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# *On a Capacitability Problem Raised in Connection with the Gauss Variational Problem*

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## § 1. Introduction with definitions and problem setting

In a locally compact Hausdorff space, there are many ways to consider a set function for compact sets which is similar to the capacity in the classical sense. Starting from such a set function, we can define an inner quantity and an outer quantity. The problem of capacitability is to discuss when they coincide. A very useful tool is the general theory of capacitability which was estabilished by G. Choquet [2].

In this paper we shall examine the capacitability problem in relation to the Gauss variational problem. More precisely, let $\Omega$ be a locally compact Hausdorff space and $\Phi(x, y)$ be a lower semicontinuous function on $\Omega\times\Omega$. Throughout this paper, we shall assume that $\Phi$ takes values in $[0, +\infty]$. A measure $\mu$ will be always a non-negative Radon measure and $S\mu$ the support of $\mu$. The potential of $\mu$ is defined by

$$\Phi(x, \mu)=\int\Phi(x, y)d\mu(y)$$

and the mutual energy of $\mu$ and $\nu$ is defined by

$$(\nu, \mu)=\int\Phi(x, \mu)d\nu(x).$$

We call $(\mu, \mu)$ simply the energy of $\mu$. Let $\mathscr{E}$ be the class of all measures with finite energy and put

$$\mathscr{E}_A=\{\mu;\ \mu\in\mathscr{E},\ S\mu \text{ is compact and } S\mu\subset A\}.$$

We note that each measure in $\mathscr{E}_\Omega$ has a compact support and hence $\mathscr{E}_\Omega\neq\mathscr{E}$ in general. The kernel $\Phi$ is assumed, unless otherwise stated, to be *of positive type*, i.e. $\Phi(x, y)=\Phi(y, x)$ for all $x, y\in\Omega$ and $(\mu, \mu)+(\nu, \nu)-2(\nu, \mu)\geqq 0$ for all $\mu, \nu\in\mathscr{E}$. The pseudo-metric $\|\mu-\nu\|=[(\mu, \mu)+(\nu, \nu)-2(\nu, \mu)]^{1/2}$ defines *the strong topology* in $\mathscr{E}$.

A class of measures is called *strongly complete* if any strong Cauchy net in the class converges strongly to an element of the class.

We shall recall the quantities which are related to the capacity and were used by M. Ohtsuka [7]. For a nonzero measure $\mu$, put $V(\mu)=\sup\{\Phi(x, \mu);\ x\in S\mu\}$ and for a nonempty set $A$ define $\mathscr{U}_A$ by $\{\mu;\ S\mu$ is compact, $S\mu\subset A$ and

$\mu(\Omega)=1\}$. Set $V_i(A)=\inf\{V(\mu);\ \mu\in\mathcal{U}_A\}$ if $A\neq\phi$ and $V_i(\phi)=\infty$ ($\phi$ denotes the empty set). Define $V_e(A)$ by $\sup\{V_i(G);\ G$ is open and $G\supset A\}$. We shall say that a property holds *n.e.*, or *nearly everywhere* (*q.e.*, or *quasi everywhere* resp.) on a set $A$ if the $V_i$-value ($V_e$-value resp.) of the exceptional set in $A$ is infinite.

Let $f$ be an extended real valued function on $\Omega$ which is universally measurable[1] and integrable with respect to any measure with compact support and finite energy (i.e., any measure in $\mathcal{E}_\Omega$). The problem of minimizing the expression

$$I(\nu)=(\nu,\nu)-2\int f\,d\nu$$

for $\nu\in\mathcal{E}_A$ is called *the (unconditional) inner Gauss variational problem.* We put

$$I_A^i=I_A^i(f)=\inf\{I(\nu);\ \nu\in\mathcal{E}_A\}.$$

It is easily seen that $I_A^i(f)=\inf\{I_K^i(f);\ K$ is compact and $K\subset A\}$. As *the (unconditional) outer Gauss variational problem*, we discuss the quantity

$$I_A^e(f)=\sup\{I_G^i(f);\ G \text{ is open and } G\supset A\}.$$

REMARK 1. In case $f=1$, the following relations are well-known: $I_A^i(1)=-V_i(A)^{-1}$, $I_A^e(1)=-V_e(A)^{-1}$.

Next, we introduce two classes of measures:

$$\Gamma_A^i(f)=\{\nu\in\mathcal{E};\ \Phi(x,\nu)\geqq f(x) \text{ n.e. on } A\}$$

and

$$\Gamma_A^e(f)=\{\nu\in\mathcal{E};\ \Phi(x,\nu)\geqq f(x) \text{ q.e. on } A\}.$$

B. Fuglede [3] considered these classes in case $f=1$. If there is no confusion from the context, we shall denote $I_A^i(f)$, $\Gamma_A^i(f)$ etc. by $I_A^i$, $\Gamma_A^i$ etc. We set

$$c_f^i(A)=\inf\{(\nu,\nu);\ \nu\in\Gamma_A^i\} \quad \text{if } \Gamma_A^i\neq\phi$$

and

$$c_f^i(A)=\infty \qquad\qquad \text{if } \Gamma_A^i=\phi.$$

We define $c_f^e(A)$ for $\Gamma_A^e$ in the same way.

In this paper, we shall be concerned with the following four problems:

*Problem* I. *Is it valid that* $-I_A^i=c_f^i(A)$ *for any set?*

*Problem* II. *Is it valid that* $-I_A^e=c_f^e(A)$ *for any set?*

*Problem* III. *For what set* $A$ *is it valid that* $c_f^i(A)=c_f^e(A)$*?*

*Problem* IV. *For what set* $A$ *is it valid that* $I_A^i=I_A^e$*?*

The last two problems are capacitability problems.

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1) A function is universally measurable if it is measurable with respect to all Radon measures.

B. Fuglede [3], M. Kishi [4] and the author [8] examined these problems in case $f=1$, and M. Ohtsuka [7] examined Problem IV in case $f$ is upper semicontinuous. In case $f$ is a potential of a measure with finite energy, S. Ogawa [5; 6] studied Problem IV.

In the present paper, we investigate our problem under the assumptions that $\Phi$ is of positive type and and often that $\mathscr{E}$ is strongly complete. Answers to our problem will be given in the following cases: the case where $f$ is upper semicontinuous (§4), the case where $f$ is lower semicontinuous (§5), the case where $f$ is a potential of a measure with finite energy (§6) and the case where $f$ is quasi upper semicontinuous (§7).

## § 2. General theory

In this section, we assume that $f$ is a universally measurable function and study properties of $c_f^i(A)$, $I_A^i$, etc. as set functions and their general relation. First we give the following useful propositions obtained by Ohtsuka.

Proposition 1.[2] *Let $\{B_n\}$ be a sequence of universally measurable sets (i.e., measurable for every measure on $\Omega$) and $A$ be an arbitrary set. Then we have* $V_i(\bigcup_n B_n \cap A)^{-1} \leqq \sum_n V_i(B_n \cap A)^{-1}$.

Proposition 2.[3] *Let $\{A_n\}$ be a sequence of arbitrary sets. Then we have* $V_e(\bigcup_n A_n)^{-1} \leqq \sum_n V_e(A_n)^{-1}$.

