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On the Torsion Submodule of a Module of Type (F_1)

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Introduction

In the paper [3] E. Kunz has proved the following:

Let R be an integral domain with quotient field K, S be a subring of R and k be its quotient field. If the module of S-differentials of R is finitely generated and if the module of k-differentials of k has rank r, then the $(r-1)$-th Kähler different of R over S vanishes and the r-th Kähler different does not.

In connection with this fact we introduce the following notion. A finitely generated module (over a commutative ring with unity element) is said to be a module of type (F_r) if its $(r-1)$-th Fitting ideal (i.e. Determinanten-ideal in [2]) is the zero ideal and its r-th Fitting ideal is a regular ideal.

The purpose of this paper is mainly to study the torsion submodule of a module of type (F_1). In §1 we give the definition of a module of type (F_r) and state some properties of modules of this type. In §2 we study the torsion submodule of a module of type (F_1), and in §3 we prove that, for a noetherian domain R of Krull dimension one, an R-module of type (F_1) is the direct sum of its torsion submodule and a free module of rank one if, and only if, its dual module is a free module of rank one. In §4 we apply the results of the preceding sections to the module of differentials on an affine curve defined over a perfect field.

Throughout this paper, all rings will be assumed to be commutative with unity element and all modules to be unitary.

The writer wishes to express his thanks to Professor Y. Nakai, who has often helped him with discussion out of which the ideas in this work developed.

§1. The module of type (F_r)

Let R be a ring and M be a finitely generated R-module. For a system $\{x_1, \cdots, x_n\}$ of generators of M there is an exact sequence

$$(1) \qquad 0 \longrightarrow N \longrightarrow R^n \overset{\varphi}{\longrightarrow} M \longrightarrow 0,$$

where R^n is a free R-module with a system $\{e_1, \cdots, e_n\}$ of basis, the R-homomorphism φ is defined by $\varphi(e_j)=x_j$ and N is the kernel of φ. Let N be generated by $u_\lambda = f_{\lambda 1}e_1 + \cdots + f_{\lambda n}e_n$, with λ in some index set Λ. We shall denote by $\mathfrak{F}_t(M)$ the ideal which is generated by all the $(n-t)\times(n-t)$ minors of the matrix

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{\lambda 1} & \cdot & \cdot & \cdot & f_{\lambda n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

For $t \geq n$ $\mathfrak{F}_t(M)$ is defined as the unit ideal, and for $t<0$ $\mathfrak{F}_t(M)$ is defined as the zero ideal. The ideal $\mathfrak{F}_t(M)$ will be called the t-th Fitting ideal of the module M. It is known that $\mathfrak{F}_t(M)$ is the invariant ideal determined by M, that is, it is determined uniquely by M and it does not depend on the choice of the system of generators of M (cf. [2]). It follows from the definition of $\mathfrak{F}_t(M)$ that $\mathfrak{F}_t(M) \subseteq \mathfrak{F}_{t+1}(M)$. Moreover, it can be shown directly that $\mathfrak{F}_0(M) \subseteq \operatorname{Ann} M$ and $(\operatorname{Ann} M)^m \subseteq \mathfrak{F}_0(M)$ for sufficiently large m, where $\operatorname{Ann} M$ is the annihilator of M.

We shall say that a finitely generated R-module M is of type (F_r) if the $(r-1)$-th Fitting ideal $\mathfrak{F}_{r-1}(M)$ is the zero ideal and the r-th Fitting ideal $\mathfrak{F}_r(M)$ is a regular ideal.[1)]

Let R be a ring and M be an R-module. An element in M will be called a torsion element in M if it is annihilated by a non-zerodivisor in R. The submodule of M, which consists of all the torsion elements in M, will be called the torsion submodule of M. If M coincides with its torsion submodule, M will be called a torsion module.

