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Boundary Value Problems for the Equation $\Delta u - qu = 0$ with respect to an Ideal Boundary

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Introduction

It is by now a classical result that linear boundary value problems for a second order elliptic linear partial differential equation with sufficiently smooth coefficients are uniquely solvable if boundary conditions and the boundary itself are sufficiently regular. While discussions for equations with non-smooth coefficients have been tried by many people, not much investigations of boundary value problems for non-smooth boundary have been made except for the Dirichlet problem.

As to the Dirichlet problem for the Laplace equation $\Delta u = 0$, there is the method of Perron-Brelot (see [3], [7], etc.), which is also applied to more general equations (see e.g. [1], [4], [13]). For boundary value problems other than the Dirichlet problem, there appears the notion of normal derivatives, which is originally defined only with respect to a smooth boundary. Therefore, as long as we try to consider problems like Neumann problem and the third boundary value problem with respect to a non-smooth boundary, it is necessary to generalize the notion of normal derivatives in some way. This has been done by L. Doob [11] with respect to the Martin boundary, by C. Constantinescu and A. Cornea [7] with respect to the Kuramochi boundary and by the author [20] with respect to a general resolutive ideal boundary. In these works, linear boundary value problems involving normal derivatives are treated for the Laplace equation, i.e., for harmonic functions. In the present treatise, we apply the techniques developed in these works to the equation $\Delta u - qu = 0$ $(q \geqq 0)$ and consider general linear boundary value problems with respect to a general ideal boundary.

We shall take a locally Euclidean space as the base space on which the equation is considered. It may be possible, however, to extend our theory to more general elliptic partial differential equations considered on a C^∞-manifold (cf. [12], [15], [16]). In fact, a locally Euclidean space is a special kind of C^∞-manifold and our theory may suggest how it is extended to a theory on a C^∞-manifold. Also, we can justify the restriction to the equation $\Delta u - qu = 0$ by noting that this is the canonical form of self-adjoint equations (cf. [12], [14]).

This paper consists of the following six chapters:

Chapter I. q-harmonic structures. In this chapter, we first give known properties of the solutions of the equation $\Delta u-qu=0$ (called q-harmonic functions) and remark that the sheaf of q-harmonic functions forms a harmonic space in the sense of M. Brelot [4]. Then we investigate the properties of corresponding superharmonic functions (called q-superharmonic functions).

Chapter II. Green functions. Existence of the Green function for the equation $\Delta u-qu=0$ (called the q-Green function) is proved in this chapter and the dependence of the q-Green function on q is studied.

Chapter III. Dirichlet problems. In this chapter, we discuss the Dirichlet problem for $\Delta u-qu=0$ with respect to an ideal boundary in the method of Perron-Brelot. We shall be particularly concerned with resolutivity of boundary functions for different q's.

Chapter IV. Normal derivatives. Definitions of "normal derivatives" on an ideal boundary, which are generalizations of those in [11], [7] and [20], are given. Then normal derivatives of q-harmonic functions are studied and properties that will be used in the next chapter are established.

Chapter V. Boundary value problems. This chapter contains the main results of this paper. In the first half, a general boundary value problem is formulated and a uniqueness theorem and an existence theorem are proved. In the second half, the properties of solutions, in particular the dependence of solutions on boundary conditions and on q, are discussed in various forms.

Chapter VI. Green functions for general mixed problems. This title means the fundamental solutions (for $\Delta u-qu=0$) which satisfy general homogeneous boundary conditions. Construction of such a Green function is an application of the existence theorem in the previous chapter. Expression of solutions of the problem in terms of such Green functions is also given.

Throughout these chapters, we use only standard methods in potential theory; in particular an elementary theory of Hilbert spaces is the main tool in the proof of the existence theorem in Chapter V.

CHAPTER I q-harmonic Structures.

§1.1. Preliminaries

Throughout this paper, let X be a connected d-dimensional ($d\geqq 2$) locally Euclidean space, i.e., a connected d-dimensional manifold for which each coordinate transformation is a rigid motion (isometry). Thus, for any $x\in X$, there exists a relatively compact neighborhood V of x with a coordinate system by which V is mapped onto an open ball $\{|y|<r\}$ in the d-dimensional Euclidean space R^d and x is mapped to $y=0$. In this case the coordinate system can be extended to an open set containing the closure $\bar{V}$ of V and the

boundary ∂V of V corresponds to the sphere $\{|y| = r\}$. The radius r does not depend on the choice of a coordinate system. Thus any such V will be called a *ball* with center at x and of radius r.

We may regard the space X as a space of type $\mathfrak{S}_i$ without points at infinity in the sense of Brelot-Choquet [6]. In this connection, we can translate the whole theory in this paper to that on a Riemann surface, by making obvious modifications in the terminology (see [22], [23], [24] for treatments of the equation $\Delta u - qu = 0$ on a Riemann surface).

On a locally Euclidean space X, the Laplacian Δ is defined coordinate-free. We consider the differential equation

$$\Delta u - qu = 0 \tag{1.1}$$

on X, where q is a *non-negative* locally Hölder continuous function on X.

Remark. The condition that q is locally Hölder continuous is assumed only to obtain the local properties of the solutions of (1.1) stated in this chapter and the existence of local fundamental solutions. No explicit use of this condition will be made in the subsequent discussion. Therefore, this condition can be replaced by any other which guarantees these local properties.

Throughout this paper, every function is assumed to be extended real valued. Let Y be any open subset of X. A locally summable function u on Y is called a weak solution of (1.1) on Y if it satisfies (1.1) in the distribution sense, i.e., for any C^∞-function (infinitely differentiable function) f having a compact support in Y, we have $\int\{(\Delta f)u - qfu\}dx = 0$, where dx denotes the Lebesgue measure on Y. The following proposition is well-known (see, e.g., [2], pp. 138–139 or [14]):

Proposition 1.1. *Any weak solution of* (1.1) *is almost everywhere equal to a C^2-function* (twice continuously differentiable function) *on Y which satisfies* (1.1) *in the ordinary sense.*

We shall call a function *q-harmonic* on Y if it is a C^2-function satisfying (1.1) on Y. If $q = 0$, then q-harmonic functions are ordinary harmonic functions.

§1.2. Local properties of q-harmonic functions

The following properties of q-harmonic functions are well-known:

Proposition 1.2. (Minimum principle I) (See, e.g., [2], [9].) *Let Y be a domain in X and v be a C^2-function on Y.*

(i) *If $\Delta v \leqq 0$ on Y and v assumes minimum in Y, then v is constant.*

(ii) *If $\Delta v - qv \leqq 0$ and $q \neq 0$ on Y, then v can not assume negative minimum in Y.*

From this proposition, the following form of minimum principle follows (cf. [13]):

PROPOSITION 1.3. (Minimum principle II) *Let Y be a relatively compact domain in X.*

(i) *If v is a C^2-function satisfying $\Delta v - qv \leqq 0$ on Y, then $\underline{\lim}_{x\to y} v(x) \geqq 0$ for all $y \in \partial Y$ implies $v \geqq 0$ on Y.*

(ii) *Let $q_1 \leqq q_2$ on Y and u_i be q_i-harmonic on Y $(i=1, 2)$. If $\underline{\lim}_{x\to y} u_1(x) \geqq 0$ and $\underline{\lim}_{x\to y}[u_1(x) - u_2(x)] \geqq 0$ for all $y \in \partial Y$, then $u_1 \geqq u_2$ on Y.*

PROPOSITION 1.4. (Dirichlet solution for balls) (See, e.g., [9], Chap. IV, [14], [21].) *Let V be any ball in X and let g be any continuous function on ∂V. Then there exists a unique continuous function $H_g^{q,V}$ on $\bar{V}$ such that $H_g^{q,V} = g$ on ∂V and $H_g^{q,V}$ is q-harmonic on V. Furthermore, $g \geqq 0$ implies $H_g^{q,V} \geqq 0$ on $\bar{V}$.*

If $q=0$, then we omit the superscript q in $H_g^{q,V}$. By (ii) of Proposition 1.3, we have

PROPOSITION 1.5. (Cf. [14], [18].) *If $q_1 \leqq q_2$ on V and $g \geqq 0$ on ∂V, then $H_g^{q_1,V} \geqq H_g^{q_2,V}$. In particular $H_g^{q,V} \leqq H_g^V$.*

PROPOSITION 1.6. (See [14].) *Let V be a ball with center at x and of radius r (>0). If $g \geqq 0$ on ∂V, then*

$$H_g^V(x) \leqq e^{r\sqrt{Q}} H_g^{q,V}(x),$$

where $Q = \sup_{y\in V} q(y)$.

PROPOSITION 1.7. (Harnack's inequality) (See [13], [14], [22]; also cf. [18].) *Let Y be a domain in X and Z be a compact set contained in Y. Let $x_0 \in Y$ be given. Then there exists a constant $M = M(Y, Z, x_0, q) \geqq 1$ such that $u(x) \leqq Mu(x_0)$ for any non-negative q-harmonic function u on Y and for all $x \in Z$.*

From this proposition the following principle follows (see [13], [14], [22]):

PROPOSITION 1.8. (Harnack's principle) *Let Y be a domain in X. If $\{u_n\}$ is a monotone increasing sequence of q-harmonic functions on Y and if $\{u_n(x)\}$ is bounded (above) for some $x \in Y$, then $\lim_{n\to\infty} u_n = \sup u_n$ defines a q-harmonic function on Y. Furthermore the convergence is locally uniform in Y.*

§1.3. q-superharmonic functions

By Propositions 1.4 and 1.8, we see that the sheaf of q-harmonic functions satisfies the axioms of harmonic spaces introduced by M. Brelot (see [1], [4]; also [13], [18]). Hence we can define such notions as q-superharmonic functions and q-potentials.

