Thus $M_i(A)=M_e(A)$.

LEMMA 7. *Assume condition (B) and that f is quasi-continuous and let G be an open set. If $\Phi(\nu, y)\geqq f(y)$ n.e. on G, then we have $\Phi(\nu, y)\geqq f(y)$ q.e. on G.*

PROOF. It is enough to show that the M_e-value of $N=\{y\in G;\ \Phi(\nu, y)<f(y)\}$ is zero. Given $\varepsilon>0$, by condition (B) and the quasi-continuity of f, there is an open set G_ε such that $M_i(G_\varepsilon)<\varepsilon$ and both $\Phi(\nu, y)$ and f are finite and continuous as functions on $Y-G_\varepsilon$. Write

$$N_\varepsilon=\{y\in Y-G_\varepsilon;\ \Phi(\nu, y)<f(y)\} \qquad \text{and} \qquad B_\varepsilon=(N_\varepsilon\cup G_\varepsilon)\cap G\ .$$

Then $N\subset B_\varepsilon\subset N\cup G_\varepsilon$ and $N_\varepsilon\cup G_\varepsilon$ is open, because $Y-(N_\varepsilon\cup G_\varepsilon)=\{y\in Y-G_\varepsilon;\ \Phi(\nu, y)\geqq f(y)\}$ is closed. It follows that B_ε is open, $M_e(N-B_\varepsilon)=M_e(\phi)=0$ and $M_e(B_\varepsilon-N)\leqq M(G_\varepsilon)<\varepsilon$. Evidently $M_i(B_\varepsilon)=M_e(B_\varepsilon)$. By Lemma 6, we obtain $M_e(N)=M_i(N)=0$.

Now we can easily prove

THEOREM 9. *If one of the following conditions (a) and (b) is satisfied, then $\gamma_i(G)=\gamma_e(G)$ for every open set G:*
(a) *Either condition (H_1) or condition (H_2) is satisfied and any open set in Y is a K_σ-set.*
(b) *Condition (B) is satisfied and f is quasi-continuous.*

§8. Summary from §5 to §7

We shall sum up our results.

THEOREM 10. *We have the relation*

$$M_i(A)=\gamma_i(A)\leqq\gamma_e(A)=M_e(A)$$

under the following hypotheses:
(1) *Condition (A).*
(2) *Either condition (H_1) or condition (H_2).*
(3) *g is positive and lower semicontinuous.*

(4) *Either* (4–1) *any open set in* Y *is a* K_σ*-set or* (4–2) *condition* (B) *and* f *is quasi-continuous.*

PROOF. By means of Theorems 6, 8 and 9, we have

$$M_e(A)\leqq\gamma_e(A)\leqq\gamma_e(G)=\gamma_i(G)=M_i(G)$$

for any open set $G\supset A$. Thus $M_e(A)=\gamma_e(A)$.

§9. Convergence theorem

We shall give

THEOREM 11.[9] *Assume conditions* (A) *and* (B) *and either condition* (H_1) *or condition* (H_2). *If* $\{\nu_n\}$ *is a sequence of measures on* X *which converges vaguely to* ν_0, *then we have*

$$\varliminf_{n\to\infty}\Phi(\nu_n, y)\leqq\Phi(\nu_0, y) \qquad q.e. \text{ on } Y .$$

PROOF. Write $h_n(y)=\inf\{\Phi(\nu_k, y); k\geqq n\}$. Then $h_n(y)$ increases to $\varliminf_{n\to\infty}\Phi(\nu_n, y)$. Given $\varepsilon>0$, for each n $(n=1, 2,\cdots)$, we can find, by condition (B), an open set $G_\varepsilon^{(n)}$ such that $M_i(G_\varepsilon^{(n)})<2^{-n}\varepsilon$ and the restriction of $\Phi(\nu_n, y)$ to $Y-G_\varepsilon^{(n)}$ is finite and continuous. If we set $G_\varepsilon=\bigcup_{n=1}^{\infty}G_\varepsilon^{(n)}$, then G_ε is open, $M_i(G_\varepsilon)<\varepsilon$ and the restriction of $\Phi(\nu_n, y)$ to $Y-G_\varepsilon$ is finite and continuous for each n. For a positive number t, we put