Corollary. *Let $A_1$ and $A_2$ be arbitrary sets. If $V_e(A_2)=\infty$, then it holds that $V_e(A_1-A_2)=V_e(A_1)=V_e(A_1 \cup A_2)$.*

Proposition 3.[4] *For any compact set $K$, we have $V_i(K)=V_e(K)$.*

We shall prove

Lemma 1. *Let $\{B_j\}$ $(j=1, 2, \cdots, n)$ be a family of universally measurable sets and let $B=\bigcup_{j=1}^n B_j$. Then we have*

$$c_f^i(B)^{1/2} \leqq \sum_{j=1}^n c_f^i(B_j)^{1/2}.$$

Proof. We may assume that each $c_f^i(B_j)$ is finite. Given $\varepsilon>0$, we can find a measure $\nu_j$ for each $j$ such that $\|\nu_j\|<c_f^i(B)^{1/2}+2^{-j}\varepsilon$ and $\Phi(x, \nu_j) \geqq f(x)$ n.e. on $B_j$. Let $N_j=\{x \in B_j;\ \Phi(x, \nu_j)<f(x)\}$, $\nu=\sum_{j=1}^n \nu_j$ and $N=\{x \in B;\ \Phi(x, \nu)<f(x)\}$. Then $N \subset \bigcup_{j=1}^n N_j$ and $V_i(N_j)=\infty$. Since each $N_j$ is universally mea-

2) [7], Proposition 1, p. 139.
3) [7], Proposition 2, p. 140.
4) [7], Theorem 1.14, p. 207.

surable, we see by Proposition 1

$$V_i(N) \geqq V_i(\bigcup_{j=1}^{n} N_j) = \infty.$$

Hence $\nu$ belongs to $\Gamma_B^i$ and it follows that

$$c_f^i(B)^{1/2} \leqq \|\nu\| \leqq \sum_{j=1}^{n} \|\nu_j\| < \sum_{j=1}^{n} c_f^i(B_j)^{1/2} + \varepsilon.$$

By the arbitrariness of $\varepsilon$, we obtain the desired inequality.

LEMMA 2. *Assume that $f$ is integrable with respect to any measure in $\mathscr{E}_\Omega$ and $-I_A^i = c_f^i(A)$ for any open set $A$. Let $\{G_n\}$ be a sequence of open sets and $G = \bigcup_{n=1}^{\infty} G_n$. Then we have*

$$[-I_G^i]^{1/2} \leqq \sum_{n=1}^{\infty} [-I_{G_n}^i]^{1/2}.$$

PROOF. We may suppose that $-I_G^i$ is positive. For any positive number $\alpha$ smaller than $[-I_G^i]^{1/2}$, there is a compact set $K$ such that $K \subset G$ and $[-I_K^i]^{1/2} > \alpha$. Since $K$ is compact, it is covered by a finite subfamily $\{G_j\}$ $(j=1, 2, \cdots, n_0)$. Then by Lemma 1 and by our assumption that $-I_A^i = c_f^i(A)$ for open sets, it holds that

$$\alpha < [-I_K^i]^{1/2} \leqq [-I_{G_0}^i]^{1/2} \leqq \sum_{j=1}^{n_0} [-I_{G_j}^i]^{1/2}$$

$$\leqq \sum_{n=1}^{\infty} [-I_{G_n}^i]^{1/2}, \text{ where } G_0 = \bigcup_{j=1}^{n_0} G_j.$$

This completes the proof.

LEMMA 3. *Let $\{A_n\}$ be a sequence of arbitrary sets and $A = \bigcup_{n=1}^{\infty} A_n$. Under the same assumptions as in Lemma 2, we have*

$$[-I_A^e]^{1/2} \leqq \sum_{n=1}^{\infty} [-I_{A_n}^e]^{1/2}.$$

PROOF. We may suppose that each $I_{A_n}^e$ is finite. Given $\varepsilon > 0$, for each $n$, there is an open set $G_n$ such that $A_n \subset G_n$ and $[-I_{G_n}^i]^{1/2} < [-I_{A_n}^e]^{1/2} + 2^{-n}\varepsilon$. Writing $G = \bigcup_{n=1}^{\infty} G_n$, $G$ is open and contains $A$. It follows from Lemma 2 that

$$[-I_A^e]^{1/2} \leqq [-I_G^i]^{1/2} \leqq \sum_{n=1}^{\infty} [-I_{G_n}^i]^{1/2} < \sum_{n=1}^{\infty} [-I_{A_n}^e]^{1/2} + \varepsilon$$

By the arbitrariness of $\varepsilon$, we have the desired inequality.

LEMMA 4. *Let $f$ be bounded from above on any compact set and $N$ be a* *relatively compact set. Then* $V_e(N)=\infty$ *implies* $I_N^e=0$.

Proof. If we put $W_i(A)=\inf\{(\nu,\nu);\ \nu\in\mathscr{U}_A\}$ for $A\neq\phi$ and $W_i(\phi)=\infty$, then it is well-known that $W_i(A)=V_i(A)$.[5] There is a relatively compact open set $G_0$ such that $N\subset G_0$ and $V_i(G)>0$ for any open set $G$, $N\subset G\subset G_0$. For any nonzero measure $\mu\in\mathscr{E}_G$, it holds that $\nu=\mu/\mu(G)\in\mathscr{U}_G$ and

$$I(\mu)=(\mu,\mu)-2\int f\,d\mu\geqq[\mu(G)]^2(\nu,\nu)-2[\sup\{f(x);\ x\in G_0\}]\mu(G)$$

$$\geqq[\mu(G)]^2V_i(G)-2\mu(G)\beta\geqq-\beta^2/V_i(G),$$

where $\beta$ is a positive upper bound of $f$ on $G_0$. This inequality is also true for zero measure and hence

$$I_G^i\geqq-\beta^2/V_i(G).$$

Therefore $0\geqq I_N^e\geqq-\beta^2/V_e(N)=0$. This completes the proof.

In the above proof, we have shown

Corollary. *Let $f$ be bounded from above in $\Omega$ and $\beta$ be a positive upper bound of $f$. Then for any set $A$, we have* $-I_A^e\leqq\beta^2/V_e(A)$.

Lemma 5. *Assume that $f$ is bounded from above on any compact set, that $-I_A^i=c_f^i(A)$ for any open set and that $\Omega$ is $\sigma$-compact. Then $V_e(N)=\infty$ implies $I_N^e=0$.*

Proof. By our assumption, $\Omega$ can be represented as a countable union of compact sets $\{K_n\}$. For each $n$, $N_n=N\cap K_n$ is relatively compact and $V_e(N_n)\geqq V_e(N)=\infty$. From Lemma 4, it follows that $I_{N_n}^e=0$. On account of Lemma 3, we conclude $I_N^e=0$.

Combining above lemmas, we obtain the following useful lemma.

Lemma 6. *Under the same assumptions as in Lemma 5, it is valid that $I_{A-N}^e=I_A^e$ for any set $A$ if $V_e(N)=\infty$.*

Next we give a relation between $I_A^i$ and $c_f^i(A)$ in the general case.

Theorem 1. *Let $f$ be integrable with respect to any measure in $\mathscr{E}_\Omega$. Then we have $-I_A^i\leqq c_f^i(A)$ for any set $A$.*

Proof. Let $\nu$ be any measure of $\Gamma_A^i$, i.e. $\nu\in\mathscr{E}$ and $\Phi(x,\nu)\geqq f(x)$ n.e. on $A$. Integrating both sides by $\lambda\in\mathscr{E}_A$, we have $(\lambda,\nu)\geqq\int f\,d\lambda$ (see footnote 5) and

$$(\nu,\nu)+I(\lambda)\geqq(\nu,\nu)+(\lambda,\lambda)-2(\lambda,\nu)\geqq0.$$

It follows that $(\nu,\nu)\geqq-I(\lambda)$ for any $\nu\in\Gamma_A^i$ and $\lambda\in\mathscr{E}_A$. Consequently

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5) [7], p. 222. By this fact, we see that a measure with finite energy has no mass on any universally measurable set with infinite $V_i$-value.

$c_f^i(A) \geqq -I_A^i$.

This theorem is not always true if $\Phi$ is not of positive type. To see this we give

EXAMPLE 1. Let $\Omega = A = \{x_1, x_2\}$, $f=1$, and $\Phi(x_1, x_1) = \Phi(x_2, x_2) = 0$ and $\Phi(x_1, x_2) = \Phi(x_2, x_1) = 1$. It is easily verified that $I_A^i(A) = -\infty$ and $c_f^i(A) = 2$ and that $\Phi$ is not of positive type.

