J. Sci. Hiroshima Univ. Ser. A-I
31 (1967), 59-66

An Example of Non-minimal Kuramochi Boundary Points

Fumi-Yuki Maeda

(Received February 28, 1967)

Introduction.

Z. Kuramochi [3] constructed an example of a plane domain whose Kuramochi boundary contains non-minimal points. However, he showed only the existence of non-minimal points and did not determine the distribution of such points. In this note, applying his idea, we shall give an example of a domain in the d-dimensional Euclidean space R^d $(d\geq 2)$ whose Kuramochi boundary contains non-minimal points and for which we are able to determine the distribution of non-minimal points completely. Our example is similar to, but simpler than Kuramochi's.

More precisely, let F be a compact set in R^d such that components of F cluster to the origin and F lies on the hyperplane $P=\{x=(x_1, \cdots, x_d);\ x_d=0\}$. Unber certain conditions on F, we shall see that the Kuramochi boundary of R^d-F corresponding to the origin is homeomorphic to the closed interval $[-1, 1]$, the points corresponding to 1 and -1 are minimal and the other points are non-minimal (Theorem 4.1).

One may refer to [2], [4] and [5] for the theory of Kuramochi boundary, including the notions of full-harmonic and full-superharmonic functions, those of potential type, Kuramochi kernel (denoted by N in [4], [5] and by $\tilde{g}$ in [2]), minimal points and non-minimal points. To apply the general theory, we take the domain $\Omega=\hat{R}^d-F$ (instead of R^d-F), where $\hat{R}^d$ is the one point compactification of R^d. Ω is a space of type $\mathfrak{S}$ in the sense of Brelot-Choquet. Let B be the unit ball $\{x;\ |x|<1\}$ in R^d and suppose F is contained in B. Then $K_0=\hat{R}^d-B$ is a compact set in Ω. Thus we can consider full-superharmonic functions on $\Omega_0=\Omega-K_0=B-F$ relative to Ω. The set of all harmonic full-superharmonic functions of potential type on Ω_0 will be denoted by $\mathcal{P}_b\equiv\mathcal{P}_b(\Omega_0)$ (cf. [4]). We remark here that any $u\in\mathcal{P}_b$ vanishes on $S=\{x;\ |x|=1\}$, i.e., u is continuous if it is extended by 0 on S.

For a subset A in R^d, let $\bar{A}$ and ∂A be the closure and the boundary (in R^d) of A, respectively. If $A\subset P$, let $\partial' A$ be the boundary of A relative to the $(d-1)$-dimensional spacc P.

§1. Preliminaries—some general results.

1.1. Let H^+ be the class of all non-negative harmonic functions on $B-\{0\}$ vanishing on S. The Green function g_0 of B with pole at $x=0$ belongs to H^+. Conversely, the following is well-known:

LEMMA 1.1. *If $h \in H^-$, then $h=\alpha g_0$ for some $\alpha \geq 0$.*

1.2. Let F be a relatively closed subset of B (not necessarily contained in P). We shall say that F is a regular closed set in B if $B-F$ is a domain and each point of $\partial F-\{0\}$ is regular for $B-F$. In this section, let F be a regular closed set in B. Let H_F^+ be the class of all non-negative harmonic functions on $B-F-\{0\}$ which vanish on $S\cup\partial F-\{0\}$ and are dominated by functions in H^+.

Let $V_n=\{x;\ |x|\leq 1/n\}$, $n=1, 2, \cdots$. For $h \in H^+$, let h_n be the Dirichlet solution (in the sense of Perron) on $B-V_n-F$ with boundary values h on ∂V_m-F and 0 on $\partial F\cup S-(V_n-\partial V_n)$. Then $\lim_{n\to\infty} h_n$ exists and belongs to H_F^+. We denote the limit function by $I_F(h)$.

THEOREM 1.1. *If $u \in H_F^+$, then $u=\alpha I_F(g_0)$ for some $\alpha\geq 0$.*

PROOF: Let u_n be the Dirichlet solution on $B-V_n$ with boundary values u on ∂V_n-F and 0 on $(\partial V_n\cap F)\cup S$. Then $h=\lim_{n\to\infty} u_n$ exists and $h \in H^+$. Hence by Lemma 1.1, $h=\alpha g_0$ for some $\alpha\geq 0$. On the other hand, we can show that $I_F(h)=u$. Hence $u=I_F(h)=\alpha I_F(g_0)$.

