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Dimensions of the Derivation Algebras of Lie Algebras

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Let L be a Lie algebra over a field of arbitrary characteristic. In the paper [3], J. Dozias has shown that if $L \neq [L, L]$ then the dimension of the derivation algebra $\mathfrak{D}(L)$ of L is not less than dim L. In this paper, making use of the method of constructing outer derivations of L which has been shown in [6], we shall give some effective estimates of $\dim \mathfrak{D}(L)$.

In Section 1 we shall recall some results which have been already shown in [4] and [6]. In Section 2 we shall give several estimates of $\dim \mathfrak{D}(L)$. If $Z(L)$ is the center of L and $C([L, L])$ is the centralizer of $[L, L]$ in L, then one of the estimates is that if $L \neq [L, L]$

$$\dim \mathfrak{D}(L) \geq \dim L + \max\{\dim Z(L) \dim L/[L, L] - \dim C([L, L]), 0\}.$$

In Section 3 we shall give several examples which are connected with the results in Section 2.

1. Throughout the paper we denote by Φ a field of arbitrary characteristic unless otherwise stated and by L a finite dimensional Lie algebra over a field Φ. We denote by $\mathfrak{D}(L)$ the derivation algebra of L, that is, the Lie algebra of all the derivations of L and by $\mathfrak{J}(L)$ the ideal of $\mathfrak{D}(L)$ consisting of all the inner derivations of L. We denote by $Z(L)$ the center of L and, for a subalgebra H of L, by $C(H)$ the centralizer of H in L. As usual $[L, L]$ will be denoted by L^2.

In the next section we need the following two results.

Lemma 1. *Let L be a Lie algebra over a field Φ and let M be an ideal of L of codimension* 1 *containing $Z(L)$. Then*:

(i) $[L, Z(M)] \subset Z(M)$ *and*

$$\dim Z(M) = \dim Z(L) + \dim [L, Z(M)].$$

(ii) *If $L = (e) + M$ and $Z(L) \neq (0)$, every endomorphism of L sending e to any element of $Z(M) \backslash [L, Z(M)]$ and M into* (0) *is an outer derivation of L.*

Lemma 2. *Let L be a Lie algebra over a field Φ. If L is the direct sum of the ideals L_1* and *L_2, then*

$$\dim \mathfrak{D}(L) = \dim \mathfrak{D}(L_1) + \dim \mathfrak{D}(L_2) + \dim Z(L_1) \dim L_2/L_2^2 + \\ + \dim Z(L_2) \dim L_1/L_1^2.$$

The statements (i) and (ii) in Lemma 1 have been shown respectively in the proofs of Lemma 1 and Theorem 1 in [6] (see [5]). Lemma 2 is an easy consequence of Lemma 1 in [4].

2. We first prove the following

THEOREM 1. *Let L be a Lie algebra over a field Φ such that $L\neq L^2$ and $(0)\neq Z(L)\subset L^2$. Let $L=(e_1, e_2, \dots, e_n)+L^2$ with $n=\dim L/L^2$ and let $M_i=(e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n)+L^2$.*

(i) *If we denote by G the space spanned by all $x\in L$ such that $[x, e_i]\in[e_i, Z(M_i)]$ for some i, then*

$$(1)\quad \dim\mathfrak{D}(L)\geq\dim L+\dim Z(L)(\dim L/L^2-1)-\dim C(L^2)+\dim\left(C(L^2)\cap G\right).$$

(ii) *If we put $Z_i=\left(\sum_{M\in\mathfrak{F}_i}Z(M)\right)/[e_i, Z(M_i)]$ where $\mathfrak{F}_i$ is the set of all the ideals M complementary to (e_i) and such that $[e_i, Z(M)]=[e_i, Z(M_i)]$, and if we denote by H the space spanned by all $x\in L$ such that $[x, L]\subset\bigcap_{i=1}^{n}[e_i, Z(M_i)]$, then*

$$(2)\quad \dim\mathfrak{D}(L)\geq\dim L+\max_i(\dim Z_i)+\dim Z(L)(\dim L/L^2-2)$$
$$-\dim C(L^2)+\dim\left(C(L^2)\cap H\right).$$

