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Axiomatic Treatment of Full-superharmonic Functions

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Introduction.

There is an axiomatic theory of harmonic functions or an axiomatic potential theory, developed by M. Brelot for the most part and investigated further by others. (See [5] for a bibliography.) The starting point of this theory is the introduction of a sheaf of functions on a locally compact space satisfying certain axioms (see [3], [4] and [5] for details). These axioms are abstracted from the characteristic properties of harmonic functions in the classical potential theory on a Euclidean space or on a Riemann surface. Thus the sheaf is called a harmonic structure.

On the other hand, a notion of full-superharmonic functions on a Riemann surface was introduced by Z. Kuramochi [9] and thoroughly investigated by C. Constantinescu and A. Cornea [6]. (Also, see [11] and [13].) The theory of full-superharmonic functions is, for the most part, quite parallel to that of superharmonic functions in the classical potential theory. Therefore, the axiomatic theory by Brelot, which gives a methodology to the classical theory of superharmonic functions, is readily extended to an axiomatic theory of full-superharmonic functions, once a suitable additional structure is given. In this paper, we shall show how this extension is carried out.

There are many variations in axioms to be assumed for the harmonic structure. In this paper, we choose Axioms T and H, which are Axioms 2 and 3 of Brelot ([3], [4] or [5]). We introduce an additional structure in §2 and assume two axioms (Axioms S and $\tilde{T}$) for it. The structure thus given will be called a full-harmonic structure. Besides the one on a Riemann surface introduced by Kuramochi, we have examples of full-harmonic structure defined for solutions of second order elliptic partial differential equations.

From this full-harmonic structure, we construct a theory of full-superharmonic functions. We follow the author's previous paper [11] for the construction of the theory, while we apply Brelot's methods to the proofs. Definitions and properties of full-superharmonic functions are discussed in §3 and §4. In particular, §4 is devoted to the study of full-superharmonic functions of potential type. We shall call them $\mathcal{P}$-functions. In Kuramochi's theory on Riemann surfaces, a kernel (Green function) for the full-harmonic structure is introduced and integral representation of $\mathcal{P}$-functions with respect to this kernel is discussed. (See [6], [9] [11] and [13]; the kernel is denoted by $\tilde{g}_a$ in [6] and by $N(p, z)$ in the others.) A $\mathcal{P}$-function is thus decomposed into two functions, one with its associated measure on the ideal boundary and the other with its measure inside the space. We shall show in §5 that in our axiomatic setting we can also make the corresponding decomposition, a decomposition into a $\mathcal{P}_b$-function and a $\mathcal{P}_i$-function, without introducing a kernel. Then we proceed to consider an integral representation of $\mathcal{P}_b$-functions in §6. In this paper, we omit the discussion of integral representation of $\mathcal{P}_i$-functions, since it would be similar to that of superharmonic functions, extensively studied in M. Brelot [3] and R.-M. Hervé [8].

Finally we remark that the Martin boundary as well as the Kuramochi boundary is obtained from a full-harmonic structure (Examples in §6).

§ 1. Preliminaries.

1.1 *Notation.* Let Ω be a locally compact Hausdorff space. For a subset A of Ω, we denote by ∂A, $\bar{A}$ and A^i the (relative) boundary, the closure and the interior of A, respectively. Given a function f on A and a subset B of A, the restriction of f on B will be denoted by $f|_B$. We consider only extended real valued functions.

1.2. *Brelot's harmonic structure* (cf. [3], [4] and [5]). Suppose, to every open set G of Ω, there corresponds a linear space $\mathcal{H}_G$ of (finite) continuous functions on G such that $\mathfrak{H}=\{\mathcal{H}_G\}_G$ forms a sheaf[1)] on Ω.

A relatively compact open set G is called *regular* (with respect to $\mathfrak{H}$) if for each continuous function f on ∂G there exists a unique continuous function h_f on $\bar{G}$ such that $h_f|_G \in \mathcal{H}_G$ and $h_f|_{\partial G}=f$ and if $f\geq 0$ implies $h_f\geq 0$. If G is a regular open set and $x \in G$, then there exists a positive Radon measure μ_x^G on ∂G such that $h_f(x)=\int f d\mu_x^G$. If D is a component of a regular open set G, then D is regular and $\mu_x^D=\mu_x^G$ for $x \in D$.

We assume the following two axioms of Brelot.

Axiom T: Regular domains form a base of open sets.

Axiom H: For any domain D, if $\{h_\iota\}_{\iota\in I}$ is an upper directed family of functions in $\mathcal{H}_D$, then $\sup_{\iota\in I} h_\iota$ is either $\equiv+\infty$ or $\in \mathcal{H}_D$.

For a domain D, the following properties are derived from the axioms:

1) If $h \in \mathcal{H}_D$, $h\geq 0$ and $h(x)=0$ for some $x \in D$, then $h\equiv 0$.

2) If f is a lower semi-continuous function on ∂D and if D is regular, then the function $h(x)=\int f d\mu_x^D$ is either $\equiv+\infty$ or $\in \mathcal{H}_D$.

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1) i.e., if $G_1\subset G_2$ and $h\in \mathcal{H}_{G_2}$, then $h|_{G_1}\in \mathcal{H}_{G_1}$ and if $G=\bigcup_\iota G_\iota$, h is a function on G and $h|_{G_\iota}\in \mathcal{H}_{G_\iota}$ for all ι, then $h \in \mathcal{H}_G$.

3) For any $x \in D$ and a compact subset K of D, there exists a positive constant M such that $h(y) \leq Mh(x)$ for all $y \in K$ and $h \in \mathcal{H}_D$ with $h \geq 0$.

4) ([10]) For any $x \in D$, the family $\{h \in \mathcal{H}_D; h \geq 0, h(x)=1\}$ is equicontinuous at x.

The following lemma is a consequence of properties 3) and 4):

LEMMA 1. *If D is a domain, then $\{h \in \mathcal{H}_D; h \geq 0, h(x)=1\}$ is compact with respect to the compact convergence topology.*

1.3. *Superharmonic functions* (cf. [3], [4] and [5]). Let G be an open set in Ω. Any function in $\mathcal{H}_G$ is called *harmonic* on G.

A function s on G is called *superharmonic* (with respect to $\mathfrak{H}$) on G if (i) s is lower semi-continuous on G; (ii) $s > -\infty$ and $s \not\equiv +\infty$ on any component of G and (iii) for any regular domain D such that $\bar{D} \subset G$ and for any $x \in D$, $s(x) \geq \int s\, d\mu_x^D$.

LEMMA 2. (Local criterion) *Suppose s satisfies conditions* (i) *and* (ii) *in the above definition. If for each point $x \in G$, there exists a base $\mathfrak{B}(x)$ of neighborhoods of x such that each $D \in \mathfrak{B}(x)$ is a regular domain such that $\bar{D} \subset G$ and for each $D \in \mathfrak{B}(x)$, $s(x) \geq \int s\, d\mu_x^D$, then s is superharmonic on G.*

A function s on G is called *nearly superharmonic* on G if (i) s is locally bounded below and $s \not\equiv +\infty$ on any component of G and (ii) for any regular domain D such that $\bar{D} \subset G$ and for any $x \in D$, $s(x) \geq \overline{\int} s\, d\mu_x^D$.

Let $\mathfrak{B}(x)$ be a fundamental system of neighborhoods of x consisting of regular domains. If s is nearly superharmonic on G, then $\hat{s}(x) = \lim_{D \in \mathfrak{B}(x)} \overline{\int} s\, d\mu_x^D = \inf\{\varliminf_{y \to x} s(y), s(x)\}$ for $x \in G$ defines a superharmonic function $\hat{s}$, the *regularization* of s.

Properties of superharmonic and nearly superharmonic functions:

(a) If s is superharmonic (resp. nearly superharmonic) on G and $\alpha > 0$, then αs is superharmonic (resp. nearly superharmonic) on G.

(b) If s_1 and s_2 are superharmonic (resp. nearly superharmonic) on G, then $s_1 + s_2$, min (s_1, s_2) are superharmonic (resp. nearly superharmonic) on G.

(c) If $\{s_\iota\}$ is an upper directed family of superhamonic functions on a domain D, then $\sup s_\iota$ is either $\equiv +\infty$ or superharmonic on D.

(d) If $\mathcal{S}$ is a family of nearly superharmonic functions, locally uniformly bounded below, then $\inf \mathcal{S}$ is nearly superharmonic.

(e) If s_1 and s_2 are nearly superharmonic, then $\widehat{s_1 + s_2} = \hat{s}_1 + \hat{s}_2$.

(f) No superharmonic function assumes $+\infty$ over an open set.

(g) If s is superharmonic on G and D is a regular open set such that $\bar{D} \subset G$, then $h(x) = \int s\, d\mu_x^D$ is harmonic on D.

(h) If s is non-negative superharmonic on a domain D and if $s(x)>0$ for some $x \in D$, then $s>0$ everywhere.