## § 3. Measures whose energy equals $c_f^i(A)$ or $c_f^e(A)$

We assume that $f$ is universally measurable. The following lemmas are well-known:

LEMMA 7.[6] *Let $\mathscr{F}$ be a nonempty convex set in $\mathscr{E}$ and $\nu_0$ be a measure in $\mathscr{F}$ such that $\|\nu_0\| = \inf\{\|\nu\|;\ \nu \in \mathscr{F}\}$. Then we have $\|\nu - \nu_0\|^2 \leqq \|\nu\|^2 - \|\nu_0\|^2$ for any $\nu \in \mathscr{F}$.*

LEMMA 8.[7] *If $\mu_n$ converges strongly to $\mu_0$, then we have*

$$\varliminf_{n\to\infty} \Phi(x, \mu_n) \leqq \Phi(x, \mu_0) \qquad \text{n.e. in } \Omega.$$

LEMMA 9.[8] *Assume that any open set in $\Omega$ is a $K_\sigma$-set, i.e. a countable union of compact sets and that $V_i(A) = V_e(A)$ for any $K_\sigma$-set. If $\mu_n$ converges strongly to $\mu_0$, then we have*

$$\varliminf_{n\to\infty} \Phi(x, \mu_n) \leqq \Phi(x, \mu_0) \qquad \text{q.e. in } \Omega.$$

THEOREM 2. *Assume that $\mathscr{E}$ is strongly complete. In case $c_f^i(A)$ is finite, there exists a measure $\mu_0$ in $\Gamma_A^i$ such that $c_f^i(A) = \|\mu_0\|^2$.*

PROOF. Since $\Gamma_A^i$ is not empty, there is a sequence $\{\mu_n\}$ in $\Gamma_A^i$ such that $\|\mu_n\|^2$ tends to $c_f^i(A)$ as $n \to \infty$. Because $(\mu_n + \mu_m)/2$ belongs to $\Gamma_A^i$, we have

$$\begin{aligned} \|\mu_n - \mu_m\|^2 &= 2(\|\mu_n\|^2 + \|\mu_m\|^2) - 4\|(\mu_n + \mu_m)/2\|^2 \\ &\leqq 2(\|\mu_n\|^2 + \|\mu_m\|^2) - 4c_f^i(A). \end{aligned}$$

Therefore $\{\mu_n\}$ is a strong Cauchy sequence in $\mathscr{E}$ and converges strongly to some measure $\mu_0 \in \mathscr{E}$ by our assumption. It follows from Lemma 8 that $\mu_0$ belongs to $\Gamma_A^i$ and hence $\|\mu_0\|^2 = c_f^i(A)$.

THEOREM 3. *Assume that $\mathscr{E}$ is strongly complete and that any open set in $\Omega$ is a $K_\sigma$-set. In case $c_f^e(A)$ is finite, there exists a measure $\mu_0$ in $\Gamma_A^e$ such that*

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6) [3], Lemma 4.1.1, p. 174.

7) [3], Lemma 3.2.4, p. 165 or [7], Theorem 1.18, p. 210.

8) [3], Lemma 4.3.3, p. 182 or [7], Lemma 3.7, p. 306.

$c_f^e(A)=\|\mu_0\|^2$.

On account of the corollary of Theorem 7 below, we can prove this theorem by the same reasoning as in Theorem 2, by using Lemma 9 instead of Lemma 8.

THEOREM 4. *Assume that $\mathscr{E}$ is strongly complete, that $f$ is integrable with respect to any measure in $\mathscr{E}_\Omega$ and that $-I_K^i=c_f^i(K)$ for every compact set $K$. Then we have $-I_A^i=c_f^i(A)$ for any set $A$.*

*If we assume furthermore that $c_f^i(G)=c_f^e(G)$ for any open set $G$, then we have the following relation*:

$$-I_A^i=c_f^i(A)\leqq c_f^e(A)\leqq -I_A^e.$$

PROOF. In order to obtain $-I_A^i=c_f^i(A)$, it is sufficient to show $-I_A^i\geqq c_f^i(A)$ in case $I_A^i>-\infty$ because of Theorem 1. Since $I_A^i=\inf\{I_K^i;\ K$ is compact and $K\subset A\}>-\infty$, there is a sequence $\{K_n\}$ such that $K_n$ is compact, $K_n\subset K_{n+1}\subset A$ and $I_{K_n}^i$ tends to $I_A^i$. By Theorem 2 and our assumption $-I_K^i=c_f^i(K)$, we can find a measure $\nu_n$ in $\Gamma_{K_n}^i$ such that $\|\nu_n\|^2=c_f^i(K_n)=-I_{K_n}^i\leqq -I_A^i<\infty$. We see by Lemma 7 that $\{\nu_n\}$ is a strong Cauchy sequence in $\mathscr{E}$. Since $\mathscr{E}$ is strongly complete, $\nu_n$ converges strongly to some measure $\nu_0$. Then we have $\Phi(x,\nu_0)\geqq f(x)$ n.e. on $\bigcup_{n=1}^{\infty}K_n$ by Lemma 8. We shall show $\Phi(x,\nu_0)\geqq f(x)$ n.e. on $A$. In fact, if we deny this, then there would be a compact set $K_0$ such that $V_i(K_0)<\infty$ and $\Phi(x,\nu_0)<f(x)$ on $K_0$. Writing $K_n'=K_n\cup K_0$, we see that $I_{K'_n}^i$ tends to $I_A^i$ by the inequality $I_A^i\leqq I_{K'_n}^i\leqq I_{K_n}^i$. If $\nu_0'$ is determined by the sequence $\{K_n'\}$ in the same way as $\nu_0$ was determined by the sequence $\{K_n\}$, then we have $\|\nu_0\|^2=\|\nu_0'\|^2$ and $\|\nu_0-\nu_0'\|=0$, or equivalently,[9] $\Phi(x,\nu_0)=\Phi(x,\nu_0')$ n.e. in $\Omega$. Since $\Phi(x,\nu_0')\geqq f(x)$ n.e. on $\bigcup_{n=1}^{\infty}K_n'\supset K_0$, we have $\Phi(x,\nu_0)\geqq f(x)$ n.e. on $K_0$. This is a contradiction. Thus $\nu_0\in\Gamma_A^i(A)$ and $c_f^i(A)\leqq\|\nu_0\|^2=\lim_{n\to\infty}\|\nu_n\|^2=-I_A^i$. Next we shall show $c_f^e(A)\leqq -I_A^e$. For any open set $G\supset A$, it holds by our assumption that $c_f^e(A)\leqq c_f^e(G)=c_f^i(G)=-I_G^i$. Hence $c_f^e(A)\leqq -I_A^e$. This completes the proof.

## § 4. The case where $f$ is upper semicontinuous

Now we discuss the problems raised in §1. In this section, we assume that *$\mathscr{E}$ is strongly complete* and that, except in Theorem 10 below, *$f$ is upper semicontinuous and does not take the value $+\infty$.* First we obtain

THEOREM 5. *It holds that $c_f^i(A)=c_f^e(A)$ for every $K_\sigma$-set $A$.*

PROOF. It is enough to show that $\Phi(x,\nu)\geqq f(x)$ n.e. on $A$ implies

9) [3], Lemma 3.2.1, p. 164.

$\Phi(x, \nu) \geqq f(x)$ q.e. on $A$. Write $N=\{x \in A;\ \Phi(x, \nu)<f(x)\}$. Then $N$ can be represented as a countable union of compact sets $\{K_n\}$, because $f$ is upper semicontinuous and $A$ is a $K_\sigma$-set. Since $V_e(K_n)=V_i(K_n) \geqq V_i(N)=\infty$ by Proposition 3, it follows from Proposition 2 that $V_e(N)=\infty$. Namely $\Phi(x, \nu) \geqq f(x)$ q.e. on A.

Next, we give an answer to Problem I:

THEOREM 6. *It is valid that* $-I_A^i=c_f^i(A)$ *for any set* $A$.