PROPOSITION 1. *Let R be a ring and M be a finitely generated R-module. Then M is of type (F_0) if and only if M is a torsion module.*

PROOF. Since $(\operatorname{Ann} M)^m \subseteq \mathfrak{F}_0(M) \subseteq \operatorname{Ann} M$, the radical of $\mathfrak{F}_0(M)$ coincides with the radical of $\operatorname{Ann} M$. Hence, $\mathfrak{F}_0(M)$ is regular if and only if $\operatorname{Ann} M$ is regular, that is, M is of type (F_0) if and only if $\operatorname{Ann} M$ is a regular ideal. Therefore, the only if part is obvious. Conversely, assume that M is a torsion module and let $\{x_1, \cdots, x_n\}$ be a system of generators of M. Since $\operatorname{Ann} x_i$ is a regular ideal for each i, $\operatorname{Ann} M$ is regular. Consequently, M is of type (F_0). q.e.d.

Although it can be derived directly from the result of Kunz [3], we give a proof of the following theorem for the sake of completeness.

THEOREM 1. *Let R be a local ring, M be a finitely generated R-module and r be a positive integer. Then, M is a free module of rank r, if and only if, M is of type (F_r) and the r-th Fitting ideal of M is the unit ideal.*

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1) An ideal is said to be regular if it contains a non-zerodivisor.

PROOF. Assume that M is a free module of rank r, then in the above exact sequence (1) we can put $n=r$ and $N=0$. This implies that $\mathfrak{F}_t(M)=0$ for $t<r$ and $\mathfrak{F}_r(M)$ is the unit ideal. The proof of the if part is as follows: Let $\{x_1, \cdots, x_n\}$ be a minimal generating system of M. Taking any generators $u_\lambda=\sum_{j=1}^{n} f_{\lambda j}e_j$ of N(=the kernel of φ in the sequence (1)), we have $\sum_{j=1}^{n} f_{\lambda j}x_j=0$, and hence all $f_{\lambda j}$ are in the maximal ideal of R. Hence $\mathfrak{F}_t(M)$ is a proper ideal in R for $t<n$. By the assumptions this implies that $n=r$ and, $\mathfrak{F}_{r-1}(M)=0$, and therefore, since $\mathfrak{F}_{r-1}(M)$ is generated by all $f_{\lambda j}$, $N=0$. Thus M is isomorphic to R^r. q.e.d.

§2. The torsion submodule of a module of type (F_1)

It follows from Proposition 1 that the torsion submodule of an R-module M of type (F_1) is properly contained in M. The aim of this section is to prove the following:

THEOREM 2. *Let R be an integral domain, M be an R-module of type (F_1) and T be the torsion submodule of M. Then there exists a non-zero ideal $\mathfrak{d}$ contained in the first Fitting ideal of M and an R-homomorphism Φ of M into $\mathfrak{d}$ can be defined such that the sequence*

$$0 \longrightarrow T \longrightarrow M \xrightarrow{\Phi} \mathfrak{d} \longrightarrow 0 \tag{2}$$

is exact.

Let R be a ring and M be an R-module of type (F_1). Assume that M is generated by n elements $x_1, \cdots, x_n$ and consider the exact sequence (1) in §1;

$$0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0.$$

In order to prove Theorem 2 we shall first study the module $\varphi^{-1}(T)$ where T is the torsion submodule of M.

With the same notations as in the definition of the Fitting ideal in §1, we consider the $n \times n$ determinant

$$\begin{vmatrix} a_1 & \cdot & \cdot & \cdot & a_n \\ f_{\mu_1 1} & \cdot & \cdot & \cdot & f_{\mu_1 n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{\mu_{n-1} 1} & \cdot & \cdot & \cdot & f_{\mu_{n-1} n} \end{vmatrix}$$

for a system $\{\mu_1, \cdots, \mu_{n-1}\}$ of $n-1$ elements in Λ and for an element $a_1e_1+\cdots+a_ne_n$ in R^n. Let $g_{\{\mu\},j}$ be the cofactor of this determinant with respect to a_j. Since the 0-th Fitting ideal $\mathfrak{F}_0(M)$ is the zero ideal, we have

