J. Sci. Hiroshima Univ. Ser. A-I
31 (1967), 67–74

Remarks on the Conditional Gauss Variational Problem

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(Received February 28, 1967)

Introduction

Let X and Y be compact Hausdorff spaces, $\Phi(x, z)$ and $\Psi(x, y)$ be real-valued lower semicontinuous functions on $X\times X$ and $X\times Y$ respectively which are bounded from below. We shall always consider non-negative Radon measures on X and on Y and assume that Φ is symmetric, i.e. $\Phi(x, z)=\Phi(z, x)$ for every $x, z\in X$. For a measure μ on X, we set

$$\Phi(x, \mu)=\int\Phi(x, z)d\mu(z) \qquad (\Phi\text{-potential of } \mu),$$

$$\Psi(\mu, y)=\int\Psi(x, y)d\mu(x)$$

and

$$(\mu, \mu)=\int\Phi(x, \mu)d\mu(x).$$

Set

$$\mathscr{E}=\{\mu;\,(\mu, \mu)\ \text{is finite}\}$$

and assume that $\mathscr{E}$ contains at least one non identically vanishing measure. We put

$$(\nu, \mu)=\int\Phi(x, \mu)d\nu(x)$$

and

$$(\mu-\nu, \mu-\nu)=(\mu, \mu)+(\nu, \nu)-2(\nu, \mu).$$

We say that Φ is of positive type if the quantity $(\mu-\nu, \mu-\nu)$ is non-negative for any $\mu, \nu\in\mathscr{E}$. In case Φ is of positive type, $\|\mu-\nu\|=(\mu-\nu, \mu-\nu)^{1/2}$ is a pseudo-metric on $\mathscr{E}$. An $\mathscr{E}$-Cauchy sequence is a sequence $\{\mu_n\}$ in $\mathscr{E}$ such that, for any $\varepsilon>0$, there exists an integer n_0 such that the relations $n\geqq n_0$ and $m\geqq n_0$ imply $\|\mu_n-\mu_m\|<\varepsilon$.

Let f be a real-valued function on X which is measurable for every measure on X and let h be a real-valued function on Y for which the following classes of measures are not empty:

$$\mathscr{F}=\{\mu\in\mathscr{E};\ \Psi(\mu,\ y)=h(y)\ \text{ on }\ Y\},$$

$$\mathscr{M}=\{\mu\in\mathscr{E};\ \Psi(\mu,\ y)\leqq h(y)\ \text{ on }\ Y\}.$$

In case $\int f d\mu$ is defined for all $\mu\in\mathscr{F}$ ($\mu\in\mathscr{M}$ resp.), we are interested in the problem of minimizing the quantity

$$I(\mu)=(\mu,\ \mu)-2\int f d\mu$$

for $\mu\in\mathscr{F}$ ($\mu\in\mathscr{M}$ resp.) and denote inf $I(\mu)$ by F (M resp.). This may be regarded as a problem in quadratic programming. We shall discuss whether there exists a measure $\mu^*\in\mathscr{F}$ ($\mu^*\in\mathscr{M}$ resp.) which satisfies $I(\mu^*)=F$ ($I(\mu^*)=M$ resp.). We call μ^* an optimal measure for F (M resp.).

In case $\Phi=0$, the problem for M is reduced to finding an optimal measure in the theory of linear programming. This was studied by M. Ohtsuka [3].

In case Y consists of a finite number of points, the problem for F is the conditional Gauss variational problem raised in [2], p. 213.

Our results will be given in the case where f is an upper semicontinuous function or the Φ-potential of a measure in $\mathscr{E}$. As an application of the existence theorem in §1, we shall discuss a duality problem in the theory of quadratic programming as in [1].

§1. The case where f is upper semicontinuous

First we observe that an optimal measure for M does not necessarily exist even if Ψ, f and h are non-negative and continuous. In fact, taking $\Phi=0$, we see that Example 1 in [4] shows this fact.

