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Distinguished Normal Operators on Open Riemann Surfaces

Hiroshi Yamaguchi

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Introduction

Given a Riemann surface W, let $\mathcal{V}$ be the collection of open sets of W whose relative boundary consists of a finite number of closed analytic curves. For $V \in \mathcal{V}$ principal operators L_{0V} and $(P)L_{1V}$ were introduced by L. Sario (see [1]) and both share the property:

$$D_V(L_V f, L_V g) = \int_{\partial V} f(dL_V g)^*.$$

In this paper normal operator with this property will be called distinguished. We consider a system $\boldsymbol{L} = \{L_V\}_{V \in \mathcal{V}}$ of distinguished normal operators L_V defined with respect to V. The system $\boldsymbol{L}$ is said to be consistent if the following condition is fulfilled:

$$L_{V_2}(L_{V_1} f) = L_{V_1} f$$

for any $V_1 \supset V_2$ and any continuous function f on ∂V_1.

Consider the Kerékjártó-Stoilow compactification W^* of W and denote the boundary by $\beta(W) = W^* - W$. Partition $\beta(W)$ into two disjoint sets α and γ where α is non-empty closed. The purpose of this paper is to investigate the following boundary value problems:

Suppose that the closure of $W_0 \in \mathcal{V}$ in W^* contains α and that f is continuously differentiable in W_0 and has $D_{W_0}(f) < \infty$. Then is there uniquely a function H_f satisfying the following conditions?

(I) H_f is harmonic in W and has $D_W(H_f) < \infty$,

(II) $H_f = L_V(H_f)$ for any $V \in \mathcal{V}$ such that the intersection of $\beta(W)$ with the closure of V is contained in γ,

(III) $\lim_{\substack{z \to \alpha \\ z \in \tau}} H_f(z) = \lim_{\substack{z \to \alpha \\ z \in \tau}} f(z)$ for almost all curves τ where each τ is a locally rectifiable curve in W, starting from a point of W and tending to α.

Roughly speaking, a solution H_f is to have $\boldsymbol{L}$-behavior on γ and assume given boundary values f on α.

We know by M. Ohtsuka [7], [9] that for the system $\boldsymbol{L}_0 = \{L_{0V}\}_{V \in \mathcal{V}}$ we have the existence and uniqueness of H_f. We shall show that if the set α is relatively open and closed on $\beta(W)$, then for any system $\boldsymbol{L}$ we have the same result as above. When α is not necessarily isolated, we shall study the existence and uniqueness of H_f under some additional conditions (Theorems 3, 4, 6).

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§1. Preliminaries

1.1 *Notation and terminology*

Given an open Riemann surface W, we denote by $\mathcal{V}$ the collection of open sets V of W such that V and its exterior have the same nonempty relative boundary whlch consists of a finite number of paiwise disjoint simple closed analytic curves. For each $V \in \mathcal{V}$, we denote by ∂V and $\bar{V}$ the relative boundary and the closure of V respectively. We orient ∂V positively with respect to V. We assume that a function defined on a subset of W is always real valued. Let V be in $\mathcal{V}$ and let u be a continuously differentiable function in V. The integral

$$D_V(u) = \iint_V \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right\} dx\, dy$$

is known as the Dirichlet integral of u over V. If u, v are continuously differentiable in V and $D_V(u)$, $D_V(v)$ are both finite, then we define the mixed Dirichlet integral over V by

$$D_V(u, v) = \iint_V \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx\, dy.$$

We consider the Kerékjártó-Stoilow compactification W^* of W, in which each ideal boundary component becomes a point. We denote by $\beta(W)$ the set of all boundary components of W, i.e., $\beta(W) = W^* - W$. For each $V \in \mathcal{V}$, the set $\beta(V)$ will mean the intersection of $\beta(W)$ with the closure of V in W^*. We observe that the set $\beta(V)$ is relatively open and closed on $\beta(W)$. We say, briefly, that a relatively open and closed subset of $\beta(W)$ is isolated on $\beta(W)$.

If α is a non-empty closed set on $\beta(W)$, then there exists a sequence of $\Omega_n \in \mathcal{V}$ such that $\Omega_{n+1} \supset \Omega_n$, $\bigcup_{n=1}^{\infty} \Omega_n = W$ and $\bigcup_{n=1}^{\infty} \beta(\Omega_n) = \beta(W) - \alpha$. We shall call such a sequence $\{\Omega_n\}$ an approximation of W toward α. Let functions f, g be defined and continuously differentiable in $V_0 \in \mathcal{V}$ with $\beta(V_0) \supset \alpha$. If $\lim_{n\to\infty} \int_{\partial\Omega_n} f(dg)^*$ exists for any approximation $\{\Omega_n\}$ toward α, then we write $\int_\alpha f(dg)^* = \lim_{n\to\infty} \int_{\partial\Omega_n} f(dg)^*$. The value $\int_\alpha f(dg)^*$ is independent of the choice of approximation $\{\Omega_n\}$.

Let α be any subset of $\beta(W)$ and let f be a function defined in a $V_0 \in \mathfrak{V}$ with $\beta(V_0) \supset \alpha$. Furthermore, suppose τ is a curve in W, starting from a point of W and tending to a point in α, that is, τ is a continuous mapping of $0 \leqq t < 1$ into W such that $\lim_{t\to 1} \tau(t)$ exists in W^* and is contained in α. Then there exists t_0 such that $0 \leqq t_0 < 1$ and the image of $t_0 \leqq t < 1$ by τ is included in V_0. If $\lim_{t\to 1} f(\tau(t))$ exists, it will be denoted by $f(\tau)$.

Given two non-empty subsets α_1, α_2 of $\beta(W)$, we denote by $\Gamma_{\alpha_1,\alpha_2}$ the family of locally rectifiable curves in W connecting points in α_1 to points in α_2. Also we denote by Γ_{α_1} the family of locally rectifiable curves in W, each starting from a point of W and tending to a point in α_1.

1.2 *Normal operators*

Let $V \in \mathfrak{V}$. The class of functions on ∂V having a continuously differentiable extension to a neighborhood of ∂V will be denoted by $C^1(\partial V)$. Let L be an operator such that it acts on $C^1(\partial V)$ and that Lf is continuously differentiable on $\bar{V}$ and is harmonic in V. The operator L is called a *normal operator* defined with respect to V, if it satisfies the following conditions:

$$Lf = f \quad \text{on} \quad \partial V, \tag{1}$$

$$L(c_1 f_1 + c_2 f_2) = c_1 L f_1 + c_2 L f_2, \tag{2}$$

$$L1 = 1, \tag{3}$$

$$Lf \geqq 0 \quad \text{if} \quad f \geqq 0, \tag{4}$$

$$\int_{\beta(V)} (dLf)^* = 0. \tag{5}$$

Normal operators are defined and investigated in Chapter III of L. Ahlfors and L. Sario [1]. Conditions (2), (3) and (4) yield the validity of maximum principle, that is, $m \leqq f \leqq M$ on ∂V implies $m \leqq Lf \leqq M$ on $\bar{V}$. Therefore, if f_n converges uniformly to f on ∂V, then Lf_n converges uniformly to Lf on $\bar{V}$. When we construct a harmonic function with prescribed boundary behavior, we use the following existence theorem, as was used in L. Ahlfors and L. Sario [1]:

Existence theorem. *Let $V \in \mathfrak{V}$ such that $W - V$ is compact and let L be a normal operator defined with respect to V. Given a harmonic function s on $\bar{V}$, a necessary and sufficient condition that there exists a harmonic function p in W which satisfies $p - s = L(p - s)$ in V is that $\int_{\beta(V)} (ds)^* = 0$. The function p is uniquely determined up to an additive constant.*

1.3 *Extremal length*

Let Γ be a family of locally rectifiable curves τ in W. The extremal length $\lambda(\Gamma)$ of Γ is defined as

$$\lambda(\Gamma)=\sup_{\rho}\frac{L(\Gamma, \rho)^2}{A(\rho)}$$

where $A(\rho)=\iint_W \rho^2 dx\,dy$, $L(\Gamma, \rho)=\inf_{\tau}\int_{\tau}\rho|dz|$ and $\rho|dz|$ ranges over all Borel measurable linear densities which are non-negative and for which $0<A(\rho)<\infty$.

We say that a statement concerning curves $\tau\in\Gamma$ is true for *almost all* curves in Γ if the subfamily Γ' of Γ consisting of curves τ for which the statement is false has $\lambda(\Gamma')=\infty$.

The following lemma will be used in the present paper, as was used effectively by A. Marden and B. Rodin [5], [6].

Fuglede's lemma [4]. *Let $\{\rho_n|dz|\}$ be a sequence of Lebesgue measurable linear densities such that $A(\rho_n)$ tends to zero as $n\to\infty$. Then there is a subsequence $\{\rho_{n_k}|dz|\}$ which satisfies $\lim_{k\to\infty}\int\rho_{n_k}|dz|=0$ for almost all locally rectifiable curves τ in W.*

We shall have occasions to use the following properties of extremal length:

(E1) *If for any $\tau\in\Gamma$ there exists a $\tau'\in\Gamma'$ contained in τ, then $\lambda(\Gamma)\geqq\lambda(\Gamma')$.*

(E2) $$\frac{1}{\lambda\left(\bigcup_{n=1}^{\infty}\Gamma_n\right)}\leqq\sum_{n=1}^{\infty}\frac{1}{\lambda(\Gamma_n)}.$$

(E3) *If Γ_n increases (i.e., $\Gamma_{n+1}\supset\Gamma_n$ for all n), then $\lim_{n\to\infty}\lambda(\Gamma_n)=\lambda(\bigcup_{n=1}^{\infty}\Gamma_n)$.*

(E4) *Let f be a continuously differentiable function in W with $D_W(f)<\infty$. Then $f(\tau)$ exists and is finite for almost all $\tau\in\Gamma_{\beta(W)}$.*

(E5) *Let $V\in\mathcal{V}$ and $f\in C^1(\partial V)$. Denote by H_f^V the Dirichlet solution with respect to V with the boundary values f on ∂V and 0 on $\beta(V)$. Then $H_f^V(\tau)=0$ for almost all $\tau\in\Gamma_{\beta(V)}$. Conversely, a harmonic function F in V which satisfies $F=f$ on ∂V, $D_V(F)<\infty$ and $F(\tau)=0$ for almost all $\tau\in\Gamma_{\beta(V)}$ is equal to H_f^V.*

As to all of these results except for (E3) we refer to M. Ohtsuka [7], [8], [9]. As to (E3), see N. Suita [11] or W. Ziemer [12].