(3) $$f_{\lambda 1} g_{\{\mu\};1} + \cdots + f_{\lambda n} g_{\{\mu\};n} = 0$$

for all λ in Λ and for all systems $\{\mu_1, \cdots, \mu_{n-1}\}$. From the relations $f_{\lambda 1}x_1 + \cdots + f_{\lambda n}x_n = 0$ ($\lambda \in \Lambda$) in M, we have

(4) $$g_{\{\mu\};k} x_j = g_{\{\mu\};j} x_k$$

for i, $k=1, \cdots, n$ and for all systems $\{\mu_1, \cdots, \mu_{n-1}\}$.

Let L be the submodule of R^n generated by the elements $a_1e_1 + \cdots + a_ne_n$ such that $a_1 g_{\{\mu\},1} + \cdots + a_n g_{\{\mu\};n} = 0$ for all systems $\{\mu_1, \cdots, \mu_{n-1}\}$. Then, by the relations (3), the module L contains N (=the kernel of φ in (1)).

LEMMA. *Let M, N and L be the same as above and T be the torsion submodule of M. Then the sequence*

(5) $$0 \longrightarrow N \longrightarrow L \overset{\varphi'}{\longrightarrow} T \longrightarrow 0$$

is exact, where the map φ' is the restriction on L of the map φ in the sequence (1).

PROOF. First we shall show that $\varphi'(L) \subseteq T$. Let $x = \sum_{j=1}^{n} a_j x_j$ be an element of $\varphi'(L)$. Then, since $\sum_{j=1}^{n} a_j g_{\{\mu\};j} = 0$, we have

$$g_{\{\mu\}\cdot k} x = \sum_{j \neq k} a_j (g_{\{\mu\};k} x_j - g_{\{\mu\};j} x_k).$$

Hence, by the relations (4), we have $g_{\{\mu\};k} x = 0$ for $k = 1, \cdots, n$ and for all systems $\{\mu_1, \cdots, \mu_{n-1}\}$. Since $\mathfrak{F}_1(M)$ is generated by all $g_{\{\mu\};k}$, this implies that $\mathfrak{F}_1(M)$ is contained in $\operatorname{Ann} x$, and hence $\operatorname{Ann} x$ is a regular ideal. Consequently, x is a torsion element in M.

Next we shall show that the map $\varphi' : L \to T$ is surjective. Taking any element $x = \sum_{j=1}^{n} a_j x_j$ in T, then $ax = 0$ for some non-zerodivisor a in R. This means that the element $a(\sum_{j=1}^{n} a_j e_j)$ in R^n is in N. Hence we can write $aa_j = \sum_i c_i f_{\lambda_i j}$ ($j = 1, \cdots, n$; $\lambda_i \in \Lambda$; $c_i \in R$). Therefore, we have

$$a \sum_{j=1}^{n} a_j g_{\{\mu\};j} = \sum_i c_i (\sum_{j=1}^{n} f_{\lambda_i j} g_{\{\mu\};j}).$$

Hence, by the relations (3), we have $\sum_{j=1}^{n} a_j g_{\{\mu\};j} = 0$ for all systems $\{\mu_1, \cdots, \mu_{n-1}\}$. This shows that the element $\sum_{j=1}^{n} a_j e_j$ is in L and $\varphi'(\sum_{j=1}^{n} a_i e_j) = x$.

Finally, since $N \subseteq L$, it is obvious that the kernel of φ' is N.

q.e.d.

As a direct consequence of the proof of this lemma, we have the following:

Corollary. *Let R be a ring and M be an R-module of type (F_1). Then the first Fitting ideal of M is contained in the annihilator of the torsion submodule of M.*

From now on we assume that R is *an integral domain.* Let $\{\nu_1, \cdots, \nu_{n-1}\}$ be a fixed system of $n-1$ elements in Λ such that at least one of $g_{\{\nu\};j}$ $(j=1, \cdots, n)$ is not the zero element, and put $g_j = g_{\{\nu\};j}$.