We begin with

LEMMA 1. *Assume that Φ is of positive type. Assume that F (M resp.) is finite and that $\{\mu_n\}$ is a sequence in $\mathscr{F}$ ($\mathscr{M}$ resp.) for which $I(\mu_n)$ tends to F (M resp.). Then $\{\mu_n\}$ is an $\mathscr{E}$-Cauchy sequence.*

PROOF. We have

$$I((\mu_n+\mu_m)/2)=I(\mu_n)/2+I(\mu_m)/2-\|\mu_n-\mu_m\|^2/4.$$

Since $\mathscr{F}$ ($\mathscr{M}$ resp.) is convex and F (M resp.) is finite, our assertion is easily verified from the above equality.

LEMMA 2. *Assume that M is finite. Let $\{\mu_n\}$ be a sequence in $\mathscr{M}$ such that $I(\mu_n)$ tends to M. Then the boundedness of total masses $\mu_n(X)$ follows from one of the following conditions:*

(H. 1) *there is $y_0\in Y$ such that $\Psi(x,\ y_0)>0$ for all $x\in X$ and $h(y_0)<\infty$.*

(H. 2) *Φ is of positive type and $f>0$.*

(H. 3) $(\mu, \mu) \geqq 0$ *for all* $\mu \in \mathscr{E}$ *and* $\sup f < 0$.
(H. 4) $W = \inf \{(\mu, \mu); \mu(X) = 1\}$ *is positive and* f *is bounded from above.*

PROOF. From (H. 1), it follows that

$$\mu_n(X)[\inf_X \Psi(x, y_0)] \leqq \Psi(\mu_n, y_0) \leqq h(y_0) < \infty.$$

From (H. 3), it follows that

$$I(\mu_n) \geqq 2[\inf_X (-f)]\mu_n(X).$$

From (H. 4), it follows that

$$I(\mu_n) \geqq W\mu_n(X)^2 - 2\beta\mu_n(X),$$

where β is an upper bound of f.

Now we assume (H. 2). Since $\{\mu_n\}$ is an $\mathscr{E}$-Cauchy sequence in $\mathscr{M}$ by Lemma 1, (μ_n, μ_n) are bounded, i.e. $0 \leqq (\mu_n, \mu_n) \leqq L < \infty$. Suppose that $\{\mu_n(X)\}$ is not bounded. We may assume that $\mu_n(X)$ tends to ∞ with n. Writing $\lambda_n = \mu_n/\mu_n(X)$, we can find a vaguely convergent subsequence of $\{\lambda_n\}$. Denote it again by $\{\lambda_n\}$ and let λ_0 be the vague limit. Then we have

$$0 \leqq (\lambda_0, \lambda_0) \leqq \varliminf_{n\to\infty} (\lambda_n, \lambda_n) \leqq \lim_{n\to\infty} L/\mu_n(X)^2 = 0.$$

Since Φ is of positive type,

$$0 \leqq (k\lambda_0 \pm \mu_n, k\lambda_0 \pm \mu_n) = \pm 2k(\lambda_0, \mu_n) + (\mu_n, \mu_n)$$

for any $k \geqq 0$. We infer that $(\lambda_0, \mu_n) = 0$ for each n. Furthermore,

$$\Psi(\lambda_0, y) \leqq \varliminf_{n\to\infty} \Psi(\lambda_n, y) \leqq \lim_{n\to\infty} h(y)/\mu_n(X) = 0 \qquad \text{if} \quad h(y) < \infty.$$

Hence $n\lambda_0 + \mu_n \in \mathscr{M}$. It holds that

$$M \leqq I(n\lambda_0 + \mu_n) = I(\mu_n) - 2n\int f\,d\lambda_0.$$

Letting $n \to \infty$, we arrive at a contradiction because $f > 0$ and $\lambda_0(X) = 1$.