COROLLARY 1.1. *If u is a harmonic function on $B-F-\{0\}$ such that it vanishes on $S\cup\partial F-\{0\}$ and $|u|\leq g_0$ on $B-F-\{0\}$, then $u=\alpha I_F(g_0)$ for some α with $|\alpha|\leq 1$.*

PROOF: Since $-u\leq g_0$ and $-u$ vanishes on $S\cup\partial F-\{0\}$, we have $-u\leq I_F(g_0)$. Hence $u+I_F(g_0) \in H_F^+$. By the above theorem, $u+I_F(g_0)=\alpha' I_F(g_0)$ for some $\alpha'\geq 0$. Since $u+I_F(g_0)\leq 2g_0$, $0\leq\alpha'\leq 2$. Hence $u=(\alpha'-1)I_F(g_0)$ and $|\alpha'-1|\leq 1$.

1.3. THEOREM 1.2. *Let $0 \in \partial F$. Then $I_F(g_0)>0$ on $B-F$ if and only if 0 is an irregular point for $B-F$.*

PROOF: Let $(g_0)_F$ be the reduced function of g_0 on F in B. (It is denoted by $\mathscr{B}_{g_0}^F$ in [1].) It is easy to see that $g_0-I_F(g_0)=(g_0)_F$ on $B-F$. On the other hand, 0 is irregular for $B-F$ if and only if F is thin at 0, or equivalently, $(g_0)_F\neq g_0$ ([1], Chap. VII and VIII). Hence 0 is irregular for $B-F$ if and only if $I_F(g_0)>0$.

§2. Functions which are full-harmonic except at the origin.

2.1. Now, we turn to the case where F lies on the hyperplane P. Thus, in what follows, we assume that F satisfies the following conditions: 1) F is a compact set in B such that $0 \in F \subset P$; 2) $\{0\}$ is a component of F and components of $F-\{0\}$ are isolated but cluster to 0; 3) $F'=\overline{P-F}\cap B$ is a regular closed set in B (in the sense defined in 1.2); 4) $\partial' F$ is a polar set in B.

For example, if $d=2$, then $F-\{0\}$ consists of a countable number of closed intervals on the real axis clustering nowhere except to 0. (Cf. the example in [3].)

Let $\mathring{F}$ be the interior points of F relative to P, i.e., $\mathring{F}=(P-F')\cap B$ and let $\mathring{F}'=(P-F)\cap B$. Conditions 2) and 3) imply $\mathring{F}\neq\emptyset$ and $\mathring{F}'\neq\emptyset$. By condition 2), $0\in\partial' F\subset F'$, so that $0\notin\mathring{F}$ and $0\notin\mathring{F}'$.

By condition 1), $\Omega_0=B-F$ is a domain. Let $B^+=\{x\in B;\, x_d>0\}$ and $B^-=\{x\in B;\, x_d<0\}$. Obviously, $\Omega_0=B^+\cup\mathring{F}'\cup B^-$. For $x=(x_1,\cdots,x_d)$, let $\hat{x}=(x_1,\cdots,x_{d-1},-x_d)$, i.e., the symmetric point of x with respect to P.

2.2. Let D be a domain in B such that $\partial D\cap F=\emptyset$ and D is symmetric with respect to P. The family of all such domains will be denoted by $\mathcal{D}_s$. For a function f defined on $D-P$, we define functions $\tilde{f}$, $\underset{\sim}{f}$ and $\hat{f}$ on $D-P$ by

$$\tilde{f}(x)=\begin{cases} f(x) & \text{if } x\in B^+\cap D \\ f(\hat{x}) & \text{if } x\in B^-\cap D \end{cases},\quad \underset{\sim}{f}(x)=\begin{cases} f(\hat{x}) & \text{if } x\in B^+\cap D \\ f(x) & \text{if } x\in B^-\cap D \end{cases}$$

and $\hat{f}(x)=f(x)+f(\hat{x})=\tilde{f}(x)+\underset{\sim}{f}(x)$. Obviously, $\tilde{f}, \underset{\sim}{f}, \hat{f}$ are symmetric with respect to P.