LEMMA 3. (Minimum principle) *Let G be a relatively compact open set and suppose there exists a continuous superharmonic function s_0 such that $s_0 \geq \delta > 0$ on G. If s is a superharmonic function on G and if* $\lim_{x\to\xi, x\in G} s(x) \geq 0$ *for all $\xi \in \partial G$, then $s \geq 0$ on G.*

1.4. *Perron's family* (Saturated family; see [4]). Let G be an open set and let $\mathcal{S}$ be a family of superharmonic functions on G. $\mathcal{S}$ is called a Perron's family if (i) $\mathcal{S} \neq \emptyset$ and $\mathcal{S}$ is lower directed; (ii) for any regular domain D such that $\bar{D} \subset G$, the superharmonic function

$$s_1(x) = \begin{cases} s(x) & \text{if } x \in G-D \\ \int s\, d\mu_x^D & \text{if } x \in D \end{cases}$$

belongs to $\mathcal{S}$ whenever $s \in \mathcal{S}$ and (iii) $\mathcal{S}$ is locally uniformly bounded below.

Perron's theorem. If $\mathcal{S}$ is a Perron's family on G, then $\inf \mathcal{S}$ *is harmonic on G.*

§ 2. Full-harmonic structure

2.1. *Axioms.* Let $\mathfrak{D}$ be the family of domains D in Ω such that D is not relatively compact and ∂D is compact. In order to assure that $\mathfrak{D}$ is not empty, we hereafter assume that Ω is not compact. Let $\mathfrak{G}$ be the family of open sets G in Ω such that ∂G is compact.

Suppose, to each $D \in \mathfrak{D}$, there corresponds a linear subspace $\tilde{\mathcal{H}}_D$ of $\mathcal{H}_D$. Let $\tilde{\mathfrak{H}} = \{\tilde{\mathcal{H}}_D\}_{D\in\mathfrak{D}}$. For $G \in \mathfrak{G}$, let

$$\tilde{\mathcal{H}}_G = \{u \in \mathcal{H}_G;\ u|_{D_\iota} \in \tilde{\mathcal{H}}_{D_\iota} \text{ for each component } D_\iota \text{ of } G \text{ such that } D_\iota \in \mathfrak{D}.\}$$

Then $\tilde{\mathcal{H}}_G$ is a linear subspace of $\mathcal{H}_G$. If G is relatively compact, then $\tilde{\mathcal{H}}_G = \mathcal{H}_G$.

In the previous section, a regular domain was assumed to be relatively compact. We extend the notion of regular domains for not relatively compact domains.

A domain $D \in \mathfrak{D}$ is called *regular* (with respect to $\tilde{\mathfrak{H}}$) if, for any continuous function f on ∂D, there exists a unique continuous function u_f on $\bar{D}$ such that $u_f|_D \in \tilde{\mathcal{H}}_D$, and $u_f|_{\partial D} = f$ and if $f \geq 0$ implies $u_f \geq 0$. In this case, there exists a positive Radon measure μ_x^D on ∂D for each $x \in D$ such that $u_f(x) = \int f\, d\mu_x^D$. An open set $G \in \mathfrak{G}$ is called *regular* if its components are all regular either in the above sense or in the sense defined in §1. For $x \in G$, we define $\mu_x^G = \mu_x^D$, where D is the component of G containing x. We denote by $\mathfrak{D}_r$ (resp. $\mathfrak{G}_r$) the set of all regular domains in $\mathfrak{D}$ (resp. regular open sets in $\mathfrak{G}$).

We assume the following two axioms for $\tilde{\mathfrak{H}}$:

Axiom S(Sheaf): Let $D \in \mathfrak{D}$. (i) If $u \in \tilde{\mathscr{H}}_D$, $D' \subset D$ and $D' \in \mathfrak{D}$, then $u|_{D'} \in \tilde{\mathscr{H}}_{D'}$. (ii) If $u \in \mathscr{H}_D$ and if there exists a compact set K such that $K^i \supset \partial D$ and $u|_{D-K} \in \tilde{\mathscr{H}}_{D-K}$, then $u \in \tilde{\mathscr{H}}_D$.

Axiom $\tilde{T}$: For any compact set K in Ω, there exists another compact set K_1 such that $K_1^i \supset K$ and $\Omega - K_1 \in \mathfrak{G}_r$.

We call $\tilde{\mathfrak{H}}$ a *full-harmonic structure* subordinate to $\mathfrak{H}$. In this paper, we fix $\mathfrak{H}$ and $\tilde{\mathfrak{H}}$ once for all.

Remark: Here, we first considered a harmonic structure $\{\mathscr{H}_G\}$ and then defined a full-harmonic structure $\{\tilde{\mathscr{H}}_D\}$ in terms of $\{\mathscr{H}_G\}$. Instead, we may define a full-harmonic structure directly by giving a sheaf of linear spaces $\tilde{\mathscr{H}}_D$, considered for all domains D with compact boundary. In this case, $\mathscr{H}_G$ is defined as the family of all functions u on G such that $u|_D \in \tilde{\mathscr{H}}_D$ for all D in a family of relatively compact domains which covers G. If we assume Axioms T, H, S and $\tilde{T}$ for these spaces, then we have a full-harmonic structure.

2.2. *Consequences of axioms.*

Lemma 4. *Let $G \in \mathfrak{G}$, $D \in \mathfrak{D}_r$ and $\bar{D} \subset G$. Then, for any $u \in \tilde{\mathscr{H}}_G$ and $x \in D$, $\int u \, d\mu_x^D = u(x)$. Conversely, if $u \in \mathscr{H}_G$ and $\int u \, d\mu_x^D = u(x)$ for all $D \in \mathfrak{D}_r$ such that $\bar{D} \subset G$ and for all $x \in D$, then $u \in \tilde{\mathscr{H}}_G$.*

Proof: $u|_{\bar{D}}$ is continuous and $u|_D \in \tilde{\mathscr{H}}_D$ by Axiom S, (i). Since D is regular, the first part of the lemma follows from the uniqueness of u_f. Suppose $u \in \mathscr{H}_G$ and $\int u \, d\mu_x^D = u(x)$ for all $D \in \mathfrak{D}_r$ with $\bar{D} \subset G$. By Axiom $\tilde{T}$, there exists a compact set K such that $K^i \supset \partial G$ and $\Omega - K \in \mathfrak{G}_r$. If D is a component of $G-K$, then it is a component of $\Omega - K$, so that D is regular and $\bar{D} \subset G$. If D is relatively compact, then $\int u \, d\mu_x^D = u(x)$, since u is harmonic on G; otherwise $D \in \mathfrak{D}_r$, so that $\int u \, d\mu_x^D = u(x)$ by assumption. Hence $\int u \, d\mu_x^{G-K} = u(x)$ for all $x \in G-K$, so that $u \in \tilde{\mathscr{H}}_{G-K}$. It follows from Axiom S, (ii) that $u \in \tilde{\mathscr{H}}_G$.

Lemma 5. *Let $D \in \mathfrak{D}$ and let $\{u_\iota\}_{\iota \in I}$ be an upper directed family of functions in $\tilde{\mathscr{H}}_D$. Then $\sup_{\iota \in I} u_\iota$ is either $\equiv +\infty$ or $\in \tilde{\mathscr{H}}_D$.*

Proof: Let $u_0 = \sup_\iota u_\iota$ and suppose $u_0 \not\equiv +\infty$. Then, by Axiom H, $u_0 \in \mathscr{H}_D$. Let $D' \in \mathfrak{D}_r$ and $\bar{D}' \subset D$. By Lemma 1, we see that $u_\iota \to u_0$ uniformly on $\partial D'$. Since $\int u_\iota \, d\mu_x^{D'} = u_\iota(x)$ for $x \in D'$ by the above lemma, we have $\int u_0 d\mu_x^{D'} = u_0(x)$ for $x \in D'$. Hence, again by the above lemma, we have $u_0 \in \tilde{\mathscr{H}}_D$.

Corollary. *Let $D \in \mathfrak{D}_r$ and let f be a function on ∂D.*

(i) *If f is lower semi-continuous and $> -\infty$, then the function $u(x) = \int f \, d\mu_x^D$*

is either $\equiv +\infty$ *or* $u \in \tilde{\mathscr{H}}_D$.

(ii) *If* f *is arbitrary, then the function* $u(x)=\overline{\int} f\, d\mu_x^D$ *is either* $\equiv +\infty$, $\equiv -\infty$ *or* $u \in \tilde{\mathscr{H}}_D$.