We have

THEOREM 1. *Assume that f is upper semicontinuous and does not take the value $+\infty$. If we assume one of conditions* (H. 1), (H. 2), (H. 3), (H. 4) *and that M is finite, then there exists an optimal measure for M.*

PROOF. Let $\{\mu_n\}$ be a sequence in $\mathscr{M}$ for which $I(\mu_n)$ tends to M. Then the total masses $\mu_n(X)$ are bounded by Lemma 2. We can find a vaguely convergent subsequence of $\{\mu_n\}$. Denote it again by $\{\mu_n\}$ and let μ_0 be the vague limit. On account of the lower semicontinuity of Φ, Ψ and $-f$, we have

$$\Psi(\mu_0, y) \leqq \varliminf_{n\to\infty} \Psi(\mu_n, y) \leqq h(y) \qquad \text{on } Y,$$

$$(\mu_0, \mu_0) \leqq \varliminf_{n\to\infty} (\mu_n, \mu_n) \quad \text{and} \quad \varlimsup_{n\to\infty} \int f\, d\mu_n \leqq \int f\, d\mu_0.$$

Therefore $\mu_0 \in \mathscr{M}$ and

$$M = \varliminf_{n\to\infty} I(\mu_n) \geqq \varliminf_{n\to\infty} (\mu_n, \mu_n) - 2\varlimsup_{n\to\infty} \int f\, d\mu_n$$

$$\geqq (\mu_0, \mu_0) - 2\int f\, d\mu_0 = I(\mu_0) \geqq M.$$

Namely μ_0 is an optimal measure for M.

Similarly, we have the following results for F.

LEMMA 3. *Assume that F is finite and let $\{\mu_n\}$ be a sequence in $\mathscr{F}$ such that $I(\mu_n)$ tends to F. Then the boundedness of total masses $\mu_n(X)$ follows from one of conditions* (H. 1), (H. 3), (H. 4) *and*

(H. 2)′ *Φ is of positive type, $f > 0$ and Ψ is non-negative.*

THEOREM 2. *Let f be an upper semicontinuous function which does not take the value $+\infty$. Assume one of conditions* (H. 1), (H. 2)′, (H. 3) *and* (H. 4), *that Ψ is finite and continuous and that F is finite. Then there exists an optimal measure for F.*

In the above theorem, the continuity of Ψ can not be omitted in general. This was shown in Example 2 in [2], p. 226.

§2. The case where f is a potential

In the case where f is lower semicontinuous and does not take the value $-\infty$, the existence of an optimal measure for F or M is not necessarily assured. This is shown by

EXAMPLE 1. (Example 5 in [2], p. 226) Let X be the unit ball $\{|x| \leqq 1\}$ in the 3-dimensional Euclidean space, Y consist of one point, $\Phi(x, z) = 1/|x-z|$, $\Psi = 1$, $h = 1$ and $f(x) = 1 + |x|$ for $|x| < 1$, $= 1$ for $|x| = 1$.

For $\mu \in \mathscr{M}$, we have $\mu(X) \leqq 1$ and

$$I(\mu) = (\mu, \mu) - 2\mu(X) - 2\int_{|x|<1} |x|\, d\mu(x)$$

$$\geqq \mu(X)^2 - 4\mu(X) \geqq -3.$$

If μ is the unit measure distributed uniformly on a sphere close to $\{|x| = 1\}$, then $I(\mu)$ is close to -3. Thus the infimum is equal to -3, but there is no measure of $\mathscr{M}$ which gives -3. It is also shown that there is no optimal measure for F.

In this section, we always assume that f is the Φ-potential of $\nu\in\mathscr{E}$, i.e. $f(x)=\Phi(x, \nu)$, and that Φ is of positive type.

THEOREM 3. *Assume one of conditions* (H. 1), (H. 2), *and* (H. 4), *that Φ is consistent*[1] *and that M is finite. Then there exists an optimal measure for M.*

PROOF. Let $\{\mu_n\}$ be a sequence in $\mathscr{M}$ such that $I(\mu_n)$ tends to M. Then $\{\mu_n\}$ is an $\mathscr{E}$-Cauchy sequence by Lemma 1. On account of Lemma 2, the total masses $\mu_n(X)$ are bounded. We can find a subsequence which converges vaguely to some measure μ_0. We see $\mu_0\in\mathscr{M}$ and that $\|\mu_n-\mu_0\|$ tends to 0 by the consistency of Φ. It holds that

$$M=\lim_{n\to\infty} I(\mu_n)=\lim_{n\to\infty}\{(\mu_n, \mu_n)-2(\nu, \mu_n)\}$$
$$=(\mu_0, \mu_0)-2(\nu, \mu_0)=I(\mu_0)\geqq M.$$