We shall denote by $HS(D, F)$ the class of all harmonic functions u on $D-F$ such that $\tilde{u}$ and $\underset{\sim}{u}$ can be extended to be harmonic on $D-F'$. The extended functions are also denoted by $\tilde{u}$ and $\underset{\sim}{u}$. If $u\in HS(D, F)$ and u is bounded on $D-P-V_n$ for any n, then $\hat{u}$ can be extended to be harmonic on $D-\{0\}$, since $\partial' F$ is assumed to be a polar set.

2.3. LEMMA 2.1. *Let $D\in\mathcal{D}_s$. If u is full-harmonic on $D-F$, then $u\in HS(D, F)$.*

PROOF: Let D' be any domain in $\mathcal{D}_s$ such that $\partial D'$ is smooth, $D'\supset D\cap F$ and $\bar{D}'\subset D$. Let u^* (resp. u_*) be the Dirichlet solution for $D'-F'$ with boundary values $\tilde{u}$ (resp. $\underset{\sim}{u}$) on $\partial D'$ and u on $\mathring{F}'\cap D'$ (any finite values on $\partial' F\cap D'$). By the Dirichlet principle, $\|u^*\|_{D'-F'}\leq\|\tilde{u}\|_{D'-P}$ and $\|u_*\|_{D'-F'}\leq\|\underset{\sim}{u}\|_{D'-P}$ where $\|\ \|_G$ denotes the Dirichlet norm on G. Let $v(x)=u^*(x)$ if $x\in D'\cap B^+$, $=u_*(x)$ if $x\in D'\cap B^-$. Then v is continuous on $D'-F$ if it is extended by u on $\mathring{F}'\cap D'$. Since $v=u$ on $\partial D'$, v and u have the same boundary values. We have

$$\|v\|^2_{D'-F}=\tfrac{1}{2}(\|u^*\|^2_{D'-F'}+\|u_*\|^2_{D'-F'})$$

$$\leq\tfrac{1}{2}(\|\tilde{u}\|^2_{D'-P}+\|\underset{\sim}{u}\|^2_{D'-P})=\|u\|^2_{D'-F}.$$

Since u is full-harmonic on $D-F$, it follows that $v=u$ (see [2], [4] or [5]), i.e., $u^*=\tilde{u}$ and $u_*=\underset{\sim}{u}$. Therefore, $u\in HS(D, F)$.

We say that $u\in\mathcal{P}_b(\Omega_0)$ is full-harmonic except at 0, if, for any $D\in\mathfrak{D}_s$ such that $0\notin D$, u is full-harmonic on $D-F$. Let $\mathcal{P}_h\equiv\mathcal{P}_h(\Omega_0)=\{u\in\mathcal{P}_b;\ u$ is full-harmonic except at $0\}$.

COROLLARY 2.1. $\mathcal{P}_h(\Omega_0)\subset HS(B, F)$.

PROOF: By condition 2) for F, we see that a harmonic function u on Ω_0 belongs to $HS(B, F)$ if and only if $u\in HS(D, F)$ for any $D\in\mathfrak{D}_s$ such that $0\notin D$. Therefore, this corollary follows from the above lemma.

2.4. We define functions u_0 and g_α (α: real) on Ω_0 as follows:

$$u_0=\begin{cases} I_{F'}(g_0) & \text{on } B^+ \\ 0 & \text{on } \mathring{F}' \\ -I_{F'}(g_0) & \text{on } B^- \end{cases} \quad\text{and}\quad g_\alpha=g_0+\alpha u_0.$$

It is easy to see that u_0 and g_α are harmonic on Ω_0 and vanish on S. By Theorem 1.2, $g_\alpha\neq g_0$ for $\alpha\neq 0$ if and only if 0 is an irregular point for $B-F'$.

THEOREM 2.1. *If* $|\alpha|\leq 1$, *then* $g_\alpha\in\mathcal{P}_h(\Omega_0)$.

PROOF: Let D be any domain in $\mathfrak{D}_s$ such that $\bar{D}\subset B$ and ∂D is smooth. Let $\mu_x\equiv\tilde{\mu}_x^{D-F}$ be the full-harmonic measure (cf. [2] or [4]) for the domain $D-F$. μ_x is a measure on ∂D. We shall show that $g_\alpha(x)\geq\int g_\alpha\,d\mu_x$ for all $x\in D-F$ and that the equality holds if $0\notin D$. Since $g_\alpha\geq 0$ (for $|\alpha|\leq 1$), g_α vanishes on S and is harmonic on Ω_0, it follows that $g_\alpha\in\mathcal{P}_h(\Omega_0)$.

