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Order of the Identity Class of a Loop Space

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Introduction.

For a topological space X with base point, its loop space ΩX is a homotopy-associative H-space with a homotopy-inverse, and so the set of homotopy classes of continuous maps from a topological space Y into ΩX, fixing base point, forms a group $\pi_0(Y; \Omega X)$. Consider the class

$$\iota_{\Omega X}=[1_{\Omega X}] \in \pi_0(\Omega X; \Omega X)$$

of the identity map $1_{\Omega X}$ of ΩX onto itself, and call the order of $\iota_{\Omega X}$ simply the *loop-order* of X.

The loop-order is clearly a homotopy type invariant. In this note, we discuss its general properties, where the dual situation to the suspension-order of Toda [2] may be seen.

1. Preliminary and definition.

For each topological space X, we always associate a point $*$, called the base point. (Continuous) maps and homotopies considered are base point preserving.

The set of the homotopy classes of maps $f:(X, *)\rightarrow(Y, *)$ is denoted by

$$\pi_0(X; Y).$$

Let $\alpha \in \pi_0(X; Y)$ and $\beta \in \pi_0(Y; Z)$ be the classes of maps $f: X\rightarrow Y$ and $g: Y\rightarrow Z$ respectively, then the composition $\beta\circ\alpha \in \pi_0(X; Z)$ is the class of the composition $g\circ f$ of maps. The formula $\beta\circ\alpha=f^*(\beta)=g_*(\alpha)$ defines two mappings

$$f^*: \pi_0(Y; Z)\rightarrow\pi_0(X; Z) \quad \text{and} \quad g_*: \pi_0(X; Y)\rightarrow\pi_0(X; Z).$$

The *loop space* ΩX of X is the space of all loops $w:(I, \dot{I})\rightarrow(X, *)$ ($I=[0, 1]$, $\dot{I}=\{0, 1\}$) with compact-open topology, and the constant loop is its base point. In this note, we assume that spaces are simply connected whenever their loop spaces are considered, and so ΩX is arcwise connected.

The product $\mu: \Omega X\times\Omega X\rightarrow\Omega X$ of the H-space ΩX is defined by

$$(w_1, w_2)(t)=w_1(2t) \quad (0\leqq t\leqq 1/2), \quad =w_2(2t-1) \quad (1/2\leqq t\leqq 1),$$

for $w_1, w_2 \in \Omega X$. The group multiplication $\alpha+\beta$ of $\alpha, \beta \in \pi_0(Y; \Omega X)$ is defined to be the homotopy class of the composition

$$\mu\circ(f\times g)\circ d : Y \xrightarrow{d} Y\times Y \xrightarrow{f\times g} \Omega X\times\Omega X \xrightarrow{\mu} \Omega X,$$

where f, g are maps of the classes α, β respectively, and d is the diagonal map.

A map $f: X\rightarrow Y$ defines a map $\Omega f: \Omega X\rightarrow\Omega Y$ by

$$(\Omega f)(w)=f\circ w \qquad (w\in\Omega X),$$

and Ωf commutes with the multiplications of loop spaces, i.e., the following diagram is commutative:

$$\begin{tikzcd}
\Omega X\times\Omega X \arrow[r, "\mu"] \arrow[d, "\Omega f\times\Omega f"] & \Omega X \arrow[d, "\Omega f"] \\
\Omega Y\times\Omega Y \arrow[r, "\mu"] & \Omega Y.
\end{tikzcd}$$

Also, a class $\Omega\alpha\in\pi_0(\Omega X; \Omega Y)$ of Ωf is determined by the class $\alpha\in\pi_0(X; Y)$ of f, and the mapping

$$\Omega: \pi_0(X; Y)\rightarrow\pi_0(\Omega X; \Omega Y)$$

is defined. From the definitions, we have easily

LEMMA 1. $$(\beta_1+\beta_2)\circ\alpha=\beta_1\circ\alpha+\beta_2\circ\alpha$$

for $\alpha\in\pi_0(X; Y)$, $\beta_1, \beta_2\in\pi_0(Y; \Omega Z)$; *and*

$$\beta\circ(\alpha_1+\alpha_2)=\beta\circ\alpha_1+\beta\circ\alpha_2$$

for $\alpha_1, \alpha_2\in\pi_0(X; \Omega Y)$, $\beta\in\Omega\pi_0(Y; Z)$ $(\subset\pi_0(\Omega Y; \Omega Z))$. *Therefore*

$$f^*: \pi_0(Y; \Omega Z)\rightarrow\pi_0(X; \Omega Z), \quad (\Omega g)_*: \pi_0(X; \Omega Y)\rightarrow\pi_0(X; \Omega Z)$$

are homomorphisms for any maps $f: X\rightarrow Y$ *and* $g: Y\rightarrow Z$.