By Proposition 1.4, any ball is a regular open set; we have the q-harmonic measure $\omega_x^{q,V}$ on ∂V with respect to $x \in V$ satisfying

$$H_g^{q,V}(x)=\int g\,d\omega_x^{q,V}$$

for any (finite) continuous function g on ∂V.

A q-superharmonic function v on an open set Y is then defined as a function satisfying the following conditions:

(i) $v(x)>-\infty$ at each $x \in Y$; $v\not\equiv+\infty$ on any component of Y;

(ii) v is lower semi-continuous on Y;

(iii) For any ball V such that $\overline{V}\subset Y$,

$$v(x)\geqq\int v\,d\omega_x^{q,V} \qquad \text{for all} \quad x \in V.$$

In condition (iii), we may restrict V to those which belong to a family forming a base of open sets in Y. Thus we see that q-superharmonicity is a local property.

As for comparison of q-superharmonicity for different q, we have the following immediately from the definition and Proposition 1.5:

Proposition 1.9. *If $q_1\leqq q_2$, then any non-negative q_1-superharmonic function is q_2-superharmonic and any non-positive q_2-superharmonic function is q_1-superharmonic. In particular, the constant function $v(x)\equiv 1$ is q-superharmonic for any q.*

The following properties of q-superharmonic functions are consequences of the general theory on harmonic spaces (cf. [1], [4], [13] and [18]).

Proposition 1.10. (i) *If v_1, v_2 are q-superharmonic on Y and if λ_1, λ_2 are positive numbers, then $\lambda_1 v_1+\lambda_2 v_2$ and $\min(v_1, v_2)$ are q-superharmonic on Y.*

(ii) *If $\{v_\iota\}_\iota$ is an upper directed family of q-superharmonic* (resp. *q-harmonic*) *functions on a domain Y, then $\sup_\iota v_\iota$ is either $\equiv+\infty$ or q-superharmonic* (resp. *q-harmonic*) *on Y.*

Proposition 1.11. (Minimum principles) (i) *If v is a non-negative q-superharmonic function on a domain Y and if $v(x)=0$ at some point $x \in Y$, then $v\equiv 0$ on Y.*

(ii) *If v is a q-superharmonic function on an open set Y and if $\underline{\lim}_{x\to \mathrm{id}(Y)} v(x)\geqq 0$,* i.e., *for any $\varepsilon>0$ there exists a compact set Z in Y such that $v(x)>-\varepsilon$ on $Y-Z$, then $v\geqq 0$ on Y. In particular, if Y is relatively compact and $\underline{\lim}_{x\to y} v(x)\geqq 0$ for all $y \in \partial Y$, then $v\geqq 0$ on Y.*

Proposition 1.12. *Let v be a q-superharmonic function on an open set Y and let V be a ball such that $\overline{V}\subset Y$. Then $u(x)=\int v\,d\omega_x^{q,V}$ is q-harmonic on V and*

$$v_V = \begin{cases} v & on \quad Y-V \\ u & on \quad V \end{cases}$$

is q-superharmonic on Y.

PROPOSITION 1.13. (Perron) *Let $\mathcal{V}$ be a family of q-superharmonic functions on an open set Y satisfying the following two conditions:*

(i) *$\mathcal{V}$ is a non-empty, lower directed family and is locally uniformly bounded below;*

(ii) *If $v \in \mathcal{V}$, then $v_V \in \mathcal{V}$ for any ball V such that $\bar{V} \subset Y$, where v_V is the function defined in the previous proposition.*

Then $\inf \mathcal{V}$ *is q-harmonic on Y.*

From this proposition, we see that any non-negative q-superharmonic function v on X has the greatest q-harmonic minorant on X, which is also the largest among the functions w such that $w \leqq v$ on X and $-w$ is q-superharmonic on X. A non-negative q-superharmonic function on X whose greatest q-harmonic minorant is zero is called a *q-potential.* As a corollary to Proposition 1.9, we have

PROPOSITION 1.14. *If $q_1 \leqq q_2$, then any q_1-potential is a q_2-potential.*

Hereafter, if $q=0$ on X, then we shall omit the index q in the terminology and notation.

§1.4. Local fundamental solutions

Let V be a ball in X. It is known (cf., e.g., [14], [17], [21]; also [12]) that there exists a (symmetric) fundamental solution $F^q(x, y)$ of the equation (1.1) on V, i.e., a function on $V \times V$ such that $F^q(x, y) = F^q(y, x)$ for any x, $y \in V$ and $F^q_y(x) = F^q(x, y)$ is locally summable on V, continuous on $V - \{y\}$ and satisfies

$$\Delta F^q_y - qF^q_y = -c_d \delta_y \tag{1.2}$$

in the distribution sense for any $y \in V$, where δ_y is the Dirac measure at the point y and c_d is the constant equal to 2π if $d=2$, to $(d-2)\times$ the surface area of the unit sphere in R^d, if $d \geqq 3$. In case $q=0$, a fundamental solution is given by $F(x, y) = -\log|x-y|$ if $d=2$ and $F(x, y) = |x-y|^{2-d}$ if $d \geqq 3$. It is known ([17], [21]; cf. [12]) that

$$F^q(x, y) - F(x, y) = O(|x-y|^{\lambda+2-d}), \tag{1.3}$$

where $\lambda = 0$ if $d=2$, $\lambda > 0$ if $d \geqq 3$.

PROPOSITION 1.15. *If $F^q(x, y)$ is a fundamental solution of* (1.1) *on V, then $F^q_y(x) = F^q(x, y)$ is q-superharmonic on V, provided that we define $F^q(x, x) = +\infty$ for each $x \in V$.*

PROOF. By (1.2) and Proposition 1.1, F_y^q is q-harmonic on $V-\{y\}$. By (1.3), $\lim_{x\to y}F_y^q(x)=+\infty$, so that F_y^q is lower semi-continuous at $x=y$. Then it is obvious that F_y^q is q-superharmonic on V.

COROLLARY. *If μ is a positive Radon measure on V whose support is compact in V, then the function*

$$F_\mu^q(x)=\int F_y^q(x)\,d\mu(y)$$

is q-superharmonic on V. Furthermore, $\Delta F_\mu^q-qF_\mu^q=-c_d\mu$ in the distribution sense on V.

The proof of this corollary is quite analogous to its proof in the special case $q=0$, which is classical (cf. [3], [7] or [25]).

§1.5. Characterization of q-superharmonic functions

It is well-known that a superharmonic function s is locally summable and $\Delta s\leqq 0$ in the distribution sense and conversely any locally summable function s such that $\Delta s\leqq 0$ in the distribution sense is equal to a superharmonic function almost everywhere (cf. e.g., [3] and [25]). We shall show similar results for q-superharmonic functions. First we prove:

LEMMA 1.1. *Any q-superharmonic function is locally summable.*

PROOF. Let v be a q-superharmonic function on a domain Y and let

$$Y_1=\{x\in Y;\ v \text{ is summable on a neighborhood of } x\}.$$

Obviously Y_1 is an open set. Let $x_0\in Y-Y_1$ and let V be a ball with center at x_0 such that $\bar{V}\subset Y$. Let r be the radius of V and let V_1 be the concentric ball of radius $r/2$. Since v is bounded below on $\bar{V}$, there exists a constant $c\geqq 0$ such that $w=v+c$ is non-negative on V. w is again q-superharmonic (Proposition 1.9). Take any $x\in V_1$ and let W_t be the ball with center at x and of radius t with $0<t\leqq r/2$. Let $Q=\sup_{y\in V}q(y)$. Then, by Proposition 1.6, we have

$$\int w\,d\omega_x^{W_t}\leqq e^{t\sqrt{Q}}\int w\,d\omega_x^{q,W_t}.$$

Since w is q-superharmonic, $\int w\,d\omega_x^{q,W_t}\leqq w(x)$. Hence

$$\int w\,d\omega_x^{W_t}\leqq e^{t\sqrt{Q}}\,w(x).$$

On the other hand, $\omega_x^{W_t}$ is a constant $(=1/c_d')$ times the surface element of ∂W_t. Hence

$$\int_{W_{r/2}} w(y)dy = c'_d\left(\frac{2}{r}\right)^d\int_0^{r/2}\left(\int w\, d\omega_x^{W_t}\right)t^{d-1}dt$$

$$\leqq c'_d\left(\frac{2}{r}\right)^d w(x)\int_0^{r/2} t^{d-1}e^{t\sqrt{Q}}\,dt.$$

Since $W_{r/2}$ is a neighborhood of x_0 and since $x_0 \in Y-Y_1$, $\int_{W_{r/2}} v(y)dy=+\infty$. Hence $\int_{W_{r/2}} w(y)dy=+\infty$, which implies $w(x)=+\infty$ by the above inequality. Since x is arbitrary, $w(x)\equiv+\infty$ on V_1, so that $v(x)\equiv+\infty$ on V_1. Therefore $Y-Y_1$ is also open and $v\equiv+\infty$ on $Y-Y_1$. Since $v\not\equiv+\infty$ by definition, we must have $Y=Y_1$. Hence v is locally summable.