Since the domain D and the function g_0 are symmetric with respect to P, so is $\varphi(x)=\int g_0\,d\mu_x$, i.e., $\tilde{\varphi}=\varphi$. φ is a bounded function and, since φ is full-harmonic on $D-F$, $\varphi\in HS(D, F)$ by Lemma 2.1. Hence φ can be extended to be harmonic on D (cf. 2.2). Let $\gamma=g_0-\varphi$ on D. Since g_0 and φ have the same boundary values, $\gamma(x)\equiv 0$ if $0\notin D$ and γ is the Green function of D with pole at $x=0$ if $0\in D$.

Next, let $\psi(x)=\int u_0\,d\mu_x$ for $x\in D-F$. Since $u_0(\hat{x})=-u_0(x)$ and D is symmetric with respect to P, we have $\psi(\hat{x})=-\psi(x)$. Hence $\tilde{\psi}=-\underset{\sim}{\psi}$ and $\psi=0$ on $\mathring{F}'$. On the other hand, ψ is full-harmonic, so that $\tilde{\psi}$ is harmonic on $D-F'$ by

Lemma 2.1. Let $u=I_{F'}(g_0)-\tilde{\psi}$. Then u is harmonic on $D-F'$ and $u=0$ on $(\mathring{F}'\cap D)\cup\partial D$. Since $\tilde{\psi}$ is bounded, $u\geq 0$ by the minimum principle. Also, we see that $u(x)=|u_0(x)-\psi(x)|$ for all $x\in D-P$.

If $0\notin D$, then u is bounded. Hence $u=0$, i.e., $\tilde{\psi}=I_{F'}(g_0)$. Hence, $\psi=u_0$, so that $g_\alpha(x)=g_0(x)+\alpha u_0(x)=\varphi(x)+\alpha\psi(x)=\int g_0\, d\mu_x+\alpha\int u_0\, d\mu_x=\int g_\alpha\, d\mu_x$.

If $0\in D$, then we compute $\delta(x)=g_\alpha(x)-\int g_\alpha\, d\mu_x=g_0(x)+\alpha u_0(x)-\varphi(x)-\alpha\psi(x)=\gamma(x)+\alpha(u_0(x)-\psi(x))$. Our theorem is prvoed if $\delta(x)\geq 0$ for all $x\in D-F$. Since $u\leq g_0-\tilde{\psi}$, $\gamma-u=(\gamma-g_0)+(g_0-u)\geq\gamma-g_0+\tilde{\psi}$. Since $\gamma-g_0$ is bounded on D, $\gamma-u$ is bounded below. Also, $\gamma-u=0$ on ∂D and ≥ 0 on $\mathring{F}'\cap D$. Hence, by the minimum principle, we have $\gamma-u\geq 0$ on $D-F'$. If $x\in D-P$, then $\delta(x)=\gamma(x)+\alpha(u_0(x)-\psi(x))\geq\gamma(x)-|\alpha|u(x)\geq\gamma(x)-u(x)\geq 0$. If $x\in\mathring{F}'\cap D$, then $\delta(x)=\gamma(x)\geq 0$.

2.5. We now prove our main theorem in this section:

THEOREM 2.2. $\mathcal{P}_h(\Omega_0)=\{\beta g_\alpha;\ \beta\geq 0$ and $|\alpha|\leq 1\}$.

PROOF: By the above theorem, we only have the show that any $v\in\mathcal{P}_h(\Omega_0)$ is of the form βg_α with $\beta\geq 0$ and $|\alpha|\leq 1$. By Corollary 2.1, $v\in HS(B, F)$. Hence, as remarked in 2.2, $\hat{v}$ is harmonic on $B-\{0\}$. Since v vanishes on S, if follows that $\hat{v}\in H^+$. By Lemma 1.1, $\hat{v}=2\beta g_0$ for some $\beta\geq 0$. Let $u=v-\beta g_0$. Then $u\in HS(B, F)$ and $\hat{u}=\hat{v}-2\beta g_0=0$. Hence $u=0$ on $\mathring{F}'$. Obviously, $u=0$ on S. Since v and g_0 are bounded on Ω_0-V_n and since each point of $F'-\{0\}$ is regular for $B-F'$, $\tilde{u}$ is a harmonic function on $B-F'$ vanishing on $S\cup F'-\{0\}$. Also, $|\tilde{u}|=|\tilde{v}-\hat{v}/2|=|\tilde{v}-\underset{\smile}{v}|/2\leq\hat{v}/2=\beta g_0$. Hence, by Corollary 1.1, $\tilde{u}=\alpha\beta I_{F'}(g_0)$ with $|\alpha|\leq 1$. Hence $u(x)=\tilde{u}(x)=\alpha\beta I_{F'}(g_0)(x)$ if $x\in B^+$ and $u(x)=-u(\hat{x})=-\alpha\beta I_{F'}(g_0)(x)$ if $x\in B^-$. Thus, $u=\alpha\beta u_0$, so that $v=\alpha\beta u_0+\beta g_0=\beta g_\alpha$.