2.3. *Examples of full-harmonic structure.*

Example 1. (Kuramochi, Constantinescu and Cornea). Let Ω be an open Riemann surface and let $\mathfrak{H}=\{\mathscr{H}_G\}$ be the classical harmonic structure on Ω. If K is a compact set with sufficiently smooth boundary ∂K and if f is a C^∞-function near ∂K, then, by [9], [13] or [6], there exists a unique harmonic function f^K on $\Omega - K$ which has the smallest Dirichlet norm among functions on $\Omega - K^i$ assuming the value f on ∂K. Kuramochi's full-harmonic structure is given by

$$(*) \quad \tilde{\mathscr{H}}_D = \left\{ u \in \mathscr{H}_D;\ u = u^{\partial K \cap D} \text{ on } D-K \text{ for some compact set } K \text{ such that } K^i \supset \partial D. \right\}.$$

We can carry out similar definitions if Ω is a space of type $\mathfrak{S}$ in the sense of Brelot-Choquet (see [11]).

Example 2. *Full-harmonic structure for solutions of* $\Delta u = qu$.

If we replace harmonic functions by solutions of a second order self-adjoint elliptic differential equation, then we have a full-harmonic structure by applying the method of Example 1, under certain condititions imposed on the coefficients of the equation (cf. [12]). To avoid inessential complications, we here restrict ourselves to the equation $Lu \equiv \Delta u - qu = 0$ on a domain Ω in the n-dimensional Euclidean space ($n \geq 2$), where Δ is the Laplacian and q is a non-negative C^∞-function on Ω. Let $\mathscr{H}_G = \{u;\ Lu = 0 \text{ on } G\}$. Then, it is known by Hervé [8] that $\mathfrak{H} = \{\mathscr{H}_G\}$ is a harmonic structure. A full-harmonic structure subordinate to $\mathfrak{H}$ is defined as follows.

Let G be an open subset of Ω and let $\mathfrak{S}(G)$ be the set of all BLD-functions f (see [11]; they are called "fonctions (BL) précisées" in [7]) in G such that $\int_G qf^2 dx < \infty$. Let $E_G(f, g) = D_G(f, g) + \int_G qfg\, dx$ for $f, g \in \mathfrak{S}(G)$, where $D_G(f, g)$ is the mutual Dirichlet integral of f and g on G. Identifying functions which are equal almost everywhere, $\mathfrak{S}(G)$ becomes a Hilbert space with the inner product $E_G(f, g)$ (cf. [7]). Next, let K be a compact set in Ω such that its boundary ∂K consists of a finite number of closed C^∞-surfaces. Let $\mathfrak{S}_K = \{u \in \mathfrak{S}(\Omega);\ u = 0 \text{ q.p. on } K\}$. We can show that $\mathfrak{S}_K$ is a closed subspace of $\mathfrak{S}(\Omega)$. If f is a C^∞-function defined in a neighborhood of ∂K, then there exists at least one C^∞-function u_f on Ω such that $u_f \in \mathfrak{S}(\Omega)$ and $u_f = f$ on ∂K. Let v_f be the projection of u_f onto $\mathfrak{S}_K$. We can show that v_f can be chosen to be continuous (cf. [12]) and that the value of $u_f - v_f$ on $\Omega - K$ is independent of the choice of u_f. Thus, we write $f^K = u_f - v_f$ on $\Omega - K$. The following proposition can be proved by the same methods as Satz 15.1 of [6] (also see [7]).

Proposition. (i) $Lf^K=0$ on $\Omega-K$, i.e., $f^K \in \mathscr{H}_{\Omega-K}$.

(ii) $f\to f^K$ is a linear mapping of $\mathfrak{S}(\Omega)$ into $\mathfrak{S}(\Omega-K)$.

(iii) If $f\geq 0$, then $f^K\geq 0$.

(iv) If $K\subset K_1$, then $(f^K)^{K_1}=f^K$ on $\Omega-K_1$.

(v) If D is a component of $\Omega-K$, then $f^K=f^{\partial D}$ on D.

For $D\in\mathfrak{D}$, we define $\tilde{\mathscr{H}}_D$ by (*) in Example 1. Then the above proposition implies that $\{\tilde{\mathscr{H}}_D\}_{D\in\mathfrak{D}}$ satisfies Axioms S and $\tilde{T}$.

Similar discussions hold in case Ω is an open Riemann surface.

Example 3. Let Ω be an open Riemann surface (or a non-compact space of type $\mathfrak{S}$) and let $\mathfrak{H}=\{\mathscr{H}_G\}$ be the classical harmonic structure. For an open set $G\in\mathfrak{G}$ and a continuous function f on ∂G, let H_f^G be the Dirichlet solution on G with boundary values f on ∂G and 0 on the ideal boundary (cf. [6] and [14]). If we take

$$\tilde{\mathscr{H}}_D=\left\{\begin{array}{l} u\in\mathscr{H}_D;\text{ there exists a compact set } K \text{ such that} \\ \qquad K^i\supset\partial D \text{ and } u=H_u^{D-K} \text{ on } D-K. \end{array}\right\},$$

then $\tilde{\mathfrak{H}}=\{\tilde{\mathscr{H}}_D\}_{D\in\mathfrak{D}}$ is a full-harmonic structure subordinate to $\mathfrak{H}$.

Example 4. *Full-harmonic structure associated with an L_1-operator.*

Let Ω be an open Riemann surface or a non-compact locally Euclidean space. If $D\in\mathfrak{D}$ has a sufficiently smooth boundary, then the L_1-operator (with respect to the canonical partition of the ideal boundary) in the sense of L. Sario is defined for D (see [1] and [15]). We denote it by $L_{1,D}$. We take the classical harmonic structure $\mathfrak{H}=\{\mathscr{H}_G\}$ and we define

$$\tilde{\mathscr{H}}_D=\left\{\begin{array}{l} u\in\mathscr{H}_D;\text{ there exists a compact set } K \text{ such that } \partial K \text{ is smooth,} \\ \qquad K^i\supset\partial D \text{ and } u=L_{1,D'}u \text{ on } D' \text{ for any component } D' \\ \qquad \text{of } D-K. \end{array}\right\}$$

Then we can see that $\{\tilde{\mathscr{H}}_D\}_{D\in\mathfrak{D}}$ is a full-harmonic structure subordinate to $\mathfrak{H}$.

§ 3. Full-superharmonic functions

3.1. *Full-superharmonic functions.*

Definition. Let $G\in\mathfrak{G}$. A superharmonic function u on G is called *full-superharmonic* (with respect to $\tilde{\mathfrak{H}}$) if, for any $D\in\mathfrak{D}_r$ such that $\bar{D}\subset G$, $u(x)\geq \int u\,d\mu_x^D$ for all $x\in D$.

By Lemma 4, we see that a function u on G belongs to $\tilde{\mathscr{H}}_G$ if and only if u and $-u$ are both full-superharmonic on G. A function of $\tilde{\mathscr{H}}_G$ will be called *full-harmonic* on G.

The following properties are immediate consequences of the definition and the corresponding properties of superharmonic functions.

(α) If u is full-superharmonic on G and if $\alpha>0$, then αu is full-superharmonic on G.

(β) If u_1 and u_2 are full-superharmonic on G, then u_1+u_2, $\min(u_1, u_2)$ are full-superharmonic on G.

(γ) If $\{u_\iota\}_\iota$ is an upper directed family of full-superharmonic functions on $D \in \mathfrak{D}$, then $\sup u_\iota$ is either $\equiv +\infty$ or full-superharmonic on D.

(δ) If u is full-superharmonic on G and if $G' \in \mathfrak{G}_r$, $\bar{G}' \subset G$, then $h(x)=\int u\, d\mu_x^{G'}$ is full-harmonic on G'.

THEOREM 1. (Minimum principle) *Let $G \in \mathfrak{G}$ and suppose there exists a non-negative continuous full-superharmonic function s_0 on G such that $s_0 \geq \delta > 0$ near ∂G,* i.e. *on $V \cap G$ for a neighborhood V of ∂G. If u is a full-superharmonic function on G such that $\lim_{x\to\xi, x\in G} u(x) \geq 0$ for all $\xi \in \partial G$, then $u \geq 0$.*

PROOF: It is enough to consider on each component of G. By virtue of Lemma 3, we may assume that $G \in \mathfrak{D}$. Let K be a compact set in Ω such that $K^i \supset \partial G$ and $G-K \in \mathfrak{G}_r$ (Axiom $\tilde{T}$). Let $\alpha = \inf_{x\in\partial K\cap G}(u(x)/s_0(x))$. Since u/s_0 is lower semi-continuous, $\alpha > -\infty$. Put $\beta = \inf_{x\in G}(u(x)/s_0(x))$.