Denote by $1_X: X\rightarrow X$ the identity map of X onto itself, and by $\iota_X\in\pi_0(X; X)$ its homotopy class. The *loop-order* $l(X)$ of a topological space X is the order of the class $\iota_{\Omega X}=\Omega\iota_X\in\pi_0(\Omega X; \Omega X)$. Obviously the loop-order is a homotopy type invariant of spaces.

The class $l\iota_{\Omega X}$ is represented by the composition

$$\mu_l\circ d_l : \Omega X\xrightarrow{d_l}(\Omega X)^l\xrightarrow{\mu_l}\Omega X,$$

where $(\Omega X)^l$ is the product space $\Omega X\times\cdots\times\Omega X$ of l copies of ΩX, d_l is the diagonal map and μ_l is the map defined inductively by $\mu_l=\mu\circ(\mu_{l-1}\times 1_{\Omega X})$. Therefore, the loop-order $l=l(X)$ of X is the least positive integer l such that

$\mu_l \circ d_l$ is homotopic to zero; if no such integer exists, $l(X)=\infty$.

2. Product spaces and fiber spaces.

Concerning the product spaces, we have the following

THEOREM 2. *The loop-order of the product spaces $X_1 \times X_2$ is the least common multiple of that of X_1 and X_2:*

$$l(X_1 \times X_2) = l.c.m. \text{ of } l(X_1) \text{ and } l(X_2).$$

PROOF. Let $p_i: X = X_1 \times X_2 \to X_i$ $(i=1, 2)$ be the natural projections, and $i_1: X_1 = X_1 \times * \subset X$, $i_2: X_2 = * \times X_2 \subset X$ be the inclusion maps. Then it is easy to see that the two mappings

$$\pi_0(Z; \Omega X) \underset{(\Omega i_1)_* + (\Omega i_2)_*}{\overset{((\Omega p_1)_*, (\Omega p_2)_*)}{\rightleftarrows}} \pi_0(Z; \Omega X_1) + \pi_0(Z; \Omega X_2)$$

are isomorphisms and one of them is the inverse of the other. Therefore

$$\begin{aligned} \iota_{\Omega X} &= ((\Omega i_1)_* + (\Omega i_2)_*)((\Omega p_1)_* \iota_{\Omega X}, (\Omega p_2)_* \iota_{\Omega X}) \\ &= (\Omega i_1)_* (\Omega p_1)^* \iota_{\Omega X_1} + (\Omega i_2)_* (\Omega p_2)^* \iota_{\Omega X_2}, \end{aligned}$$

and $l' \iota_{\Omega X} = (\Omega i_1)_* (\Omega p_1)^* (l' \iota_{\Omega X_1}) + (\Omega i_2)_* (\Omega p_2)^* (l' \iota_{\Omega X_2}) = 0$ by Lemma 1, where l' is the l.c.m. of $l(X_1)$ and $l(X_2)$. This shows that l' is a multiple of the loop-order $l(X)$.

On the other hand, $(\Omega p_1) \circ 1_{\Omega X} \circ \Omega i_1 = 1_{\Omega X_1}$ shows that

$$l \iota_{\Omega X_1} = l(\Omega p_1)_* (\Omega i_1)^* \iota_{\Omega X} = (\Omega p_1)_* (\Omega i_1)^* (l \iota_{\Omega X}) = 0,$$

where $l = l(X)$. Thus $l(X)$ is a multiple of $l(X_1)$, and of $l(X_2)$ in the same way, and the theorem is proved, q.e.d.