$$E_n(t)=\{y\in Y; h_n(y)-\Phi(\nu_0, y)\geqq t\}$$

and

$$E_n(\varepsilon, t)=\{y\in Y-G_\varepsilon; h_n(y)-\Phi(\nu_0, y)\geqq t\} .$$

Since the restriction of $h_n(y)-\Phi(\nu_0, y)$ to $Y-G_\varepsilon$ is upper semicontinuous, $E_n(\varepsilon, t)$ is a compact set. We recall $M_i(E_n(\varepsilon, t))=M_e(E_n(\varepsilon, t))$ by Theorem 1. If $M_i(E_n(\varepsilon, t))$ were positive, by means of condition (A), we could find a unit measure μ such that $S\mu\subset E_n(\varepsilon, t)$ and $\Phi(x, \mu)$ is finite and continuous in X. It would follow that

$$t\leqq\int[h_n(y)-\Phi(\nu_0, y)]d\mu(y)=\int h_n d\mu-\int\Phi(\nu_0, y)d\mu(y)$$

$$\leqq\int\Phi(\nu_k, y)d\mu(y)-\int\Phi(\nu_0, y)d\mu(y)$$

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9) cf. [2], Théorème 7.3.

$$= \int \Phi(x, \mu) d\nu_k(x) - \int \Phi(x, \mu) d\nu_0(x) \qquad (k \geqq n),$$

and the right side tends to 0 as $k \to \infty$. This is a contradiction. Consequently

$$M_e(E_n(\varepsilon, t)) = M_i(E_n(\varepsilon, t)) = 0 \ .$$

It is valid that

$$0 \leqq M_e(E_n(t)) \leqq M_e(E_n(\varepsilon, t) \cup G_\varepsilon) \leqq M_e(E_n(\varepsilon, t)) + M_e(G_\varepsilon)$$
$$= M_i(G_\varepsilon) < \varepsilon \ .$$

Thus $M_e(E_n(t))=0$. By the relation

$$N = \{ y \in Y; \overline{\lim_{n\to\infty}} \Phi(\nu_n, y) - \Phi(\nu_0, y) > 0\} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_n(1/k)$$

and by Theorem 3 we see $M_e(N)=0$. This completes the proof.

THEOREM 12. *Assume conditions (A) and (B) and either condition* (H_1) *or condition* (H_2). *Further assume that g is positive and lower semicontinuous and either that f is quasi-continuous or that any open set in Y is a* K_σ*-set. If* $M_e(A)$ *is finite, then there exists a measure* ν_0 *such that*

$$\Phi(\nu_0, y) \geqq f(y) \ \text{q.e. on } A \qquad \text{and} \qquad M_e(A) = \int g d\nu_0 \ .$$

PROOF. We can find a sequence $\{G_n\}$ of open sets such that $\lim_{n\to\infty} M_i(G_n) = M_e(A)$, $M_i(G_n)$ is finite and $G_n \supset G_{n+1} \supset A$. On account of Theorem 8, we can find a measure ν_n on X such that $\Phi(\nu_n, y) \geqq f(y)$ n.e. on $G_n \supset A$ and $M_i(G_n) = \int g d\nu_n$. It is valid that $\Phi(\nu_n, y) \geqq f(y)$ q.e. on G_n (see Lemma 7 and the proof of Theorem 4). Since $s = \inf\{g(x); x \in X\} > 0$, the total masses $\nu_n(X)$ are bounded. The rest of the proof is carried out in the same way as that of Theorem 7. We have only to note that we use Theorem 11 instead of Lemma 5.

THEOREM 13. *Let* $\{A_n\}$ *be an increasing sequence of arbitrary sets and* $A = \bigcup_{n=1}^{\infty} A_n$. *Then, under the same assumptions as in Theorem 12, we have*

$$M_e(A) = \lim_{n\to\infty} M_e(A_n) \ .$$

PROOF. Since $M_e(A_n) \leqq M_e(A_{n+1}) \leqq M_e(A)$, it holds that $\lim_{n\to\infty} M_e(A_n) \leqq M_e(A)$.