PROOF. It suffices to show $-I_K^i \geqq c_f^i(K)$ for every compact set $K$ in case $I_K^i>-\infty$, because of Theorems 1 and 4. Assume that $K$ is an arbitrary compact set with $I_K^i>-\infty$. We may suppose that $\sup\{f(x);\ x \in K\}$ is positive. In fact, otherwise, $I_K^i=0$ and $\mu=0$ satisfies the inequality $\Phi(x, \mu) \geqq f(x)$ on $K$ and hence $c_f^i(K)=0$. In case $V_i(K)$ is positive, it is a known result that there exists a measure $\mu_K \in \mathscr{E}_K$ such that $I_K^i=I(\mu_K)$, $\Phi(x, \mu_K) \leqq f(x)$ on $S\mu_K$ and $\Phi(x, \mu_K) \geqq f(x)$ n.e. on $K$; see, for instance, [7]. Then it follows that $-I_K^i=(\mu_K, \mu_K)=c_f^i(K)$. Next we shall show the inequality in case $V_i(K)=0$. Write $F=\{x \in K; f(x)>0\}$ and $F_n=\{x \in K; f(x) \geqq 1/n\}$. Then $\{F_n\}$ is an increasing sequence of compact sets whose union equals $F$. We shall show that $V_i(F_n)$ is positive for all $n$. In fact, if we deny this, there would be an integer $n_0$ and a measure $\mu_0 \in \mathscr{U}_{F_{n_0}}$ such that $(\mu_0, \mu_0)=W_i(F_{n_0})=V_i(F_{n_0})=0$. It follows that $I_K^i \leqq I_{F_{n_0}}^i \leqq I(t\mu_0) \leqq -2t/n_0$ for any positive number $t$ and hence $I_K^i=-\infty$. This contradicts our assumption $I_K^i>-\infty$. Since $V_i(F_n)>0$ for each $n$, we can find a measure $\mu_n \in \mathscr{E}_{F_n}$ such that $I_{F_n}^i=-c_f^i(F_n)=\|\mu_n\|^2$ and $\Phi(x, \mu_n) \geqq f(x)$ n.e. on $F_n$. We see that $\{\mu_n\}$ is a strong Cauchy sequence by Lemma 7 and therefore it converges strongly to some measure $\mu^* \in \mathscr{E}$ by the strong completeness of $\mathscr{E}$. Then we have $\lim_{n\to\infty}(-I_{F_n}^i)=\|\mu^*\|^2$ and $\Phi(x, \mu^*) \geqq f(x)$ n.e. on $F$ by Lemma 8. Since $f \leqq 0$ on $K-F$, it is valid that $\Phi(x, \mu^*) \geqq f(x)$ n.e. on $K$. Therefore $\|\mu^*\|^2 \geqq c_f^i(K)$. It is always valid that $I_K^i \leqq I_{F_n}^i$ and hence $I_K^i \leqq \lim_{n\to\infty} I_{F_n}^i$. It follows that $-I_K^i \geqq \lim_{n\to\infty}(-I_{F_n}^i)=\|\mu^*\|^2 \geqq c_f^i(K)$.

In the above proof, we see easily

COROLLARY.[10] *In case* $I_K^i>-\infty$, *there exists a measure* $\mu^*$ *such that* $I_K^i=-(\mu^*, \mu^*)$ *and* $\Phi(x, \mu^*) \geqq f(x)$ *n.e. on* $K$.

Thus Problem I is solved in this case. But for Problem II, we have a negative answer in general. In fact, we give

EXAMPLE 2. Let $\Omega$ be the real line, $K=\{x=0\}$, $\Phi(x, y)=|x|\,|y|$ and $\mu$ be the unit point measure at $x=1$. We take $f(x)=\Phi(x, \mu)=|x|$. Then $I_K^i=c_f^i(K)=c_f^e(K)=0$ and $I_K^e=-1$. In fact, let $G$ be any open set containing $K$. For any measure $\nu$ of $\mathscr{E}_G$, put $a=a(\nu)=\int |x|\, d\nu(x)$. Then $I(\nu)=a^2-2a$ with

10) cf. [7], Theorem 3.35, p. 342.

$0 \leqq a < \infty$. We see $I_G^i = \inf\{I(\nu);\ \nu \in \mathscr{E}_G\} = -1$. Thus $I_K^e = -1$.

However, we can prove

THEOREM 7. *Assume that any open set in $\Omega$ is a $K_\sigma$-set. If we assume either $f>0$ or $\Phi>0$, then we have $-I_A^e = c_f^e(A)$ for any set $A$.*

PROOF. Since we obtain $-I_A^e \geqq c_f^e(A)$ by Theorem 4, let us show that $\Phi(x, \nu) \geqq f(x)$ q.e. on A implies $-I_A^e \leqq (\nu, \nu)$. Let $N = \{x \in A;\ \Phi(x, \nu) < f(x)\}$. Then $V_e(N) = \infty$. For any numbers $t$, $0<t<1$, and $s>1$, put $G_{ts} = \{x \in \Omega;\ s\Phi(x, \nu) > tf(x)\}$. By our assumption that $f>0$ or $\Phi>0$, $G_{ts}$ contains $A-N$. Since $f$ is upper semicontinuous, $G_{ts}$ is an open set. Because $(s/t)\nu$ belongs to $\Gamma_{G_{ts}}^i$, it follows from Lemma 6 and Theorem 6 that $-I_A^e = -I_{A-N}^e \leqq -I_{G_{ts}}^i = c_f^i(G_{ts}) \leqq (s/t)^2(\nu, \nu)$. Letting $t$ and $s$ tend to 1, we have $-I_A^e \leqq (\nu, \nu)$.

By combining Remark 1 with Theorems 5, 6 and 7, we have

COROLLARY. *It holds that $V_i(A) = V_e(A)$ for any $K_\sigma$-set $A$.*

THEOREM 8. *Assume that any open set in $\Omega$ is a $K_\sigma$-set. Let $\{A_n\}$ be an increasing sequence of arbitrary sets and let $A = \bigcup_{n=1}^{\infty} A_n$. Then it holds that $\lim_{n\to\infty} c_f^e(A_n) = c_f^e(A)$.*

*Consequently, under the same assumptions as in Theorem 7, we have $\lim_{n\to\infty} I_{A_n}^e = I_A^e$.*

PROOF. It suffices to show that $\lim_{n\to\infty} c_f^e(A_n) \geqq c_f^e(A)$ in case $\lim_{n\to\infty} c_f^e(A_n)$ is finite. There is a measure $\nu_n$ such that $c_f^e(A_n) = \|\nu_n\|^2$ and $\Phi(x, \nu_n) \geqq f(x)$ q.e. on $A_n$ by Theorem 3. Since $\Gamma_{A_n}^e \supset \Gamma_{A_{n+1}}^e$, we have by Lemma 7

$$\|\nu_n - \nu_m\|^2 \leqq \|\nu_n\|^2 - \|\nu_m\|^2 \qquad \text{for } n \geqq m.$$

Therefore $\{\nu_n\}$ is a strong Cauchy sequence and converges strongly to a measure $\nu_0 \in \mathscr{E}$. It follows from Lemma 9 that $\Phi(x, \nu_0) \geqq f(x)$ q.e. on $A$ and hence

$$c_f^e(A) \leqq \|\nu_0\|^2 = \lim_{n\to\infty} \|\nu_n\|^2.$$

On account of Theorems 5, 6 and 7, we have $I_K^i = I_K^e$ for any compact set $K$ in case $\Phi>0$ or $f>0$. By this fact and Theorem 8, we can apply Choquet's theorem (Théorème 30.1 in [2]). Thus we have an answer to Problem IV:

THEOREM 9.[11] *If we assume that any open set in $\Omega$ is a $K_\sigma$-set and either $\Phi>0$ or $f>0$, then it is valid that $I_A^i = I_A^e$ for every analytic set $A$.*

Combining this theorem and Remark 1, we have

COROLLARY. *It holds that $V_i(A) = V_e(A)$ for every analytic set $A$.*

Using this result, we have an answer to Problem III:

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11) cf. [7], Theorem 3.48, p. 345.