LEMMA 1. *Let f be a Dirichlet function*[1] *on W. Then f is a Dirichlet potential*[1] *if and only if $f(\tau)=0$ for almost all $\tau\in\Gamma_{\beta(W)}$.*

PROOF. First, suppose that f is a Dirichlet potential. It follows from Theorem 7.5 and Lemma 7.8 in C. Constantinescu and A. Cornea [3] that

1) For these notions, see C. Constantinescu and A. Cornea [3].

there exists a sequence $\{f_n\}$ of continuously differentiable functions with compact supports such that $\lim_{n\to\infty} D_W(f_n - f) = 0$ and $\lim_{n\to\infty} f_n(z) = f(z)$ for any $z \in W - A$ where A is a polar set in W. We see that the subfamily Γ_1 of $\Gamma_{\beta(W)}$ which consists of curves $\tau \in \Gamma_{\beta(W)}$ starting from points of A has $\lambda(\Gamma_1) = \infty$ and that there is a subfamily Γ_2 of $\Gamma_{\beta(W)}$ such that the functions f_n, f are absolutely continuous on τ for all $\tau \in \Gamma_{\beta(W)} - \Gamma_2$ and all n (see B. Fuglede [4]). We write $\Gamma^*_{\beta(W)} = \Gamma_{\beta(W)} - \Gamma_1 - \Gamma_2$. Applying Fuglede's lemma with $\{\rho_n |dz|\} = \{|\text{grad}\,(f_n - f)|\,|dz|\}$, we find a subsequence $\{\rho_{n_k}|dz|\}$ such that $\lim_{k\to\infty}\int_\tau |\text{grad}(f_{n_k} - f)|\,|dz| = 0$ for almost all $\tau \in \Gamma^*_{\beta(W)}$. It follows from $f_{n_k}(\tau) = 0$ and $\lim_{k\to\infty} f_{n_k}(\tau(0)) = f(\tau(0))$ that $0 = \lim_{k\to\infty}\int_\tau |\text{grad}\,(f_{n_k} - f)|\,|dz| \geqq \lim_{k\to\infty}\int_\tau |df_{n_k} - df| \geqq \lim_{k\to\infty} |\{f_{n_k}(\tau) - f_{n_k}(\tau(0))\} - \{f(\tau) - f(\tau(0))\}| = |f(\tau)|$ for almost all $\tau \in \Gamma^*_{\beta(W)}$. Observing $\lambda(\Gamma_{\beta(W)} - \Gamma^*_{\beta(W)}) = \infty$, we have thus $f(\tau) = 0$ for almost all $\tau \in \Gamma_{\beta(W)}$.

To prove the converse, suppose that a Dirichlet function f has $f(\tau) = 0$ for almost all $\tau \in \Gamma_{\beta(W)}$. We consider the Royden decomposition (see [3]) of f : $f = u + f_0$ where u is harmonic in W with $D_W(u) < \infty$ and f_0 is a Dirichlet potential. Since $f_0(\tau) = 0$ for almost all $\tau \in \Gamma_{\beta(W)}$, our assumption implies that $u(\tau) = 0$ for almost all $\tau \in \Gamma_{\beta(W)}$. It follows from Corollary 1 in M. Ohtsuka [9] that $u = 0$ in W. Therefore, f is equal to the Dirichlet potential f_0.

§2. Distinguished normal operators

2.1 *Definition*

Definition. A normal operator L defined with respect to V is said to be *distinguished*, if it satisfies the following additional condition:

(6) $D_V(Lf) < \infty$ and $\int_{\beta(V)} (Lf)(dLg)^* = 0$ for all $f, g \in C^1(\partial V)$.

Let L be a normal operator defined with respect to V and let $f, g \in C^1(\partial V)$. Now, take any approximation $\{\Omega_n\}$ toward $\beta(V)$ such that $\Omega_1 \supset \partial V$ and $V \supset \partial\Omega_1$. Since $Lf = f$ on ∂V, using Green's formula, we obtain

$$D_{\Omega_n \cap V}(Lf, Lg) = \int_{\partial\Omega_n} (Lf)(dLg)^* + \int_{\partial V} f(dLg)^*.$$

Hence $\lim_{n\to\infty}\int_{\partial\Omega_n} (Lf)(dLg)^*$ exists and is equal to $D_V(Lf, Lg) - \int_{\partial V} f(dLg)^*$, namely, $\int_{\beta(V)} (Lf)(dLg)^* = D_V(Lf, Lg) - \int_{\partial V} f(dLg)^*$. It follows that the normal operator L is distinguished if and only if

$$(*) \qquad D_V(Lf, Lg) = \int_{\partial V} f(dLg)^*$$

for all $f, g \in C^1(\partial V)$.

Principal operators L_{0V} and $(P)L_{1V}$ defined in Chapter III of L. Ahlfors and L. Sario [1] are typical examples of distinguished normal operators defined with respect to V. Other examples will be given in 2.3. We shall give a normal operator which is not distinguished:

Let W be an open Riemann surface and take a $V \in \mathfrak{V}=\mathfrak{V}(W)$ which is not relatively compact. Assume that a distinguished normal operator L defined with respect to V is given. Choose a relatively compact regular region Ω in W such that $\Omega \supset \partial V$. We consider Ω as a given open Riemann surface. Obviously, $\Omega \cap V \in \mathfrak{V}(\Omega)$, which we denote by V'. Then $C^1(\partial V)=C^1(\partial V')$. We let $L'f$ be the restricted function to V' of Lf for each $f \in C^1(\partial V')$. The operator $L': f \in C^1(\partial V') \to L'f$ is a normal operator defined with respect to V' but not distinguished.

REMARK. In general, the operator $H^V: f \in C^1(\partial V) \to H_f^V$ does not satisfy conditions (3), (5) in 1.2 and hence H^V is not normal. However, it has the following property:

$$\int_{\beta(V)} H_f^V (dF)^* = 0$$

where F is any harmonic function in V with $D_V(F) < \infty$. In particular, the operator H_f^V satisfies condition (6) in 2.1. It is clear that if $f \geqq 0$, $f \in C^1(\partial V)$, then $Lf \geqq H_f^V$ for any normal operator L defined with respect to V.

The characterizations of L_{0V} and $(P)L_{1V}$ in K. Oikawa [10], together with the above property of H^V, imply the following lemma:

LEMMA 2. *Let $V \in \mathfrak{V}$. Assume that a distinguished normal operator L_V defined with respect to V satisfies the additional condition that, for any $f \in C^1(\partial V)$, $\int_\beta (dL_V f)^* = 0$ for every dividing cycle β of W which are contained in V and does not separate components of ∂V. For any $f \in C^1(\partial V)$, we set $u_1 = H_f^V$, $u_2 = (I)L_{1V}f$, $u_3 = (Q)L_{1V}f$, $u_4 = L_V f$ and $u_5 = L_{0V}f$. Then $D_V(u_i - u_j) = D_V(u_i) - D_V(u_j) \geqq 0$ for any $1 \leqq i < j \leqq 5$.*

PROOF. If $i \leqq j$, we see that $\int_{\beta(V)} u_i (du_j)^* = 0$. It follows that $D_V(u_i, u_j) = \int_{\beta(V)+\partial V} u_i (du_j)^* = \int_{\partial V} f (du_j)^* = \int_{\beta(V)+\partial V} u_j (du_j)^* = D_V(u_j)$ and that $0 \leqq D_V(u_i - u_j) = D_V(u_i) - D_V(u_j)$.

2.2 *Consistent system of distinguished normal operators*

Definition. For each $V \in \mathfrak{V}$, let a distinguished normal operator L_V defined with respect to V be given. The system $\boldsymbol{L} = \{L_V\}_{V \in \mathfrak{V}}$ is said to be *consistent*, if for any V_1 and V_2 such that $V_1 \supset V_2$

$$(**) \qquad L_{V_2}(L_{V_1}f) = L_{V_1}f$$

in V_2 for all $f \in C^1(\partial V_1)$.

It is a consequence of the definition that $\int_\beta (dL_V f)^* = 0$ for every dividing cycle of W which is contained in V and does not separate components of ∂V. Hence, in general, the system $\{(I)L_{1V}\}_{V \in \boldsymbol{v}}$ cannot be consistent. The systems $\boldsymbol{L}_0 = \{L_{0V}\}_{V \in \boldsymbol{v}}$ and $\boldsymbol{L}_1 = \{(Q)L_{1V}\}_{V \in \boldsymbol{v}}$ are typical examples of consistent systems. The following fact is well known (for example, see L. Ahlfors and L. Sario [1], IV, 1D):

If W is of class $O_{KD}(=O_{H_1 D})$, then $L_{0V} = (Q)L_{1V}$ for any $V \in \boldsymbol{v}$. Conversely, $W \in O_{KD}$, if $L_{0V} = (Q)L_{1V}$ for some $V \in \boldsymbol{v}$ such that $W - V$ is compact.

It follows from Lemma 1 that *a surface W is of class O_{KD} if and only if all the $\boldsymbol{L}$ coincide.*

2.3 *Examples of consistent systems*

Example 1. Let W be an open Riemann surface. Partition the ideal boundary $\beta(W)$ into two disjoint closed subsets γ_0 and γ_1. That is, γ_0 and γ_1 are isolated sets on $\beta(W)$ and $\beta(W) = \gamma_0 \cup \gamma_1$. For each $f \in C^1(\partial V)$, there exists a unique function $L_V^{\gamma_0,\gamma_1}f (= L_V f)$ which is continuously differentiable on $\bar{V}$, is harmonic in V and satisfies the following conditions:

(i) $L_V f = f$ on ∂V,

(ii) $L_V f = L_{0V'}(L_V f)$ for any $V' \in \boldsymbol{v}$ such that $V' \subset V$ and $\beta(V') = \gamma_0 \cap \beta(V)$,

(iii) $L_V f = (Q)L_{1V'}(L_V f)$ for any $V' \in \boldsymbol{v}$ such that $V' \subset V$ and $\beta(V') = \gamma_1 \cap \beta(V)$.