Now, we consider the fiber spaces. In this note, the fiber space is assumed to be in the strong sense that it has the covering homotopy property with base point for any spaces. Precisely speaking, (E, p, B) is a fiber space if the following covering homotopy property is satisfied: for any space X, a map $f: (X, *) \to (E, *)$ and a homotopy $g_t: (X, *) \to (B, *)$ such that $p \circ f = g_0$, there exists a homotopy $f_t: (X, *) \to (E, *)$ such that $f_0 = f$ and $p \circ f_t = g_t$.

If (E, p, B) is a fiber space, then it is easy to see that $(\Omega E, \Omega p, \Omega B)$ is also a fiber space.

THEOREM 3. *Let (E, p, B) be a fiber space and $F = p^{-1}(*)$ be its fiber. Then the loop-order $l(E)$ is a divisor of the multiple $l(B) \cdot l(F)$.*

PROOF. Set $l = l(B)$, then the composition $\mu_l \circ d_l: \Omega E \to (\Omega E)^l \to \Omega E$ satisfies $\Omega p \circ \mu_l \circ d_l = \mu_l \circ d_l \circ \Omega p \sim 0: \Omega E \to \Omega B$ by the definition of the loop-order $l(B)$.

Thus $\mu_l \circ d_l$ is homotopic to a map $f: \Omega E \rightarrow \Omega F$ (preserving base point) by the covering homotopy property.

Also, it is clear that

$$\mu_{lm} \circ d_{lm} \sim \mu_m \circ d_m \circ \mu_l \circ d_l : \Omega E \rightarrow \Omega E.$$

Therefore, we have

$$\mu_{lm} \circ d_{lm} \sim \mu_m \circ d_m \circ f = (\mu_m \circ d_m | \Omega F) \circ f \sim 0,$$

where $m = l(F)$. This shows that lm is a multiple of $l(E)$, q.e.d.

3. Homotopy groups.

THEOREM 4. *Assume that the loop-order $l = l(X)$ of a space X is finite, then $l\pi_i(X) = lH_i(\Omega X) = lH^i(\Omega X) = 0$ for $i > 0$, and*

$$l(\Omega\pi_0(X; Y)) = l\pi_0(Z; \Omega X) = 0$$

for any spaces Y and Z.

PROOF. It is clear that the induced homomorphism

$$(\mu_l \circ d_l)_* : H_i(\Omega X) \rightarrow H_i(\Omega X)$$

satisfies $(\mu_l \circ d_l)_* \alpha = l\alpha$ for each element $\alpha \in H_i(\Omega X)$. By the assumption, $\mu_l \circ d_l$ is homotopic to zero, and so $lH_i(\Omega X) = 0$ for all $i > 0$. The proof of $lH^i(\Omega X) = 0$ is similar.

By Lemma 1 and $l\iota_{\Omega X} = 0$ of the assumption, we have

$$l\alpha = l(\alpha \circ \iota_{\Omega X}) = \alpha \circ (l\iota_{\Omega X}) = 0$$

for $\alpha \in \Omega\pi_0(X; Y)$, and

$$l\beta = l(\iota_{\Omega X} \circ \beta) = (l\iota_{\Omega X}) \circ \beta = 0$$

for $\beta \in \pi_0(Z; \Omega X)$. Thus $l(\Omega\pi_0(X; Y)) = l\pi_0(Z; \Omega X) = 0$. In particular, $l\pi_i(X) = l\pi_0(S^i; X) = l\pi_0(S^{i-1}; \Omega X) = 0$, where S^i is the i-dimensional sphere, q.e.d.

In the rest of this note, spaces are assumed to be the same homotopy type of CW-complexes; then their loop spaces are also so [1].

If X is an Eilenberg-MacLane space $\mathcal{K}(\pi, n)$, where π is an abelian group and $n > 1$, then its loop space ΩX is $\mathcal{K}(\pi, n-1)$ and it is well known that there are the isomorphisms

$$\pi_0(\Omega X; \Omega X) \approx H^{n-1}(\pi, n-1; \pi) \approx \mathrm{Hom}(\pi, \pi).$$

Also, the image of $\iota_{\Omega X} \in \pi_0(\Omega X; \Omega X)$ under these isomorphisms is the identity isomorphism $1_\pi : \pi \rightarrow \pi$. Clearly, the order of $1_\pi \in \mathrm{Hom}(\pi, \pi)$ is equal to the least positive integer l such that $l\pi=0$, which will be denoted by $l(\pi)$.