THEOREM 10. *Assume that $f$ is a Borel measurable function and that any open set in $\Omega$ is a $K_\sigma$-set. Then it is valid that $c_f^i(A)=c_f^e(A)$ for every analytic set $A$.*

REMARK 2. Fuglede [3] called a kernel *consistent* if it is of positive type and any strong Cauchy net converging vaguely to a measure converges strongly to the same measure. Without the strong completeness of $\mathscr{E}$, he obtained a result similar to this corollary. It is restated as follows:

PROPOSITION 4.[12] *If the kernel is consistent and any open set in $\Omega$ is a $K_\sigma$-set, then it holds that $V_i(A)=V_e(A)$ for every analytic set $A$.*

If we use this fact, we have a result similar to Theorem 10:

*Assume that $f$ is a Borel measurable function, that any open set in $\Omega$ is a $K_\sigma$-set and that $\Phi$ is consistent. Then it is valid that $c_f^i(A)=c_f^e(A)$ for every analytic set $A$.*

## § 5. The case where $f$ is lower semicontinuous

In this section, we are interested in the case where $f$ *is lower semicontinuous and does not take the value* $-\infty$. As for Problem I, we have

THEOREM 11. *Assume that $\mathscr{E}$ is strongly complete. Then it holds that $-I_A^i=c_f^i(A)$ for any set $A$.*

PROOF. It is enough to show that $-I_K^i \geqq c_f^i(K)$ for every compact set $K$ in case $I_K^i>-\infty$. Let $D=\{\alpha\}$ be a directed set, and $\{f_\alpha; \alpha\in D\}$ be a net of continuous functions increasing to $f$. For each $\alpha$, there exists a measure $\nu_\alpha\in\Gamma_K^i(f_\alpha)$ such that $c_{f_\alpha}^i(K)=\|\nu_\alpha\|^2$. If $f_\alpha\leqq f_\beta$ then $\Gamma_K^i(f_\alpha)\supset\Gamma_K^i(f_\beta)$ and by Lemma 7

$$\|\nu_\alpha-\nu_\beta\|^2\leqq\|\nu_\beta\|^2-\|\nu_\alpha\|^2.$$

Hence $\{\nu_\alpha\}$ is a strong Cauchy net and converges strongly to some measure $\nu_0$. We see that $\nu_0$ belongs to $\Gamma_K^i(f)$. In fact, put $N=\{x\in K; \Phi(x,\nu_0)<f(x)\}$. If $V_i(N)$ were finite, then there would be a nonzero measure $\mu\in\mathscr{E}_N$. Obviously $(\mu,\nu_0)<\int f d\mu$ and $(\mu,\nu_\alpha)\geqq\int f_\alpha d\mu$ for any $\alpha\in D$. By the latter inequality, we have $\int f d\mu=\lim_{\alpha\in D}\int f_\alpha d\mu\leqq\lim_{\alpha\in D}(\mu,\nu_\alpha)=(\mu,\nu_0)$. This is absurd. It follows that

$$c_f^i(K)\leqq\|\nu_0\|^2=\lim_{\alpha\in D}\|\nu_\alpha\|^2=\lim_{\alpha\in D}c_{f_\alpha}^i(K)=\lim_{\alpha\in D}(-I_K^i(f_\alpha))\leqq-I_K^i(f)$$

This completes the proof.

12) [3], Theorem 4.5, p. 184.

We have a negative answer to Problems II and IV even if $f>0$ and $\Phi>0$. It is shown in

EXAMPLE 3. Let $\Omega$ be the real line, $K=\{|x|\leqq 1\}$, $\Phi=1$ and $f(x)=1$ if $|x|\leqq 1$ and $=2$ if $|x|>1$. Then we see $I_K^i=-1$, $c_f^e(K)=1$ and $I_G^i=-4$ for any open set $G$ containing $K$. Hence $I_K^e=-4$. We observe that in this example $\mathscr{E}$ is strongly complete.

## § 6. The case where $f$ is a potential

We now consider the case where $f$ *is the potential of a fixed nonzero measure* $\mu$ *of* $\mathscr{E}$: $f(x)=\Phi(x, \mu)$. Since it is lower semicontinuous, Problem I is solved as proved in §5. However for Problem II, we have a negative answer as Example 2 shows. The assumption $\Phi>0$ is not sufficient (cf. Theorem 7) to have an affirmative answer either. It is shown in

EXAMPLE 4. Let $\Omega$ be the real line, $K=\{x=0\}$, $\Phi(x, y)=1/|x||y|$ and $\mu$ be the unit point measure at $x=1$. Then $f(x)=1/|x|$ and $c_f^i(K)=c_f^e(K)=0$. Since $I_G^i=-1$ for any open set $G$ containing $K$, we have $I_K^e=-1$.

In order to obtain an affirmative answer in a special case, we shall consider the generalized capacity introduced by Cartan in the classical case. We begin with some preparations.

For any set $A$ and $\nu\in\mathscr{E}_A$,

$$I(\nu)=(\nu, \nu)-2\int f d\nu=(\nu, \nu)-2(\nu, \mu)=\|\nu-\mu\|^2-\|\mu\|^2.$$

Therefore it is valid that $0\geqq I_A^i(f)\geqq-\|\mu\|^2>-\infty$. The following proposition was proved by Cartan [1] in the classical case.

PROPOSITION 5. *Let $\mathscr{F}$ be a nonempty strongly complete convex set in $\mathscr{E}$ and let $\Lambda(\mathscr{F}; \mu)=\{\mu'\in\mathscr{F}; \|\mu'-\mu\|=\inf\{\|\mu-\nu\|; \nu\in\mathscr{F}\}\}$. Then $\Lambda(\mathscr{F}; \mu)\neq\phi$ and $\|\mu_1'-\mu_2'\|=0$ for any $\mu_1', \mu_2'\in\Lambda(\mathscr{F}; \mu)$.*

$\mathscr{F}$ is called *a cone* if $t\nu\in\mathscr{F}$ for any measure $\nu\in\mathscr{F}$ and any number $t\geqq 0$. Like Cartan, we have

LEMMA 10. *Let $\mathscr{F}$ be a nonempty strongly complete convex cone in $\mathscr{E}$. A measure $\mu'\in\mathscr{F}$ belongs to $\Lambda(\mathscr{F}; \mu)$ if and only if $(\mu'-\mu, \nu)\geqq 0$ for all $\nu\in\mathscr{F}$ and $(\mu'-\mu, \mu')=0$.*

LEMMA 11. *Let $K$ be a compact set and assume that $\mathscr{E}_K$ is strongly complete. Then any measure $\mu_K'$ in $\Lambda(\mathscr{E}_K; \mu)$ satisfies $(\mu, \mu_K')=\|\mu_K'\|^2$ and $\Phi(x, \mu_K')\geqq\Phi(x, \mu)$ n.e. on $K$.*

PROOF. Since $\mathscr{E}_K$ is a nonempty strongly complete convex cone in $\mathscr{E}$ by our assumption, $\Lambda(\mathscr{E}_K; \mu)$ is not empty because of Proposition 5. Take any

$\mu'_K \in \Lambda(\mathscr{E}_K; \mu)$. By the above lemma, $(\mu, \mu'_K) = \|\mu'_K\|^2$. We shall show that the set $N$ defined by $\{x \in K; \Phi(x, \mu'_K) < \Phi(x, \mu)\}$ has infinite $V_i$-value. If we deny this, there would be a nonzero measure $\nu \in \mathscr{E}_N \subset \mathscr{E}_K$. Then

$$(\nu, \mu'_K) < (\nu, \mu).$$

This contradicts $\mu'_K \in \Lambda(\mathscr{E}_K; \mu)$ on account of the above lemma.

THEOREM 12. *Let $K$ be a compact set and assume that $\mathscr{E}_K$ is strongly complete. Then we have* $-I^i_K = c^i_f(K) = \|\mu'_K\|^2$.

PROOF. First we recall that $I(\nu) = \|\nu - \mu\|^2 - \|\mu\|^2$. It follows that $I^i_K = \inf I(\nu) = \|\mu'_K - \mu\|^2 - \|\mu\|^2 = -\|\mu'_K\|^2$. We have $-I^i_K = c^i_f(K)$ as shown in the proof of Theorem 11.