The operator $L_V^{\gamma_0,\gamma_1}$ is a distinguished normal operator defined with respect to V and the the system $\{L_V^{\gamma_0,\gamma_1}\}_{V \in \boldsymbol{v}}$ is consistent. If $\gamma_0 = \phi$ (resp. $\gamma_1 = \phi$), then the system $\{L_V^{\gamma_0,\gamma_1}\}_{V \in \boldsymbol{v}}$ is equal to $\boldsymbol{L}_1$ (resp. $\boldsymbol{L}_0$).

Example 2. Let W be an open Riemann surface. Partition $\beta(W)$ into two disjoint sets γ_0 and γ_1 where γ_1 is closed. Choose a sequence of isolated sets $\gamma_1^{(n)}$ on $\beta(W)$ such that $\gamma_1^{(n)} \supset \gamma_1^{(n+1)}$ and $\bigcap_{n=1}^{\infty} \gamma_1^{(n)} = \gamma_1$. Write $\gamma_0^{(n)} = \beta(W) - \gamma_1^{(n)}$ and $L_V^{(n)} = L_V^{\gamma_0(n),\gamma_1(n)}$ (see Example 1). For each $f \in C^1(\partial V)$, we have

$$D_V(L_V^{(n)}f - L_V^{(m)}f) = D_V(L_V^{(n)}f) - D_V(L_V^{(m)}f) \geqq 0$$

for $m > n$. Since $D_V(L_V^{(n)}f) \leqq D_V(H_f^V) < \infty$ for all n, $L_V^{(n)}f$ tends to a continuously differentiable function on $\bar{V}$ which is harmonic in V. We denote the limit function by $L_V^{\gamma_0,\gamma_1}f$. The operator $L_V^{\gamma_0,\gamma_1}$ is a distinguished normal operator defined with respect to V and the system $\{L_V^{\gamma_0,\gamma_1}\}_{V \in \boldsymbol{v}}$ is consistent. This is a generalization of Example 1.

Example 3. Suppose that W is the interior of a compact bordered Riemann surface $\bar{W}$. Divide the border ∂W into two disjoint sets E_0 and E_1 in such a way that, for each contour C on ∂W, $E_0\cap C$ (resp. $E_1\cap C$) consists of a finite number of open (resp. closed) subarcs on C. ($E_0\cap C$ or $E_1\cap C$ may be empty.) For $V\in \mathcal{V}$, we denote by $\hat{V}$ the closure of V in $\bar{W}$. For each $f\in C^1(\partial V)$, there exists a unique function $L_V^{E_0,E_1}f(=L_Vf)$ on $\hat{V}$ which is continuously differentiable on $\bar{V}$ and harmonic in V and satisfies the following conditions:

(i) $L_Vf=f$ on ∂V,

(ii) the normal derivative of L_Vf vanishes on $E_0\cap\hat{V}$,

(iii) for each connected component c of $E_1\cap\hat{V}$, the function L_Vf is constant on c and has $\int_c (dL_Vf)^*=0$.

Then $L_V^{E_0,E_1}$ is distinguished and $\{L_V^{E_0,E_1}\}_{V\in\mathcal{V}}$ is consistent.

Example 4. Let an open Riemann surface W be a rectangle with vertical sides A, A' and horizontal sides B, B'. Identify A, B with A', B' respectively so that the resulting manifold becomes a compact Riemann surface S of genus 1. For each $f\in C^1(\partial V)$, consider the Dirichlet solution $H_f^{S-(W-V)}$ with respect to the relatively compact open set $S-(W-V)$ in S. We denote by L_Vf the restriction to V of the function $H_f^{S-(W-V)}$. Then the system $\{L_V\}_{V\in\mathcal{V}}$ is a consistent system of distinguished normal operators.

2.4 ***L*-behavior**

Let $\boldsymbol{L}=\{L_V\}_{V\in\mathcal{V}}$ be a consistent system and let γ be a relatively open set on $\beta(W)$. Assume that a function u is defined in the intersection of W with a neighborhood Ω_1^* of γ in W^*.

Definition. We say that the function u has *$\boldsymbol{L}$-behavior* on γ, if there exists a neighborhood Ω_0^* of γ such that $\Omega_0^*\subset\Omega_1^*$ and $L_Vu=u$ for all $V\in\mathcal{V}$ which satisfy $\beta(V)\subset\gamma$ and $V\subset\Omega_0^*\cap W$.

Conditions (2), (3) in 1.2 imply that if u has $\boldsymbol{L}$-behavior on γ, then c_1u+c_2 has also $\boldsymbol{L}$-behavior on γ for any real numbers c_1 and c_2. It is clear from the maximum principle that a harmonic function in W has $\boldsymbol{L}$-behavior on $\beta(W)$ if and only if it is a constant.

It is known that the system $\{H^V\}_{V\in\mathcal{V}}$ has the consistency (**) in 2.2. Therefore we can define $\{H^V\}_{V\in\mathcal{V}}$-behavior in the same way as above. We shall call this simply *0-behavior*.

LEMMA 3. *Let $V_0\in\mathcal{V}$ and let u be a harmonic function on $\bar{V}_0$. Then u has $\boldsymbol{L}$-behavior on $\beta(V_0)$ if and only if $u=L_{V_1}u$ in V_1 for some $V_1\in\mathcal{V}$ such that $V_1\subset V_0$ and $\beta(V_1)=\beta(V_0)$. Moreover, in this case, $u=L_{V_0}u$ on $\bar{V}_0$.*

PROOF. Suppose that u has $\boldsymbol{L}$-behavior on $\beta(V_0)$. It follows from the definition of $\boldsymbol{L}$-behavior that there exists a $V_1\in\mathcal{V}$ such that $V_1\subset V_0$, $\beta(V_1)=\beta(V_0)$ and $u=L_{V_1}u$ in V_1.

To prove the converse, suppose that there exists a $V_1\in\mathcal{V}$ with the above property. We observe by (**) in 2.2 that $u-L_{V_0}u=L_{V_1}(u-L_{V_0}u)$ on $\bar{V}_1$. It follows from the maximum principle for L_{V_1} that $\max_{z\in\partial V_1}(u-L_{V_0}u)(z)=\sup_{z\in V_1}(u-L_{V_0}u)(z)$ and $\min_{z\in\partial V_1}(u-L_{V_0}u)(z)=\inf_{z\in V_1}(u-L_{V_0}u)(z)$. On the other hand, $\beta(V_1)=\beta(V_0)$ implies that $\bar{V}_0-V_1$ is compact. If $u-L_{V_0}u$ is positive at some point in V_0, the maximum is taken at some point on ∂V_1. Hence $u-L_{V_0}u$ is constantly equal to a positive value. This is impossible because $u-L_{V_0}u=0$ on ∂V_0. Consequently $u-L_{V_0}u\leqq 0$ in V_0. Similarly, we see that $u-L_{V_0}u\geqq 0$ in V_0. Therefore $u=L_{V_0}u$ in V_0. For each $V\in \mathfrak{V}$ such that $V\subset V_0$, we obtain by (**) $L_V u=L_V(L_{V_0}u)=L_{V_0}u=u$ in V. Hence the function u has $\boldsymbol{L}$-behavior on $\beta(V_0)$.

Lemma 4. *Let α be a closed set on $\beta(W)$ and let $V\in\mathfrak{V}$ be such that $\beta(V)\supset\alpha$ or $V=W$. Let $\{\Omega_n\}$ be an approximation of W toward α. We write $\Omega_n\cap V=V_n$. Suppose that u_n is a harmonic function in V_n which has $\boldsymbol{L}$-behavior on $\beta(V_n)$. Furthermore suppose that u_n converges to u uniformly on any compact set in V. Then the function u in V has $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$.*

Proof. Let V' be any set in $\mathfrak{V}$ such that $\beta(V')\subset\beta(V)-\alpha$ and $\bar{V}'\subset V$. For sufficiently large n, the set $\bar{V}'$ is contained in V_n. Since the function u_n has $\boldsymbol{L}$-behavior on $\beta(V_n)(\supset\beta(V'))$, we have $u_n=L_{V'}u_n$ on $\bar{V}'$. Observing that $L_{V'}u_n$ converges to $L_{V'}u$ uniformly on $\bar{V}'$, we obtain, by letting $n\to\infty$, $u=L_{V'}u$ on $\bar{V}'$. Consequently the function u has $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$.

§3. L-harmonic measures, L-Green functions and L-null sets

3.1 *L-harmonic measures*

Let $\boldsymbol{L}=\{L_V\}_{V\in\mathfrak{V}}$ be a consistent system and a set α be a closed subset of $\beta(W)$. Let $V\in\mathfrak{V}$ be such that $\beta(V)\supset\alpha$. We shall define the $\boldsymbol{L}$-harmonic measure of α with respect to V. Let w be the function on $\bar{V}$ which is equal to 0 on ∂V and 1 in V. Let $\{\Omega_n\}$ be an approximation of W toward α such that $\Omega_1\supset\partial V$ and $\partial\Omega_1\subset V$. We set $\Omega_n\cap V=V_n$ and $\partial\Omega_n=\alpha_n$. Then $\bigcup_{n=1}^{\infty}V_n=V$ and $\partial V_n=\partial\Omega_n\cup\partial V$. It follows from the maximum principle for L_{V_n} that the function $L_{V_n}w$ decreases. Hence the limit function

$$\omega_{\boldsymbol{L}}(z;\alpha,V)=\lim_{n\to\infty}L_{V_n}w(z)$$

exists and is harmonic in V. It is independent of the choice of approximation $\{\Omega_n\}$. It is clear that $0\leqq\omega_{\boldsymbol{L}}(z;\alpha,V)<1$. We see by Lemma 4 that $\omega_{\boldsymbol{L}}(z;\alpha,V)$ has $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$. We say that $\omega_{\boldsymbol{L}}(z;\alpha,V)$ $(=\omega_{\boldsymbol{L}}(\alpha,V))$ is the ***L*-*harmonic measure*** of α with respect to V.