PROOF. (i): Put $m=\dim Z(L)$. Then by (i) of Lemma 1 we have $\dim\left(Z(M_i)/[e_i, Z(M_i)]\right)=m$ for each i. Choose the elements z_{ij}, $j=1, 2, \dots, m$, in such a way that

$$Z(M_i)=(z_{i1}, \cdots, z_{im})+[e_i, Z(M_i)].$$

By (ii) of Lemma 1 every endomorphism D_{ij} of L sending e_i to z_{ij} and M_i into (0) is an outer derivation of L. The set of D_{ij} with $i=1, 2, \dots, n$ and $j=1, 2, \dots, m$ is linearly independent. If we denote by $\mathfrak{M}$ the space spanned by all the D_{ij}, then $\dim\mathfrak{M}=\dim Z(L)\dim L/L^2$. Let $x\in L$ be such that ad x is a non-zero element of $\mathfrak{I}(L)\cap\mathfrak{M}$. Then $[x, L^2]=(0)$ and $[x, e_i]\notin[e_i, Z(M_i)]$ for $i=1, 2, \dots, n$. It follows that $x\in C(L^2)\setminus\left(C(L^2)\cap G\right)$. Hence

$$\dim\left(\mathfrak{I}(L)\cap\mathfrak{M}\right)\leq\dim C(L^2)-\dim\left(C(L^2)\cap G\right).$$

Consequently we have

$$\begin{aligned}\dim\mathfrak{D}(L)&\geq\dim\left(\mathfrak{I}(L)+\mathfrak{M}\right)\\&=\dim\mathfrak{I}(L)+\dim\mathfrak{M}-\dim\left(\mathfrak{I}(L)\cap\mathfrak{M}\right)\\&\geq\dim L+\dim Z(L)(\dim L/L^2-1)-\dim C(L^2)+\dim\left(C(L^2)\cap G\right).\end{aligned}$$

(ii): We may assume that $\dim Z_1=\max(\dim Z_i)$. Put $h=\dim Z_1$. Then we can write

$$\sum_{M\in\mathfrak{F}_1} Z(M)=(z_{11}, \cdots, z_{1m}, z_{1,m+1}, \cdots, z_{1m_1}, \cdots, z_{1,m_{k-1}+1}, \cdots, z_{1m_k})+[e_1, Z(M_1)],$$

where $m_k=h$, $(z_{11}, \cdots, z_{1m})\subset Z(M_1)$, $(z_{1,m+1}, \cdots, z_{1m_1})\subset Z(M_{11})$, $\cdots$, $(z_{1,m_{k-1}+1}, \cdots, z_{1m_k})\subset Z(M_{1k})$ and $M_{1j}\in\mathfrak{F}_1$ for $j=1, 2, \cdots, k$. Making a convention that $m=m_0$, for any j such that $m_{i-1}<j\leq m_i$ we denote by D_{1j} the endomorphism of L sending e_1 to z_{1j} and M_{1i} into (0). Then by (ii) of Lemma 1 D_{1j} is an outer derivation of L. For $i=1, \cdots, n$ and $j=1, 2, \cdots, m$, we define D_{1j} as in the proof of (i). Then the set of all D_{ij} is linearly independent. In fact, if $\sum_{i,j}\lambda_{ij}D_{ij}=0$, then $\sum_{j=1}^{h}\lambda_{1j}D_{1j}e_1=\sum_{j=1}^{h}\lambda_{1j}z_{1j}=0$. Hence $\lambda_{1j}=0$ for $j=1, 2, \cdots, h$. It is now immediate that $\lambda_{ij}=0$ for $i=2, 3, \cdots, n$ and $j=1, 2, \cdots, m$. Thus these D_{ij} are linearly independent. If we denote by $\mathfrak{M}$ the space spanned by all of these D_{ij}, then

$$\dim\mathfrak{M}=h+\dim Z(L)\ (\dim L/L^2-1).$$

Let $x\in L$ be such that ad x is a non-zero element of $\mathfrak{J}(L)\cap\mathfrak{M}$. Then $[x, L^2]=(0)$ and $[x, e_i]$ belongs to a subspace of L complementary to $[e_i, Z(M_i)]$ for $i=1, 2, \cdots, n$. Hence $[x, L]\cap(\bigcap_{i=1}^{n}[e_i, Z(M_i)])=(0)$. It follows that $x\in C(L^2)\backslash(C(L^2)\cap H)$. Therefore

$$\dim(\mathfrak{J}(L)\cap\mathfrak{M})\leq\dim C(L^2)-\dim(C(L^2)\cap H).$$

Consequently we have

$$\begin{aligned}\dim\mathfrak{D}(L)&\geq\dim(\mathfrak{J}(L)+\mathfrak{M})\\ &\geq\dim L-\dim Z(L)+h+\dim Z(L)\ (\dim L/L^2-1)\\ &\qquad-\dim C(L^2)+\dim(C(L^2)\cap H)\\ &=\dim L+h+\dim Z(L)\ (\dim L/L^2-2)-\dim C(L^2)+\dim(C(L^2)\cap H).\end{aligned}$$