*Hereafter in this section we assume, unless otherwise stated, that $\mathscr{E}$ is strongly complete.* We shall define $\mathscr{E}^i_A$ and $\mathscr{E}^e_A$ for any set $A$ like Cartan [1]:

$$\mathscr{E}^i_A = \text{the strong closure of } \{\nu \in \mathscr{E};\ \tilde{\nu}(\Omega - A) = 0\}^{13)}$$

and

$$\mathscr{E}^e_A = \cap \{\mathscr{E}^i_G;\ \text{is open and } G \supset A\}.$$

We observe that $\mathscr{E}^i_A$ is equal to the strong closure of $\cup\{\mathscr{E}_K; K$ is compact and $K \subset A\}$. Since $\mathscr{E}$ is strongly complete by our assumption, $\mathscr{E}^i_A$ and $\mathscr{E}^e_A$ are strongly complete convex cones in $\mathscr{E}$. Given $\mu \in \mathscr{E}$, taking $\mathscr{E}^i_A$ ($\mathscr{E}^e_A$ resp.) for $\mathscr{F}$ in Proposition 5, there exists a measure $\mu^i_A \in \mathscr{E}^i_A (\mu^e_A \in \mathscr{E}^e_A$ resp.) which minimizes $I(\nu)$ for $\nu \in \mathscr{E}^i_A (\mathscr{E}^e_A$ resp.). This extremal measure is not always determined uniquely, but the corresponding values of energy are unique by Proposition 5. We put

$$C^i_\mu(A) = (\mu^i_A, \mu^i_A) \quad \text{and} \quad C^e_\mu(A) = (\mu^e_A, \mu^e_A).$$

We shall study the relation between these quantities and those in the previous sections. As stated at the begining of this section, we take $\Phi(x, \mu)$ for $f(x)$. First we shall prove

THEOREM 13. *Let $f(x) = \Phi(x, \mu)$. If $\mathscr{E}_K$ is strongly complete, then we have*

$$I^i_K = c^i_f(K) = C^i_\mu(K)$$

*for any compact set $K$.*

PROOF. For any $\nu \in \mathscr{E}^i_K$, there is a sequence $\{\nu_n\}$ in $\mathscr{E}_K$ which converges strongly to $\nu$. The measure $\mu'_K$ obtained in Lemma 11 satisfies the following relations: $\mu'_K \in \mathscr{E}_K \subset \mathscr{E}^i_K$, $(\mu'_K - \mu, \nu_n) \geqq 0$ and $(\mu'_K - \mu, \mu'_K) = 0$. We let $n \to \infty$ and see $(\mu'_K - \mu, \nu) \geqq 0$. By the arbitrariness of $\nu$ and Lemma 10, we conclude that $\mu'_K \in \Lambda(\mathscr{E}^i_K; \mu)$ and hence $-I^i_K = \|\mu'_K\|^2 = C^i_\mu(K)$ (Theorem 12).

13) $\tilde{\nu}$ represents the outer measure of $\nu$, i.e. $\tilde{\nu}(A) = \inf\{\nu(G); G$ is open and $G \supset A\}$.

Like in the classical case, we have

LEMMA 12.[14] *Let $\{\mathscr{F}_\alpha;\ \alpha\in D\}$ be an increasing net of nonempty strongly convex cones and $\mathscr{F}$ be the strong closure of $\bigcup_{\alpha\in D}\mathscr{F}_\alpha$. Then $\mu'_\alpha\in\Lambda(\mathscr{F}_\alpha;\ \mu)$ converges strongly to some measure $\mu'\in\Lambda(\mathscr{F};\ \mu)$.*

In this lemma, by taking $D=\{K;\ K$ is compact and contained in $A\}$ and $\{\mathscr{E}^i_K;\ K\in D\}$ for $\{\mathscr{F}_\alpha;\ \alpha\in D\}$, and observing that $\mathscr{E}^i_A$ is also equal to the strong closure of $\cup\{\mathscr{E}^i_K;\ K\in D\}$, we obtain

LEMMA 13. $C^i_\mu(A)=\sup\{C^i_\mu(K);\ K$ is compact and $K\subset A\}$.

Combining this lemma with Theorem 13 and taking account of the equality $I^i_A=\inf\{I^i_K;\ K$ is compact and $K\subset A\}$, we obtain

THEOREM 14. *Let $f(x)=\Phi(x,\ \mu)$. If $\mathscr{E}_K$ is strongly complete for every compact set K, then we have $C^i_\mu(A)=-I^i_A$ for any set A.*

LEMMA 14.[15] *Let $\{\mathscr{F}_\alpha;\ \alpha\in D\}$ be a decreasing net of nonempty strongly closed convex cones in $\mathscr{E}$ and let $\mathscr{F}=\bigcap_{\alpha\in D}\mathscr{F}_\alpha$. Then $\mu'_\alpha\in\Lambda(\mathscr{F}_\alpha;\ \mu)$ converges strongly to some measure $\mu'\in\Lambda(\mathscr{F};\ \mu)$.*

By taking $D=\{G;\ G$ is open and contains $A\}$ and $\{\mathscr{E}^i_G;\ G\in D\}$ for $\{\mathscr{F}_\alpha;\ \alpha\in D\}$ in the above lemma, we see $C^e_\mu(A)=\inf\{C^i_\mu(G);\ G$ is open and $G\supset A\}$. Thus we obtain

LEMMA 15. *If $\mathscr{E}_K$ is strongly complete for every compact set K, then we have $C^e_\mu(A)=-I^e_A$ for any set A.*

Thus in case $f(x)=\Phi(x,\ \mu)$ with $\mu\in\mathscr{E}$, Problem II is reduced to the question as to the coincidence of $c^e_f(A)$ with $C^e_\mu(A)$. In order to obtain an affirmative answer, we give the following propositions which were proved by Ohtsuka:

PROPOSITION 6.[16] *Assume that the kernel $\Phi$ satisfies the continuity principle[17] and $\Phi>0$. Then for every $\mu\in\mathscr{E}$, there is a sequence $\{\mu_n\}\subset\mathscr{E}$ such that $\Phi(x,\ \mu_n)$ is finite and continuous in $\Omega$ and $\mu_n$ converges strongly to $\mu$.*

PROPOSITION 7.[18] *Under the same assumptions as in the above proposition, $\mathscr{E}_K$ is strongly complete for every compact set K.*

REMARK 3. We do not need in the above two propositions that $\mathscr{E}$ is strongly complete.

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14) [1], Proposition 1.

15) [1], Proposition 2.

16) [7], Lemma 1.4, p. 190.

17) Continuity principle: If the potential $\Phi(x,\ \mu)$ of a measure $\mu$ with compact support $S\mu$ is finite and continuous as a function on $S\mu$, then $\Phi(x,\ \mu)$ is continuous in $\Omega$.

18) [7], Theorem 1.7, p. 196.

The following two lemmas are easy to see.

LEMMA 16. *Let $\mathscr{F}$ be a nonempty strongly complete convex cone in $\mathscr{E}$, $\mu_1$ and $\mu_2$ be nonzero measures of $\mathscr{E}$ and $\mu_j' \in \Lambda(\mathscr{F}; \mu_j)$ $(j=1, 2)$. Then we have*

$$\|\mu_1' - \mu_2'\| \leqq \|\mu_1 - \mu_2\|.$$

LEMMA 17. *Let $\{\mu_n\}$ be a sequence of nonzero measures in $\mathscr{E}$ and $\mathscr{F}$ be a nonempty strongly complete convex cone in $\mathscr{E}$. If $\mu_n$ converges strongly to $\mu$, then $\mu_n' \in \Lambda(\mathscr{F}; \mu_n)$ converges strongly to some $\mu' \in \Lambda(\mathscr{F}, \mu)$.*