Let L be the above defined submodule of R^n. Then it is defined by *the one relation* $\sum_{j=1}^{n} a_j g_j = 0$,, i.e., $L = \{\sum_{j=1}^{n} a_j e_j \mid \sum_{j=1}^{n} a_j g_j = 0\}$.

In fact, let $\{\mu_1, \cdots, \mu_{n-1}\}$ be another system which has the same property as $\{\nu_1, \cdots, \nu_{n-1}\}$. Then, since $\mathfrak{F}_0(M)$ is the zero ideal and since R is a domain, we have

$$c_{\mu_q} u_{\mu_q} = c_{\mu_q 1} u_{\nu_1} + \cdots + c_{\mu_q n-1} u_{\nu_{n-1}} \qquad (q=1, \cdots, n-1),$$

where c_{μ_q} and $c_{\mu_q i}$ are the elements in R, $c_{\mu_q} \neq 0$, and $u_\lambda = \sum_{j=1}^{n} f_{\lambda j} e_j$. Hence, we have $c' g_{\{\mu\};j} = c g_j$ $(j=1, \cdots, n)$, where c is the $(n-1)\times(n-1)$ determinant $|c_{\mu_q i}|$ and $c' = c_{\mu_1} \cdots c_{\mu_{n-1}}$. It is clear that neither c nor c' is the zero element in R and they do not depend on the index j. Therefore, $\sum_{j=1}^{n} a_j g_j = 0$ if and only if $\sum_{j=1}^{n} a_j g_{\{\mu\};j} = 0$. Consequently, L is defined by the relation $\sum_{j=1}^{n} a_j g_j = 0$.

Let $\mathfrak{d}$ be the ideal in R generated by $g_1, \cdots, g_n$. Then the *non-zero* ideal $\mathfrak{d}$ is contained in $\mathfrak{F}_1(M)$. Let ψ be the R-homomorphism of R^n into $\mathfrak{d}$ defined by $\psi(e_j) = g_j$, then the map ψ is surjective and the kernel of ψ is L. Therefore, we have the exact sequence

$$(6) \qquad 0 \longrightarrow L \longrightarrow R^n \longrightarrow \mathfrak{d} \longrightarrow 0.$$

Remark. If the module N in the sequence (1) is generated by $n-1$ elements, then the ideal $\mathfrak{d}$ coincides with $\mathfrak{F}_1(M)$.

Proof of Theorem 2: From the exact sequence (1), (5) and (6) we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & N & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & R^n & \xrightarrow{\psi} & \mathfrak{d} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \sigma} & & \\
0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & M/T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the map σ is defined by $\sigma(\sum_{j=1}^{n} a_j g_j)$ = residue class of $\sum_{j=1}^{n} a_j x_j$ module T. Since in the above diagram three rows, the first and the second column are all exact, the map σ is an isomorphism. Therefore, we have the exact sequence (2). This completes the proof.

§3. The free dual of a module of type (F_1)

PROPOSITION 2. *Let R be a noetherian ring of Krull dimension one and $\mathfrak{a}$ be an ideal in R. If $\mathrm{Hom}_R(\mathfrak{a}, R)$ is a free R-module of rank one, then $\mathfrak{a}$ is a regular principal ideal and conversely.*

PROOF. Let f be a free base of $\mathrm{Hom}_R(\mathfrak{a}, R)$. Assume that $\mathfrak{a}$ is not regular, then there exists a non-zero element x in R which annihilates $\mathfrak{a}$. Hence $xf=0$. This is a contradiction. Therefore, $\mathfrak{a}$ is a regular ideal. Let $\{a_1, \cdots, a_n\}$ be a system of generators of $\mathfrak{a}$ such that a_1 is a non-zerodivisor and let $\mathfrak{b}$ be the ideal in R generated by $f(a_1), \cdots, f(a_n)$. Then since $f(a_1)$ is a non-zerodivisor, $\mathfrak{b}$ is a regular ideal.