Suppose that $\beta<0$. If $\alpha\geq 0$, then $u\geq 0$ on $\partial K\cap G$. Hence, $u(y)\geq \int u\, d\mu_y^{G-K} \geq 0$ for all $y \in G-K$, Hence $u/s_0 \geq 0 > \beta$ on $G-K^i$. If $\alpha<0$, then $u-\alpha s_0$ is full-superharmonic on G and non-negative on $\partial K\cap G$. Hence $u(y)-\alpha s_0(y) \geq \int (u-\alpha s_0) d\mu_y^{G-K} \geq 0$ for all $y\in G-K$. Therefore, $u\geq \alpha s_0$ on $G-K^i$ or $u/s_0 \geq \alpha \geq \beta$ on $G-K^i$. Thus, we have seen that either $\inf_{G-K^i}(u/s_0) > \beta$ or $\alpha=\beta$. By assumption, $\lim_{x\to\xi, x\in G}(u(x)/s_0(x)) \geq 0$ for all $\xi \in \partial G$. Hence, it follows that $\beta > -\infty$ and u/s_0 attains β inside G, say at $x_0 \in G$. Then the superharmnic function $u-\beta s_0$ is non-negative on the domain G and vanishes at x_0. Hence $u-\beta s_0 \equiv 0$ on G or $u/s_0 \equiv \beta < 0$, which is impossible. Therefore, $\beta \geq 0$ and the theorem is proved.

LEMMA 6. *Let $G \in \mathfrak{G}$. If u is superharmonic on G and if, for each compact set K in Ω, there exists another compact set K_1 such that $K_1^i \supset K \cup \partial G$, $G-K_1 \in \mathfrak{G}_r$ and $u(y) \geq \int u\, d\mu_y^{G-K_1}$ for all $y \in G-K_1$, then u is full-superharmonic on G.*

PROOF: Let $D \in \mathfrak{D}_r$ and $\bar{D} \subset G$. Let g be any continuous function on ∂D such that $g \leq u$ on ∂D and let $u_g(x) = u(x) - \int g d\mu_x^D$ for $x \in D$. Then $\lim_{x\to\xi\in\partial D, x\in D} u_g(x) \geq 0$. By assumption, there exists a compact set K_1 such that $K_1^i \supset \partial G \cup \partial D$, $G-K_1 \in \mathfrak{G}_r$ and $u(y) \geq \int u \, d\mu_y^{G-K_1}$ for all $y \in G-K_1$. Then $D-K_1 \in \mathfrak{G}_r$ and $\mu_y^{G-K_1} = \mu_y^{D-K_1}$ for $y \in D-K_1$. Hence $u(y) \geq \int u \, d\mu_y^{D-K_1}$ for all $y \in D-K_1$. Then it follows that $u_g(y) \geq \int u_g \, d\mu_y^{D-K_1}$ for all $y \in D-K_1$. Since D is regular, there exists a positive full-harmonic function h_0 on D such that $h_0 \geq \delta > 0$ near ∂D(e.g., $h_0(x) = \int d\mu_x^D$). Then, by arguments similar to the proof of the above theorem, we conclude that $u_g \geq 0$ on D. Hence $u(x) \geq \int g \, d\mu_x^D$ for all $x \in D$. Since $u = \sup_{g \leq u} g$, it follows that $u(x) \geq \int u \, d\mu_x^D$ for all $x \in D$. Hence u is full-superharmonic.

Corollary. *Let $G \in \mathfrak{G}$ and let u be a superharmonic function on G. If there exists a compact set K such that u is full-superharmonic on $G-K$, then u is full-superharmonic on G.*

3.2. *Theorems on full-superharmonic functions.*

Theorem 2. *Let $G \in \mathfrak{G}$, $G' \in \mathfrak{G}_r$ and $\bar{G}' \subset G$. For a full-superharmonic function u on G, let*

$$u_1 = \begin{cases} u(x) & \text{if } x \in G-G' \\ \int u \, d\mu_x^{G'} & \text{if } x \in G'. \end{cases}$$

Then u_1 is full-superharmonic on G.

Proof: It is easy to see that u_1 is superharmonic on G (cf. Lemma 2). Let K be a compact set such that $K^i \supset \partial G \cup \partial G'$ and $G-K \in \mathfrak{G}_r$. If $x \in G-G'-K$, then $u_1(x) = u(x) \geq \int u \, d\mu_x^{G-K} \geq \int u_1 \, d\mu_x^{G-K}$. If $x \in G'-K$, then Lemma 4 implies $\int u_1 \, d\mu_x^{G-K} = \int u_1 \, d\mu_x^{G'-K} = u_1(x)$, since u_1 is full-harmonic on G'. Hence u_1 is full-superharmonic by the above lemma.

If K is a compact set such that $K^i \supset \partial G$ and $G-K \in \mathfrak{G}_r$, then the full-superharmonic function u_1 defined in the above theorem for $G' = G-K$ will be denoted by u_K.

Theorem 3. (Perron) *Let $\mathfrak{U}$ be a family of full-superharmonic functions on $G \in \mathfrak{G}$. Suppose that $\mathfrak{U}$ is a Perron's family on G and that, for each compact set K with $K^i \supset \partial G$ and $G-K \in \mathfrak{G}_r$, $u \in \mathfrak{U}$ implies $u_K \in \mathfrak{U}$. Then* $\inf \mathfrak{U}$ *is full-harmonic on G.*

The proof is similar to that of Perron's theorem.

Theorem 4. *Let $G \in \mathfrak{G}$ and u be a full-superharmonic function on G. If there exists a full-superharmonic function v_1 on G such that $u \geq -v_1$, then there exists the smallest function v_0 among full-superharmonic functions v such* *that* $u \geq -v$. *Furthermore,* v_0 *is full-harmonic on* G, i.e., $-v_0$ *is the greatest full-harmonic minorant of* u.

PROOF: Let $\mathcal{U}=\{v;$ full-superharmonic on G, $u \geq -v\}$. Since $v_1 \in \mathcal{U}$, $\mathcal{U} \neq \emptyset$. It is easy to see that $\mathcal{U}$ satisfies the conditions of Theorem 3. Hence $v_0 = \inf \mathcal{U}$ is full-harmonic and $u \geq -v_0$.

3.3. *Nearly full-superharmonic functions.* A function g on $G \in \mathfrak{G}$ is called *nearly full-superharmonic* if it is nearly superharmonic on G and for any $D \in \mathfrak{D}_r$ such that $\bar{D} \subset G$, $g(x) \geq \bar{\int} g \, d\mu_x^D$ for all $x \in D$.

Obviously, a full-superharmonic function is nearly full-superharmonic. If g is nearly full-superharmonic, then its regularization $\hat{g}$ is full-superharmonic.

The following properties are easy to see:

(α) If g is nearly full-superharmonic and if $\alpha > 0$, then αg is nearly full-superharmonic.

(β) If g_1 and g_2 are nearly full-superharmonic, then $g_1 + g_2$, $\min(g_1, g_2)$ are nearly full-superharmonic.

(γ) If $\mathcal{U}$ is a family of nearly full-superharmonic functions locally uniformly bounded below, then $\inf \mathcal{U}$ is nearly full-superharmonic.

§ 4. Full-superharmonic functions of potential type

4.1. *Full-superharmonic functions of potential type.* We consider a domain $\Omega_0 \in \mathfrak{D}$ such that there exists a positive continuous full-superharmonic function u_0 on Ω_0 which satisfies $0 < \delta \leq u_0 \leq M < +\infty$ near $\partial\Omega_0$, i.e., on $V \cap \Omega_0$ for some neighborhood V of $\partial\Omega_0$. The existence of such a function u_0 is assured if Ω_0 is contained in a regular domain. *We take such a domain* Ω_0 *and fix it throughout the rest of this paper.* A compact set K of Ω will be called *admissible* (for Ω_0) if $K^i \supset \partial\Omega_0$.

Definition. A non-negative full-superharmonic function on Ω_0 is called *of potential type* (on Ω_0) if its greatest full-harmonic minorant on Ω_0 is zero. We denote by $\mathcal{P}$ the family of all full-superharmonic functions of potential type.

Any non-negative full-superharmonic function u has a unique decomposition $u = h + v$ with h full-harmonic on Ω_0 and $v \in \mathcal{P}$; h is the greatest full-harmonic minorant of u.

If $v \in \mathcal{P}$ and if u is a non-negative full-superharmonic function such that $u \leq v$, then $u \in \mathcal{P}$. Hence, if $v_1, v_2 \in \mathcal{P}$, then $\min(v_1, v_2) \in \mathcal{P}$.

LEMMA 7. *Let* u *be a non-negative full-superharmonic function on* Ω_0 *and let* K *be an admissible compact set. If there exists* $v \in \mathcal{P}$ *such that* $u \leq v$ *on* $K^i \cap \Omega_0$, *then* $u \in \mathcal{P}$.

PROOF: Let h be a full-harmonic minorant of u. Then $v \geq u \geq h$ on $K^i \cap \Omega_0$ and $v-h$ is full-superharmonic on Ω_0. Hence, by Theorem 1, $v-h \geq 0$ on Ω_0. Since $v \in \mathcal{P}$, $h \leq 0$. Hence $u \in \mathcal{P}$.