Now we shall prove

THEOREM 15. *Let $f(x)=\Phi(x, \mu)$. Assume that $\Phi$ satisfies the continuity principle, that $\Phi>0$ and that any open set in $\Omega$ is a $K_\sigma$-set. Then we have*

$$-I_A^e = C_\mu^e(A) = c_f^e(A)$$

*for any set $A$.*

PROOF. By means of Proposition 7, it follows from Theorems 4, 10 and 11 and Lemma 15 that $c_f^e(A) \leqq -I_A^e = C_\mu^e(A)$. Hence it is sufficient to show that $c_f^e(A) \geqq C_\mu^e(A)$. First we consider the case where $\Phi(x, \mu)$ is finite and continuous in $\Omega$. Since $\Phi>0$, on account of Theorem 7 and Lemma 15, we have $c_f^e(A) = -I_A^e = C_\mu^e(A)$. Next, in the general case, there is a sequence $\{\mu_n\} \subset \mathscr{E}$ such that $f_n(x)=\Phi(x, \mu_n)$ is finite and continuous in $\Omega$ and $\mu_n$ converges strongly to $\mu$ by Proposition 6. We can choose $\{\mu_n\}$ in such a way that $f_n(x)$ increases with $n$. Since $\Gamma_A^e(f_n) \supset \Gamma_A^e(f)$, it follows that $C_{\mu_n}^e(A) = c_{f_n}^e(A) \leqq c_f^e(A)$. Taking $\mathscr{E}_A^e$ for $\mathscr{F}$ in Lemma 17, we see that $C_{\mu_n}^e(A)$ tends to $C_\mu^e(A)$ and hence $C_\mu^e(A) \leqq c_f^e(A)$,

Summing up Proposition 7, Theorems 10, 14 and 15, we have

THEOREM 16. *Suppose that $\Phi$ satisfies the continuity principle, that $\Phi>0$ and that any open set in $\Omega$ is a $K_\sigma$-set. Then it holds that $C_\mu^i(A) = C_\mu^e(A)$ for every analytic set $A$.*

COROLLARY. *Under the same assumptions as in the theorem, we have $\mathscr{E}_A^i = \mathscr{E}_A^e$ for every analytic set $A$.*

PROOF. It is always true that $\mathscr{E}_A^i \subset \mathscr{E}_A^e$. Let $\mu \in \mathscr{E}_A^e$. Then we can find a measure $\mu' \in \Lambda(\mathscr{E}_A^i; \mu)$ by Proposition 5. Since $\mu \in \Lambda(\mathscr{E}_A^e; \mu)$, we see $\|\mu-\mu'\|^2 \leqq \|\mu\|^2 - \|\mu'\|^2 = C_\mu^e(A) - C_\mu^i(A)$. If $A$ is an analytic set, then it follows from Theorem 16 that $\|\mu-\mu'\|=0$. Because $\mathscr{E}_A^i$ is strongly closed and $\mu' \in \mathscr{E}_A^i$, we conclude that $\mu \in \mathscr{E}_A^i$ and hence $\mathscr{E}_A^e \subset \mathscr{E}_A^i$.

Ogawa [5; 6] proved this theorem under a stronger condition that the kernel satisfies the domination principle.

We give an example such that $\mathscr{E}$ is strongly complete and the kernel does not satisfy the domination principle.

EXAMPLE 5. Let $\Omega$ be the 3-dimensional Euclidean space and $g(x)=|x|$ if $|x|\leqq 3$, $=6-|x|$ if $3<|x|\leqq 6$ and $=0$ if $|x|>6$. If we take $\Phi(x, y)=\frac{1}{|x-y|}+g(x)g(y)$, then $\Phi$ is of positive type and satisfies the continuity principle. We can verify that $\mathscr{E}$ is strongly complete by using the property of the Newtonien kernel. Let $\lambda$ be the unit uniform measure on $\{|x|=1\}$ and $\mu=2\varepsilon_0$ ($\varepsilon_0$ is the unit point measure at $x=0=$the origin). Then $\Phi(x, \mu)=2|x|^{-1}$. It holds that $\Phi(x, \lambda)\leqq\Phi(x, \mu)$ on $S\lambda$ and $\lambda\in\mathscr{E}$. However it is not valid that $\Phi(x, \lambda)\leqq\Phi(x, \mu)$ in $\Omega$.

If we do not assume both the continuity principle and $\Phi>0$, it is not necessarily true that $\mathscr{E}_K^i=\mathscr{E}_K^e$ even for a compact set $K$. Actually $\mathscr{E}_K^i\neq\mathscr{E}_K^e$ in Examples 2 and 4. Ogawa [6] stated that $\mathscr{E}_K^i=\mathscr{E}_K^e$ if $\Phi$ satisfies the continuity principle and $\Phi>0$[19] *without the strong completeness of* $\mathscr{E}$. In its proof, he asserted that any measure of $\mathscr{E}_K^i$ is supported by $K$. However, even if $\mathscr{E}$ is strongly complete, a measure of $\mathscr{E}_K^i$ is not always supported by $K$. In fact, we give

EXAMPLE 6. Let $\Omega=(-\infty, \infty)$, $K=\{|x|\leqq 1\}$ and $\Phi=1$. Then it holds that $\mathscr{E}_K\neq\mathscr{E}_K^i=\mathscr{E}_K^e=\mathscr{E}$.

We give here a correct proof for the above statement. First, without assuming that $\mathscr{E}$ is strongly complete, we observe

LEMMA 18. *Let $\mathscr{F}$ be a subset of $\mathscr{E}$. If $\mathscr{F}$ is strongly complete, then the strong closure $\bar{\mathscr{F}}$ of $\mathscr{F}$ in $\mathscr{E}$ is strongly complete.*

PROOF. Let $\{\mu_\alpha\}$ be any strong Cauchy net in $\bar{\mathscr{F}}$. Given $\eta>0$, we can find $\alpha_0$ and a measure $\nu_\alpha\in\mathscr{F}$ such that $\|\mu_\alpha-\nu_\alpha\|<\eta$ for each $\alpha$ and $\|\mu_\alpha-\mu_\beta\|<\eta$ for any $\alpha, \beta\geqq\alpha_0$. Then $\{\nu_\alpha\}$ is a strong Cauchy net in $\mathscr{F}$. Since $\mathscr{F}$ is strongly complete, $\nu_\alpha$ converges strongly to some $\mu_0\in\mathscr{F}$. It follows that $\mu_\alpha$ converges strongly to $\mu_0$.

For any compact set $K$, without the strong completeness of $\mathscr{E}$, we shall prove

THEOREM 17. *Assume that $\Phi$ satisfies the continuity principle and $\Phi>0$. Then $\mathscr{E}_K^i=\mathscr{E}_K^e$.*

PROOF. Since $\mathscr{E}_K$ is strongly complete (Proposition 7), $\mathscr{E}_K^i$ is strongly complete by the above lemma. $\mathscr{E}_K^e$ is strongly closed and contained in $\mathscr{E}_{G_0}^i$, which is strongly complete, where $G_0$ is a relatively compact open set containing $K$. Therefore $\mathscr{E}_K^e$ is strongly complete. $C_\mu^i(K)$ and $C_\mu^e(K)$ can be considered as before. It is enough to show $C_\mu^i(K)=C_\mu^e(K)$ in order to obtain our theorem (cf. the proof of the corollary of Theorem 16). This is carried out in the same way as the proof of Theorem 15.

19) [6], Corollaire du Théorème C, p. 229 and Lemme 9, p. 237.

## § 7. The case where *f* is quasi upper semicontinuous

A function $f$ is called *quasi upper semicontinuous* if, for any positive number $\varepsilon>0$, there is an open set $G_\varepsilon$ such that $V_i(G_\varepsilon)>1/\varepsilon$ and the restriction of $f$ to $\Omega-G_\varepsilon$ is upper semicontinuous and does not take the value $+\infty$. In case $f$ and $-f$ are quasi upper semicontinuous, we say that $f$ is quasi continuous.