Lemma 5. $\lim_{n\to\infty}D_{V_n}(L_{V_n}w-\omega_{\boldsymbol{L}}(\alpha,V))=0.$

Proof. If $m>n$, $L_{V_m}w=L_{V_n}(L_{V_m}w)$ in V_n. It follows from (*) in 2.1 that $D_{V_n}(L_{V_n}w, L_{V_m}w)=D_{V_n}(L_{V_n}w, L_{V_n}(L_{V_m}w))=\int_{\alpha_n\cup\partial V}w(dL_{V_m}w)^*=\int_{\alpha_n}(dL_{V_m}w)^*$. Condition (5) in 1.2 implies that $0=\int_{\beta(V_n)}(dL_{V_n}(L_{V_m}w))^*=-\int_{\alpha_n\cup\partial V}(dL_{V_n}(L_{V_m}w))^*$ $=-\int_{\alpha_n\cup\partial V}(dL_{V_m}w)^*$ and $0=\int_{\beta(V_m)}(dL_{V_m}w)^*=-\int_{\alpha_m\cup\partial V}(dL_{V_m}w)^*$, this is, $\int_{\alpha_n}(dL_{V_m}w)^*=\int_{\alpha_m}(dL_{V_m}w)^*$. Consequently $D_{V_n}(L_{V_n}w, L_{V_m}w)=\int_{\alpha_m}(dL_{V_m}w)^*=$ $\int_{\partial V_m}w(dL_{V_m}w)^*=D_{V_m}(L_{V_m}w)$. We have thus $0\leqq D_{V_n}(L_{V_n}w-L_{V_m}w)=D_{V_n}(L_{V_n}w)$ $-D_{V_m}(L_{V_m}w)$. Hence $D_{V_n}(L_{V_n}w)$ decreases and $\lim_{n\to\infty}D_{V_n}(L_{V_n}w)$ exists and is finite. Given $\varepsilon>0$, choose n such that $D_{V_n}(L_{V_n}w-L_{V_m}w)<\varepsilon$ for all $m>n$. By Fatou's lemma we obtain $D_{V_n}(L_{V_m}w-w_{\boldsymbol{L}}(\alpha, V)))\leqq\underline{\lim}_{m\to\infty}D_{V_n}(L_{V_n}w-L_{V_m}w)\leqq\varepsilon$ proving $\lim_{n\to\infty}D_{V_n}(L_{V_n}w-\omega_{\boldsymbol{L}}(\alpha,V))=0$.

3.2. ***L**-Green functions*

Let $V\in\mathfrak{V}$. Then the $\boldsymbol{L}$-Green function of V with pole at $\zeta\in V$ is defined as the harmonic function in $V-\{\zeta\}$ with singularity $-\log|z-\zeta|$ at ζ, which vanishes continuously on the relative boundary ∂V of V and has $\boldsymbol{L}$-behavior on $\beta(V)$. Its uniqueness is evident. Its existence is proved by the same way as in L. Ahlfors and L. Sario [1], III, 4C. We denote it by $g_{\boldsymbol{L}}(z;\zeta,\partial V)$. It is clear that $g_{\boldsymbol{L}}(z;\zeta,\partial V)>0$ in the component of V which contains ζ.

Let α be a closed set on $\beta(W)$. We shall define the $\boldsymbol{L}$-Green function of W with respect to α. Let $\{\Omega_n\}$ be an approximation of W toward α such that $\Omega_1\ni\zeta$. We see by the maximum principle for L_{Ω_n} that $g_{\boldsymbol{L}}(z;\zeta,\partial\Omega_n)$ increases with n. Therefore

$$g_{\boldsymbol{L}}(z;\zeta,\alpha)=\lim_{n\to\infty}g_{\boldsymbol{L}}(z;\zeta,\partial\Omega_n)$$

is either identically equal to $+\infty$ or finite and positive for all $z\neq\zeta$. In the second case we say that the ***L**-Green function* of W with pole at ζ with respect to α exists. The above definition does not depend on the choice of approximation $\{\Omega_n\}$. If it exists, it is clear that $g_{\boldsymbol{L}}(z;\zeta,\alpha)$ is harmonic in $W-\{\zeta\}$ with singularity $-\log|z-\zeta|$ at ζ and has $\boldsymbol{L}$-behavior on $\beta(W)-\alpha$. Moreover, for any $V\in\mathfrak{V}$ with $\beta(V)\supset\alpha$, we have $\int_{\beta(V)}(dg_{\boldsymbol{L}}(z;\zeta,\alpha))^*=-2\pi$.

LEMMA 6. *If the **L**-Green function $g_{\boldsymbol{L}}(z;\zeta,\alpha)$ exists, then $\lim_{n\to\infty}D_{\Omega_n}(g_{\boldsymbol{L}}(z;\zeta,\partial\Omega_n)-g_{\boldsymbol{L}}(z;\zeta,\alpha))=0$. In particular, $D_{W-\Delta}(g_{\boldsymbol{L}}(z;\zeta,\alpha))<\infty$ where Δ is any neighborhood of ζ.*

PROOF. We write $g_{\boldsymbol{L}}(z;\zeta,\partial\Omega_n)=g_n$ and $g_{\boldsymbol{L}}(z;\zeta,\alpha)=g$. Let Δ_0 be a disk with center at ζ. For $m\geqq n$, we have $D_{\Omega_n}(g_m-g_n)\leqq D_{\Omega_m-\Delta_0}(g_m)-2D_{\Omega_n-\Delta_0}(g_m, g_n)$

$+ D_{\Omega_n - \Delta_0}(g_n) + D_{\Delta_0}(g_m - g_n) = \int_{-\partial\Delta_0} g_m(dg_m)^* - 2\int_{-\partial\Delta_0} g_n(dg_m)^* + \int_{\partial\Delta_0} g_n(dg_n)^* + \int_{\partial\Delta_0} (g_m - g_n)(d(g_m - g_n))^*$. Since g_n converges to g uniformly on a neighborhood of $\partial\Delta_0$, we obtain

$$D_{\Omega_n}(g - g_n) = \lim_{m\to\infty} D_{\Omega_n}(g_m - g_n) \leqq \int_{\partial\Delta_0} g_n(dg)^* - \int_{\partial\Delta_0} g(dg_n)^*.$$

It follows that $\lim_{n\to\infty} D_{\Omega_n}(g - g_n) = 0$.

3.3 ***L**-null sets*

Definition. A closed set α on $\beta(W)$ is said to be an ***L**-null* set, if there is a $V \in \mathcal{V}$ such that $\beta(V) \supset \alpha$ and the $\boldsymbol{L}$-harmonic measure $\omega_{\boldsymbol{L}}(\alpha, V)$ vanishes.

If the closed set α is $\boldsymbol{L}_1$ (resp. $\boldsymbol{L}_0$)-null, then α is called "*schwach*" (resp. "*halbschwach*") in C. Constantinescu [2].

PROPOSITION 1. *Every $\boldsymbol{L}_1$-null set is an **L**-null set. Every **L**-null set is an $\boldsymbol{L}_0$-null set.*

PROOF. Suppose that α is an $\boldsymbol{L}_1$-null set. Choose $V \in \mathcal{V}$ such that $\beta(V) \supset \alpha$ and $\omega_{\boldsymbol{L}_1}(\alpha, V) = 0$. Let w and V_n be the same as in 3.1. Lemma 2 implies that

$$D_{V_n}((Q)L_{1V_n}w) = D_{V_n}(L_{V_n}w) + D_{V_n}((Q)L_{1V_n}w - L_{V_n}w).$$

Letting $n \to \infty$, we obtain by Lemma 5

$$D_V(\omega_{\boldsymbol{L}_1}(\alpha, V)) = D_V(\omega_{\boldsymbol{L}}(\alpha, V)) + D_V(\omega_{\boldsymbol{L}_1}(\alpha, V) - \omega_{\boldsymbol{L}}(\alpha, V)).$$

It follows from $\omega_{\boldsymbol{L}_1}(\alpha, V) = 0$ that $\omega_{\boldsymbol{L}}(\alpha, V) = 0$. Hence every $\boldsymbol{L}_1$-null set is $\boldsymbol{L}$-null.

Similarly, we see that every $\boldsymbol{L}$-null set is $\boldsymbol{L}_0$-null.

The next proposition follows from Theorem III in 3.1 of A. Marden and B. Rodin [5]:

PROPOSITION 2. *A closed set α is $\boldsymbol{L}_0$-null if and only if $\lambda(\Gamma_\alpha) = \infty$ where Γ_α is the family of curves in W defined in* 1.1.

THEOREM 1. *Let α be a closed set on $\beta(W)$. The following four conditions are equivalent:*

($N1$) *The set α is an **L**-null set, that is, for some V with $\beta(V) \supset \alpha$, $\omega_{\boldsymbol{L}}(\alpha, V)$ vanishes;*

($N2$) *For some $V \in \mathcal{V}$ with $\beta(V) \supset \alpha$, any bounded harmonic function u on $\bar{V}$ which has **L**-behavior on $\beta(V) - \alpha$ is equal to $L_V u$ on $\bar{V}$. That is, u has **L**-behavior on $\beta(V)$;*

($N3$) *For some $V \in \mathcal{V}$ with $\beta(V) \supset \alpha$, any harmonic function u on $\bar{V}$ which* *has $D_V(u)<\infty$ and $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$ is equal to $L_V u$ on $\bar{V}$;*

(*N*4) *For some $\zeta\in W$, the $\boldsymbol{L}$-Green function $g_{\boldsymbol{L}}(z;\zeta,\alpha)$ does not exist.*

PROOF. (N1)→(N2). Assume that α is an $\boldsymbol{L}$-null set. Namely, we can find a $V\in\mathfrak{V}$ such that $\beta(V)\supset\alpha$ and $\omega_{\boldsymbol{L}}(\alpha, V)=0$ on $\bar{V}$. Let $\{V_n\}$ be as in 3.1. Let u be any bounded harmonic function on $\bar{V}$ which has $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$. We write $M=\sup_{z\in V}|u(z)|$. Then

$$L_V u-2Mw\leqq u\leqq L_V u+2Mw$$

on ∂V_n. Operating L_{V_n} to this inequality, we have

$$L_{V_n}(L_V u-2Mw)\leqq L_{V_n}u\leqq L_{V_n}(L_V u+2Mw)$$

in V_n. Since u has $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$, $L_{V_n}u=u$ in V_n. This, together with (**) in 2.2, implies that

$$L_V u-2ML_{V_n}w\leqq u\leqq L_V u+2ML_{V_n}w$$

in V_n. Letting $n\to\infty$, we have

$$L_V u-2M\omega_{\boldsymbol{L}}(\alpha, V)\leqq u\leqq L_V u+2M\omega_{\boldsymbol{L}}(\alpha, V)$$

in V. It follows from $\omega_{\boldsymbol{L}}(\alpha, V)=0$ that $L_V u=u$ in V.