Given a Borel function f defined on a neighborhood of a point $x_0 \in X$, let $\mathcal{M}_f^r(x_0)$ (resp. $\mathcal{A}_f^r(x_0)$) be the surface mean (resp. the volume mean) of f with respect to the ball V_r with center at x_0 and of radius r (cf. [3]). In fact, we can write

$$\mathcal{M}_f^r(x_0)=\int f\,d\omega_{x_0}^{V_r} \quad \text{and} \quad \mathcal{A}_f^r(x_0)=\frac{d}{r^d}\int_0^r \mathcal{M}_f^t(x_0)t^{d-1}dt.$$

LEMMA 1.2. *If v is a q-superharmonic function on a ball V with center at $x_0 \in X$, then*

$$\lim_{r\to 0}\mathcal{M}_v^r(x_0)=v(x_0) \quad \textit{and} \quad \lim_{r\to 0}\mathcal{A}_v^r(x_0)=v(x_0).$$

PROOF. The second assertion immediately follows from the first. Since v is lower semi-continuous, we see that $\underline{\lim}_{r\to 0}\mathcal{M}_v^r(x_0)\geqq v(x_0)$. On the other hand, if $v\geqq 0$ on V, then Proposition 1.6 implies that

$$\mathcal{M}_v^r(x_0)\leqq e^{r\sqrt{Q}}\int v\,d\omega_{x_0}^{q,V_r}\leqq e^{r\sqrt{Q}}\,v(x_0),$$

where $Q=\sup_{x\in V} q(x)$. Hence $\overline{\lim}_{r\to 0}\mathcal{M}_v^r(x_0)\leqq v(x_0)$. Since v, in general, is bounded below near x_0, this conclusion holds for any v.

By Lemma 1.1, any q-superharmonic function v can be regarded as a distribution. We shall show that $\Delta v-qv\leqq 0$ in the distribution sense. In case v is a C^2-function, this is well-known (cf. [13]); in fact we have the following lemma as an easy consequence of Proposition 1.3:

LEMMA 1.3. *If v is a C^2-function on a domain Y, then it is q-superharmonic on Y if and only if $\Delta v-qv\leqq 0$ on Y* (in the ordinary sense, hence in the distribution sense as well).

Using this lemma, we prove

THEOREM 1.1. *If v is a q-superharmonic function on a domain Y, then $\Delta v-qv\leqq 0$ on Y in the distribution sense.*

PROOF. Let V be any ball such that $\bar{V}\subset Y$ and r be its radius. Let V_1 be the concentric ball of radius $r/2$. Since v is bounded below on $\bar{V}$, there exists a constant $c>0$ such that $v+H_c^{q,V}\geqq 0$ on $\bar{V}$ (cf. Propositions 1.4 and 1.11). Since $H_c^{q,V}$ is q-harmonic on V, we may *assume that v is non-negative on $\bar{V}$.* Let $\{f_n\}$ be a δ-sequence of non-negative C^∞-functions on R^d such that $S(f_n)\subset\{|x|<r/(2n)\}$, where $S(f_n)$ denotes the support of f_n. Then the convolution $(v*f_n)(x)$ makes sense for $x\in V_1$ and is a C^∞-function on V_1. Furthermore, $v*f_n\to v$ $(n\to\infty)$ weakly as distributions.

Since q is uniformly continuous on $\bar{V}$, given $\varepsilon>0$ there exists n_0 such that $|x-x'|<r/(2n_0)$ and $x, x'\in\bar{V}$ imply $|q(x)-q(x')|<\varepsilon$. Let $q_\varepsilon(x)=\sup_{|y-x|<r/(2n_0)} q(y)$ for $x\in V_1$. Also, for each $z\in R^d$ such that $|z|<r/(2n_0)$, let $q_z(x)=q(x-z)$ for $x\in V_1$. Then q_ε and q_z are non-negative Hölder continuous on V_1, $q_z\leqq q_\varepsilon$ and $0\leqq q_\varepsilon(x)-q(x)<\varepsilon$ for all $x\in V_1$.

Let W be any open ball such that $\bar{W}\subset V_1$. By Proposition 1.5, we have

$$\int v(x-z)\,d\omega_y^{q_\varepsilon,W}(x)\leqq\int v(x-z)\,d\omega_y^{q_z,W}(x)$$

for any z with $|z|<r/(2n_0)$ and $y\in W$. Obviously, $d\omega_y^{q_z,W}(x)=d\omega_{y-z}^{q,W-z}(x')$, where $x'=x-z$ and $W-z=\{w-z\,;\,w\in W\}\subset V$. Therefore, v being q-superharmonic,

$$\int v(x-z)d\omega_y^{q_\varepsilon,W}(x)\leqq\int v(x')d\omega_{y-z}^{q,W-z}(x')\leqq v(y-z).$$

Hence, for $n\geqq n_0$,

$$\int(v*f_n)(x)\,d\omega_y^{q_\varepsilon,W}(x)\leqq\int v(y-z)f_n(z)dz=(v*f_n)(y).$$

Therefore, $v*f_n$ is q_ε-superharmonic on V_1. By Lemma 1.3,

$$\Delta(v*f_n)-q_\varepsilon(v*f_n)\leqq 0$$

on V_1, i.e., for any C^∞-function g such that $g\geqq 0$ and $S(g)\subset V_1$,

$$\int[\Delta g(x)-q_\varepsilon(x)g(x)](v*f_n)(x)dx\leqq 0.$$

Now, letting $n\to\infty$, we have

$$\int[\Delta g(x)-q_\varepsilon(x)g(x)]v(x)dx\leqq 0.$$

Hence

$$\int[\Delta g-qg]v\,dx=\int[\Delta g-q_\varepsilon g]v\,dx+\int(q_\varepsilon-q)gv\,dx$$

$$\leqq\varepsilon\int gv\,dx.$$

Since $\int_{V_1} v\,dx$ is finite and ε is arbitrary, we have $\int(\Delta g-qg)v\,dx\leqq 0$, i.e., $\Delta v-qv\leqq 0$ on V_1 in the distribution sense.

As a converse, we have

THEOREM 1.2. *If v is a locally summable function such that $\Delta v-qv\leqq 0$ in the distribution sense on a domain Y, then there exists a q-superharmonic function w on Y such that $v=w$ almost everywhere. If, in addition $\mathcal{A}_v^r(x)\to v(x)$ $(r\to 0)$ for any $x\in Y$, then v itself is q-superharmonic on Y.*

PROOF. Let V and V_1 be as in the previous proof. The distribution $-(1/c_d)(\Delta v-qv)$ is non-negative, so that it can be regarded as a Radon measure on Y. Let μ be its restriction on V_1. Let $F_y^q(x)=F^q(x, y)$ be a local fundamental solution of $\Delta u-qu=0$ on V and consider $F_\mu^q(x)=\int F_y^q(x)d\mu(y)$. By the corollary to Proposition 1.15, F_μ^q is q-superharmonic on V and $\Delta F_\mu^q-qF_\mu^q=-c_d\mu=\Delta v-qv$ on V_1. Hence, by Proposition 1.1, there exists a q-harmonic function u on V_1 such that $v-F_\mu^q=u$ almost everywhere on V_1. Let $w=u+F_\mu^q$ on V_1. Then w is q-superharmonic and $w=v$ almost everywhere on V_1. Since Y is covered by such balls V_1, we conclude the first assertion of the theorem. Now the second assertion follows from Lemma 1.2.

CHAPTER II Green functions.

§2.1. Definition and uniqueness

A q-Green function for X is an extended real valued function $G_y^q(x)=G^q(x, y)$ on $X\times X$ such that for each $y\in X$

(i) G_y^q is a q-potential on X;

(ii) $\Delta G_y^q-qG_y^q=-c_d\delta_y$ in the distribution sense.

By condition (i), $G^q(x, y)\geqq 0$. Condition (ii) is equivalent to say that G_y^q is q-harmonic on $X-\{y\}$ and $G_y^q-F_y^q$ is q-harmonic on any ball V such that $y\in V$, where F_y^q is a local fundamental solution of $\Delta u-qu=0$ on V.

LEMMA 2.1. *The q-Green function is uniquely determined by conditions* (i) *and* (ii) *for each y. Furthermore, if there is a positive q-potential w on X such that it is q-harmonic on $X-\{y\}$, then the function G_y^q satisfying* (i) *and* (ii) *exists and $w=\lambda G_y^q$ for some $\lambda>0$.*

PROOF. Let v_i $(i=1, 2)$ satisfy (i) and (ii), i.e., each v_i is a q-potential and $\Delta v_i-qv_i=-c_d\delta_y$ in the distribution sense. It follows that v_1 and v_2 are q-harmonic on $X-\{y\}$ and $\Delta(v_1-v_2)-q(v_1-v_2)=0$ in the distribution sense. Hence there exists a q-harmonic function u on X such that $v_1=v_2+u$ on $X-\{y\}$ (Proposition 1.1). By Lemma 1.2, $v_1=v_2+u$ everywhere on X. Since v_1, v_2 are both q-potentials, it follows that $u=0$, i.e., $v_1=v_2$.

If w is a positive q-potential on X which is q-harmonic on $X - \{y\}$, then $\mu = -(1/c_d)(\Delta w - qw)$ is a positive Radon measure (Theorem 1.1) and is supported by the point set $\{y\}$. Hence $\mu = \lambda\delta_y$ for some $\lambda > 0$. Then $G_y^q = w/\lambda$ satisfies (i) and (ii).

§2.2. Existence of the q-Green function

If $q = 0$, then it is a classical result that the existence of the Green function for X is equivalent to the existence of a non-constant positive superharmonic function. We shall call X a Green space if it has the Green function. There are non-compact locally Euclidean spaces which are not Green spaces. If $q \neq 0$, then we shall see that the q-Green function always exists (even if X is compact). For its proof, we rely on local existence theorems which are known. We may, for example, start with the following result (cf. [15]; also [21], [22]):

If Y is a relatively compact domain in X such that ∂Y consists of a finite number of closed C^∞-hypersurfaces, then there exists the q-Green function $G_y^{q,Y}(x) = G^{q,Y}(x, y)$ for Y. Furthermore, it has the following properties:

(a) For each $y \in Y$, $G_y^{q,Y}$ vanishes on ∂Y (i.e., it can be continuously extended on $\bar{Y}$ with vanishing values on ∂Y);

(b) For each $y \in Y$, the inner normal derivative $\partial G_y^{q,Y}/\partial n$ exists and is non-negative at every point on ∂Y.

(c) $G^{q,Y}(x, y) = G^{q,Y}(y, x)$ for any $x, y \in Y$.