Throughout this section, we assume that *$f$ is Borel measurable, quasi upper semicontinuous and integrable with respect to any measure in $\mathscr{E}_\Omega$*, and that *$\mathscr{E}$ is strongly complete.* We shall examine Problems I, II and IV in this case.[20] As for Problem I, we have

THEOREM 18. *It holds that $-I_A^i=c_f^i(A)$ for any set $A$.*

PROOF. Let us show first $-I_K^i\geqq c_f^i(K)$ for every compact set $K$. We may suppose that $I_K^i$ is finite. For each $n$, there is an open set $G_n$ such that $V_i(G_n)>n$ and $f$ is upper semicontinuous and does not take the value $+\infty$ as a function on $\Omega-G_n$. We may assume that $G_n$ decreases as $n$ increases. Put $K_n=K-G_n$ and $N=\bigcap_{n=1}^{\infty}G_n$. Then $\bigcup_{n=1}^{\infty}K_n=K-N$ and $f$ is upper semicontinuous in $K_n$ and does not take the value $+\infty$. On account of Theorem 6 and its corollary, we can find a measure $\nu_n$ such that $-I_{K_n}^i=c_f^i(K_n)=\|\nu_n\|^2$ and $\Phi(x,\nu_n)\geqq f(x)$ n.e. on $K_n$. By using Lemma 7 (in §3), we see that $\{\nu_n\}$ is a strong Cauchy sequence and $\nu_n$ converges strongly to some measure $\nu_0$. By Lemma 8, we have $\Phi(x,\nu_0)\geqq f(x)$ n.e. on $K-N$. Since $V_i(N)\geqq V_i(G_n)>n$ for any $n$, it follows that $V_i(N)=\infty$ and $\Phi(x,\nu_0)\geqq f(x)$ n.e. on $K$. Thus $\nu_0\in\Gamma_K^i$ and $c_f^i(K)\leqq\|\nu_0\|^2=\lim_{n\to\infty}\|\nu_n\|^2=-\lim_{n\to\infty}I_{K_n}^i\leqq-I_K^i$. Theorem 1 yields $c_f^i(K)=-I_K^i$ and hence $c_f^i(A)=-I_A^i$ follows from Theorem 4.

Next we shall give an answer to Problem II:

THEOREM 19. *Assume either $f>0$ or $\Phi>0$ and that any open set in $\Omega$ is a $K_\sigma$-set and that $f$ is bounded from above. Then we have $-I_A^e=c_f^e(A)$ for any set $A$.*

PROOF. It suffices to show that $\Phi(x,\nu)\geqq f(x)$ q.e. on $A$ implies $-I_A^e\leqq(\nu,\nu)$ by Theorem 4. Given $\varepsilon>0$, on account of the quasi upper semicontinuity of $f$, there is an open set $B_\varepsilon$ such that $V_i(B_\varepsilon)>1/\varepsilon$ and $f$ is upper semicontinuous as a function on $\Omega-B_\varepsilon$. Write

$$N=\{x\in A;\ \Phi(x,\nu)<f(x)\}$$

and

$$G_{ts}=\{x\in\Omega-B_\varepsilon;\ t\Phi(x,\nu)>sf(x)\}$$

20) This extension was suggested by Professor M. Ohtsuka. The author considered first the case that $f$ is quasi continuous and bounded from above.

for positive numbers $t>1$ and $0<s<1$. Then $G_{ts}\cup B_\varepsilon$ is an open set and contains $A-N$ because of our assumption that $\Phi>0$ or $f>0$. It follows from Lemmas 2 and 6, Theorem 18 and the corollary of Lemma 4, that

$$[-I_A^e]^{1/2}=[-I_{A-N}^e]^{1/2}\leqq[-I_{G_{ts}\cup B_\varepsilon}^i]^{1/2}$$
$$\leqq[-I_{G_{ts}}^i]^{1/2}+[-I_{B_\varepsilon}^i]^{1/2}<c_f^i(G_{ts})^{1/2}+\beta\varepsilon^{1/2}.$$

Since $(t/s)\nu$ belongs to $\Gamma_{G_{ts}}^i$, we have $c_f^i(G_{ts})\leqq(t/s)^2\|\nu\|^2$ so that $[-I_A^e]^{1/2}<(t/s)\|\nu\|+\beta\varepsilon^{1/2}$. Letting $t\to1$, $s\to1$ and $\varepsilon\to0$, we obtain the desired inequality.

Finally we give an answer to Problem IV:

THEOREM 20. *Under the same assumptions as in Theorem* 19, *we have* $I_A^i=I_A^e$ *for every analytic set* $A$.

In the last two theorems, we can not omit in general the condition that $f$ is bounded from above. In fact, we refer to Example 4. The function $f$ in this example is quasi continuous, because $K$ and $\phi$ are only sets of infinite $V_e$-value.

## § 8. Conditional Gauss variational problem

Let $g$ be a positive continuous function on $\Omega$ and $t$ be a positive number. We introduce a class of measures: $\mathscr{E}_A(g, t)=\{\nu\in\mathscr{E}_A;\ \int g d\nu=t\}$. The minimizing problem of $I(\nu)$ for $\nu\in\mathscr{E}_A(g, t)$ is called *the conditional inner Gauss variational problem.* Put $I_A^i=I_A^i(f, g, t)=\inf\{I(\nu);\ \nu\in\mathscr{E}_A(g, t)\}$ and $I_A^e=\sup\{I_G^i;\ G$ is open and $G\supset A\}$. Ohtsuka examined the capacitability problem for these quantities. Can we apply the method of Fuglede in this case? Namely, is there any advantage to introduce two classes of measures similar to $\Gamma_A^i$ and $\Gamma_A^e$?

In case $A$ is a compact set $K$, Ohtsuka obtained the following result.[21] *Let* $\mu_t$ *be a measure of* $\mathscr{E}_K(g, t)$ *such that* $I(\mu_t)=\inf\{I(\nu);\ \nu\in\mathscr{E}_K(g, t)\}$ *is finite and let* $\gamma_t(K)$ *be defined by* $t\gamma_t(K)=(\mu_t, \mu_t)-\int f d\mu_t$. *Then it holds that* $\Phi(x, \mu_t)\geqq f(x)+\gamma_t(K)g(x)$ *n.e. on* $K$ *and* $\Phi(x, \mu_t)\leqq f(x)+\gamma_t(K)g(x)$ *on* $S\mu_t$.

We can define $\Gamma_K^i$ as the class

$$\{\nu\in\mathscr{E};\ \Phi(x, \nu)\geqq f(x)+\gamma_t(K)g(x) \text{ n.e. on } K\}.$$

We do not know how to define $\Gamma_A^i$ for a noncompact set $A$. We shall show with an example that $\gamma_t(K)$ does not have a monotone property with respect to $K$. Accordingly, none of $\pm\gamma_t(K)$ is suitable to be regarded as a kind of capacity.

21) [7], Theorem 2.7, p. 221.

EXAMPLE 7. Let $\Omega$ be the 3-dimensional Euclidean space, $f=0$, $g=1$, $t=1$, $\Phi(x, y)=1/|x-y|$, $K_1=\{|x|\leqq 1\}$ and $K_2=\{|x|\leqq 2\}$. If we denote the uniform unit measure on $\{|x|=r\}$ by $\mu_r$, then $I^i_{K_1}=I(\mu_1)=1$, $I^i_{K_2}=I(\mu_2)=1/2$, $\gamma_t(K_1)=1$ and $\gamma_t(K_2)=1/2$.

EXAMPLE 8. Let $\Omega=K_2$ be the interval $[0, 1]$ in the real line, $K_1=\{x=0\}$, $g(x)=2-x$, $f(x)=x/2$, $t=1$ and $\Phi=1$. Then it holds that $\gamma_t(K_1)=1/4<1/2=\gamma_t(K_2)$. We observe that $I^i_{K_1}=1/4$ and $I^i_{K_2}=I(\varepsilon_1)=0$, where $\varepsilon_1$ is the unit point measure at $x=1$. In fact, let $\nu\in\mathscr{E}_{K_2}(g, t)$. Then $I(\nu)=[\nu(\Omega)]^2-\int x\,d\nu(x)$ and $\int g d\nu=1$ means that $-1+2\nu(\Omega)=\int x\,d\nu(x)$. If we put $\nu(\Omega)=s$, then $I(\nu)=s^2-2s+1=(s-1)^2$ and hence $I^i_{K_2}=0$. Let $\nu\in\mathscr{E}_{K_1}(g, t)$. Then $\int g d\nu=1$ implies $\nu(K_1)=1/2$. Therefore $I^i_{K_1}=1/4$.

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