4.2. *Reduced functions.* Let u be a non-negative full-superharmonic function on Ω_0 and let F be a subset of Ω. Then the function

$$R_F u = \inf\{v;\ \text{full-superharmonic} \geq 0 \text{ on } \Omega_0,\ v \geq u \text{ on } F \cap \Omega_0\}$$

is nearly full-superharmonic on Ω_0 and $0 \leq R_F u \leq u$. Hence $u_F = \widehat{R_F u}$ is full-superharmonic on Ω_0 and $0 \leq u_F \leq u$. We call u_F the reduced function of u on F (with respect to Ω_0).

The following properties are easy to prove.

1) $u_F = u_{F \cap \Omega_0}$; $u_F = u$ on $F^i \cap \Omega_0$; $u_\phi = 0$.

2) u_F is harmonic on $\Omega_0 - \bar{F}$; u_F is full-harmonic on $\Omega_0 - \bar{F}$ if F is relatively compact in Ω (by Theorem 3).

3) $u_1 \leq u_2$ implies $(u_1)_F \leq (u_2)_F$.

4) $F_1 \subset F_2$ implies $u_{F_1} \leq u_{F_2}$.

5) $(u_1)_F + (u_2)_F \geq (u_1 + u_2)_F$.[2)]

Remark: If $\mathcal{P} = \{0\}$, then all non-negative full-superharmonic functions are full-harmonic and proportional to each other. Therefore, in this case, $F^i \cap \Omega_0 \neq \emptyset$ implies $u = u_F$.

LEMMA 8. *Let u be a non-negative full-superharmonic function on Ω_0 and let K be an admissible compact set such that $\Omega_0 - K \in \mathfrak{G}_r$. Then $u_K(x) = \int u\, d\mu_x^{\Omega_0 - K}$ for $x \in \Omega_0 - K$.* (Thus the notation after Theorem 2 does not conflict with the present one.)

PROOF: By Theorem 2,

$$u_1(x) = \begin{cases} u(x) & \text{if } x \in K \cap \Omega_0 \\ \int u\, d\mu_x^{\Omega_0 - K} & \text{if } x \in \Omega_0 - K \end{cases}$$

is full-superharmonic on Ω_0. Obviously, $u_1 \geq 0$. Hence $u_K \leq u_1$. On the other hand, if v is a non-negative full-superharmonic function on Ω_0 such that $v \geq u$ on $K \cap \Omega_0$, then $v(x) \geq \int v\, d\mu_x^{\Omega_0 - K} \geq \int u\, d\mu_x^{\Omega_0 - K}$ for $x \in \Omega_0 - K$. Therefore, $v \geq u_1$ and hence $u_K \geq u_1$.

4.3. *Lemmas on full-superharmonic functions of potential type.*

LEMMA 9. *Let $\{K_\iota\}$ be a directed family of admissible compact sets such that $K_\iota \supset K_{\iota'}$ if $\iota < \iota'$ and $\bigcap_\iota (K_\iota \cap \Omega_0) = \emptyset$. If u is a non-negative full-super-*

2) It is possible to prove that the equality holds in 5). Cf. [2] and [8].

harmonic function on Ω_0, *then* $\lim_{\iota} u_{K_\iota}$ *is equal to the greatest full-harmonic minorant of* u.

PROOF: Each u_{K_ι} is full-harmonic on $\Omega_0 - K_\iota$. For any ι_0, $\{u_{K_\iota}; \iota \geq \iota_0\}$ is a lower directed family of full-harmonic functions on $\Omega_0 - K_{\iota_0}$. Hence $h = \inf_{\iota} u_{K_\iota}$ is full-harmonic on $\Omega_0 - K_{\iota_0}$ by Lemma 5. Hence h is full-harmonic on Ω_0. Obviously, $0 \leq h \leq u$. Now, let h_1 be any full-harmonic minorant of u on Ω_0. Since $u_{K_\iota} = u$ on $K_\iota^i \cap \Omega_0$, $u_{K_\iota} \geq h_1$ on $K_\iota^i \cap \Omega_0$. Applying Theorem 1 to the full-superharmonic function $u_{K_\iota} - h_1$, we see that $u_{K_\iota} \geq h_1$. Hence $h \geq h_1$. Therefore, h is the greatest full-harmonic minorant of u.

COROLLARY. *If* $u_1, u_2 \in \mathcal{P}$, *then* $u_1 + u_2 \in \mathcal{P}$. *Hence* $\mathcal{P}$ *is a cone.*

PROOF. Let $\{K_\iota\}$ be as in the above lemma. Then $\lim_{\iota} (u_1)_{K_\iota} = 0$ and $\lim_{\iota} (u_2)_{K_\iota} = 0$. Hence $0 \leq \lim_{\iota} (u_1 + u_2)_{K_\iota} \leq \lim_{\iota} (u_1)_{K_\iota} + \lim_{\iota} (u_2)_{K_\iota} = 0$. Therefore, $u_1 + u_2 \in \mathcal{P}$.

LEMMA 10. *Suppose* $\mathcal{P} \neq \{0\}$. *If* $\bar{F} \cap \partial\Omega_0 = \emptyset$, *then* $u_F \in \mathcal{P}$ *for any non-negative full-superharmonic function* u *on* Ω_0.

PROOF: By assumption, there exists $v_0 \in \mathcal{P}$ such that $v_0 > 0$ on Ω_0. Let K be an admissible compact set such that $F \cap \Omega_0 \cap K = \emptyset$ and let $\alpha = \inf_{\partial K \cap \Omega_0} v_0$. Then $\alpha > 0$. Let $u = h + v$ with h full-harmonic and $v \in \mathcal{P}$. If $h = 0$, then $u_F = v_F \in \mathcal{P}$. Suppose $h \neq 0$ and let $\beta = \sup_{\partial K \cap \Omega_0} h$. Then $0 < \beta < +\infty$. $(\beta/\alpha)v_0 - h$ is full-superharmonic on $\Omega_0 - K$ and $\varliminf_{x \to \xi, x \in \Omega_0 - K} [(\beta/\alpha)v_0(x) - h(x)] \geq 0$ for all $\xi \in \partial K \cap \Omega_0$. Hence, by Theorem 1, $(\beta/\alpha)v_0 \geq h$ on $\Omega_0 - K$, and hence on $F \cap \Omega_0$. Therefore, $(\beta/\alpha)v_0 \geq h_F$. Since $(\beta/\alpha)v_0 \in \mathcal{P}$, $h_F \in \mathcal{P}$. Obviously, $v_F \in \mathcal{P}$. Hence $h_F + v_F \in \mathcal{P}$ by the above corollary. Since $u_F \leq h_F + v_F$, it follows that $u_F \in \mathcal{P}$.

LEMMA 11. *Let* u *be a non-negative full-superharmonic function on* Ω_0.

(i) *If* $u \in \mathcal{P}$ *and if* u *is harmonic on* $K_0^i \cap \Omega_0$ *for some admissible compact set* K_0, *then, for any admissible compact set* K *such that* $K \subset K_0^i$, $u_{\partial K} = u$ *on* $K^i \cap \Omega_0$.

(ii) *Suppose* $\mathcal{P} \neq \{0\}$. *If there exists an admissible compact set* K *such that* $u_{\partial K} = u$ *on* $K^i \cap \Omega_0$, *then* $u \in \mathcal{P}$.

PROOF: (i) Let v be any non-negative full-superharmonic function on Ω_0 such that $v \geq u$ on $\partial K \cap \Omega_0$ and let

$$v_1 = \begin{cases} 0 & \text{on } \Omega_0 - K^i \\ \inf(v - u, 0) & \text{on } K^i \cap \Omega_0. \end{cases}$$

Then it is easy to see that v_1 is a non-positive full-superharmonic function on Ω_0. Since $-v_1 \leq u$ everywhere, $u \in \mathcal{P}$ implies $-v_1 \leq 0$, so that $v_1 = 0$. Hence

$v \geq u$ on $K^i \cap \Omega_0$. Then it follows that $u = u_{\partial K}$ on $K^i \cap \Omega_0$.

(ii) By Lemma 10, $u_{\partial K} \in \mathcal{P}$. Hence, Lemma 7 implies that $u \in \mathcal{P}$.

Lemma 12. *Let K be an admissible compact set. Then there exists a positive constant M_K such that*

$$\sup_{x \in K \cap \Omega_0} |u_1(x) - u_2(x)| \leq M_K \sup_{x \in \partial K \cap \Omega_0} |u_1(x) - u_2(x)|$$

for any u_1, $u_2 \in \mathcal{P}$ which are harmonic on an open subset of Ω_0 containing $K \cap \Omega_0$.