It is enough to show the converse inequality in the case that $\lim_{n\to\infty} M_e(A_n)$ is finite. For each n, by Theorem 12, we can find a measure ν_n such that

$$M_e(A_n)=\int g d\nu_n \quad \text{and} \quad \Phi(\nu_n, y)\geqq f(y) \qquad \text{q.e. on } A_n .$$

Since $s=\inf\{g(x);\ x\in X\}>0$ and $s\nu_n(X)\leqq\int g d\nu_n\leqq\lim_{n\to\infty} M_e(A_n)<\infty$, the total masses $\nu_n(X)$ are bounded. We choose a subsequence of $\{\nu_n\}$ which converges vaguely to a measure ν_0. We shall denote it again by $\{\nu_n\}$. By means of Theorem 11, we deduce $\Phi(\nu_0, y)\geqq f(y)$ q.e. on A and hence $\nu_0\in\Gamma_A^e$. Since g is lower semicontinuous, we have by Theorem 10

$$M_e(A)=\gamma_e(A)\leqq\int g d\nu_0\leqq\underline{\lim}_{n\to\infty}\int g d\nu_n\leqq\lim_{n\to\infty} M_e(A_n) .$$

§ 10. Equality $M_i(A)=M_e(A)$ for analytic sets

Because of Theorems 1 and 13 we can apply Choquet's theorem.[10)]
Thus we have

THEOREM 14.[11)] *Under the same assumptions as in Theorem* 12, *it is valid that* $M_i(A)=M_e(A)$ *for every analytic set.*

Fuglede proved this theorem in case $g=1$ and $f=1$.

§ 11. Comparison with Fuglede's capacity

We shall mention the relation between $M_i(A)$ ($M_e(A)$ resp.) and cap_*A (cap^*A resp.). In this section, we assume that f is a universally measurable function. First, we remark that there is no general relation between them. In fact, $M(K)=0<\infty=\mathrm{cap}\,K$ in Example 1 and $M(K)=1>0=\mathrm{cap}\,K$ in Example 4.

However we obtain

THEOREM 15. *Assume that g is bounded from above and let α (β resp.) be a positive upper bound of g (f resp.). Then we have*

$$M_i(A)\leqq\alpha\beta(\mathrm{cap}_*A) \qquad \textit{and} \qquad M_e(A)\leqq\alpha\beta(\mathrm{cap}^*A) .$$

PROOF. It is enough to show $M(K)\leqq\alpha\beta(\mathrm{cap}\ K)$. By the relation

$$\mathscr{M}_K\subset\{\mu;\ S\mu\subset K \text{ and } \Phi(x,\mu)\leqq\alpha \text{ on } X\}=\mathscr{F}_K ,$$

10) [1], Théorème 30.1.
11) cf. [2], Théorème 7.8.

we have

$$M(K) \leqq \sup\left\{\int f d\mu;\ \mu \in \mathscr{F}_K\right\} \leqq \beta \sup\{\mu(Y);\ \mu \in \mathscr{F}_K\} = \alpha\beta(\mathrm{cap}\ K) \ .$$

THEOREM 16. *Let* $s = \inf\{g(x);\ x \in X\}$ *and* $t = \inf\{f(y);\ y \in Y\}$ *be positive. Then we have*

$$st(\mathrm{cap}_* A) \leqq M_i(A) \qquad \textit{and} \qquad st(\mathrm{cap}^* A) \leqq M_e(A) \ .$$

PROOF. It is sufficient to prove $st(\mathrm{cap}\, K) \leqq M(K)$. For any measure μ such that $S\mu \subset K$ and $\Phi(x, \mu) \leqq 1$ on X, we have $\Phi(x, s\mu) \leqq s \leqq g(x)$ on X and hence

$$M(K) \geqq \int f d(s\mu) \geqq st\mu(Y) \ .$$

Thus we have $M(K) \geqq st(\mathrm{cap}\ K)$.

THEOREM 17. *Let* $s = \inf\{g(x);\ x \in X\} > 0$ *and* $f > 0$. *Then* $M_i(A) = 0$ *implies* $\mathrm{cap}_* A = 0$.

PROOF. For any measure such that $S\mu \subset A$ and $\Phi(x, \mu) \leqq 1$ on X, we have

$$\Phi(x, s\mu) \leqq s \leqq g(x) \qquad\qquad \text{on } X$$

and hence

$$M_i(A) \geqq s\int f d\mu \ .$$

Since $f > 0$, $M_i(A) = 0$ implies $\mu = 0$. Thus $\mathrm{cap}_* A = 0$.

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