We shall now show that $Ra:\mathfrak{b}=Ra$ for any non-zerodivisor a in R.[2] Let y be an element in $Ra:\mathfrak{b}$, then there exist n elements $b_1, \cdots, b_n$ in R such that $yf(a_j)=b_j a$ $(j=1, \cdots, n)$, and then we can define an R-homomorphism g of $\mathfrak{a}$ into R by $g(\sum_{j=1}^{n} r_j a_j)=\sum_{j=1}^{n} r_j b_j$.[3] Since f is a base, we have $g=bf$ for some element b in R, and hence $b_j=bf(a_j)$ $(j=1, \cdots, n)$. Therefore, since $f(a_1)$ is a non-zerodivisor, we have $y=ba$. This shows that y is in Ra.

Next we shall show that $\mathfrak{b}$ is the unit ideal. Suppose that $\mathfrak{b}$ is not the unit ideal, then there is a maximal ideal $\mathfrak{m}$ in which $\mathfrak{b}$ is contained. Taking a non-zerodivisor c in $\mathfrak{m}$, then $\mathfrak{m}$ is an associated prime ideal of the principal ideal Rc. Therefore, we see $Rc:\mathfrak{b}\neq Rc$. This is a contradiction.

From the fact that $\mathfrak{b}$ is the unit ideal, we can deduce the existence of elements $c_1, \cdots, c_n$ in R such that $\sum_{j=1}^{n} c_j f(a_j)=1$. Put $d=\sum_{j=1}^{n} c_j a_j$, then it is easy to see that $\mathfrak{a}$ is generated by the element d.

The converse is evident. q.e.d.

The following example shows that Proposition 2 is not true if Krull dimension of R is greater than one.

EXAMPLE.[4] Let $R=k[X, Y]$ be a polynomial ring in two indeterminates X and Y over a field k and $\mathfrak{a}$ be the ideal generated by X and Y. Then

2) The proof of this part is due to Y. Nakai.

3) In fact, let $\sum s_j a_j$ be another representation of $\sum r_j a_j$, then we have $\sum r_j f(a_j)=\sum s_j f(a_j)$. Multiplying this relation by y, we have $a\sum r_j b_j=a\sum s_j b_j$, and hence $\sum r_j b_j=\sum s_j b_j$. This shows that g is well defined.

4) This example is due to H. Yanagihara.

$\mathrm{Hom}_R(\mathfrak{a}, R)$ is a free module of rank one.

In fact, let f be any element in $\mathrm{Hom}_R(\mathfrak{a}, R)$, then we see $f(XY)=Xf(Y)=Yf(X)$. Hence, there is an element C in R such that $f(X)=CX$ and $f(Y)=CY$. Therefore, $f(AX+BY)=C(AX+BY)$ for any $AX+BY$ in $\mathfrak{a}$. This shows that $\mathrm{Hom}_R(\mathfrak{a}, R)=Ri$, where i is the inclusion map of $\mathfrak{a}$ into R.

Let R be a noetherian ring with total quotient ring K, then the following facts are known (cf. [1]).

a) Let $\mathfrak{a}$ be a regular ideal in R and put $\mathfrak{a}^{-1}=\{x \in K \mid x\mathfrak{a}\subseteq R\}$, then $\mathfrak{a}^{-1}$ is a finitely generated R-module in K and is isomorphic to $\mathrm{Hom}_R(\mathfrak{a}, R)$.

b) Let $\mathfrak{f}$ be an R-module in K such that $\mathfrak{f}K=K$, then $\mathfrak{f}$ is invertible if and only if $\mathfrak{f}$ is a finitely generated R-module and $\mathfrak{f}\otimes_R R_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module of rank one for any maximal ideal $\mathfrak{m}$ in R, where $R_\mathfrak{m}$ is the quotient ring of R with respect to $\mathfrak{m}$.