Thus the proof is complete.

In order to obtain further estimates of $\dim\mathfrak{D}(L)$, we first consider the case where L has no non-zero abelian direct summands. If $L=L^2$, then

$$\begin{aligned}\dim\mathfrak{D}(L)&\geq\dim\mathfrak{J}(L)=\dim L-\dim Z(L)\\ &=\dim L+\dim Z(L)\dim L/L^2-\dim C(L^2).\end{aligned}$$

If $L\neq L^2$ and $Z(L)=(0)$,

$$\begin{aligned}\dim\mathfrak{D}(L)&\geq\dim\mathfrak{J}(L)=\dim L\\ &=\dim L+\max\{\dim Z(L)\dim L/L^2-\dim C(L^2), 0\}.\end{aligned}$$

Now assume that $L\neq L^2$ and $Z(L)\neq(0)$. Then $Z(L)\subset L^2$. With the notations of Theorem 1, $Z(L)\subset C(L^2)\cap G$. Hence by (i) of Theorem 1 we have

$$\dim \mathfrak{D}(L) \geq \dim L + \dim Z(L) \dim L/L^2 - \dim C(L^2).$$

Let D_{1j} and z_{1j} for $j=1, 2, \cdots, m$ be the same ones as defined in the first part of the proof of Theorem 1. If we denote by $\mathfrak{M}_1$ the space spanned by the set of all D_{1j}, then $\dim \mathfrak{M}_1 = \dim Z(L)$. We assert that $\mathfrak{M}_1 \cap \mathfrak{J}(L) = (0)$. In fact, suppose that $\mathrm{ad}\, x \in \mathfrak{M}_1 \cap \mathfrak{J}(L)$. Then $\mathrm{ad}\, x = \sum_{j=1}^{m} \lambda_j D_{1j}$. If $\sum_{j=1}^{m} \lambda_j z_{1j} \neq 0$, $\sum_{j=1}^{m} \lambda_j D_{1j}$ must be an outer derivation of L by (ii) of Lemma 1. Hence $\sum_{j=1}^{m} \lambda_j z_{1j} = 0$. It follows that $\lambda_j = 0$ for all j. Thus $\mathrm{ad}\, x = 0$. We now have

$$\begin{aligned} \dim \mathfrak{D}(L) \geq \dim (\mathfrak{J}(L) + \mathfrak{M}_1) &= \dim \mathfrak{J}(L) + \dim \mathfrak{M}_1 \\ &= \dim \mathfrak{J}(L) + \dim Z(L) = \dim L. \end{aligned}$$

Next we consider the case where L has a non-zero abelian direct summand. If L is abelian, then

$$\dim \mathfrak{D}(L) = (\dim L)^2 = \dim L + \dim Z(L) \dim L/L^2 - \dim C(L^2).$$

Assume that L is not abelian. Then L is the direct sum of a non-zero abelian ideal L_1 and an ideal L_2 which has no non-zero abelian direct summands. Put $k = \dim L_1$. If $L_2 = L_2^2$, by Lemma 2 we have

$$\begin{aligned} \dim \mathfrak{D}(L) &= \dim \mathfrak{D}(L_1) + \dim \mathfrak{D}(L_2) + \dim L_1 \dim Z(L_2) \\ &\geq k^2 + \dim L_2 - \dim Z(L_2) + k \dim Z(L_2) \\ &= \dim L + \dim Z(L) \dim L/L^2 - \dim C(L^2) \\ &= \dim L + (k-1) \dim Z(L), \end{aligned}$$