(N2)→(N1). The harmonic measure $\omega_{\boldsymbol{L}}(\alpha, V)$ is a bounded harmonic function on $\bar{V}$ which is 0 on ∂V and has $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$. Hence under hypothesis (N2) we have $\omega_{\boldsymbol{L}}(\alpha, V)=L_V\omega_{\boldsymbol{L}}(\alpha, V)=L_V 0=0$.

(N1)→(N3). Let $V\in\mathfrak{V}$ satisfy the condition in (N1). In order to prove that the set V also satisfies the condition in (N3), it is sufficient to show that a harmonic function u on $\bar{V}$ such that $D_V(u)<\infty$, $u=0$ on ∂V and u has $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$ reduces to zero. To this end we begin with showing that the function u is decomposed as $u=u^+ + u^-$ in V, where u^+, $u^-=0$ on ∂V, u^+, $-u^-\geqq 0$ on $\bar{V}$. $D_V(u^+)$, $D_V(u^-)<\infty$ and u^+, u^- have $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$.

Let $\{\Omega_n\}$ be an approximation of W toward α such that $\Omega_1\supset\partial V$ and $V\supset\partial\Omega_1$. Write $\partial\Omega_n=\alpha_n$ and $\Omega_n\cap V=V_n$. Consider the functions $L_{V_n}\max(u, 0)$ and $L_{V_n}\min(u, 0)$ in V_n, which we denote by u_n^+ and u_n^- respectively. Since u has $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$, $u=L_{V_n}u=u_n^+ + u_n^-$ in V_n. Obviously, u_n^+, $u_n^-=0$ on ∂V, u_n^+, $-u_n^-\geqq 0$ on $\bar{V}$ and u_n^+, u_n^- have $\boldsymbol{L}$-behavior on $\beta(V_n)$. Furthermore, $D_{V_n}(u_n^+)$, $D_{V_n}(u_n^-)\leqq D_{V_n}(u)\leqq D_V(u)<\infty$. In fact, we obtain by (*) in 2.1 that $D_{V_n}(u_n^+)=\int_{\partial V_n}\max(u,0)(du_n^+)^*=\int_{\alpha_n\cap\{u>0\}}u(du_n^+)^*$. Since $(du_n^+)^*=(\partial u_n^+/\partial n)ds\leqq 0$ on $\alpha_n\cap\{u<0\}$, we have $D_{V_n}(u_n^+)\leqq\int_{\alpha_n}u(du_n^+)^*=\int_{\partial V_n}L_{V_n}u(du_n^+)^*=D_{V_n}(L_{V_n}u, u_n^+)$ $=D_{V_n}(u, u_n^+)$. Consequently $D_{V_n}(u_n^+)\leqq D_{V_n}(u)$. Similarly, we have $D_{V_n}(u_n^-)\leqq D_n^V(u)$.

It follows that $\{u_n^+(z),\ u_n^-(z)\}$ is bounded on any compact subset of $\bar{V}$ where n is so large that $\bar{V}_n$ contains the compact set. Hence we can choose a subsequence $\{n_k\}$ such that $u_{n_k}^+$ and $u_{n_k}^-$ converge uniformly on any compact subset of $\bar{V}$. We write $u^+=\lim_{k\to\infty} u_{n_k}^+$ and $u^-=\lim_{k\to\infty} u_{n_k}^-$. It is easy to see that $u=u^++u^-$ on $\bar{V}$, u^+, $u^-=0$ on ∂V, u^+, $-u^-\geqq 0$ on $\bar{V}$ and u^+, u^- have $\boldsymbol{L}$-behavior on $\beta(V)-\alpha$. Moreover, by Fatou's lemma, we obtain $D_V(u^+)\leqq \varliminf_{k\to\infty} D_V(u_{n_k}^+)\leqq D_V(u)<\infty$. Similarly, we obtain $D_V(u^-)\leqq D_V(u)<\infty$. Hence we have a required decomposition of u.

Since $\omega_{\boldsymbol{L}}(\alpha, V)=0$, we see by Lemma 5 and (*) in 2.1 that $0= D_V(u^+, \omega_{\boldsymbol{L}}(\alpha, V))=\lim_{n\to\infty} D_{V_n}(u^+, L_{V_n}w)=\lim_{n\to\infty} D_{V_n}(L_{V_n}u^+, L_{V_n}w)=\lim_{n\to\infty}\int_{\partial V_n} w(dL_{V_n}u^+)^*$ $=\lim_{n\to\infty}\int_{\alpha_n}(dL_{V_n}u^+)^*=-\lim_{n\to\infty}\int_{\partial V}(dL_{V_n}u^+)^*=-\int_{\partial V}(du^+)^*$. On the other hand, because $u^+=0$ on ∂V and $u^+\geqq 0$ on $\bar{V}$, we have $(du^+)^*=(\partial u^+/\partial n)ds\leqq 0$ on ∂V. Hence $(du^+)^*=0$ on ∂V. It follows that $u^+=0$ on $\bar{V}$. Similarly, we have $u^-=0$ on $\bar{V}$. Consequently $u=u^++u^-=0$ on $\bar{V}$.

(N3)$\to$(N1). Since the harmonic measure $\omega_{\boldsymbol{L}}(\alpha, V)$ satisfies the condition in (N3), we have $\omega_{\boldsymbol{L}}(\alpha, V)=L_V\omega_{\boldsymbol{L}}(\alpha, V)=0$.

(N1)$\to$(N4). Assume that the closed set α is $\boldsymbol{L}$-null. That is, there exists a $V\in \mathfrak{V}$ such that $\beta(V)\supset\alpha$ and $\omega_{\boldsymbol{L}}(V, \alpha)=0$ on $\bar{V}$. Now, suppose that for some $\zeta\in W$, the $\boldsymbol{L}$-Green function $g_{\boldsymbol{L}}(z;\zeta,\alpha)$ exists. Choose $V'\in\mathfrak{V}$ such that $V'\subset V$, $\bar{V}'\not\ni\zeta$ and $\beta(V')=\beta(V)$. Then it follows that $0=\omega_{\boldsymbol{L}}(\alpha, V)\geqq\omega_{\boldsymbol{L}}(\alpha, V')\geqq 0$. Lemma 6 shows $D_{V'}(g_{\boldsymbol{L}}(z;\zeta,\alpha))<\infty$. Hence property (N3) implies $g_{\boldsymbol{L}}(z;\zeta,\alpha)=L_{V'}g_{\boldsymbol{L}}(z;\zeta,\alpha)$. We obtain by (5) in 1.2 a contradiction as follows:

$$0=\int_{\beta(V')}(dL_{V'}g_{\boldsymbol{L}}(z;\zeta,\alpha))=\int_{\beta(V')}(dg_{\boldsymbol{L}}(z;\zeta,\alpha))^*=-2\pi.$$

Consequently if α is an $\boldsymbol{L}$-null set, then the $\boldsymbol{L}$-Green function $g_{\boldsymbol{L}}(z;\zeta,\alpha)$ does not exist for any $\zeta\in W$.

(N4)$\to$(N1). Assume that the closed set α is not an $\boldsymbol{L}$-null set. That is $\omega_{\boldsymbol{L}}(\alpha, V)$ does not vanish for any $V\in\mathfrak{V}$ such that $\beta(V)\supset\alpha$. Let ζ be any point in W. Take a $V\in\mathfrak{V}$ such that $\bar{V}\not\ni\zeta$ and $\beta(V)\supset\alpha$. Then we observe that $c=\int_{\beta(V)}(d\omega_{\boldsymbol{L}}(\alpha, V))^*=-\int_{\partial V}(d\omega_{\boldsymbol{L}}(\alpha, V))^*\neq 0$. Choose a $V'\in\mathfrak{V}$ such that $\bar{V}\cap\bar{V}'=\phi$, $\bar{V}'\not\ni\zeta$ and $\beta(V')=\beta(W)-\beta(V)$. Let Δ be an open disk with center at ζ which is contained in $W-\bar{V}\cup\bar{V}'$. We take $W-\{\zeta\}$ as W and $V\cup V'\cup\Delta-\{\zeta\}$ as V in the existence theorem in 1.2 and we define L by $L_{V\cup V'}$ in $V\cup V'$ and H^{Δ} in $\Delta-\{\zeta\}$. We apply the theorem with $s=-\log|z-\zeta|$ in $\Delta-\{\zeta\}$, $s=-2\pi c^{-1}\omega_{\boldsymbol{L}}(\alpha, V)$ and $s=0$ in V'. This function s has the total flux 0. Hence there exists a harmonic function p in $W-\{\zeta\}$ such that $p-s=L(p-s)$ in $V\cup V'\cup\Delta-\{\zeta\}$. Namely,

$$p=-\log|z-\zeta|+H^{\Delta}_{p-s} \quad \text{in} \quad \Delta-\{\zeta\},$$

$$p=-\frac{2\pi}{c}\,\omega_{\boldsymbol{L}}(\alpha, V)+L_V p \quad \text{in} \quad V,$$

$$p=L_{V'}p \quad \text{in} \quad V'.$$

It follows that the function p is a harmonic function in $W-\{\zeta\}$ with singularity $-\log|z-\zeta|$ at ζ such that p has $\boldsymbol{L}$-behavior on $\beta(W)-\alpha$ and is bounded from below. Therefore the function $p+a$ is positive in $W-\{\zeta\}$ for sufficiently large positive number a and has the above properties. We see by the maximum principle that $p+a \geqq g_{\boldsymbol{L}}(z; \zeta, \Omega_n)$ and hence that the $\boldsymbol{L}$-Green function $g_{\boldsymbol{L}}(z; \zeta, \alpha)$ with pole at ζ exists.