From these a) and b) and from Proposition 2 we can easily deduce the following:

Proposition 3. *Let R be a noetherian ring of Krull dimension one and $\mathfrak{a}$ be a regular ideal in R. Then, $\mathfrak{a}$ is an invertible ideal if and only if* $\mathrm{Hom}_R(\mathfrak{a}, R)$ *is a projective module.*

Proof. The only if part is evident. Assume that $\mathrm{Hom}_R(\mathfrak{a}, R)$ is projective, then the R-module $\mathfrak{a}^{-1}$ is invertible, and hence $\mathfrak{a}^{-1}\otimes_R R_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module of rank one, that is, $\mathrm{Hom}_{R_\mathfrak{m}}(\mathfrak{a}R_\mathfrak{m}, R_\mathfrak{m})$ is a free $R_\mathfrak{m}$-module of rank one for any maximal ideal $\mathfrak{m}$ in R. Therefore, by Proposition 2, $\mathfrak{a}R_\mathfrak{m}$ is a regular principal ideal in $R_\mathfrak{m}$ for any $\mathfrak{m}$, whence $\mathfrak{a}$ is invertible. q.e.d.

We now state and prove the main theorem in this paper.

Theorem 3. *Let R be a noetherian domain of Krull dimension one and M be an R-module of type (F_1). Then the following conditions are equivalent:*

i) *The module M is the direct sum of its torsion submodule and a free module of rank one (resp. a projective module).*

ii) *The module* $\mathrm{Hom}_R(M, R)$ *is a free module of rank one (resp. a projective module).*

Proof. Let T be the torsion submodule of M. By Theorem 2, there exist a non-zero ideal $\mathfrak{b}$ in R and a map Φ: $M\to\mathfrak{b}$ such that the sequence

$$0 \longrightarrow T \longrightarrow M \xrightarrow{\Phi} \mathfrak{b} \longrightarrow 0$$

is exact. Dualizing of this sequence, since $\mathrm{Hom}_R(T, R)=0$, we have $\mathrm{Hom}_R(M, R)\simeq\mathrm{Hom}_R(\mathfrak{b}, R)$. If ii) is true, then $\mathrm{Hom}_R(\mathfrak{b}, R)$ is a free module of rank one (resp. a projective module). Hence, by Proposition 2 (resp. Proposition 3), $\mathfrak{b}$ is a principal ideal (resp. an invertible ideal). Therefore, the above sequence splits. This implies i). Since $\mathrm{Hom}_R(T, R)=0$, it is obvious that i) implies ii). q.e.d.

REMARK. Let $\mathfrak{d}$ be the same ideal as that in Theorem 2. Then the proof of Theorem 3 shows that the conditions i), ii) and the following condition iii) are all equivalent.

iii) *The ideal* $\mathfrak{d}$ *is a principal ideal (resp. an invertible ideal).*

§4. Applications

Let V be an r-dimensional irreducible affine variety defined over a perfect field k and W be an irreducible subvariety of V/k. We assume that V/k is embedded in an affine n-space, that is, V is defined by a prime ideal $\mathfrak{p}$ in the polynomial ring $A=k[X_1, \cdots, X_n]$. Let $\mathfrak{q}$ be the prime ideal in A which corresponds to W, then the local ring R of W on V is the ring $A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}$, where $A_{\mathfrak{q}}$ is the quotient ring of A with respect to $\mathfrak{q}$. The prime ideal $\mathfrak{p}A_{\mathfrak{q}}$ is generated by at least $n-r$ elements in $A_{\mathfrak{q}}$. In particular, if $\mathfrak{p}A_{\mathfrak{q}}$ is generated by $n-r$ elements, we shall say that V is a complete intersection locally at W. Let $\{f_1, \cdots, f_m\}$ be a system of generators of $\mathfrak{p}A_{\mathfrak{q}}$ and f_{ij} be the $\mathfrak{p}A_{\mathfrak{q}}$-residue of the partial derivative $\frac{\partial f_i}{\partial X_j}$. Then, since k is a perfect field, the rank of the Jacobian matrix