Similarly we have

THEOREM 4. *Assume that Φ is consistent, that Ψ is finite and continuous and that F is finite. Then under one of conditions* (H. 1), (H. 2)$'$ *and* (H. 4), *there exists an optimal measure for F.*

We show by an example that the consistency of Φ can not be dropped in the above two theorems:

EXAMPLE 2. Let X be the interval $\{0\leqq x\leqq 1\}$ in the real line, $Y=\{y\}$, $\Psi=1$, $h=1$, $\Phi(x, z)=f(x)f(z)$, $f(x)=x$ for $0\leqq x<1$, $=0$ for $x=1$. Then Φ is not consistent and $f(x)$ is the Φ-potential of the point measure at $x=1/2$ with total mass 2. For $\mu\in\mathscr{M}$, we have $\mu(X)\leqq 1$ and

$$I(\mu)=\left(\int f(x)d\mu(x)-1\right)^2-1\geqq -1.$$

If we take the unit point measure μ_n at $x=1-1/n$, then $I(\mu_n)=1/n^2-1$. Therefore the infimum of $I(\mu_n)$ is equal to -1. However, we see easily that $I(\mu)>-1$ for every $\mu\in\mathscr{M}$ and hence there is no optimal measure. It is also shown that there is no optimal measure for F.

§3. A duality problem for M

In this section, we always assume that f and $-h$ are upper semicontinuous and do not take the value $+\infty$. As in [1] and [3], we consider the fol-

1) Φ is called consistent if any $\mathscr{E}$-Cauchy sequence converging vaguely to a measure converges in the pseudo-metric to the same measure.

lowing dual problem for M:

Let $\mathscr{M}'$ be the totality of pairs $[\mu, \nu]$ of measures $\mu \in \mathscr{E}$ and ν on Y satisfying

$$\Psi(x, \nu)+\Phi(x, \mu) \geqq f(x) \qquad \text{on } X.$$

In case $\mathscr{M}'$ is not empty, we consider the problem of minimizing the quantity

$$J(\mu, \nu)=(\mu, \mu)+2\int h\,d\nu$$

for $[\mu, \nu] \in \mathscr{M}'$. We put

$$M'=\inf\{J(\mu, \nu);\ [\mu, \nu] \in \mathscr{M}'\} \qquad \text{if } \mathscr{M}' \neq \phi,$$

$$M'=\infty \qquad \text{if } \mathscr{M}'=\phi.$$

Our interest lies in the problem to find when the equality $-M=M'$ holds. First we have

THEOREM 5. *If Φ is of positive type, then it holds that $-M \leqq M'$.*

PROOF. We may suppose that $\mathscr{M}' \neq \phi$. Let $[\mu, \nu] \in \mathscr{M}'$ and $\lambda \in \mathscr{M}$. Then

$$\begin{aligned} J(\mu, \nu) &= (\mu, \mu)+2\int h(y)\,d\nu(y) \\ &\geqq (\mu, \mu)+2\int \Psi(\lambda, y)\,d\nu(y) \\ &= (\mu, \mu)+2\int \Psi(x, \nu)\,d\lambda(x) \\ &\geqq (\mu, \mu)+2\int [f(x)-\Phi(x, \mu)]\,d\lambda(x) \\ &= -I(\lambda)+(\mu-\lambda, \mu-\lambda). \end{aligned}$$

Since Φ is of positive type, $(\mu-\lambda, \mu-\lambda) \geqq 0$ and hence

$$J(\mu, \nu) \geqq -I(\lambda).$$

We give a characterization of an optimal measure for M.