§3. Kuramochi kernel for Ω_0.

3.1. Let F and Ω_0 be as in the previous section. We denote by $N_p(x)\equiv N(p, x)$ the Kuramochi kernel (N-Green function) for Ω_0 (see [2], [4] or [5]). For a domain D, let $G_p^D(x)\equiv G^D(p, x)$ be the Green function for D.

THEOREM 3.1.

$$N(p, x)=\begin{cases} G^B(\hat{p}, x)+G^{B-F'}(p, x) & \text{if } p, x\in B^+ \text{ or } p, x\in B^-,\\ G^B(\hat{p}, x)-G^{B-F'}(\hat{p}, x) & \text{if } p\in B^+, x\in B^- \text{ or } p\in B^-, x\in B^+,\\ G^B(\hat{p}, x)\equiv G^B(p, x) & \text{if } x\in\mathring{F}' \text{ or } p\in\mathring{F}'.\end{cases}$$

PROOF: Let $N_p^* \equiv N^*(p, x)$ be the function defined by the right hand side of the equation. It is easy to see that, for each p, N_p^* is harmonic on $\Omega_0 - \{p\}$ and has the same singularity as G_p^B at $x = p$. Therefore, $w_p = N_p - N_p^*$ is bounded harmonic on Ω_0. We shall show that $w_p \equiv 0$. Since N_p is full-harmonic on $\Omega_0 - \{p\}$, Lemma 2.1 implies that $N_p \in HS(B - \{p, \hat{p}\}, F)$, so that $\hat{N}_p$ is harmonic on $B - \{p, \hat{p}\}$. On the other hand, $\hat{N}_p^*(x) = N^*(p, x) + N^*(p, \hat{x}) = G^B(\hat{p}, x) + G^B(\hat{p}, \hat{x}) = G^B(\hat{p}, x) + G^B(p, x)$. Hence $\hat{N}_p^*$ is also harmonic on $B - \{p, \hat{p}\}$. Hence $\hat{w}_p$ is harmonic on B. Since $w_p = 0$ on S, $\hat{w}_p \equiv 0$. In particular, $w_p = 0$ on $\mathring{F}'$.

Let $p \in B^+$. $\underset{\sim}{N}_p$ is harmonic on $B^+ \cup B^-$ and since $N_p \in HS(B - \{p, \hat{p}\}, F)$, $\underset{\sim}{N}_p$ is harmonic on $B - F'$. On the other hand, since $G_p^B - G_p^{B-F'}$ is bounded harmonic on $B - F'$ and continuous everywhere on $B - \{0\}$, it is symmetric with respect to P, i.e., $G^B(p, \hat{x}) - G^{B-F'}(p, \hat{x}) = G^B(p, x) - G^{B-F'}(p, x)$. Hence, $\underset{\sim}{N}^*(p, x) = G^B(p, x) - G^{B-F'}(p, x)$ for any $x \in B^+ \cup B^-$. It follows that $\underset{\sim}{N}_p^*$ is also harmonic on $B - F'$; hence so is $\underset{\sim}{w}_p$. Since $\underset{\sim}{w}_p$ is bounded and vanishes on $\mathring{F}' \cup S$, we have $\underset{\sim}{w}_p \equiv 0$. Since $\hat{w}_p \equiv 0$, it follows that $\tilde{w}_p \equiv 0$ and hence $w_p \equiv 0$.

Similarly, we obtain $\tilde{w}_p \equiv 0$ for $p \in B^-$, which implies $w_p \equiv 0$ for $p \in B^-$.