Now we prove the existence theorem on X using this result. The method of the following proof is due to L. Myrberg [23].

Theorem 2.1. *Suppose $q \neq 0$ on X. Then the q-Green function for X exists.*

Proof. Let $\{X_n\}$ be an exhaustion of X such that each ∂X_n consists of a finite number of closed C^∞-hypersurfaces. By Green's formula, we have

$$0 \leq \int_{\partial X_n} \frac{\partial G_y^{q,X_n}}{\partial n} dS = c_d - \int_{X_n} q(x) G_y^{q,X_n}(x) dx \qquad (y \in X_n).$$

Hence

$$\frac{1}{c_d} \int_{X_n} q(x) G_y^{q,X_n}(x) dx \leq 1. \tag{2.1}$$

The minimum principle (Proposition 1.3) implies that $\{G_y^{q,X_n}\}_n$ is monotone increasing. If $q(x) \neq 0$ on X, then (2.1) implies that $u_0(x) = \lim_{n\to\infty} G_y^{q,X_n}(x)$ is finite at some $x \in X$. Since each G_y^{q,X_n} is q-superharmonic on X_n and q-harmonic on $X_n - \{y\}$, u_0 is q-superharmonic on X and q-harmonic on $X - \{y\}$. It is easy to see that u_0 satisfies condition (ii) for the q-Green function. In fact, for each n, there exists a q-harmonic function u_n on X_n such that $u_0=G_y^{q,X_n}+u_n$ ($u_n\geqq 0$). If v is a q-harmonic minorant of u_0, then $G_y^{q,X_n}\geqq v-u_n$ on X_n, so that $v\leqq u_n$. Since $u_n\rightarrow 0$, $v\leqq 0$. Therefore u_0 is a q-potential, i.e., u_0 satisfies also condition (i). Hence $u_0=G_y^q(x)$.

COROLLARY. (i) $G^q(x, y)=G^q(y, x)$ *for any* $x, y\in X$.

(ii) $(1/c_d)\int q(x)G_y^q(x)dx\leqq 1$ *for any* $y\in X$.

PROOF. (i) follows from the property (c) for $G^{q,Y}$ and (ii) follows from (2.1) in the above proof.

REMARK. In the proof of the above theorem, we used the existence of $G^{q,Y}(x, y)$, which is rather a strong assumption. We give here an alternative proof which requires only the existence of local fundamental solutions: Let $y\in X$ be fixed and let $F_y^q(x)$ be a local fundamental solution of (1.1) on a ball V with center at y. Let r be the radius of V and let V' be the concentric ball of radius $r/2$. As in the proof of Theorem 1.1, we may assume that $F_y^q\geqq 0$ on $\bar{V}'$ by adding a suitable positive q-harmonic function. We consider the family

$$\mathcal{U}_y=\left\{u\,;\ \begin{array}{l} q\text{-superharmonic, } \geqq 0 \text{ on } X \text{ and there exists a } q\text{-super-} \\ \text{harmonic function } w_u \text{ on } V \text{ such that } u=F_y^q+w_u \text{ on } V-\{y\} \end{array}\right\}.$$

We first show that $\mathcal{U}_y$ is non-empty.

Let V'' be the ball concentric with V and of radius $r''<r/2$ such that $q\neq 0$ on $X-\bar{V}''$. Since $q\neq 0$ on X by assumption, it is possible to find such V''. Let

$$v_1=\inf\{v;\ q\text{-superharmonic, } \geqq 0 \text{ on } X,\ \geqq 1 \text{ on } \bar{V}''\}.$$

By a general theory of harmonic spaces (see [1], [4], [13]), the regularization $\hat{v}_1$ of v_1 is q-superharmonic on X and $\hat{v}_1=1$ on V''. Hence $v_1>0$ on $X-\bar{V}''$. Also, we see that v_1 is q-harmonic on $X-\bar{V}''$. Since $q\neq 0$ on $X-\bar{V}''$, 1 is *not* q-harmonic. Hence $v_1<1$ on $X-\bar{V}''$. Next let $u_1=H_{v_1}^{q,V'}$. Then $\sigma=\inf_{x\in V''}[\hat{v}_1(x)-u_1(x)]>0$. Put $\lambda=(1/\sigma)\sup_{x\in\partial V''}F_y^q(x)$. Then $\lambda>0$ and we see that the function

$$u(x)=\begin{cases} F_y^q(x)+\lambda u_1(x) & \text{for } x\in V'' \\ \inf[F_y^q(x)+\lambda u_1(x), \lambda v_1(x)] & \text{for } x\in V'-V'' \\ \lambda v_1(x) & \text{for } x\in X-V' \end{cases}$$

is q-superharmonic, $\geqq 0$ on X. Also $u-F_y^q$ is q-superharmonic on V. Hence $u\in\mathcal{U}_y$ and $\mathcal{U}_y$ is non-empty.

Now let $u_0(x)=\inf\{u(x);\ u\in\mathcal{U}_y\}$ ($x\in X$) and $w(x)=\inf\{w_u(x);\ u\in\mathcal{U}_y\}$ ($x\in V$). Obviously $u_0\geqq 0$. We apply Proposition 1.13 to the class $\mathcal{U}_y$ considered on the domain $X-\{y\}$ and we see that u_0 is q-harmonic on $X-\{y\}$. On the other hand, F_y^q is bounded on ∂V_1 for any ball V_1 such that $y \in Y_1$ and $\bar{V}_1 \subset V$. Hence $\{w_u;\ u \in \mathcal{U}_y\}$ is uniformly bounded below on $\bar{V}_1$. Therefore, again by Proposition 1.13, we see that w is q-harmonic on V. Since $u_0=F_y^q+w$ on V, u_0 is q-superharmonic on X. If v is any q-harmonic minorant of u_0, then $u_0-v \in \mathcal{U}_y$. Hence $u_0-v \geqq u_0$ or $v \leqq 0$. Hence u_0 is a q-potential and it follows that $u_0=G_y^q$ (Lemma 2.1).

§2.3. G^q-potentials

In this section, if $q=0$, then we suppose that X is a Green space.

Let μ be a positive Radon measure on X. Then we see that $G_\mu^q(x)=\int G^q(x, y)d\mu(y)$ is either $\equiv +\infty$ or a q-superharmonic function on X (Proposition 1.10, (ii)). If it is not $\equiv +\infty$, then we call it a G^q-potential (of μ). As is the case of classical Green potentials (the case $q=0$), we can show that any q-harmonic minorant of a G^q-potential is non-positive (cf. [3] and [7]), so that a G^q-potential is a q-potential. This fact can be seen also by a general theory by R.- M. Hervé ([13], Chap. III). By the corollary to Proposition 1.15, $\Delta G_\mu^q - qG_\mu^q = -c_d\mu$ as distributions. Thus we have the following Riesz decomposition theorem for q-superharmonic functions (cf. [3]):

THEORFM 2.2. *If u is a non-negative q-superharmonic function, then* $u(x) = G_\mu^q(x)+u_0(x)$, *where* $\mu=-(1/c_d)(\Delta u-qu)$(as distributions) *and u_0 is the greatest q-harmonic minorant of u.*

§2.4. Dependence on q

THEOREM 2.3. *If $q_1 \leqq q_2$ on X, then $G_y^{q_1} \geqq G_y^{q_2}$ for any $y \in X$.* (In case $q_1=0$, we assume that X is a Green space.)

PROOF. Let $u(x)=G_y^{q_1}(x)-G_y^{q_2}(x)$ for $x \neq y$ and let $u(y)=\lim_{r\to 0} A_u^r(y)$. Since

$$\Delta u - q_2 u = (q_1-q_2)G_y^{q_1} \leqq 0$$

in the distribution sense, u is q_2-superharmonic by Theorem 1.2. Since $-u \leqq G_y^{q_2}$ and $G_y^{q_2}$ is a q_2-potential, we have $-u \leqq 0$ or $u \geqq 0$. Hence $G_y^{q_1} \geqq G_y^{q_2}$.

THEOREM 2.4. (Resolvent equation) *If $q_1 \leqq q_2$, then*

$$G_y^{q_1}(x)-G_y^{q_2}(x) = \frac{1}{c_d}\int [q_2(z)-q_1(z)]G_x^{q_1}(z)G_y^{q_2}(z)dz$$

$$= \frac{1}{c_d}\int [q_2(z)-q_1(z)]G_y^{q_1}(z)G_x^{q_2}(z)dz$$

for $x\neq y$. (If $q_1=0$, then we assume that X is a Green space.)

PROOF. In the distribution sense, we have

$$\Delta(G_y^{q_1}-G_y^{q_2})-q_1(G_y^{q_1}-G_y^{q_2})=(q_1-q_2)G_y^{q_2}\leqq 0.$$

Hence there exists a q_1-superharmonic function u such that $u=G_y^{q_1}-G_y^{q_2}$ on $X-\{y\}$ (Theorem 1.2). By the above theorem, we have $u\geqq 0$. Since $u\leqq G_y^{q_1}$, u is a q_1-potential. Hence it follows from Theorem 2.2 that for $x\neq y$

$$G_y^{q_1}(x)-G_y^{q_2}(x)=\frac{1}{c_d}\int[q_2(z)-q_1(z)]G_y^{q_2}(z)G_z^{q_1}(x)dz.$$

Since G^{q_i}, $i=1, 2$, are symmetric, we have the theorem.

COROLLARY 1. *If q_n increases to q, then $G_y^{q_n}$ decreases to G_y^q for each $y\in X$.*

PROOF. By Theorem 2.3, $\{G_y^{q_n}\}$ is a decreasing sequence. By the above theorem, for $x\neq y$

$$0\leqq G_y^{q_n}(x)-G_y^q(x)=\frac{1}{c_d}\int[q(z)-q_n(z)]G_x^{q_n}(z)G_y^q(z)dz.$$

Since $[q(z)-q_n(z)]G_x^{q_n}(z)$ decreases to 0 for all $z\neq x$, we have the assertion.