$$J=\begin{pmatrix} f_{11} & \cdot & \cdot & \cdot & f_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{m1} & \cdot & \cdot & \cdot & f_{mn} \end{pmatrix}$$

is equal to $n-r$ (cf. [7]). Hence the ideal $\mathfrak{J}$, which is generated by all the $(n-r)\times(n-r)$ minors of J, is not the zero ideal and the ideal, which is generated by all the $(n-r+1)\times(n-r+1)$ minors of J, is the zero ideal in R. We shall call the ideal $\mathfrak{J}$ the Jacobian ideal of R. It is well known that the Jacobian ideal $\mathfrak{J}$ of R is the unit ideal if and only if the subvariety W is simple on V.

With the above notations and assumptions, let $D_k(R)$ be the R-module of k-differentials of R, let R^n be a free R-module with a system $\{e_1, \cdots, e_n\}$ of basis and let N be the submodule of R^n generated by the m elements $u_i=f_{i1}e_1+\cdots+f_{in}e_n$ $(i=1, \cdots, m)$, then it is known that the module $D_k(R)$ is isomorphic to the residue module of R^n by N, that is, the sequence

$$0\longrightarrow N\longrightarrow R^n\longrightarrow D_k(R)\longrightarrow 0$$

is exact. (For the definition of the module $D_k(R)$ and the above mentioned property see [3] or [5].) Therefore, the r-th Fitting ideal of $D_k(R)$ is equal to the Jacobian ideal $\mathfrak{J}$ of R and the $(r-1)$-th Fitting ideal is the zero ideal. This shows that *the module* $D_k(R)$ *is of type* (F_r). Moreover, the fact that the Fitting ideal of $D_k(R)$ is the invariant ideal of $D_k(R)$ means that the Jacobian ideal $\mathfrak{J}$ of R is independent of the choice of the affine embedding of V/k.

From the definition of $D_k(R)$ the dual module $D_k^*(R)=\mathrm{Hom}_R(D_k(R), R)$ of $D_k(R)$ may be identified with the module of k-derivations of R into itself. Since the defining field k of V is perfect, it is known that if $D_k^*(R)$ is free then the rank of $D_k^*(R)$ is equal to the dimension of V/k (cf. [4]). Therefore, if $D_k(R)$ is free then the rank of $D_k(R)$ is equal to the dimension of V/k.

Applying these facts to Theorem 1 in §1, we have the following well known:

THEOREM 4. *Let V be an irreducible affine variety defined over a perfect field k, W be an irreducible subvariety of V/k and R be the local ring of W on V. Then, the module $D_k(R)$ of k-differentials of R is a free R-module if and only if the subvariety W is simple on V. Moreover, the rank of $D_k(R)$ is equal to the dimension of V/k.*

From now on we will restrict the variety V to *a curve*. Let P be a point of an irreducible affine curve V and R be a local ring of P on V. If V is a complete intersection locally at P and if we put $M=D_k(R)$ in Theorem 2 in §2, then the ideal $\mathfrak{d}$ in Theorem 2 coincides with the Jacobian ideal of R (cf. Remark in §2). Therefore, as a corollary of Theorem 3 in §3, we have the following:

THEOREM 5. *Let V be an irreducible affine curve defined over a perfect field k, P be a point of V/k and R be the local ring of P on V. Then the following conditions are equivalent.*

i) *The module $D_k(R)$ of k-differentials of R is the direct sum of its torsion submodule and a free R-module.*

ii) *The module $D_k^*(R)$ of k-derivations of R into itself is a free R-module.*

Moreover, if V is a complete intersection locally at P, then i), ii) *and the following condition* iii) *are all equivalent.*

iii) *The Jacobian ideal $\mathfrak{J}$ of R is a principal ideal.*

REMARK. Clearly Theorem 5 is valid if we replace R by the affine ring of V/k.