3.2. From the expression of $N(p, x)$ in the above theorem, we see: If $p_i \to \xi \in F - \{0\}$ with $p_i \in B^+$(resp. $p_i \in B^-$), then $\{N(p_i, x)\}$ converges to

$$N(\xi^+, x)(\text{resp. } N(\xi^-, x)) = \begin{cases} G^B(\xi, x) + G^{B-F'}(\xi, x) & \text{if } x \in B^+ \text{ (resp. } x \in B^-) \\ G^B(\xi, x) - G^{B-F'}(\xi, x) & \text{if } x \in B^- \text{ (resp. } x \in B^+) \\ G^B(\xi, x) & \text{if } x \in \mathring{F}'. \end{cases}$$

We note that $G^{B-F'}(\xi, x) \neq 0$ for $\xi \in \mathring{F}$. If $p_i \to 0$ and $\{N(p_i, x)\}$ converges, then the limit function $u(x)$ belongs to $\mathcal{P}_h(\Omega_0)$. Hence $u = \beta g_\alpha$ for some $\beta \geq 0$ and $|\alpha| \leq 1$ by Theorem 2.2. By Theorem 3.1, we see that $u(x) = g_0(x)$ if $x \in \mathring{F}'$. It follows that $\beta = 1$. Thus, we have

THEOREM 3.2. *To each $\xi \in \mathring{F}$, there correspond two Kuramochi boundary points ξ^+ and ξ^- and to each $\xi \in \partial' F - \{0\}$, there corresponds one point* (denoted by ξ again). *If $p_i \to 0$ and $\{N(p_i, x)\}$ converges, then the limit function is of the form g_α ($|\alpha| \leq 1$) and is different from any Kuramochi kernel corresponding to $\xi \in F - \{0\}$.*

Thus the Kuramochi boundary Δ of Ω consists of two parts Δ' and Δ°, where $\Delta' = \{\xi^+, \xi^-; \xi \in \mathring{F}\} \cup \{\xi; \xi \in \partial' F - \{0\}\}$ and Δ° is the set of points defined by fundamental sequences tending to the origin.

§4. Kuramochi boundary at the origin.

4.1. For any $\xi \in \Delta^\circ$, let $N_\xi(x) \equiv N(\xi, x)$ be the corresponding Kuramochi kernel. ξ is a minimal point if N_ξ is minimal in $\mathcal{P}_b$, i.e., if $N_\xi = u_1 + u_2$ with $u_i \in \mathcal{P}_b (i=1, 2)$ implies $u_1 = \lambda N_\xi$ for some constant λ.

LEMMA 4.1. *$u \in \mathcal{P}_h$ is minimal in $\mathcal{P}_b$ if and only if it is minimal in $\mathcal{P}_h$.*

PROOF: Since $\mathcal{P}_h \subseteq \mathcal{P}_b$, the "only if" part is trivial. Suppose $u \in \mathcal{P}_h$ is minimal in $\mathcal{P}_h$ and let $u = u_1 + u_2$ with $u_1, u_2 \in \mathcal{P}_b$. It is enough to show that $u_1, u_2 \in \mathcal{P}_h$. Take any $D \in \mathcal{D}_s$ such that $0 \notin D$. Since u is full-harmonic on $D-F$ and u_1, u_2 are full-superharmonic on $D-F$, u_1, u_2 must be full-harmonic there. Then it follows that $u_1, u_2 \in \mathcal{P}_h$.

LEMMA 4.2. *If $u_0 \not\equiv 0$, then $u = g_\alpha$ $(|\alpha| \leq 1)$ is minimal in $\mathcal{P}_h$ if and only if $|\alpha| = 1$.*

PROOF: Since $u_0 \not\equiv 0$, $g_1 \neq g_{-1}$. If $g_1 = \lambda g_{-1}$, then $2g_0 = \hat{g}_1 = \lambda \hat{g}_{-1} = 2\lambda g_0$ implies $\lambda = 1$, which is impossible. Hence g_1 and g_{-1} are not proportional. If $|\alpha| < 1$, then $g_\alpha = [(1+\alpha)/2] g_1 + [(1-\alpha)/2] g_{-1}$ and $(1+\alpha)/2$, $(1-\alpha)/2 \neq 0$. Hence g_α is non-minimal in $\mathcal{P}_h$.