COROLLARY 2. *For any q_1 and q_2,*

$$|G_y^{q_1}(x)-G_y^{q_2}(x)|\leqq\frac{1}{c_d}\int|q_1(z)-q_2(z)|G_x^{q_1}(z)G_y^{q_2}(z)dz \qquad (x\neq y).$$

PROOF. Let $\hat{q}=\max(q_1, q_2)$. By the above theorems,

$$G_y^{q_1}(x)-G_y^{\hat{q}}(x)=\frac{1}{c_d}\int[\hat{q}(z)-q_1(z)]G_x^{q_1}(z)G_y^{\hat{q}}(z)dz$$
$$\leqq\frac{1}{c_d}\int[\hat{q}(z)-q_1(z)]G_x^{q_1}(z)G_y^{q_2}(z)dz$$

and

$$G_y^{q_2}(x)-G_y^{\hat{q}}(x)=\frac{1}{c_d}\int[\hat{q}(z)-q_2(z)]G_x^{\hat{q}}(z)G_y^{q_2}(z)dz$$
$$\leqq\frac{1}{c_d}\int[\hat{q}(z)-q_2(z)]G_x^{q_1}(z)G_y^{q_2}(z)dz.$$

Hence

$$|G_y^{q_1}(x)-G_y^{q_2}(x)|\leqq[G_y^{q_1}(x)-G_y^{\hat{q}}(x)]+[G_y^{q_2}(x)-G_y^{\hat{q}}(x)]$$
$$\leqq\frac{1}{c_d}\int[2\hat{q}(z)-q_1(z)-q_2(z)]G_x^{q_1}(z)G_y^{q_2}(z)dz$$
$$=\frac{1}{c_d}\int|q_1(z)-q_2(z)|G_x^{q_1}(z)G_y^{q_2}(z)dz.$$

Now we suppose that X is a Green space and consider the class

$$Q_0 \equiv Q_0(X) = \left\{ \begin{array}{ll} q; & \text{locally Hölder continuous}, \geqq 0 \text{ on } X, \\ & \int G_x(z) q(z) dz < \infty \quad \text{for some } x \in X \end{array} \right\}.$$

Since q is continuous, $\int G_x(z)q(z)dz < \infty$ for some $x \in X$ implies the same for all $x \in X$. The class Q_0 contains non-zero functions; in fact any non-negative C^1-function with compact support in X belongs to Q_0. We easily see that if $q_1, q_2 \in Q_0$, then $\lambda_1 q_1 + \lambda_2 q_2$ ($\lambda_1, \lambda_2 \geqq 0$), $\max(q_1, q_2)$, $\min(q_1, q_2)$ and $|q_1 - q_2|$ belong to Q_0.

THEOREM 2.5. *Let $q, q^*, q_n \in Q_0$ ($n = 1, 2, \cdots$). If $q_n \leqq q^*$ for all n and if $q_n(z) \to q(z)$ ($n \to \infty$) for every $z \in X$, then $G_y^{q_n}(x) \to G_y^q(x)$ ($n \to \infty$) for any x, y ($x \neq y$).*

PROOF. Fix x, y ($x \neq y$). Let V be a ball with center at y such that $x \notin \bar{V}$ and let $Q = \sup_{z \in V} q^*(z) + 1$. Given $\varepsilon > 0$, there exists a neighborhood W of y such that $W \subset V$ and

$$(2.2) \qquad \frac{1}{c_d} \int_W G_y(z) G_x(z) dz < \frac{\varepsilon}{2Q}.$$

Let $M = \sup_{z \in X-W} G_y(z)$. Then $0 < M < \infty$. Since $q_n \leqq q^* \in Q_0$ and $q_n(z) \to q(z)$, the Lebesgue convergence theorem implies

$$\frac{1}{c_d} \int |q(z) - q_n(z)| G_x(z) dz \to 0 \qquad (n \to \infty).$$

Hence there exists n_0 such that $n \geqq n_0$ implies

$$(2.3) \qquad \frac{1}{c_d} \int |q(z) - q_n(z)| G_x(z) dz < \frac{\varepsilon}{2M}.$$

By Theorem 2.3 and Corollary 2 to Theorem 2.4, we have

$$|G_y^{q_n}(x) - G_y^q(x)| \leqq \frac{1}{c_d} \int |q(z) - q_n(z)| G_y(z) G_x(z) dz.$$

By (2.2),

$$\frac{1}{c_d} \int_W |q(z) - q_n(z)| G_y(z) G_x(z) dz \leqq \frac{Q}{c_d} \int_W G_y(z) G_x(z) dz < \frac{\varepsilon}{2}.$$

If $n \geqq n_0$, then (2.3) implies

$$\frac{1}{c_d} \int_{X-W} |q(z) - q_n(z)| G_y(z) G_x(z) dz$$
$$\leqq \frac{M}{c_d} \int |q(z) - q_n(z)| G_x(z) dz < \frac{\varepsilon}{2}$$

Hence $|G_y^{q_n}(x)-G_y^q(x)|<\varepsilon$ for $n\geqq n_0$, i.e., $G_y^{q_n}(x)\to G_y^q(x)$ as $n\to\infty$.

COROLLARY. *If $q_1\in Q_0$ and q_n decreases to q, then $G_y^{q_n}$ increases to G_y^q for each $y\in X$.*

REMARK. The condition $q_1\in Q_0$ in the above corollary can be weakened to $\int G_x^q(z)[q_1(z)-q(z)]dz<\infty$ (cf. Corollary 1 to Theorem 2.4).

CHAPTER III Dirichlet Problems.

§3.1. Perron-Brelot's method

Hereafter we shall always assume that X is non-compact. Let $\hat{X}$ be an arbitrary compactification of X, i.e., a compact Hausdorff space such that there exists a homeomorphism τ of X into $\hat{X}$ such that $\tau(X)$ is dense in $\hat{X}$. We identify $\tau(X)$ with X and let $\Gamma=\hat{X}-X$. In this way, we consider an ideal boundary Γ of X and discuss the Dirichlet problem for the equation $\Delta u-qu=0$ with respect to this boundary. We shall apply Perron-Brelot's method.

Let φ be an extended real valued function on Γ. We define

$$\bar{\mathfrak{S}}_\varphi^q=\left\{\begin{array}{ll} v;\ q\text{-superharmonic, bounded below on } X, & \\ \varliminf_{x\to\xi} v(x)\geqq\varphi(\xi) \qquad \text{for all } \xi\in\Gamma & \end{array}\right\}\cup\{\infty\}$$

and

$$\underline{\mathfrak{S}}_\varphi^q=\{-v;\ v\in\bar{\mathfrak{S}}_{-\varphi}^q\},$$

where ∞ means the function which is equal to $+\infty$ everywhere on X. We further define

$$\bar{H}_\varphi^q(x)=\inf\{v(x);\ v\in\bar{\mathfrak{S}}_\varphi^q\} \quad\text{and}\quad \underline{H}_\varphi^q(x)=\sup\{v(x);\ v\in\underline{\mathfrak{S}}_\varphi^q\}$$

for each $x\in X$. Then, by Propositions 1.10, 1.11, 1.12 and 1.13, we have (cf. [1], [8]; also [4] for a general theory)

LEMMA 3.1. (i) $\bar{H}_\varphi^q$ (resp. $\underline{H}_\varphi^q$) *is either $\equiv+\infty$ or $\equiv-\infty$ or q-harmonic on X.*

(ii) $\underline{H}_\varphi^q\leqq\bar{H}_\varphi^q$.

LEMMA 3.2. *If $q_1\leqq q_2$ and $\varphi\geqq 0$, then $\bar{H}_\varphi^{q_1}\geqq\bar{H}_\varphi^{q_2}$ and $\underline{H}_\varphi^{q_1}\geqq\underline{H}_\varphi^{q_2}$.*

PROOF. By Proposition 1.9, $\bar{\mathfrak{S}}_\varphi^{q_1}\subset\bar{\mathfrak{S}}_\varphi^{q_2}$ and $\{v\in\underline{\mathfrak{S}}_\varphi^{q_1};\ v\geqq 0\}\supset\{v\in\underline{\mathfrak{S}}_\varphi^{q_2};\ v\geqq 0\}$. Hence we have the lemma.

LEMMA 3.3. *If $\bar{H}_\varphi^q$ (resp. $\underline{H}_\varphi^q$) is finite, then there exists a non-negative q-superharmonic function v on X such that $\bar{H}_\varphi^q+\varepsilon v\in\bar{\mathfrak{S}}_\varphi^q$ (resp. $\underline{H}_\varphi^q-\varepsilon v\in\underline{\mathfrak{S}}_\varphi^q$) for all $\varepsilon>0$.*

The proof of this lemma is analogous to the case $q=0$ (cf. Hilfssatz 3.1 of [7]). If X is a Green space and $q \in Q_0$, then we have the following stronger form for bounded functions:

LEMMA 3.4. *Suppose X is a Green space and $q \in Q_0$. If φ is bounded, then there exists a non-negative superharmonic function s on X such that $\bar{H}_\varphi^q + \varepsilon s \in \bar{\mathfrak{S}}_\varphi^q$ for all $\varepsilon > 0$.*

PROOF. First, we prove that, for any compact set Z in X and for any $\delta > 0$, there exists $u \in \bar{\mathfrak{S}}_\varphi^q$ such that $u \leqq \bar{H}_\varphi^q + \delta$ on Z. For any $x \in Z$, let V_x be a ball with center at x and let V_x' be the concentric ball of radius one half of that of V_x. By Proposition 1.7, there exists $M_x \geqq 1$ such that $u(y) \leqq M_x u(x)$ for any non-negative q-harmonic function u on V_x and for any $y \in V_x'$. Given $\delta > 0$, there exists $v_x \in \bar{\mathfrak{S}}_\varphi^q$ such that $v_x(x) < \bar{H}_\varphi^q(x) + \delta / M_x$. Let $u_x = (v_x)_{V_x}$ in the notation in Proposition 1.12. Then $u_x \in \bar{\mathfrak{S}}_\varphi^q$ and $u_x(x) \leqq v_x(x) < \bar{H}_\varphi^q(x) + \delta / M_x$. Since $u_x - \bar{H}_\varphi^q$ is non-negative q-harmonic on V_x, $u_x(y) - \bar{H}_\varphi^q(y) \leqq M_x(u_x(x) - \bar{H}_\varphi^q(x)) < \delta$ for any $y \in V_x'$. Since Z is compact, there exist a finite number of points $x_1, \cdots, x_n \in Z$ such that $\bigcup_{i=1}^n V_{x_i}' \supset Z$. Let $u = \min(u_{x_1}, \cdots, u_{x_n})$. Then $u \in \bar{\mathfrak{S}}_\varphi^q$ and $u \leqq \bar{H}_\varphi^q + \delta$ on Z.