Proof: Let u_0 be a positive continuous full-superharmonic function on Ω_0 such that u_0 is bounded near $\partial\Omega_0$. Let $\beta = \sup_{K \cap \Omega_0} u_0$ and $\alpha = \inf_{\partial K \cap \Omega_0} u_0$. Then $0 < \alpha \leq \beta < +\infty$. For any u_1, $u_2 \in \mathcal{P}$ satisfying the condition of the lemma, let $\lambda = \sup_{x \in \partial K \cap \Omega_0} |u_1(x) - u_2(x)|$. Then $\lambda < +\infty$ and $|u_1 - u_2| \leq \lambda \leq (\lambda/\alpha) u_0$ on $\partial K \cap \Omega_0$. $u_1 \leq (\lambda/\alpha) u_0 + u_2$ on $\partial K \cap \Omega_0$ implies that $(u_1)_{\partial K} \leq (\lambda/\alpha) u_0 + u_2$. Since $u_1 \in \mathcal{P}$, Lemma 11, (i) implies $u_1 \leq (\lambda/\alpha) u_0 + u_2$ on $K^i \cap \Omega_0$. Hence $u_1 - u_2 \leq (\lambda/\alpha) u_0 \leq (\beta/\alpha)\lambda$ on $K \cap \Omega_0$. Similarly, we have $u_2 - u_1 \leq (\beta/\alpha)\lambda$ on $K \cap \Omega_0$. Hence it is enough to take $M_K = \beta/\alpha$.

Remark: (i) If we take $u_1 = u$ and $u_2 = 0$ in the above lemma, we have

$$u \leq M_K \sup_{x \in \partial K \cap \Omega_0} u(x)$$

on $K \cap \Omega_0$.

(ii) if $u \equiv 1$ is a full-superharmonic function, then we can take $M_K = 1$.

§ 5 Decomposition of full-superharmonic functions of potential type

5.1. *Space $\mathcal{P}_b$*. Let $\mathcal{P}_b$ be the set of all harmonic full-superharmonic functions of potential type, i.e., $\mathcal{P}_b = \mathcal{P} \cap \mathcal{H}_{\Omega_0}$. Obviously, $\mathcal{P}_b$ is a cone. Hereafter, we assume that $\mathcal{P}_b \neq \{0\}$.

Let v_1, $v_2 \in \mathcal{P}$. If there exists $w \in \mathcal{P}$ such that $v_1 + w = v_2$, then we write $v_1 \prec v_2$. The relation $\prec$ is an order relation in $\mathcal{P}$. If v_1, $v_2 \in \mathcal{P}_b$, then $v_1 \prec v_2$ if and only if $v_2 - v_1 \in \mathcal{P}_b$. Hence the relation $\prec$ restricted on $\mathcal{P}_b$ is the order induced by $\mathcal{P}_b$ itself.

Lemma 13. *$\mathcal{P}_b$ is a lattice with respect to the order $\prec$.*

Proof: Let w_1, $w_2 \in \mathcal{P}_b$. Consider the family $\mathcal{U} = \{u \in \mathcal{P};\ u \succ w_1$ and $u \succ w_2\}$. Since $w_1 + w_2 \in \mathcal{U}$, $\mathcal{U} \neq \emptyset$. Let $w_0 = \inf \mathcal{U}$. It follows from (γ) in 3.3 that w_0 is non-negative nearly full-superharmonic on Ω_0. Since w_1, w_2 are harmonic on Ω_0, we see that $\mathcal{U}$ is a Perron's family on Ω_0. Hence, w_0 is harmonic and full-superharmonic. Since $0 \leq w_0 \leq w_1 + w_2 \in \mathcal{P}$, $w_0 \in \mathcal{P}_b$. Similarly, we see that $w_0 - w_1 \in \mathcal{P}_b$ and $w_0 - w_2 \in \mathcal{P}_b$, i.e., $w_0 \succ w_1$ and $w_0 \succ w_2$.

If $w' \in \mathcal{P}_b$ satisfies $w' \succ w_1$ and $w' \succ w_2$, then $w' \in \mathcal{U}$. Hence $w' \geq w_0$. We shall show that $w'-w_0$ is full-superharmonic. Then, $w'-w_0 \in \mathcal{P}_b$ or $w' \succ w_0$ and we conclude that w_0 is the least upper bound of w_1 and w_2.

To show that $w'-w_0$ is full-superharmonic, it is enough to prove that $w'(x)-w_0(x) \geq \int (w'-w_0) d\mu_x^{\Omega_0-K}$ for any admissibile compact set K such that $\Omega_0-K \in \mathfrak{G}_r$ and for all $x \in \Omega_0-K$. Let

$$g(x)=\begin{cases} \inf\{w'(x)-\int (w'-w_0) d\mu_x^{\Omega_0-K}, w_0(x)\} & \text{if } x \in \Omega_0-K \\ w_0(x) & \text{if } x \in K \cap \Omega_0. \end{cases}$$

Then, we see that g is non-negative full-superharmonic on Ω_0. Since $0 \leq g \leq w_0 \in \mathcal{P}$, $g \in \mathcal{P}$. Similarly, replacing w' by $w'-w_i$ $(i=1, 2)$ and w_0 by w_0-w_i $(i=1, 2)$, we see that $g-w_i \in \mathcal{P}$ $(i=1, 2)$. Hence $g \in \mathcal{U}$, so that $w_0 \leq g$. Hence $w'(x)-w_0(x) \geq \int (w'-w_0)\, d\mu_x^{\Omega_0-K}$.

We have shown that $\mathcal{P}_b$ is an upper semi-lattice with respect to the order induced by itself. Since $\mathcal{P}_b$ is a cone in the linear space $\mathcal{H}_{\Omega_0}$, it follows that $\mathcal{P}_b$ is a lattice with respect to this order.

We shall write $w_0 = w_1 \vee w_2$.

5.2. *Decomposition theorem*. We denote by $\mathcal{P}_i$ the set of $v \in \mathcal{P}$ such that there is no non-zero $w \in \mathcal{P}_b$ satisfying $v \succ w$. The purpose of this section is to show that any $v \in \mathcal{P}$ has a unique decomposition $v = v_b + v_i$ with $v_b \in \mathcal{P}_b$ and $v_i \in \mathcal{P}_i$.

For any $v \in \mathcal{P}$, let $\mathcal{B}(v) = \{w \in \mathcal{P}_b;\, v \succ w\}$. Since $0 \in \mathcal{B}(v)$, $\mathcal{B}(v)$ is not empty. $v \in \mathcal{P}_i$ if and only if $\mathcal{B}(v) = \{0\}$. For each $w \in \mathcal{B}(v)$, $w \prec v$. Hence $Bv = \sup \mathcal{B}(v)$ exists and $Bv \leq v$.

LEMMA 14. *$Bv \in \mathcal{B}(v)$ and Bv is the least upper bound of $\mathcal{B}(v)$ with respect to $\prec$.*

PROOF: If $w_1, w_2 \in \mathcal{B}(v)$, then $w_1 \vee w_2 \in \mathcal{P}_b$ and an argument similar to the proof of Lemma 13 shows that $w_1 \vee w_2 \prec v$. Hence, $w_1 \vee w_2 \in \mathcal{B}(v)$. It follows that $\mathcal{B}(v)$ is an upper directed family of harmoic full-superharmonic functions dominated by $v \in \mathcal{P}$. Hence $Bv \in \mathcal{P}_b$. Since $v-w \in \mathcal{P}$ for any $w \in \mathcal{B}(v)$, $v-Bv$ is nearly full-superharmonic. It then follows that $v-Bv \in \mathcal{P}$, i.e., $v \succ Bv$. Hence $Bv \in \mathcal{B}(v)$. For any $w \in \mathcal{B}(v)$, $Bv \vee w \in \mathcal{B}(v)$ by the above argument. Hence $Bv \vee w \leq Bv$, so that $Bv \vee w = Bv$, or $Bv \succ w$. Therefore, Bv is the least upper bound of $\mathcal{B}(v)$.

THEOREM 5. *If $v \in \mathcal{P}$, then $Bv \in \mathcal{P}_b$ and $v-Bv \in \mathcal{P}_i$. Conversely, if $v = v_b + v_i$ with $v_b \in \mathcal{P}_b$ and $v_i \in \mathcal{P}_i$, then $v_b = Bv$,* i.e., *the decomposition of v into a sum of a $\mathcal{P}_b$-function and a $\mathcal{P}_i$-function is unique.*

PROOF: By the above lemma, $Bv \in \mathcal{P}_b$. If $w' \in \mathcal{P}_b$ satisfies $w' \prec v-Bv$, then $w'+Bv < v$, i.e., $w'+Bv \in \mathcal{B}(v)$. Hence $w'+Bv \leq Bv$ or $w'=0$. Hence $v-Bv \in \mathcal{P}_i$. Now suppose $v=v_b+v_i$ with $v_b \in \mathcal{P}_b$ and $v_i \in \mathcal{P}_i$. Since $v_b \in \mathcal{B}(v)$, $v_b < Bv$ by the above lemma. Since $v_i=(Bv-v_b)+(v-Bv)$, $v_i > Bv-v_b \in \mathcal{P}_b$. Hence, by the definition of $\mathcal{P}_i$, $Bv-v_b=0$ or $Bv=v_b$.