Next, let $g_1 = u_1 + u_2$ with $u_i = \beta_i g_{\alpha_i}$, $\beta_i \geq 0$, $|\alpha_i| \leq 1$ $(i=1, 2)$. Since $2g_0 = \hat{g}_1 = \hat{u}_1 + \hat{u}_2 = 2\beta_1 g_0 + 2\beta_2 g_0 = 2(\beta_1 + \beta_2) g_0$, $\beta_1 + \beta_2 = 1$. It follows that $u_0 = \beta_1 \alpha_1 u_0 + \beta_2 \alpha_2 u_0$, or $\beta_1 \alpha_1 + \beta_2 \alpha_2 = 1$. These equalities can hold only when $\alpha_1 = \alpha_2 = 1$. Hence g_1 is minimal in $\mathcal{P}_h$. Similarly, we see that g_{-1} is minimal in $\mathcal{P}_h$.

4.2. Now, we are able to determine the part Δ°. Our final and main result is:

THEOREM 4.1. *If 0 is a regular point for $B-F'$, then Δ° consists of a single point and the corresponding Kuramochi kernel is equal to g_0. If 0 is an irregular point for $B-F'$, then Δ° is homeomorphic to the closed interval $[-1, 1]$ in such a way that the points corresponding to -1 and 1 are minimal and other points are non-minimal; the Kuramochi kernel corresponding to $\alpha \in [-1, 1]$ is equal to g_α.*

PROOF: Let $\{p_i\}$ be a sequence of points in $B-F$ tending to 0 such that $\{N(p_i, x)\}$ is convergent. The limit function is of the form g_α $(|\alpha| \leq 1)$ by Theorem 3.2. If 0 is a regular point for $B-F'$, the $u_0 \equiv 0$ by Theorem 1.2, so that $g_\alpha = g_0$ for any α. Hence g_0 can be the only limit function of $\{N(p_i, x)\}$.

If 0 is an irregular point for $B-F'$, then $u_0 \not\equiv 0$ by Theorem 1.2. It is generally known (see [2], [4] or [5]) that any $\mathcal{P}_b$-minimal function is a constant multiple of $N(\xi, x)$ for some $\xi \in \Delta$. Thus, it follows from Theorem 3.2, Lemma 4.1 and Lemma 4.2 that there exist sequences $\{p_i\}$ and $\{q_i\}$ such that $p_i \to 0$, $q_i \to 0$, $N(p_i, x) \to g_1(x)$ and $N(q_i, x) \to g_{-1}(x)$. Now, we shall show that, for each α with $|\alpha| < 1$, there exists a sequence $\{p_i^{(\alpha)}\}$ such that $p_i^{(\alpha)} \to 0$ and $N(p_i^{(\alpha)}, x) \to g_\alpha(x)$. We may assume that p_i, $q_i \in V_i - F$. We can connect p_i

and q_i by a curve Γ_i in $V_i - F$. Fix $x_0 \in \Omega_0 - P$. There exists $p_i^{(\alpha)} \in \Gamma_i$ such that $N(p_i^{(\alpha)}, x_0) = [(1+\alpha)/2]N(p_i, x_0) + [(1-\alpha)/2]N(q_i, x_0)$. Subtracting a subsequence, if necessary, we may assume that $\{N(p_i^{(\alpha)}, x)\}$ is convergent. Then it is easy to see that $N(p_i^{(\alpha)}, x) \to g_\alpha(x)$. Thus, there is a one-to-one mapping φ of $[-1, 1]$ onto Δ° such that $N(\varphi(\alpha), x) = g_\alpha(x)$. From the definition of g_α, we see that φ is a homeomorphism. Now our theorem follows from Lemmas 4.1. and 4.2.

References

[1] M. Brelot, *Eléments de la théorie classique du potentiel*, 3e éd., Paris, C.D.U., 1965.

[2] C. Constantinescu and A. Cornea, *Ideale Ränder Riemannscher Flächen*, Berlin-Göttingen-Heidelberg, 1963.

[3] Z. Kuramochi, *Examples of non minimal points on Riemann surfaces of planar character*, J. Fac. Sci. Hokkaido Univ., Ser. I, **19** (1965), 28-48.

[4] F-Y. Maeda, *On full-superharmonic functions*, Proceedings of Symposium on Kuramochi Boundary, 1965, (mimeographed).

[5] M. Ohtsuka, *An elementary introduction of Kuramochi boundary*, J. Sci. Hiroshima Univ., Ser. A-I Math., **28** (1964), 271-299.

Department of Mathematics,
Faculty of Science,
Hiroshima University