Consequently if the $\boldsymbol{L}$-Green function $g_{\boldsymbol{L}}(z; \zeta, \alpha)$ of W with respect to α does not exist for some $\zeta \in W$, then the closed set α is $\boldsymbol{L}$-null.

REMARK. From the the above proof we infer that the four conditions which are obtained by replacing *some* by *any* in (N1)$\sim$(N4) are also equivalent to (N1).

THEOREM 2. *If a closed set α is an $\boldsymbol{L}$-null set, then there is no non-constant harmonic function u in W such that $D_W(u)<\infty$ and u has $\boldsymbol{L}$-behavior on $\beta(W)-\alpha$.*

PROOF. Suppose α is an $\boldsymbol{L}$-null set. Let u be a harmonic function in W such that $D_W(u)<\infty$ and u has $\boldsymbol{L}$-behavior on $\beta(W)-\alpha$. Then it follows from (N3) that u has $\boldsymbol{L}$-behavior on $\beta(W)$. Hence u must be a constant.

This theorem includes the relation $0_G \subset 0_{HD}$ as a special case.

§4. Boundary value problems

4.1 *The statement of problems*

Let W be an open Riemann surface and let $\boldsymbol{L}=\{L_V\}_{V\in\boldsymbol{V}}$ be a consistent system of distinguished normal operators. Assume that α is a non-empty closed set on $\beta(W)$ and write $\gamma=\beta(W)-\alpha$. That is, γ is a relatively open set on $\beta(W)$ and $\beta(W)=\alpha\cup\gamma$ (disjoint union). We fix $\boldsymbol{L}$ and α once for all.

Suppose that a function f defined near α satisfies the following condition (A):

(A) There is a $W_0 \in \boldsymbol{V}$ with $\beta(W_0)\supset\alpha$ such that f is continuously differentiable in W_0 and has $D_{W_0}(f)<\infty$.

For such a function f, we investigate the existence and uniqueness of a function H_f which satisfies the following conditions:

(I) H_f is harmonic in W and has $D_W(H_f)<\infty$,

(II) H_f has $\boldsymbol{L}$-behavior on γ,

(III) $H_f(\tau)=f(\tau)$ for almost all $\tau \in \Gamma_\alpha$, where Γ_α is the family of curves defined in 1.1.

By Theorem 2 we must confine ourselves to the case where the closed set α is not $\boldsymbol{L}$-null.

Condition (III) means that the function H_f assumes f on α in a certain sense. We find in M. Ohtsuka [9] that if a boundary component in α is realized as an analytic curve C in the plane and a boundary value function f can be continuously extended to C, then condition (III) induces that H_f is continuously extended to C and is equal to f on C.

4.2 *Existence theorem*

We see by the next example that the problem does not in general have a solution.

Let an open Riemann surface W be a circular slit disk $\{z:\ |z|<1\}-\bigcup_{n=1}^{\infty}\{z:\ |z|=1-1/n,\ 0\leqq \arg z\leqq \pi\}$. Let a closed set α on $\beta(W)$ be the buondary component of W corresponding to $\{z:\ |z|=1\}$. We take the system $\boldsymbol{L}_1$ for $\boldsymbol{L}$ and take f for a continuously differentiable function on $\{z:\ |z|<3/2\}$ which is 0 on $\{z:\ |z|=1,\ -\pi/4\leqq \arg z\leqq \pi/4\}$ and is 1 on $\{z:\ |z|=1,\ 3\pi/4\leqq \arg z\leqq 5\pi/4\}$. Then there is no function H_f with properties (I), (II), (III). In fact, if such a function H_f exists, properties (I), (III) induce $H_f=0$ on $\{z:\ |z|=1,\ -\pi/4<\arg z<0\}$ and $H_f=1$ on $\{z:\ |z|=1,\ \pi<\arg z<5\pi/4\}$. On the other hand, we obtain by (II) that the function H_f is constant on each slit $\{z:\ |z|=1-1/n,\ 0\leqq \arg z\leqq \pi\}$. We can see that the Dirichlet integral of H_f over the intersection of W with any neighborhood of $z=1$ or $z=-1$ in the plane is infinite. This is a contradiction to property (I).

We shall give two sufficient conditions for the existence of H_f. The one is obtained by imposing the following stronger condition (B) on a boundary value function f:

(B) There is a $W_0 \in \mathfrak{V}$ with $\beta(W_0)\supset\alpha$ such that

(B$_1$) f is continuously differentiable in W_0 and has $D_{W_0}(f)<\infty$,

(B$_2$) for any $V\in\mathfrak{V}$ such that $\bar{V}\subset W_0$ and $\beta(V)\subset\gamma$, the equality $\int_{\beta(V)} f(dL_V g)^*=0$ holds for all $g\in C^1(\partial V)$.

THEOREM 3. *Suppose that the closed set α is not $\boldsymbol{L}$-null. Let f be a function satisfying condition* (B). *Then* $-(2\pi)^{-1}\int_\alpha f(z)(dg_{\boldsymbol{L}}(z;\ \zeta,\ \alpha))^*$ *defines a function $H_f(\zeta)$ in W which satisfies conditions* (I), (II), (III).

PROOF. Let $\{\Omega_n\}$ be an approximation of W toward α such that $\Omega_1\supset\partial W_0$ and $W_0\supset\partial\Omega_1$, where W_0 is the set stated in condition (B). We set $\alpha_n=\partial\Omega_n$

and $L_n = L_{\Omega_n} f$.

Suppose $m > n$. We see by (**) in 2.2 that $L_m f = L_n(L_m f)$ on $\bar{\Omega}_n$. It follows from (*) in 2.1 that

$$D_{\Omega_n}(L_m f, L_n f) = D_{\Omega_n}(L_n(L_m f), L_n f) = \int_{\alpha_n} f(dL_n(L_m f))^* = \int_{\alpha_n} f(dL_m f)^*.$$

By virtue of condition (B_2) for f, we have

$$\int_{\beta(\Omega_m - \bar{\Omega}_n)} f(dL_m f)^* = \int_{\beta(\Omega_m - \bar{\Omega}_n)} f(dL_{\Omega_m - \bar{\Omega}_n}(L_m f))^* = 0.$$

Hence Green's formula implies that

$$\int_{\alpha_m - \alpha_n} f(dL_m f)^* = \int_{\beta(\Omega_m - \bar{\Omega}_n) + \alpha_m - \alpha_n} f(dL_m f)^* = D_{\Omega_m - \Omega_n}(f, L_m f).$$

This, together with $D_{\Omega_m}(L_m f) = \int_{\alpha_m} f(dL_m f)^*$, implies that $D_{\Omega_n}(L_m f, L_n f) = -\int_{\alpha_m - \alpha_n} f(dL_m f)^* + \int_{\alpha_m} f(dL_m f)^* = -D_{\Omega_m - \bar{\Omega}_n}(f, L_m f) + D_{\Omega_m}(L_m f)$. It follows that, for $m > n$,

$$\begin{aligned} 0 \leqq D_{\Omega_n}(L_m f - L_n f) &\leqq D_{\Omega_m}(L_m f) - 2D_{\Omega_n}(L_m f, L_n f) + D_{\Omega_n}(L_n f) \\ &= D_{\Omega_n}(L_n f) - D_{\Omega_m}(L_m f) + 2D_{\Omega_m - \bar{\Omega}_n}(f, L_m f) \\ &\leqq D_{\Omega_n}(L_n f) - D_{\Omega_m}(L_m f) + 2\sqrt{D_{\Omega_m}(L_m f)}\sqrt{D_{\Omega_m - \bar{\Omega}_n}(f)} \end{aligned}$$

or

$$\sqrt{D_{\Omega_m}(L_m f)} \leqq \sqrt{D_{\Omega_n}(L_n f) + D_{\Omega_m - \bar{\Omega}_n}(f)} + \sqrt{D_{\Omega_m - \bar{\Omega}_n}(f)}.$$

Therefore, letting $m \to \infty$, we obtain $\overline{\lim}_{m\to\infty} \sqrt{D_{\Omega_m}(L_m f)} \leqq \sqrt{D_{\Omega_n}(L_n f) + D_{W - \bar{\Omega}_n}(f)} + \sqrt{D_{W - \bar{\Omega}_n}(f)}$. This shows that $\{D_{\Omega_n}(L_n f)\}_n$ is bounded. Next we let $n \to \infty$ and have $\overline{\lim}_{n\to\infty} D_{\Omega_n}(L_n f) \leqq \underline{\lim}_{n\to\infty} D_{\Omega_n}(L_n f)$. Hence $\lim_{n\to\infty} D_{\Omega_n}(L_n f)$ exists and is finite. Therefore we have $D_{\Omega_n}(L_m f - L_n f) \to 0$ as n and $m(>n)$ tend to ∞.