5.3. *Additivity of the B-operation.* Let $v \in \mathcal{P}$ and let K be an admissible compact set. Let $\mathcal{B}_K(v)$ be the family of functions $u \in \mathcal{P}$ such that there exists a full-superharmonic function s (may be negative) on Ω_0-K such that $u=v+s$. Since $v \in \mathcal{B}_K(v)$, $\mathcal{B}_K(v) \neq \emptyset$. Let $B_K v=\inf \mathcal{B}_K(v)$.

LEMMA 15. (i) *$B_K v \in \mathcal{P}$ and it is harmonic on $K^i \cap \Omega_0$.*

(ii) *$B_K v < v$, in fact, $v=B_K v+w_K$ with $w_K \in \mathcal{P}$ which is full-harmonic on Ω_0-K.*

PROOF: (i) Let $\mathcal{U}$ be the family of full-superharmonic functions s on Ω_0-K such that $u=v+s$ on Ω_0-K for some $u \in \mathcal{B}_K(v)$. Then, it is easy to see that $\mathcal{U}$ satisfies the conditions of Theorem 3 for Ω_0-K, so that $s_0=\inf \mathcal{U}$ is full-harmonic on Ω_0-K. On the other hand, $B_K v$ is nearly full-superharmonic on Ω_0 and $0 \leq B_K v \leq v$. Since $B_K v=v+s_0$ on Ω_0-K, $\widehat{B_K v}=v+s_0$ on Ω_0-K. It follows that $\widehat{B_K v} \in \mathcal{B}_K(v)$ and hence $\widehat{B_K v} \geq B_K v$. Therefore, $\widehat{B_K v}=B_K v$, so that $B_K v \in \mathcal{P}$. Since $\mathcal{B}_K(v)$ is a Perron's family on $K^i \cap \Omega_0$, $B_K v$ is harmonic on $K^i \cap \Omega_0$.

(ii) Let

$$w_1=\begin{cases} v-B_K v & \text{if } B_K v<\infty \\ \infty & \text{if } B_K v=\infty. \end{cases}$$

Then, $w_1 \geq 0$. For any relatively compact regular domain D (resp. $D \in \mathfrak{D}_r$) such that $\bar{D} \subset \Omega_0$, $f(x)=\int v \, d\mu_x^D$ and $g(x)=\int B_K v \, d\mu_x^D$ are harmonic (resp. full-harmonic) on D. It follows that $\int w_1 \, d\mu_x^D=f(x)-g(x)$ for $x \in D$. Next, let φ be any continuous function such that $\varphi \leq v$ on ∂D, let $f_1(x)=\int \varphi d\mu_x^D$ for $x \in D$ and let

$$v_1=\begin{cases} \inf(v-f_1+g, B_K v) & \text{on } D \\ B_K v & \text{on } \Omega_0-D. \end{cases}$$

Then, we see that v_1 is a non-negative full-superharmonic function on Ω_0 and $v_1 \leq B_K v \in \mathcal{P}$. Hence $v_1 \in \mathcal{P}$. Similarly,

$$s=\begin{cases} \inf(-f_1+g, s_0) & \text{on } D-K \\ s_0 & \text{on } (\Omega_0-K)-D \end{cases}$$

is full-superharmonic on Ω_0-K and $s+v=v_1$ on Ω_0-K. Hence, $v_1\in\mathcal{B}_K(v)$, so that $v_1\geq B_Kv$. Therefore $v-f_1+g\geq B_Kv$ on D, or $w_1\geq f_1-g$ on D. Since $f=\sup_{\varphi\leq v}f_1$, we have $w_1(x)\geq f(x)-g(x)=\int w_1\,d\mu_x^D$ for $x\in D$. Thus, we have shown that w_1 is nearly full-superharmonic on Ω_0. It follows that $\hat{w}_1+B_Kv=v$. We take $w_K=\hat{w}_1$. Then $w_K\in\mathcal{P}$ and $w_K=-s_0$ on Ω_0-K, so that w_K is full-harmonic on Ω_0-K.

LEMMA 16. $Bv=\inf_{K:\ \text{admissible}} B_Kv$.

PROOF: Let $v_0=\inf_K B_Kv$. This is nearly full-superharmonic. If $K_1\subset K_2$, then $B_{K_1}v\geq B_{K_2}v$. Hence $\{B_Kv; K \text{ admissible}\}$ is a lower directed family. It follows from Lemma 15, (i) that v_0 is harmonic on Ω_0, and hence full-superharmonic on Ω_0. Since $v_0\leq v\in\mathcal{P}$, $v_0\in\mathcal{P}_b$. By Lemma 15, (ii), $v=v_0+\sup_K w_K$. We see that $\sup_K w_K\in\mathcal{P}$, so that $v\succ v_0$. Hence $v_0\leq Bv$. On the other hand, $u=B_Kv-Bv$ is superharmonic on Ω_0 and, since $u=-w_K+(v-Bv)$ on Ω_0-K, it is full-superharmonic on Ω_0. Since $-u\leq Bv$ and $Bv\in\mathcal{P}$, $u\geq 0$, i.e., $B_Kv\geq Bv$. Hence $v_0\geq Bv$.

THEOREM 6. $B(v_1+v_2)=Bv_1+Bv_2$.

PROOF: Since $Bv_1+Bv_2\prec v_1+v_2$ and $Bv_1+Bv_2\in\mathcal{P}_b$, $Bv_1+Bv_2\in\mathcal{B}(v_1+v_2)$. Hence $Bv_1+Bv_2\leq B(v_1+v_2)$. On the other hand, it is easy to see that $B_Kv_1+B_Kv_2\in\mathcal{B}_K(v_1+v_2)$, so that $B_Kv_1+B_Kv_2\geq B_K(v_1+v_2)$. Hence, by the above lemma, we have $Bv_1+Bv_2\geq B(v_1+v_2)$.

COROLLARY. *$\mathcal{P}_i$ is a cone.*

PROOF: It is enough to recall that $v\in\mathcal{P}_i$ if and only if $Bv=0$.

If we write $Jv=v-Bv$, then B and J are additive mappings of $\mathcal{P}$ into $\mathcal{P}_b$ and $\mathcal{P}_i$, respectively. It is easy to see that $\mathcal{P}_b=B(\mathcal{P})$, $\mathcal{P}_i=J(\mathcal{P})$ and $\mathcal{P}_b\cap\mathcal{P}_i=\{0\}$. It follows that B and J are mutually orthogonal projections, i.e., $B^2=B$, $J^2=J$ and $BJ=JB=0$.

§ 6 Integral representation of $\mathcal{P}_b$-functions.

In this section, we assume that $\mathcal{P}_b\neq\{0\}$. Furthermore, we assume that *Ω is countable at infinity.* i.e. there exists a sequence $\{K_n\}$ of compact sets such that $K_n\subset K_{n+1}^i$ for each n and $\bigcup_n K_n=\Omega$. We can choose K_n admissible. Such a sequence $\{K_n\}$ will be called an admissible compact exhaustion of Ω.

6.1. *Metric on $\mathcal{P}_b$.*

LEMMA 17. *The set $\mathcal{P}_b$ with the compact convergence topology is metrizable.*

PROOF: Let $\{K_n\}$ be an admissible compact exhaustion of Ω. For u_1,

$u_2 \in \mathcal{P}_b$, we define

$$\rho(u_1, u_2)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\sup_{x\in K_n\cap\Omega_0}|u_1(x)-u_2(x)|}{1+\sup_{x\in K_n\cap\Omega_0}|u_1(x)-u_2(x)|}.$$

Obviously, ρ is a metric on $\mathcal{P}_b$. We shall show that this metric is compatible with the compact convergence topology on $\mathcal{P}_b$.

Fix $u_0 \in \mathcal{P}_b$. For any $\varepsilon>0$, choose n so large that $1/2^{n-1}<\varepsilon$. Let $\delta=\varepsilon/2(2nM_{K_n})$, where M_{K_n} is the constant given in Lemma 12. If $u \in \mathcal{P}_b$ satisfies $|u(x)-u_0(x)|<\delta$ on $\partial K_n\cap\Omega_0$, then $|u(x)-u_0(x)|<\varepsilon/2n$ on $K_n\cap\Omega_0$. Hence $\rho(u, u_0)<\varepsilon/2+n(\varepsilon/2n)=\varepsilon$. Hence the compact convergence topology is stronger than the ρ-topology. Conversely, let a compact set F in Ω_0 and $\varepsilon>0(\varepsilon<1)$ be given. There exists m such that $K_m\supset F$. If $\rho(u, u_0)<\varepsilon/2^{m+1}$, then

$$\frac{1}{2^m}\frac{\sup_{x\in K_m\cap\Omega_0}|u(x)-u_0(x)|}{1+\sup_{x\in K_m\cap\Omega_0}|u(x)-u_0(x)|}<\rho(u, u_0)<\frac{\varepsilon}{2^{m+1}}.$$

Hence, $\sup_{x\in F}|u(x)-u_0(x)|\leq\sup_{x\in K_m\cap\Omega_0}|u(x)-u_0(x)|<\varepsilon$. Thus, we have the lemma.