LEMMA 4. (α) *Assume that M is finite. If μ^* is an optimal measure for M, then it holds that*

$$(\sharp) \qquad (\mu-\mu^*, \mu^*) \geqq \int f\,d(\mu-\mu^*)$$

for every $\mu \in \mathscr{M}$.

(β) *If $\mu^* \in \mathscr{M}$ satisfies the above relation ($\sharp$) and Φ is of positive type, then μ^* is an optimal measure for M.*

PROOF. (α) Assume $I(\mu^*)=M$ and $\mu^* \in \mathscr{M}$. Let μ be any measure of $\mathscr{M}$ such that $\int f d\mu > -\infty$ and t be a number such that $0 \leqq t \leqq 1$. Since $\mu(t)=t\mu + (1-t)\mu^*$ belongs to $\mathscr{M}$, we have

$$\begin{aligned} I(\mu^*) &\leqq I(\mu(t)) \\ &= t^2(\mu, \mu)+2t(1-t)(\mu, \mu^*)+(1-t)^2(\mu^*, \mu^*) \\ &\quad -2t\int f d\mu - 2(1-t)\int f d\mu^*. \end{aligned}$$

It follows that

$$0 \leqq \left.\frac{dI(\mu(t))}{dt}\right|_{t=0} = 2(\mu, \mu^*) - 2(\mu^*, \mu^*) - 2\int f d\mu + 2\int f d\mu^*.$$

(β) Assume that $\mu^* \in \mathscr{M}$ satisfies ($\sharp$) and let μ be any measure of $\mathscr{M}$. Then we have

$$\begin{aligned} I(\mu) &= (\mu, \mu) - 2\int f d\mu \\ &\geqq (\mu, \mu) - 2(\mu-\mu^*, \mu^*) - 2\int f d\mu^* \\ &= I(\mu^*) + (\mu-\mu^*, \mu-\mu^*) \geqq I(\mu^*). \end{aligned}$$

Finally we prove

THEOREM 6. *Assume that Φ is of positive type and one of the following conditions is true*:

(1) *condition* (H. 1),
(2) $f>0$ *and* $\Phi \geqq 0$,
(3) $\sup_{X \times Y} \Psi < 0$ *and either* $f>0$ *or* $f<0$ *or* $W>0$.

If M is finite, then $\mathscr{M}' \neq \phi$ and $-M=M'$.

PROOF. We can find by Theorem 1 an optimal measure μ^* for M. Put $f^*(x)=f(x)-\Phi(x, \mu^*)$. Then the relation ($\sharp$) can be written as

$$\int f^* d\mu^* \geqq \int f^* d\mu$$

for every $\mu \in \mathscr{M}$, i.e. $\int f^* d\mu^* = \sup\left\{\int f^* d\mu;\ \mu \in \mathscr{M}\right\}$. Writing $\mathscr{M}^{*\prime} = \{\nu; \Psi(x, \nu) \geqq f^*(x)$ on $X\}$ and making use of Ohtsuka's duality theorem (Theorem 3 in [3], p. 35), we see $\mathscr{M}^{*\prime} \neq \phi$ and

$$\int f^* d\mu^* = \inf\left\{\int h d\nu;\ \nu \in \mathscr{M}^{*\prime}\right\}.$$

Given $\varepsilon > 0$, there is a measure $\nu_\varepsilon \in \mathscr{M}^{*\prime}$ satisfying

$$\int h d\nu_\varepsilon < \int f^* d\mu^* + \varepsilon = \int f d\mu^* - (\mu^*, \mu^*) + \varepsilon.$$

Obviously $[\mu^*, \nu_\varepsilon] \in \mathscr{M}'$. It holds that

$$\begin{aligned} M' &\leqq 2\int h d\nu_\varepsilon + (\mu^*, \mu^*) \\ &< 2\int f d\mu^* - (\mu^*, \mu^*) + 2\varepsilon \\ &= -I(\mu^*) + 2\varepsilon = -M + 2\varepsilon. \end{aligned}$$

By the arbitrariness of ε, we obtain $M' \leqq -M$. The inverse inequality was shown in Theorem 5.

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