and

$$(k-1) \dim Z(L) \geq 0.$$

If $L_2 \neq L_2^2$, then by using Lemma 2 and the first case above we have

$$\begin{aligned} \dim \mathfrak{D}(L) = \dim \mathfrak{D}(L_1) &+ \dim \mathfrak{D}(L_2) + \dim L_1 \dim Z(L_2) \ + \dim L_1 \dim L_2/L_1^2 \\ &\geq k^2 + \dim L_2 + \max \{\dim Z(L_2) \dim L_2/L_2^2 - \dim (C(L_2^2) \cap L_2), 0\} \\ &\qquad + k \dim Z(L_2) + k \dim L_2/L_2^2 \\ &= \dim L + \max \{\dim Z(L) \dim L/L^2 - \dim C(L^2), \\ &\qquad k(\dim Z(L) - 1 + \dim L_2 - \dim L_2^2)\} \end{aligned}$$

and

$$k(\dim Z(L)-1+\dim L_2-\dim L_2^2)>0.$$

Therefore, in this case we obtain

$$\dim \mathfrak{D}(L)\geq \dim L+\max\{\dim Z(L)\dim L/L^2-\dim C(L^2),\ 0\}.$$

Thus we have proved the following theorems.

THEOREM 2. *Let L be a Lie algebra over a field Φ which has a non-zero abelian direct summand. Let L_1 be such a direct summand of maximal dimension.*

(i) *If $L/L_1=(L/L_1)^2$, then*

$$\dim \mathfrak{D}(L)\geq \dim L+\dim Z(L)\ (\dim L_1-1).$$

(ii) *If $L/L_1\neq(L/L_1)^2$, then*

$$\dim \mathfrak{D}(L)\geq \dim L+\max\{\dim Z(L)\dim L/L^2-\dim C(L^2),$$
$$\dim L_1(\dim Z(L)-1)+\dim L_1(\dim L/L_1-\dim(L/L_1)^2)\}.$$

THEOREM 3. *Let L be a Lie algebra over a field Φ. Then*

$$\dim \mathfrak{D}(L)\geq \dim L+\dim Z(L)\dim L/L^2-\dim C(L^2).$$

In particular, if $L\neq L^2$, then

$$\dim \mathfrak{D}(L)\geq \dim L+\max\{\dim Z(L)\dim L/L^2-\dim C(L^2),\ 0\}. \tag{3}$$

As an immediate consequence of Theorem 3 we have the following

COROLLARY 1. *Let L be a Lie algebra over a field Φ. If* $\dim C(L^2)\leq k\dim Z(L)$ *and* $\dim L/L^2>k$ *for some integer $k>0$, then*

$$\dim \mathfrak{D}(L)\geq \dim L+\dim Z(L)\ (\dim L/L^2-k).$$

COROLLARY 2. *Let L be a Lie algebra over a field Φ. If $C(L^2)=Z(L)$, or if L satisfies the condition that* $(\mathrm{ad}\,x)^2=0$ *implies* $\mathrm{ad}\,x=0$, *then*

$$\dim \mathfrak{D}(L)\geq \dim L+\dim Z(L)\ (\dim L/L^2-1).$$

PROOF. If L satisfies the condition that $(\mathrm{ad}\,x)^2=0$ implies $\mathrm{ad}\,x=0$, then for any $y\in C(L^2)$ we have $(\mathrm{ad}\,y)^2L\subset(\mathrm{ad}\,y)L^2=(0)$ and therefore $\mathrm{ad}\,y=0$, that is, $y\in Z(L)$. Hence $C(L^2)=Z(L)$. Therefore the statement follows from Corollary 1.

3. In this section we shall show several examples which are connected with the results obtained in Section 2.

Let L be the Lie algebra over a field Φ described in terms of a basis e_1, e_2, e_3, e_4, e_5 by the table:

$$[e_1, e_2]=e_5,\ [e_1, e_3]=e_3,\ [e_2, e_4]=e_4$$

([1], p. 126). Then with the notations in Theorem 1 we have $L^2=(e_3, e_4, e_5)$, $Z(L)=(e_5)$, $L=(e_1, e_2)+L^2$, $G=L$, $C(L^2)=L^2$, and $\dim \mathfrak{D}(L)=6$. Hence in this case equality holds in the inequality (1) of Theorem 1.