Now, we fix $\zeta \in W$. Since the set α is not $\boldsymbol{L}$-null, Theorem 1 implies that the $\boldsymbol{L}$-Green function $g_{\boldsymbol{L}}(z; \zeta, \alpha) = \lim_{n\to\infty} g_{\boldsymbol{L}}(z; \zeta, \alpha_n)$ exists. To simplify the notation we write $g_{\boldsymbol{L}}(z; \zeta, \alpha) = g$ and $g_{\boldsymbol{L}}(z; \zeta, \alpha_n) = g_n$. Computing $D_{\Omega_n}(g_n, L_n f)$ as Cauchy's principal value, we have

$$L_n f(\zeta) = -\frac{1}{2\pi}\int_{\alpha_n} f(dg_n)^*.$$

Let n_0 be any integer such that $\Omega_{n_0} \ni \zeta$. If $n \geqq n_0$, then we obtain, by (B_2) and by the fact that $g_n = L_{\Omega_n - \bar{\Omega}_{n_0}}(g_n)$ on $\bar{\Omega}_n - \Omega_{n_0}$,

$$D_{\Omega_n-\bar{\Omega}n_0}(f,\ g_n)=\int_{\alpha n-\alpha n_0} f(dg_n)^*,$$

that is,

$$\int_{\alpha n} f(dg_n)^* = \int_{\alpha n_0} f(dg_n)^* + D_{\Omega n-\bar{\Omega}n_0}(f,\ g_n).$$

On account of Lemma 6 we see that $\lim_{n\to\infty}\int_{\alpha_n} f(dg_n)^*$ exists and is equal to $\int_{\alpha n_0} f(dg)^* + D_{W-\bar{\Omega}n_0}(f,\ g)$. Namely,

$$\lim_{n\to\infty} L_n f(\zeta) = -\frac{1}{2\pi}\int_{\alpha n_0} f(dg)^* - \frac{1}{2\pi} D_{W-\bar{\Omega}n_0}(f,\ g).$$

Letting $n_0\to\infty$, we conclude that $-(2\pi)^{-1}\int_\alpha f(dg)^*$ exists and is equal to $\lim_{n\to\infty} L_n f(\zeta)$. It is denoted by $H_f(\zeta)$. It follows from $\lim_{n\to\infty} D_{\Omega_n}(L_m f - L_n f)=0$ that $L_n f$ converges to H_f uniformly on any compact set in W, H_f is harmonic in W and $\lim_{n\to\infty} D_W(H_f - L_n f)=0$. Hence H_f satisfies condition (I). On applying Lemma 4 with $u_n = L_n f$, we see that H_f also satisfies (II). Finally, we shall prove that H_f satisfies condition (III). Properties (E2), (E4) of extremal length in 1.3, together with $D_{W_0}(f)$, $D_W(H_f)<\infty$, imply that there is $\Gamma_\alpha^*\subset\Gamma_\alpha$ such that $\lambda(\Gamma_\alpha-\Gamma_\alpha^*)=\infty$ and $f(\tau)$, $H_f(\tau)$ exist and are finite for all $\tau\in\Gamma_\alpha^*$. Extend the function $L_n f$ to $W-\Omega_n$ by f and denote it also by $L_n f$. Applying Fuglede's lemma with $\{\rho_n|dz|\}=\{|\mathrm{grad}(L_n f - H_f)|\,|dz|\}$, we find a subsequence $\{\rho_{n_k}|dz|\}$ such that $\lim_{k\to\infty}\int_\tau |\mathrm{grad}(L_{n_k} f - H_f)|\,|dz|=0$ for almost all $\tau\in\Gamma_\alpha^*$. By $\lim_{k\to\infty}\int_\tau |\mathrm{grad}(L_{n_k} f - H_f)|\,|dz| \geqq \lim_{k\to\infty}\int_\tau |dL_{n_k} f - dH_f| \geqq |\lim_{k\to\infty}\int_\tau dL_{n_k} f - \int_\tau dH_f|$ $= |\{f(\tau)-\lim_{k\to\infty} L_{n_k} f(\tau(0))\} - \{H_f(\tau)-H_f(\tau(0))\}| = |f(\tau)-H_f(\tau)|$ where $\tau(0)$ is the intial point in W of the curve τ, we have $H_f(\tau)=f(\tau)$ for almost all $\tau\in\Gamma_\alpha^*$. It follows from $\lambda(\Gamma_\alpha-\Gamma_\alpha^*)=\infty$ that $H_f(\tau)=f(\tau)$ for almost all $\tau\in\Gamma_\alpha$. The proof of Theorem 3 is complete.

Here we show examples for which condition (B) is fulfilled:

(Cl) Suppose α is isolated. Any function f satisfying condition (A) does always satisfy condition (B).

(C2) Suppose $\boldsymbol{L}$ is the system $\boldsymbol{L}_0$. Then the same result as in (Cl) is valid.

(C3) If we can choose $W_0\in\mathcal{V}$ with $\beta(W_0)\supset\alpha$ such that f is constant on each connected component of W_0, then f satisfies condition (B).

(C4) If we can choose $W_0\in\mathcal{V}$ with $\beta(W_0)\supset\alpha$ such that f satisfies (B_1) and has $\boldsymbol{L}$ or 0-behavior on $\beta(W_0)-\alpha$, then condition (B) is fulfilled.

In fact, in the case where α is isolated, we can find $W_0'\in\mathcal{V}$ such that $\beta(W_0')=\alpha$ and $W_0'\subset W_0$. Then there is no $V\in\mathcal{V}$ such that $\bar{V}\subset W_0'$ and $\beta(V)\subset\gamma$. Hence condition (B) is fulfilled. In the case $\boldsymbol{L}=\boldsymbol{L}_0$, the result is proved by the characterization of L_{0V} in K. Oikawa [10]. (C3) follows from the fact that $\int_\beta (dL_V g)^*=0$ for any dividing cycle β of W which is contained in V and does not separate components of ∂V (see 2.2). (C4) follows from (6) in 2.1.

Let us give an other sufficient condition for the existence of a function satisfying conditions (I), (II), (III). We begin with the following lemma:

LEMMA 7. *Assume that an open Riemann surface W is hyperbolic.*[2] *Denote by ω_α the harmonic measure of α with respect to W. If $\lambda(\Gamma_{\alpha,\gamma})>0$,*[3] *then $\lambda(\Gamma_{\alpha,\gamma})=1/D_W(\omega_\alpha)$, where $\Gamma_{\alpha,\gamma}$ is the family of curves defined in* 1.1.

PROOF. Let $\{\alpha_n\}$ be a sequence of isolated sets on $\beta(W)$ such that $\alpha_{n+1}\supset\alpha_n$ and $\bigcap_{n=1}^{\infty}\alpha_n=\alpha$. We write $\gamma_n=\beta(W)-\alpha_n$. We define ω_n as follows: If $\alpha\cup\gamma_n$ is not $\boldsymbol{L}_0$-null, then $\omega_n(\zeta)=-(2\pi)^{-1}\int_{\alpha\cup\gamma_n} w_n(dg_{\boldsymbol{L}_0}(z;\zeta,\alpha\cup\gamma_n))^*$ where $w_n=1$ on a $V_0\in\mathcal{V}$ with $\beta(V_0)=\alpha_n$ and $=0$ on a V_1 with $\beta(V_1)=\gamma_n$ and $\bar{V}_0\cap\bar{V}_1=\phi$. If $\alpha\cup\gamma_n$ is $\boldsymbol{L}_0$-null, then $\omega_n(\zeta)=0$. It follows from Theorem III in 3.1 of A. Marden and B. Rodin [5] that $0<\lambda(\Gamma_{\alpha,\gamma})\leqq\lambda(\Gamma_{\alpha,\gamma_n})=1/D_W(\omega_n)$. In particular, $D_W(\omega_n)\leqq 1/\lambda(\Gamma_{\alpha,\gamma})<\infty$. Obviously, $\lim_{n\to\infty}\omega_n=\omega_\alpha$ in W. By standard approximation method we have $D_W(\omega_m-\omega_n)=D_W(\omega_m)-D_W(\omega_n)$ for $m>n$. Hence $\lim_{n\to\infty}D_W(\omega_n-\omega_\alpha)=0$ and $\lim_{n\to\infty}D_W(\omega_n)=D_W(\omega_\alpha)$. On the other hand, applying (E3) in 1.3 with $\Gamma_n=\Gamma_{\alpha,\gamma_n}$, we obtain $\lim_{n\to\infty}\lambda(\Gamma_{\alpha,\gamma_n})=\lambda(\bigcup_{n=1}^{\infty}\Gamma_{\alpha,\gamma_n})=\lambda(\Gamma_{\alpha,\gamma})$. Consequently, $\lambda(\Gamma_{\alpha,\gamma})=1/D_W(\omega_\alpha)$.

THEOREM 4. *Suppose α is not $\boldsymbol{L}$-null and there exists a sequence $\{\alpha_n\}$ of isolated sets such that $\alpha_n\supset\alpha_{n+1}$, $\bigcap_{n=1}^{\infty}\alpha_n=\alpha$ and $\overline{\lim_{n\to\infty}}\lambda(\Gamma_{\alpha_n,\gamma_n})\geqq\lambda(\Gamma_{\alpha,\gamma})>0$ where $\gamma_n=\beta(W)-\alpha_n$. Assume that f satisfies condition* (A) *and is bounded. Then there exists a function which satisfies* (I), (II), (III) *and is bounded in W.*

PROOF. Let ω_n (resp. ω_α) be the harmonic measure of α_n (resp. α) with respect to W. It is well known that $\lambda(\Gamma_{\alpha_n,\gamma_n})=1/D_W(\omega_n)$. It follows from $\overline{\lim_{n\to\infty}}\lambda(\Gamma_{\alpha_n,\gamma_n})\geqq\lambda(\Gamma_{\alpha,\gamma})>0$ that $\{D_W(\omega_n)\}_n$ is bounded. Obviously, ω_n converges to ω_α locally uniformly in W. Hence $\lim_{n\to\infty}D_K(\omega_n-\omega_\alpha)=0$ for any compact set K in W. We see by Lemma 7.4 in C. Constantinescu and A. Cornea [3] that $\lim_{n\to\infty}D_W(\omega_n, v)=D_W(\omega_\alpha, v)$ for any Dirichlet function v on W. Fatou's lemma implies that $\underline{\lim}_{n\to\infty}D_W(\omega_n)\geqq D_W(\omega_\alpha)$. By Lemma 7 and our assumption, we have $\overline{\lim_{n\to\infty}}D_W(\omega_n)=\overline{\lim_{n\to\infty}}1/\lambda(\Gamma_{\alpha_n,\gamma_n})\leqq 1/\lambda(\Gamma_{\alpha,\gamma})=D_W(\omega_\alpha)\leqq\underline{\lim}_{n\to\infty}D_W(\omega_n)$. That is,

2) If W is parabolic, $\lambda(\Gamma_{\alpha,\gamma})$ is ∞.

3) If $\lambda(\Gamma_{\alpha,\gamma})=0$, then the equality $\lambda(\Gamma_{\alpha,\gamma})=1/D_W(\omega_\alpha)$ does not necessarily hold.

$\lim_{n\to\infty} D_W(\omega_n) = D_W(\omega_\alpha)$. Hence, $\lim_{n\to\infty} D_W(\omega_n - \omega_\alpha) = 0$.