Thus we have the following theorem for the case that X is an Eilenberg-MacLane space $\mathcal{K}(\pi, n)$.

THEOREM 5. *Let X be a space having only a finite number of non-trivial homotopy groups*

$$\pi_{i_j}(X)=\pi_j \quad (j=1, \cdots, n), \quad 1<i_1<\cdots<i_n.$$

Then the loop-order of X is a multiple of the l.c.m. of $l(\pi_1), \cdots, l(\pi_n)$, and is a divisor of $\prod_j l(\pi_j)$, where $l(\pi)$ is the least positive integer l such that $l\pi=0$ for an abelian group π.

PROOF. By Theorem 4, the loop-order $l(X)$ satisfies $l(X)\pi_i(X)=0$ for $i>0$. Therefore $l(X)$ is a multiple of $l(\pi_i(X))$, and so of the l.c.m. of $l(\pi_1), \cdots, l(\pi_n)$.

The second half of the theorem is proved by the induction of n. As is well known as the Postnikov decomposition, there exists a fiber space (E, p, B) such that E is homotopy equivalent to X,

$$p_* : \pi_i(E)\approx\pi_i(B) \quad (i<i_n), \quad \pi_i(B)=0 \quad (i\geqq i_n),$$

and its fiber F is an Eilenberg-MacLane space $\mathcal{K}(\pi_n, i_n)$. The loop-order $l(F)$ is $l(\pi_n)$ as proved above, and $l(B)$ is a divisor of $\prod_{j=1}^{n-1} l(\pi_j)$ by the inductive assumption. Therefore $l(X)=l(E)$ is a divisor of $\prod_{j=1}^{n} l(\pi_j)$ by Theorem 3, q.e.d.

4. Example.

By Theorem 2, there is a space X such that its loop-order $l(X)$ is equal to the l.c.m. of $\{l(\pi_i(X)) \mid i>0\}$. We notice that there is also a space X such that $l(X)$ is equal to the multiple $\prod_{i>0} l(\pi_i(X))$, as seen by the following example.

Let $Y=S^{n-1}\bigcup_2 e^n$ be the space obtained by attaching n-cell e^n to the $n-1$ sphere S^{n-1} by the map $2: S^{n-1}\rightarrow S^{n-1}$ of degree 2. It is well known that $\pi_n(SY)=\pi_{n+1}(SY)=Z_2$ (the group of integers mod 2), where $SY=S^n\bigcup_2 e^{n+1}$ is the suspension of Y.

Also, let X be the space obtained by attaching i-cells $(i\geqq n+3)$ to SY so as to kill the homotopy groups of dim $\geqq n+2$. Then

$$\pi_n(X)=\pi_{n+1}(X)=Z_2, \quad \pi_i(X)=0 \quad (i\neq n, n+1),$$

and the loop-order of X is a divisor of 4 by Theorem 5.

Consider the group $\pi_0(Y; \Omega X)$ which is isomorphic to $\pi_0(SY; X)$, and the induced homomorphism

$$i_* : \pi_0(SY;\ SY) \to \pi_0(SY;\ X)$$

of the inclusion map $i: SY \subset X$. Since X is the space attaching i-cells $(i > n+2)$ to SY and SY is the $n+1$ dimensional CW-complex, i_* is an isomorphism as is well known.

On the other hand, the order of $\iota_{SY} \in \pi_0(SY;\ SY)$, the suspension-order of Y, is 4 by Theorem 4.1 of [2]. Therefore, $\pi_0(Y;\ \Omega X)$ has an element of order 4, and so $l(X)$ is a multiple of 4 by Theorem 4. These show that the loop-order of X is equal to $4 = l(Z_2) \cdot l(Z_2) = \prod l(\pi_i(X))$.

References

[1] J. Milnor: *On spaces having homotopy type of a CW-complex*, Trans. Amer. Math. Soc., **90** (1959), 272-280.

[2] H. Toda: *Order of the identity class of a suspension space*, Ann. of Math., **78** (1963), 300-325.

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