Now, fix a point $x_0 \in X$ and let $J = \int G(x_0, y) q(y) dy$. Since J is finite by assumption, there exists a sequence $\{Z_n\}$ of compact sets on X such that $x_0 \in Z_n$ and

$$\int_{X - Z_n} G(x_0, y) q(y) dy < \frac{1}{2^n}$$

for each n. Let $|\varphi| \leqq M$. Then $|\bar{H}_\varphi^q| \leqq M$. By the above result, there exists $u_n \in \bar{\mathfrak{S}}_\varphi^q$ such that $u_n \leqq M$ on X and $u_n \leqq \bar{H}_\varphi^q + 1/2^n$ on Z_n for each n. Let $v_n = u_n - \bar{H}_\varphi^q$. Then v_n is non-negative q-superharmonic on X and $v_n \leqq 2M$. Let

$$p_n(x) = \frac{1}{c_d} \int G(x, y) q(y) v_n(y) dy.$$

Then $\Delta(v_n + p_n) = \Delta v_n - q v_n \leqq 0$ in the distribution sense. Hence, by Theorem 1.2 and Lemma 1.2, $v_n + p_n$ is superharmonic on X. We have

$$\begin{aligned} p_n(x_0) &\leqq \frac{2M}{c_d} \int_{X - Z_n} G(x_0, y) q(y) dy + \frac{1}{c_d 2^n} \int_{Z_n} G(x_0, y) q(y) dy \\ &< \frac{1}{c_d} (2M + J) \frac{1}{2^n}. \end{aligned}$$

Also $v_n(x_0) \leqq 1/2^n$, since $x_0 \in Z_n$. Hence $v_n(x_0) + p_n(x_0) < [(1/c_d)(2M + J) + 1] \times (1/2^n)$. Therefore, $s = \sum_{n=1}^\infty (v_n + p_n)$ defines a superharmonic function on X. Obviously, $s \geqq 0$. For any $\varepsilon > 0$, choose m such that $1/m < \varepsilon$. Then

$$\bar{H}_\varphi^q + \varepsilon s \geqq \bar{H}_\varphi^q + \frac{1}{m} \sum_{n=1}^m (v_n + p_n) \geqq \bar{H}_\varphi^q + \frac{1}{m} \sum_{n=1}^m v_n = \frac{1}{m} \sum_{n=1}^m u_n.$$

Since $(1/m)\sum_{n=1}^{m} u_n \in \bar{\mathfrak{S}}_{\varphi}^{q}$, we conclude that $\bar{H}_{\varphi}^{q}+\varepsilon s \in \bar{\mathfrak{S}}_{\varphi}^{q}$.

§3.2. *q*-resolutive functions

In case $\underline{H}_{\varphi}^{q}=\bar{H}_{\varphi}^{q}$ and it is q-harmonic, we say that φ is *q-resolutive* (with respect to X) and we denote the common function by H_{φ}^{q}. This may be called the q-Dirichlet solution of φ (with respect to $\hat{X}$).

We can easily show the following properties, which are well-known in case $q=0$ (see [7]; also [1] and [4] for a general theory):

PROPOSITION 3.1. (i) *If φ_1, φ_2 are q-resolutive functions on Γ and if λ_1, λ_2 are real numbers, then $\lambda_1\varphi_1+\lambda_2\varphi_2$, $\max(\varphi_1, \varphi_2)$ and $\min(\varphi_1, \varphi_2)$ are q-resolutive and*

$$H_{\lambda_1\varphi_1+\lambda_2\varphi_2}^{q}=\lambda_1 H_{\varphi_1}^{q}+\lambda_2 H_{\varphi_2}^{q},$$

(Here, in $\lambda_1\varphi_1+\lambda_2\varphi_2$, a convention $0\cdot\infty=\infty\cdot 0=-\infty+\infty=+\infty-\infty=0$ is in force.)

$H_{\max(\varphi_1,\varphi_2)}^{q}=$ *the least q-harmonic majorant of* $\max(H_{\varphi_1}^{q}, H_{\varphi_2}^{q})$,

$H_{\min(\varphi_1,\varphi_2)}^{q}=$ *the greatest q-harmonic minorant of* $\min(H_{\varphi_1}^{q}, H_{\varphi_2}^{q})$.

(ii) *If $\varphi\geqq 0$, then $H_{\varphi}^{q}\geqq 0$; if φ is bounded, then H_{φ}^{q} is bounded and $|H_{\varphi}^{q}| \leqq \sup_{\xi\in\Gamma}|\varphi(\xi)|$.*

(iii) *If $\{\varphi_n\}$ is a monotone sequence of q-resolutive functions such that $\{H_{\varphi_n}^{q}(x)\}$ is bounded for some $x\in X$, then $\varphi=\lim_{n\to\infty}\varphi_n$ is q-resolutive and $H_{\varphi}^{q}(x)=\lim_{n\to\infty}H_{\varphi_n}^{q}(x)$ for all $x\in X$.*

PROPOSITION 3.2. *A constant function is always q-resolutive.*

PROOF. Let $\varphi(\xi)\equiv 1$. Since $1\in\mathfrak{S}_{\varphi}^{q}$, $\bar{H}_{\varphi}^{q}\leqq 1$. It follows that $\bar{H}_{\varphi}^{q}$ belongs to $\underline{\mathfrak{S}}_{\varphi}^{q}$. Therefore $\bar{H}_{\varphi}^{q}\leqq\underline{H}_{\varphi}^{q}$, so that $\varphi(\xi)\equiv 1$ is q-resolutive. By (i) of the previous proposition, we conclude that any constant function is q-resolutive.

REMARK. The function H_{1}^{q} does not depend on the compactification, that is, this function is an invariant of the pair (X, q).

LEMMA 3.5. *If φ is a non-negative q-resolutive function and if $\lambda\geqq 0$, then $\min(\varphi, \lambda)$ is q-resolutive and*

$H_{\min(\varphi,\lambda)}^{q}=$ *the greatest q-harmonic minorant of* $\min(H_{\varphi}^{q}, \lambda)$.

PROOF. By Propositions 3.2 and 3.1, (i), $\min(\varphi, \lambda)$ is q-resolutive. Let u be the greatest q-harmonic minorant of $\min(H_{\varphi}^{q}, \lambda)$. Since $\lambda\geqq 0$, we easily see that $u\geqq H_{\min(\varphi,\lambda)}^{q}$. On the other hand, $u\leqq\lambda$ implies $u\leqq H_{\lambda}^{q}$. Hence $u\leqq\min(H_{\varphi}^{q}, H_{\lambda}^{q})$. Therefore, by Proposition 3.1, (i), we have $u\leqq H_{\min(\varphi,\lambda)}^{q}$.

§3.3. *q*-resolutive compactification

We shall say that $\hat{X}$ is a *q-resolutive compactification*, if any $\varphi \in \boldsymbol{C}(\Gamma)$ (=the space of all finite continuous functions on Γ) is q-resolutive. In this case, the mapping $\varphi \in \boldsymbol{C}(\Gamma) \rightarrow H_\varphi^q(x)$ is a non-negative linear functional on $\boldsymbol{C}(\Gamma)$ for each $x \in X$ (Proposition 3.1, (i) and (ii)). Hence there exists a non-negative Radon measure ω_x^q on Γ such that

$$\int \varphi d\omega_x^q = H_\varphi^q(x)$$

for all $\varphi \in \boldsymbol{C}(\Gamma)$. If $q=0$, then $\omega_x \equiv \omega_x^0$ is the ordinary harmonic measure on Γ with respect to x. By Proposition 3.1, (ii), we see that $\omega_x^q(\Gamma) \leqq 1$.

LEMMA 3.6. *Let $\hat{X}$ be a q-resolutive compactification.*
(i) *For any extended real valued function φ on Γ,*

$$\underline{H}_\varphi^q(x) \leqq \underline{\int} \varphi d\omega_x^q \leqq \overline{\int} \varphi d\omega_x^q \leqq \overline{H}_\varphi^q(x)$$

for any $x \in X$.

(ii) *If $\overline{\int} \varphi d\omega_x^q$ (resp. $\underline{\int} \varphi d\omega_x^q$) is finite for some $x \in X$, then the function $x \rightarrow \overline{\int} \varphi d\omega_x^q$ (resp. $\underline{\int} \varphi d\omega_x^q$) is q-harmonic on X.*

We can prove this lemma in a way similar to the proof of Hilfssatz 8.3 of [7] and by using Proposition 1.10, (ii). (Also, cf. [1] and [8].)