Let $|f| \leqq M$ in W_0 where W_0 is the set defined in condition (A). We may assume that f is continuously differentiable on $\bar{W}_0$. Extend f to $W - W_0$ in such a way that $f \in C^1(W)$, $D_W(f) < \infty$ and $|f| \leqq M$ in W. Consider the Dirichlet function s_n (resp. s) $= \min(\max(f, -M\omega_n), M\omega_n)$(resp. $\min(\max(f, -M\omega_\alpha), M\omega_\alpha)$). We denote by S_n (resp. S) the harmonic part of s_n (resp. s) in the Royden decomposition. Theorem 7.4 in C. Constantinescu and A. Cornea [3], together with $\lim_{n\to\infty} D_W(\omega_n - \omega_\alpha) = 0$, implies $\lim_{n\to\infty} D_W(s_n - s) = 0$ and hence $\lim_{n\to\infty} D_W(S_n - S) = 0$. Using (E2), (E4), (E5) in 1.3, Lemma 1 and Fuglede's lemma, we have

$$\omega_n(\tau) = \omega_\alpha(\tau) = 1,$$

$$s_n(\tau) = s(\tau) = f(\tau),$$

$$S_n(\tau) = S(\tau) = f(\tau)$$

for almost all $\tau \in \Gamma_\alpha$. Furthermore, $s_n(\tau) = 0$ for almost all $\tau \in \Gamma_{\gamma_n}$, and hence

$$s(\tau) = S(\tau) = 0$$

for almost all $\tau \in \Gamma = \bigcup_{n=1}^{\infty} \Gamma_{\gamma_n}$. It follows from (E5) in 1.3 that for any $V \in \mathfrak{V}$ with $\beta(V) \subset \gamma$, the function S is equal to H_S^V. Namely, S has 0-behavior on γ. Consequently (C4) implies that Theorem 3 is applicable to S and a function H_S is obtained as in Theorem 3. Because of $S(\tau) = f(\tau)$ for almost all $\tau \in \Gamma_\alpha$, we see that H_S satisfies conditions (I), (II) and $H_S(\tau) = f(\tau)$ for almost all $\tau \in \Gamma_\alpha$. Moreover, it is easily proved that our function H_S is bounded in W. We have thus completely proved Theorem 4.

4.3 *Uniqueness theorem*

The next example shows that, for some $\boldsymbol{L}$, α and some f, the existence of a function with properties (I), (II), (III) is true but the uniqueness is not true:

Let an open Riemann surface W be the circular slit annulus $\{z: 1/2 < |z| < 1\} - \bigcup_{n=3}^{\infty} \{z: |z| = 1 - 1/n,\ 1/n \leqq \arg z \leqq 2\pi - 1/n\}$. Let a closed set α on $\beta(W)$ be two boundary components of W corresponding to $\{z: |z| = 1/2\}$ and $\{z: |z| = 1\}$. Obviously, the set α is not $\boldsymbol{L}_1$-null. We take the system $\boldsymbol{L}_1$ for $\boldsymbol{L}$ and take a boundary value function f to be 0 on $\{z: 1/2 < |z| < 2/3\}$ and 1 on $\{z: 3/4 < |z| < 1\}$. If we denote by ω the harmonic function on the annulus $\{z: 1/2 \leqq |z| \leqq 1\}$ such that $\omega = 0$ on $\{z: |z| = 1\}$ and $= 1$ on $\{z: |z| = 1/2\}$, then for any real number k the function $k\omega$ satisfies conditions (I), (II), (III).

To investigate the uniqueness theorem for H_f is equivalent to study the following problem:

Is it true that a function φ in W such that

(I′) φ is harmonic in W and has $D_W(\varphi)<\infty$,

(II′) φ has $\boldsymbol{L}$-behavior on γ,

(III′) $\varphi(\tau)=0$ for almost all $\tau \in \Gamma_\alpha$

must reduce to zero?

If α is $\boldsymbol{L}_0$-null, then (III′) is meaningless. It follows that any constant satisfies (I′), (II′), (III′). Hence in the sequel we assume that α is not $\boldsymbol{L}_0$-null, i.e., $\lambda(\Gamma_\alpha)<\infty$.

We find in M. Ohtsuka [7], [9] that the uniqueness theorem holds for the system $\boldsymbol{L}_0$. We shall prove the following uniqueness theorem:

THEOREM 5. *Suppose that the closed set α is not $\boldsymbol{L}_0$-null and is isolated on $\beta(W)$.*[4] *Then the uniqueness theorem holds for any system $\boldsymbol{L}$.*

PROOF. Let φ be any function in W with properties (I′), (II′), (III′). Since α is isolated, we can choose an approximation $\{\Omega_n\}$ of W toward α such that $\beta(\Omega_n)=\gamma$ for all n. Consider the sequence of Dirichlet solutions $H_\varphi^{\Omega_n}$ (see (E5) in 1.3). We can see $\lim_{n,m\to\infty} D_{\Omega_n}(H_\varphi^{\Omega_m}-H_\varphi^{\Omega_n})=0$ by the same reasoning as that showing $\lim_{n,m\to\infty} D_{\Omega_n}(L_m f-L_n f)=0$ in the proof of Theorem 3. Hence there is a harmonic function H in W up to an additive constant such that $\lim_{n\to\infty} D_{\Omega_n}(H-H_\varphi^{\Omega_n})=0$. Extend $H_\varphi^{\Omega_n}$ by φ to $W-\Omega_n$ and denote it by Φ_n. Properties (E2), (E4), (E5) in 1.3, together with (III′), imply that there is $\Gamma^*\subset\Gamma_{\beta(W)}$ such that $\lambda(\Gamma_{\beta(W)}-\Gamma^*)=\infty$ and $\Phi_n(\tau)=0$ for all $\tau\in\Gamma^*$ and all n. That is, $\int_\tau d\Phi_n = -\Phi_n(\tau(0))$. Applying Fuglede's lemma with $\{\rho_n|dz|\}=\{|\mathrm{grad}(\Phi_n-H)||dz|\}$, we conclude that there is a subsequence $\{\rho_{n_k}|dz|\}$ such that, for almost all $\tau\in\Gamma^*$, $0=\lim_{k\to\infty}\int_\tau |\mathrm{grad}(\Phi_{n_k}-H)||dz|\geqq\lim_{k\to\infty}\int_\tau |d\Phi_{n_k}-dH|=|\lim_{k\to\infty}\int_\tau d\Phi_{n_k}-\int_\tau dH|=$ $|-\lim_{k\to\infty}\Phi_{n_k}(\tau(0))-\int_\tau dH|$. Hence $\lim_{k\to\infty}\Phi_{n_k}(\tau(0))=\lim_{k\to\infty}H_\varphi^{\Omega_{n_k}}(\tau(0))$ exists and is finite. It follows from $\lambda(\Gamma_\alpha)<\infty$ (Proposition 2) that $\lim_{k\to\infty}H_\varphi^{\Omega_{n_k}}$ exists and is harmonic in W. We can easily infer that, for the original sequence, $\lim_{n\to\infty}H_\varphi^{\Omega_n}$ exists. Set $H=\lim_{n\to\infty}H_\varphi^{\Omega_n}$ $(=\lim_{n\to\infty}\Phi_n)$. By the above computation, $0=-H(\tau(0))-\int_\tau dH$ for almost all $\tau\in\Gamma^*$. Otherwise stated, $H(\tau)=0$ for almost all $\tau\in\Gamma_{\beta(W)}$. It follows from Lemma 1 that the function H reduces to zero.

On the other hand, since φ has $\boldsymbol{L}$-behavior on γ, $L_{\Omega_n}\varphi=\varphi$ on $\bar{\Omega}_n$. This, together with Lemma 2, implies that

4) Suppose α is isolated on $\beta(W)$. Then we easily see that α is $\boldsymbol{L}$-null if and only if α is $\boldsymbol{L}_0$-null.

$$0 \leqq D_W(\varphi) = D_W(H-\varphi) = \lim_{n\to\infty} D_{\Omega_n}(H_\varphi^{\Omega_n} - L_{\Omega_n}\varphi)$$
$$= \lim_{n\to\infty} \{D_{\Omega_n}(H_\varphi^{\Omega_n}) - D_{\Omega_n}(L_{\Omega_n}\varphi)\} = -\lim_{n\to\infty} D_{\Omega_n}(\varphi) \leqq 0,$$

or $D_W(\varphi)=0$. Hence φ is a constant. It follows from (III′) that φ reduces to 0. This completes the proof.

Finally we shall show the following uniqueness theorem:

THEOREM 6. *Suppose α satisfies the same conditions as in Theorem 4. Then any function φ which satisfies conditions* (I′), (II′), (III′) *and is bounded in W must reduce to zero.*

PROOF. Replacing by φ the extended function f in the proof of Theorem 4 and otherwise using the same notations, we have $\lim_{n\to\infty} D_W(S_n - S)=0$ and $S(\tau)=0$ for almost all $\tau \in \Gamma_{\beta(W)} = \Gamma_\alpha \cup \Gamma_\gamma$. It follows from Lemma 1 that S is identically zero. Now, fix n. Take an approximation $\{\Omega_k\}$ of W toward α_n. Then the sequence of Dirichlet solutions $H_\varphi^{\Omega_k}$ tends to S_n pointwise in W and in terms of Dirichlet norm. It follows from the Remark in 2.1 and $D_{\Omega_k}(\varphi) = \int_{\partial\Omega_k} \varphi(d\varphi)^*$ that $D_W(S_n, \varphi) = \lim_{k\to\infty} D_{\Omega_k}(H_\varphi^{\Omega_k}, \varphi) = \lim_{k\to\infty} \int_{\partial\Omega_k} H_\varphi^{\Omega_k}(d\varphi)^* = \lim_{k\to\infty} \int_{\partial\Omega_k} \varphi(d\varphi)^* = \lim_{k\to\infty} D_{\Omega_k}(\varphi) = D_W(\varphi)$. Hence $D_W(S_n) \geqq D_W(\varphi)$. On letting $n \to \infty$, we obtain $D_W(\varphi)=0$. We see by (III′) that φ is equal to zero.

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Department of Mathematics,
Faculty of Science,
Kyoto University