Let L be the Lie algebra over a field of characteristic $\neq 2$ described in terms of a basis $e_1, e_2, \cdots, e_8$ by the table:

$$[e_1, e_2]=e_3,\ [e_1, e_3]=e_4,\ [e_1, e_4]=e_5,\ [e_1, e_5]=e_6,$$

$$[e_1, e_6]=e_8,\ [e_1, e_7]=e_8,\ [e_2, e_3]=e_5,\ [e_2, e_4]=e_6,$$

$$[e_2, e_5]=e_7,\ [e_2, e_6]=2e_8,\ [e_3, e_4]=-e_7+e_8,$$

$$[e_3, e_5]=-e_8$$

([1], p. 123). Then with the notations of Theorem 1 we have $L^2=(e_3, e_4, \cdots, e_8)$, $Z(L)=(e_8)$, $L=(e_1, e_2)+L^2$, $[e_i, Z(M_i)]=(e_8)$, $\sum_{M\in\mathfrak{F}_i} Z(M)=(e_6, e_7, e_8)$, $\dim Z_i=2\ (i=1,2)$, $H=C(L^2)=(e_6, e_7, e_8)$, and $\dim \mathfrak{D}(L)=10$. Hence in this case equality holds in the inequality (2) of Theorem 1.

In the remainder of the paper, for simplicity we denote by m $\max\{\dim Z(L)\ \dim L/L^2-\dim C(L^2), 0\}$.

For every non-zero abelian Lie algebra L over Φ, equality holds in the inequality (3) of Theorem 3 and m is >0 if $\dim L>1$.

Let L be the Lie algebra over a field of characteristic $\neq 2, 3$ described in terms of a basis $e_1, e_2, \cdots, e_8$ by the table:

$$[e_1, e_2]=e_5,\ [e_1, e_3]=e_6,\ [e_1, e_4]=e_7,\ [e_1, e_5]=-e_8,$$

$$[e_2, e_3]=e_8,\ [e_2, e_4]=e_6,\ [e_2, e_6]=-e_7,$$

$$[e_3, e_4]=-e_5,\ [e_3, e_5]=-e_7,\ [e_4, e_6]=-e_8$$

([2]). Then $L^2=(e_5, e_6, e_7, e_8)$, $Z(L)=(e_7, e_8)$, $C(L^2)=L^2$ and $\dim \mathfrak{D}(L)=12$. Hence in this case equality holds in the inequality (3) of Theorem 3 and $m=4$.

Let L be the Lie algebra of $n\times n$ triangular matrices over a field of characteristic 0. Then $\dim Z(L)=1$, $\dim C(L^2)=2$ and $\dim L/L^2=n$. Hence if $n\geq 3$ then L satisfies the condition in Corollary 1 to Theorem 3 with $k=2$.

Let L be the Lie algebra over a field Φ described in terms of a basis $e_1, e_2, \cdots, e_7$ by the table:

$$[e_1, e_3]=e_4,\ [e_1, e_4]=-e_3,\ [e_2, e_5]=e_6,$$

$$[e_2, e_6]=-e_5,\ [e_3, e_4]=[e_5, e_6]=e_7.$$

Then $L^2=(e_3, e_4, e_5, e_6, e_7)$, $Z(L)=C(L^2)=(e_7)$ and $\dim L/L^2=2$. Hence L satisfies the first condition in Corollary 2 to Theorem 3 and $m=\dim Z(L)$.

Let Φ be the field of real numbers and let L be the Lie algebra given above. Then we can show that if $(\operatorname{ad} x)^2=0$ then $\operatorname{ad} x=0$. Thus L satisfies the second condition in Corollary to Theorem 3.

References

[1] N. Bourbaki, Groupes et Algèbres de Lie, Chap. I, Algèbres de Lie, Hermann, Paris, 1960.

[2] J. Dixmier and W. G. Lister, *Derivations of nilpotent Lie algebras*, Proc. Amer. Math. Soc., **8** (1957), pp. 155-158.

[3] J. Dozias, *Sur les dérivations des algèbres de Lie*, C. R. Acad. Sc. Paris, **259** (1964), pp. 2748-1750.

[4] S. Tôgô, *On the derivation algebras of Lie algebras*, Can. J. Math., **13** (1961), pp. 201-216.

[5] S. Tôgô, *Derivations of Lie algebras*, Bull. Amer. Math. Soc., **72** (1966), pp. 690-692.

[6] S. Tôgô, *Outer derivations of Lie algebras*, to appear in Trans. Amer. Math. Soc.

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