COROLLARY. (i) *Any q-resolutive function is ω_x^q-summable for any $x \in X$.*
(ii) *For any $x, x' \in X$, ω_x^q and $\omega_{x'}^q$ are equivalent measures and a function is ω_x^q-summable if and only if it is $\omega_{x'}^q$-summable.*

Thus we shall use the terminology "ω^q-summable" or "ω^q-a.e." instead of "ω_x^q-summable for some $x \in X$" or "ω_x^q-almost everywhere for some x".
The following lemma is a generalization of Satz 8.3 of [7]:

LEMMA 3.7. (i) *Given q_1 and q_2, let $\hat{X}$ be q_1- and q_2-resolutive. If φ is a bounded function on Γ, then there exists a function ψ such that it is q_1- and q_2-resolutive and $\overline{\int} \varphi d\omega_x^{q_i} = H_\psi^{q_i}(x)$ for all $x \in X$ $(i=1, 2)$.*
(ii) *If $\hat{X}$ is a q-resolutive compactification, then any ω^q-summable function is equal to a q-resolutive function ω^q-a.e.*

PROOF. (i) First, let φ be lower semicontinuous. For a fixed point $x_0 \in X$, there exists a sequence $\{\varphi_n\}$ in $\boldsymbol{C}(\Gamma)$ such that $\varphi_n \leqq \varphi$ for each n and

$\int\varphi_n\, d\omega_{x_0}^{q_i}\nearrow\int\varphi\; d\omega_{x_0}^{q_i}$ $(n\to\infty)$ for $i=1, 2$. Let $\psi_n=\max(\varphi_1, \cdots, \varphi_n)$. Then $\lim\psi_n=\psi$ exists and ψ is q_1- and q_2-resolutive by Proposition 3.1, (iii). For each $i=1, 2$, we have $\int\varphi_n\, d\omega_x^{q_i}\leqq\int\psi_n d\omega_x^{q_i}\leqq\int\varphi d\omega_x^{q_i}$ for all $x\in X$, so that $H_\psi^{q_i}(x)\leqq\int\varphi d\omega_x^{q_i}$ for all $x\in X$ and $\int\psi d\omega_{x_0}^{q_i}=\int\varphi d\omega_{x_0}^{q_i}$. Since $\psi\leqq\varphi$, it follows that $\int\varphi d\omega_x^{q_i}=H_\psi^{q_i}(x)$ for all $x\in X$ and $\psi=\varphi$ ω^{q_i}-a.e. $(i=1, 2)$.

Next, suppose φ is any bounded function. Choose a sequence $\{\varphi_n\}$ of bounded lower semi-continuous functions such that $\varphi_n\geqq\varphi$ for each n and $\int\varphi_n\, d\omega_{x_0}^{q_i}\searrow\overline{\int}\varphi d\omega_{x_0}^{q_i}$ $(i=1, 2)$. For each n, there exists a q_1- and q_2-resolutive function ψ_n such that $\int\varphi_n d\omega_x^{q_i}=H_{\psi_n}^{q_i}(x)$ for all $x\in X$ and $\varphi_n=\psi_n$ ω^{q_i}-a.e. $(i=1, 2)$. Let $\tilde{\psi}_n=\min(\psi_1, \cdots, \psi_n)$. Then $\lim\tilde{\psi}_n=\psi$ exists and ψ is q_1- and q_2-resolutive by Proposition 3.1, (iii). As above, we see that $H_\psi^{q_i}(x)\geqq\overline{\int}\varphi d\omega_x^{q_i}$ for all $x\in X$ and $\int\psi d\omega_{x_0}^{q_i}=\overline{\int}\varphi d\omega_{x_0}^{q_i}$ $(i=1, 2)$. Since $\psi\geqq\varphi$ ω^{q_i}-a.e. $(i=1, 2)$, it follows that $\overline{\int}\varphi d\omega_x^{q_i}=H_\psi^{q_i}(x)$ for all $x\in X$ and $\psi=\varphi$ ω^{q_i}-a.e. $(i=1, 2)$.

(ii) If we put $q_1=q_2=q$ in (i), then we see that any bounded ω^q-measurable function φ is equal to a q-resolutive function ψ ω^q-a.e. and $H_\psi^q(x)=\int\varphi d\omega_x^q$. If φ is any ω^q-summable function, then apply the above result to functions $\varphi_n=\max(\min(\varphi, n), -n)$. Then we can easily complete the proof using Proposition 3.1.

§3.4. Comparison of q-resolutivity

Let $\hat{X}$ be an arbitrary compactification and let $\Gamma=\hat{X}-X$.

LEMMA 3.8. *If $q_1\leqq q_2$ and $\varphi\geqq 0$, then the greatest q_2-harmonic minorant of $\underline{H}_\varphi^{q_1}$ is equal to $\underline{H}_\varphi^{q_2}$, provided that $\underline{H}_\varphi^{q_1}$ is finite.*

PROOF. By Lemma 3.2, $\underline{H}_\varphi^{q_1}\geqq\underline{H}_\varphi^{q_2}$. Let u be the greatest q_2-harmonic minorant of $\underline{H}_\varphi^{q_1}$. Then $u\geqq\underline{H}_\varphi^{q_2}$. By Lemma 3.3, there exists a non-negative q_1-superharmonic function v on X such that $\underline{H}_\varphi^{q_1}-\varepsilon v\in\underline{\mathcal{S}}_\varphi^{q_1}$ for all $\varepsilon>0$. By Proposition 1.9, v is q_2-superharmonic. Since $u\leqq\underline{H}_\varphi^{q_1}$, we see that $u-\varepsilon v\in\underline{\mathcal{S}}_\varphi^{q_2}$ for all $\varepsilon>0$. Hence $u-\varepsilon v\leqq\underline{H}_\varphi^{q_2}$ for all $\varepsilon>0$, so that $u\leqq\underline{H}_\varphi^{q_2}$. Therefore, $u=\underline{H}_\varphi^{q_2}$.

LEMMA 3.9. *Let $q_1\leqq q_2$ $(q_2\neq 0)$.*

(i) *If φ is bounded below and $\underline{H}_\varphi^{q_1}$ is finite, then*

$$\underline{H}_\varphi^{q_2}(x)=\underline{H}_\varphi^{q_1}(x)-\frac{1}{c_d}\int G^{q_2}(x, y)[q_2(y)-q_1(y)]\underline{H}_\varphi^{q_1}(y)dy.$$

(ii) *If φ is bounded above and $\bar{H}_{\varphi}^{q_1}$ is finite, then*

$$\bar{H}_{\varphi}^{q_2}(x)=\bar{H}_{\varphi}^{q_1}(x)-\frac{1}{c_d}\int G^{q_2}(x,\ y)[q_2(y)-q_1(y)]\bar{H}_{\varphi}^{q_1}(y)dy.$$

Proof. First, let $\varphi\geqq 0$. By the previous lemma and Theorem 2.2, we have

$$\underline{H}_{\varphi}^{q_1}(x)=\underline{H}_{\varphi}^{q_2}(x)-\frac{1}{c_d}\int G^{q_2}(x,\ y)\{\Delta\underline{H}_{\varphi}^{q_1}(y)-q_2(y)\underline{H}_{\varphi}^{q_1}(y)\}\,dy$$

$$=\underline{H}_{\varphi}^{q_2}(x)+\frac{1}{c_d}\int G^{q_2}(x,\ y)[q_2(y)-q_1(y)]\underline{H}_{\varphi}^{q_1}(y)dy.$$

In particular, this equality holds for $\varphi=M$ (const. >0). By Proposition 3.2, M is q-resolutive and it follows that $\underline{H}_{\varphi+M}^{q}=\underline{H}_{\varphi}^{q}+H_{M}^{q}$ for any q. Therefore, if φ is bounded below, then, by considering $\varphi+M$ for some $M>0$ such that $\varphi+M\geqq 0$, we obtain the required result. (ii) follows immediately from (i) by considering $-\varphi$.

Theorem 3.1. *Let $q_1\leqq q_2$ ($q_2\neq 0$).*

(i) *Any q_1-resolutive function on Γ is q_2-resolutive.*

(ii) *If φ is a q_1-resolutive function, then*

$$H_{\varphi}^{q_2}(x)=H_{\varphi}^{q_1}(x)-\frac{1}{c_d}\int G^{q_2}(x,\ y)[q_2(y)-q_1(y)]H_{\varphi}^{q_1}(y)dy; \tag{3.1}$$

in particular, if φ is non-negative, then the greatest q_2-harmonic minorant of $H_{\varphi}^{q_1}$ is equal to $H_{\varphi}^{q_2}$.

Proof. If φ is a bounded q_1-resolutive function on Γ, then Lemma 3.9 implies that φ is q_2-resolutive and the equality (3.1) holds. If φ is q_1-resolutive and non-negative, then consider $\varphi_n=\min(\varphi,\ n)$. By Lemma 3.5, each φ_n is q_1-resolutive, and hence q_2-resolutive. Since $H_{\varphi_n}^{q_2}\leqq H_{\varphi_n}^{q_1}\leqq H_{\varphi}^{q_1}$ and $\varphi_n\nearrow\varphi$, Proposition 3.1, (iii) implies that φ is q_2-resolutive and the equality (3.1) follows from the corresponding equalities for φ_n. Finally, if φ is an arbitrary q_1-resolutive function, then consider $\varphi^+=\max(\varphi,\ 0)$ and $\varphi^-=-\min(\varphi,\ 0)$. Since $\varphi=\varphi^+-\varphi^-$, the above results and Proposition 3.1, (i) imply the theorem.