If the characteristic of k is zero, it is shown in [4] that $D_k(R)$ is free if and only if $D_k^*(R)$ is free. However, in the case of positive characteristic, the freeness of $D_k(R)$ is not deduced from the freeness of $D_k^*(R)$. In fact: Let k be a perfect field of positive characteristic p, V/k be the plane curve defined by the equation $X^p - Y^{p+1}=0$ and $R=k[x, y]_{(x,y)}$ be the local ring of the origin on V. Then, although $D_k^*(R)$ is free, $D_k(R)\simeq R\oplus(R/Ry^p)$ (direct sum).

The following example shows that Theorem 5 is not true for higher dimensional varieties.

EXAMPLE. Let k be a perfect field of positive characteristic p. Let V/k be the irreducible surface, in an affine 3-space, defined by the equation $XY-Z^p=0$ and P be the origin. Then $D_k^*(R)$ is free and $D_k(R)$ is torsion free, however, not free.

PROOF. For the fact that $D_k^*(R)$ is free see [4]. It is clear that P is a singular point of V, and hence $D_k(R)$ is not free. The direct proof of the fact that $D_k(R)$ is torsion free[5] is as follows: Let V/k be an irreducible surface, in 3-space, defined by the equation $f(X, Y, Z)=0$ such that $f(0, 0, 0)=0$ and let $R=k[x, y, z]_{(x, y, z)}$ be the local ring of the origin on V. Then, it is easy to show that a differential $\omega=adx+bdy+cdz$ $(a, b, c \in R)$ is a torsion element if and only if $af_y=bf_x$, $bf_z=cf_y$ and $cf_x=af_z$, where $f_x=f_X(x, y, z)$ etc.

In our case, since $f_x=y, f_y=x$ and $f_z=0$, a differential $\omega=adx+bdy+cdz$ is a torsion element if and only if $ax=by$ and $c=0$. On the other hand, since $z^p=xy$, the Jacobian ideal $\mathfrak{J}=(x, y)R$ is an $\mathfrak{m}$-primary ideal where $\mathfrak{m}$ is the maximal ideal of R. Since V is a complete intersection locally at P, R is a Macaulay ring and hence both $\{x, y\}$ and $\{y, x\}$ are prime sequences. Therefore, $ax=by$ implies $b \in Rx$ and $a \in Ry$, whence there exists an element u in R such that $a=uy$ and $b=ux$. Hence, we have $\omega=u(ydx+xdy)=0$.

q.e.d.

References

[1] N. Bourbaki, Algèbre Commutative, Ch. 1, 2, Act. Scient. et Ind., 1290, Hermann, Paris, (1961).

[2] H. Fitting, Die Determinantenideale eines Moduls, Jahresbericht d. Deutschen Math.-Vereinigung, **46** (1936), 195-228.

[3] E. Kunz, Die Primidealteiler der Differenten in allgemeinen Ringen, J. Reine Angew. Math., **204** (1960), 165-182.

[4] J. Lipman, Free Derivation Modules on Algebraic Varieties, Amer. J. Math., **87** (1965), 874-898.

[5] Y. Nakai, On the theory of differentials in commutative rings, J. Math. Soc. Japan, **13** (1961), 63-84.

[6] S. Suzuki, On torsion of the module of differentials of a locality which is a complete intersection, J. Math. Kyoto Univ., **4** (1965), 471-475.

[7] A. Weil, Foundations of Algebraic Geometry, Amer. Math. Soc. Col. Pub., 29 (1946).

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5) Since P is a normal point of V/k, this fact is the direct consequence of the following: If V is a complete intersection locally at P, then the torsionfreeness of $D_k(R)$ is equivalent to the normality of R (cf. [4] or [6]).