LEMMA 18. *$\mathcal{P}_b$ is complete with respect to ρ.*

PROOF: Let $u_n \in \mathcal{P}_b$ and $\{u_n\}$ converge locally uniformly on Ω_0. Let u be the limit function. It is easy to see that u is harmonic, full-superharmonic on Ω_0. Hence it is enough to show that $u \in \mathcal{P}$. Let K be an admissible compact set and let v be a non-negative full-superharmonic function such that $v\geq u$ on $\partial K\cap\Omega_0$. Since $u_n\to u$ uniformly on $\partial K\cap\Omega_0$, there exists $n_0=n(\varepsilon)$ for any $\varepsilon>0$ such that $n\geq n_0$ implies $v\geq u_n-\varepsilon$ on $\partial K\cap\Omega_0$. Let s_0 be a positive full-superharmonic function such that $s_0\leq M<+\infty$ on $K^i\cap\Omega_0$. We may assume that $\inf_{\partial K\cap\Omega_0}s_0\geq 1$. Then $v+\varepsilon s_0\geq u_n$ on $\partial K\cap\Omega_0$, so that $v+\varepsilon s_0\geq(u_n)_{\partial K}$. By Lemma 11, $v+\varepsilon s_0\geq u_n$ on $K^i\supset\Omega_0$ $(n\geq n_0)$. Hence, $v+\varepsilon s_0\geq u$ on $K^i\cap\Omega_0$. Since ε is arbitrary and $s_0\leq M$ on $K^i\cap\Omega_0$, it follows that $v\geq u$ on $K^i\cap\Omega_0$. Hence $u_{\partial K}=u$ on $K^i\cap\Omega_0$. Then, Lemma 11 implies $u \in \mathcal{P}$.

6.2. *Representation theorem.* We fix $x_0 \in \Omega_0$ and let $\mathcal{P}_{b,0}=\{u \in \mathcal{P}_b;\ u(x_0)=1\}$. Obviously, $\mathcal{P}_{b,0}$ is closed in $\mathcal{P}_b$. Hence, Lemma 1, Lemma 17 and Lemma 18 imply that $\mathcal{P}_{b,0}$ is compact metrizable with respect to the compact convergence topology. Also, $\mathcal{P}_{b,0}$ is a convex set and is a base of the cone $\mathcal{P}_b$, which is a lattice with respect to the order induced by itself (Lemma 13). Therefore, we can apply Choquet's theorem (see [3], [4] and [11]) to the cone $\mathcal{P}_b$ and a base $\mathcal{P}_{b,0}$ and we obtain:

THEOREM 7. *For any $u \in \mathcal{P}_b$, there exists a unique Radon measure μ on the compact space $\mathcal{P}_{b,0}$ such that $\mu(\mathcal{P}_{b,0}-e(\mathcal{P}_{b,0}))=0$ and $u=\int v\,d\mu(v)$, where $e(\mathcal{P}_{b,0})$ is the set of all extreme points of the convex set $\mathcal{P}_{b,0}$. The total mass* *of μ is equal to $u(x_0)$.*

6.3. *The special case where a kernel is defined.* Let $x_0 \in \Omega_0$ be fixed. An extended real valued function $k_x(y)=k(x, y)$ on $\Omega_0 \times \Omega_0$ is called a kernel on Ω_0 (with respect to the full-harmonic structure $\tilde{\mathfrak{H}}$, normalized at x_0) if it satisfies the following three conditions:

(i) For each $x \in \Omega_0$, $k_x \in \mathcal{P}$, k_x is full-harmonic on $\Omega_0 - \{x\}$ and $k_x(x_0)=1$ for $x \in \Omega_0 - K_0$, where K_0 is an admissible compact set such that $x_0 \in K_0^i$.

(ii) For each $y \in \Omega_0$, the mapping $x \rightarrow k(x, y)$ is continuous on $\Omega_0 - \{y, x_0\}$.

(iii) For any admissible compact set K such that $K^i \supset K_0$, if $v \in \mathcal{P}$ is full-harmonic on $\Omega_0 - K$ and harmonic on $K^i \cap \Omega_0$, then there exists a Radon measure μ on $\partial K \cap \Omega_0$ such that

$$v(y)=\int_{\partial K \cap \Omega_0} k_x(y) d\mu(x)$$

for all $y \in \Omega_0$.

We assume the existence of a kernal $k_x(y)$ and consider a realization of $e(\mathcal{P}_{b,0})$ as a part of an ideal boundary of Ω_0. For each $y \in \Omega_0$, there exists a continuous function f_y on $\bar{\Omega}_0$ such that $f_y(x)=k_x(y)$ for $x \in \Omega_0 - K$ for some admissible compact set K. The property 3) in 1.2 allows us to take f_y to be bounded. Let C_0 be the space of all continuous functions on Ω with compact support and let $Q=\{f_y|_{\Omega_0};\ y \in \Omega_0\} \cup \{f|_{\Omega_0}; f \in C_0\}$. Let Ω_0^* be the Q-compactification (see [6]) of Ω_0. Then $\Omega_0^* - \Omega_0$ consists of two compact parts Δ and $\partial\Omega_0$, where Δ is characterized as follows: If $x \in \Omega_0$ tends to a point $\xi \in \Delta$, then x tends to the ideal boundary of Ω and k_x converges everywhere.

THEOREM 8. *$e(\mathcal{P}_{b,0})$ is homeomorphically embedded in Δ.*

PROOF: First, we shall show that to each $\xi \in \Delta$ there corresponds a $k_\xi \in \mathcal{P}_{b,0}$. Let $x \in \Omega_0$ tend to $\xi \in \Delta$. By the property of Δ and by Lemma 1, k_x converges locally uniformly on Ω_0. Let k_ξ be the limit function. It follows, like Lemma 18, that $k_\xi \in \mathcal{P}_b$. Obviously $k_\xi(x_0)=1$, so that $k_\xi \in \mathcal{P}_{b,0}$. By the definition of the compactification, k_ξ is uniquely determined by ξ and if $\xi_1 \neq \xi_2$ ($\xi_1, \xi_2 \in \Delta$), then $k_{\xi_1} \neq k_{\xi_2}$. Hence the mapping $\xi \rightarrow k_\xi$ is one-to-one from Δ into $\mathcal{P}_{b,0}$. Also, we see that this mapping is a homeomorphism.

Now, it is enough to show that, for each $u \in e(\mathcal{P}_{b,0})$, there exists $\xi \in \Delta$ such that $u=k_\xi$. Let $\{K_n\}$ be an admissible compact exhaustion such that $K_1^i \supset K_0$. Since $u_{K_n} \in \mathcal{P}$ and is full-harmonic on $\Omega_0 - K_n$, harmonic on $K_n^i \cap \Omega_0$, condition (iii) for k_x implies that there exists a Radon measure μ_n on $\partial K_n \cap \Omega_0$ such that $u_{K_n}(y)=\int k_x(y)\, d\mu_n(x)$. We regard μ_n as a measure on Ω_0^*. Since the total mass of μ_n is less than one, $\{\mu_n\}$ has a subsequence $\{\mu_{n_j}\}$ vaguely converging to a measure μ_0 on Ω_0^*. It is obvious that μ_0 is supported on Δ. We have $u(y)=\lim_{n\rightarrow\infty} u_{K_n}(y)=\lim_{j\rightarrow\infty}\int k_x(y)\, d\mu_{n_j}(x)=\int k_\xi(y) d\mu_0(\xi)$, since the mapping $x \rightarrow k_x(y)$ is continuous on $\Omega_0 \cup \Delta - \{x_0, y\}$. It is easy to see that, if ν is a unit measure on Δ, then $\int k_\xi d\nu(\xi) \in \mathcal{P}_{b,0}$. Since $u \in e(\mathcal{P}_{b,0})$, it follows that μ_0 is a point mass, so that $u = k_\xi$ for some $\xi \in \Delta$. (See [6] for similar discussions.)

Examples. In the case of Example 1 (§2), it is known that there exists a kernel in the above sense. The boundary Δ in this case is a part of the Kuramochi boundary of Ω. (See [11]; also [6], [9] and [13].)

For Example 3, a kernel is defined by $k_x(y) = G_x(y)/G_x(x_0)$, where $G_x(y)$ is the Green function of Ω_0, and the corresponding boundary Δ is a part of the Martin boundary of Ω (cf. [14]; also [6]).

In the above two examples, if $\Omega - \Omega_0$ is compact, then we obtain the whole of the Kuramochi boundary and the Martin boundary of Ω. Thus, we can say that these two boundaries arise from different full-harmonic structures subordinate to the same (classical) harmonic structure.

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