Corollary 1. *If $q_1\leqq q_2$, then any q_1-resolutive compactification is q_2-resolutive; in particular, any resolutive compactification is q-resolutive for any q.*

Corollary 2. *If $q_1\leqq q_2$ and if $\hat{X}$ is a q_1-resolutive compactification, then ω^{q_2} is absolutely continuous with respect to ω^{q_1} and any q_1-measurable set (or function) is q_2-measurable.*

COROLLARY 3. *Let $\hat{X}$ be a resolutive compactification of X and let $x_0 \in X$ be fixed. For each $y \in X$, there exists a bounded non-negative ω-measurable function χ_y^q such that $d\omega_y^q = \chi_y^q d\omega_{x_0}$.*

PROOF. The existence of a non-negative function χ_y^q satisfying the above relation follows from the previous corollary and the corollary to Lemma 3.6. Harnack's inequality (Proposition 1.7) implies the boundedness of χ_y^q.

The converse of the above theorem or Corollary 1 is not generally true. For example, we can construct q on a Green space such that $H_1^q = 0$. Then any bounded function on any compactification is q-resolutive. On the other hand, there are non-resolutive compactifications of X (see [7]).

However, we have the following:

LEMMA 3.10. *Let $q_1 \leqq q_2$ and suppose*

$$\int G^{q_1}(x, y)[q_2(y) - q_1(y)] H_1^{q_1}(y) dy < \infty. \tag{3.2}$$

(In case $q_1 = 0$, we assume that X is a Green space.) *Then, for any bounded function φ on Γ,*

$$\underline{H}_\varphi^{q_1}(x) = \underline{H}_\varphi^{q_2}(x) + \frac{1}{c_d} \int G^{q_1}(x, y)[q_2(y) - q_1(y)] \underline{H}_\varphi^{q_2}(y) dy$$

and

$$\bar{H}_\varphi^{q_1}(x) = \bar{H}_\varphi^{q_2}(x) + \frac{1}{c_d} \int G^{q_1}(x, y)[q_2(y) - q_1(y)] \bar{H}_\varphi^{q_2}(y) dy.$$

PROOF. Let $u_i = \underline{H}_\varphi^{q_i}$ (resp. $= \bar{H}_\varphi^{q_i}$), $i = 1, 2$. By Lemma 3.9,

$$u_2(x) = u_1(x) - \frac{1}{c_d} \int G^{q_2}(x, y)[q_2(y) - q_1(y)] u_1(y) dy.$$

Hence

$$\frac{1}{c_d} \int G^{q_1}(x, y)[q_2(y) - q_1(y)] u_2(y) dy$$
$$= \frac{1}{c_d} \int G^{q_1}(x, y)[q_2(y) - q_1(y)] \left\{ u_1(y) - \frac{1}{c_d} \int G^{q_2}(y, z)[q_2(z) - q_1(z)] u_1(z) dz \right\} dy.$$

By condition (3.2), Fubini's theorem and Theorem 2.4, this is equal to

$$\frac{1}{c_d} \int G^{q_1}(x, y)[q_2(y) - q_1(y)] u_1(y) dy$$
$$- \frac{1}{c_d} \int \left\{ \frac{1}{c_d} \int G^{q_1}(x, y) G^{q_2}(y, z)[q_2(y) - q_1(y)] dy \right\} [q_2(z) - q_1(z)] u_1(z) dz$$
$$= \frac{1}{c_d} \int G^{q_1}(x, y)[q_2(y) - q_1(y)] u_1(y) dy$$
$$- \frac{1}{c_d} \int \left\{ G^{q_1}(x, z) - G^{q_2}(x, z) \right\} [q_2(z) - q_1(z)] u_1(z) dz$$
$$= \frac{1}{c_d} \int G^{q_2}(x, y)[q_2(y) - q_1(y)] u_1(y) dy = u_1(x) - u_2(x).$$

Hence we have the lemma.

THEOREM 3.2. *Let $q_1 \leqq q_2$ and suppose that condition* (3.2) *in the above lemma is satisfied; in case $q_1 = 0$, we assume that X is a Green space.*

(i) *Any bounded q_2-resolutive function on Γ is q_1-resolutive; any q_2-resolutive function φ such that $|H_\varphi^{q_2}|$ is dominated by a q_1-harmonic function is q_1-resolutive.*

(ii) *If φ is a q_1-resolutive function on Γ, then*

$$H_\varphi^{q_1}(x) - H_\varphi^{q_2}(x) = \frac{1}{c_d}\int G^{q_1}(x, y)[q_2(y) - q_1(y)]H_\varphi^{q_2}(y)dy,$$

so that if φ is non-negative, then the least q_1-harmonic majorant of $H_\varphi^{q_2}$ is equal to $H_\varphi^{q_1}$.

The proof of this theorem is similar to that of Theorem 3.1, using Lemma 3.10 in place of Lemma 3.9.

COROLLARY 1. *Under the same assumptions as in the above theorem, any q_2-resolutive compactification is q_1-resolutive* (and vice versa).

REMARK. If $q_1 = 0$ and $q_2 = q$, then condition (3.2) is reduced to the condition $q \in Q_0$. Thus we have

COROLLARY 2. *If $q \in Q_0$, then a q-resolutive compactification is resolutive.*

LEMMA 3.11. *Let $q_1 \leqq q_2$ and let $\hat{X}$ be a q_1-resolutive compactification. Then, for any non-negative bounded function φ, the greatest q_2-harmonic minorant of*

$$\bar{h}^{q_1}(x) \equiv \overline{\int}\varphi d\omega_x^{q_1} \text{ (resp. } \underline{h}^{q_1}(x) \equiv \underline{\int}\varphi d\omega_x^{q_1})$$

is equal to

$$\bar{h}^{q_2}(x) \equiv \overline{\int}\varphi\ d\omega_x^{q_2} \text{ (resp. } \underline{h}^{q_2}(x) \equiv \underline{\int}\varphi d\omega_x^{q_2}).$$

If, in addition, condition (3.2) *is satisfied and X is a Green space in case $q_1 = 0$, then the least q_1-harmonic majorant of $\bar{h}^{q_2}$* (resp. $\underline{h}^{q_2}$) *is equal to $\bar{h}^{q_1}$* (resp. $\underline{h}^{q_1}$).

PROOF. This lemma immediately follows from Lemmas 3.7, 3.9 and 3.10.

THEOREM 3.3. *Let $q_1 \leqq q_2$ and suppose that condition* (3.2) *is satisfied; in case $q = 0$, assume that X is a Green space. If $\hat{X}$ is a q_1-resolutive compactification, then $\omega_x^{q_1}$ and $\omega_x^{q_2}$ are equivalent measures for any $x \in X$. In particular, if X is a Green space, $\hat{X}$ is a resolutive compactification and $q \in Q_0$, then ω_x^q is equivalent to ω_x.*

PROOF. Let φ be a non-negative bounded function on Γ. By the above lemma, we see that $\overline{\int}\varphi d\omega_x^{q_1}=0$ if and only if $\overline{\int}\varphi d\omega_x^{q_2}=0$. It follows that $\omega_x^{q_1}$ and $\omega_x^{q_2}$ are equivalent to each other.

From this lemma, it follows that, if $q \in Q_0$, then $\chi_y^q>0$ ω-a.e. on Γ.

§3.5. Dependence of H_φ^q on q

THEOREM 3.4. *Let q_n increase to q as $n\to\infty$.*

(i) *If φ is bounded below and $\underline{H}_\varphi^{q_1}$ is finite, then $\underline{H}_\varphi^{q_n}(x)$ tends to $\underline{H}_\varphi^q(x)$ for each $x \in X$;*

(ii) *If φ is bounded above and $\overline{H}_\varphi^{q_1}$ is finite, then $\overline{H}_\varphi^{q_n}(x)$ tends to $\overline{H}_\varphi^q(x)$ for each $x \in X$;*

(iii) *If φ is q_1-resolutive, then $H_\varphi^{q_n}(x)$ tends to $H_\varphi^q(x)$ for each $x \in X$.*

PROOF. First let φ be non-negative, $u_n=\underline{H}_\varphi^{q_n}$ (resp. $=\overline{H}_\varphi^{q_n}$, $=H_\varphi^{q_n}$) and $u=\underline{H}_\varphi^q$ (resp. $=\overline{H}_\varphi^q$, $=H_\varphi^q$). Then, by Lemma 3.9, (i) (resp. Lemma 3.9, (ii), Theorem 3.1, (ii)), we have

$$u_n(x)-u(x)=\frac{1}{c_d}\int G^q(x, y)[q(y)-q_n(y)]u_n(y)dy.$$

Since $u_1(x)$ is finite, $\{u_n\}$ is monotone decreasing and $q-q_n$ decreases to 0 as $n\to\infty$, we see that u_n tends to u. Now it is easy to show the theorem in case φ is not necessarily non-negative (cf. proofs of Lemma 3.9 and Theorem 3.1).

THEOREM 3.5. *If φ is a bounded function on Γ, then, for any q_1 and q_2,*

$$|\overline{H}_\varphi^{q_1}(x)-\overline{H}_\varphi^{q_2}(x)|\leqq\frac{||\varphi||}{c_d}\int G^q(x, y)|q_1(y)-q_2(y)|dy,$$

where $||\varphi||=\sup_{\xi\in\Gamma}|\varphi(\xi)|$ and $q^*=\max(q_1, q_2)$.

PROOF. By Lemma 3.9 and Proposition 3.1, (ii), we have

$$\begin{aligned}|\overline{H}_\varphi^{q_i}(x)-\overline{H}_\varphi^{q^*}(x)|&=\frac{1}{c_d}\left|\int G^{q^*}(x, y)\{q^*(y)-q_i(y)\}\overline{H}_\varphi^{q_i}(y)dy\right|\\&\leqq\frac{||\varphi||}{c_d}\int G^{q^*}(x, y)\{q^*(y)-q_i(y)\}dy\end{aligned}$$

for $i=1, 2$. Hence

$$\begin{aligned}|\overline{H}_\varphi^{q_1}(x)-\overline{H}_\varphi^{q_2}(x)|&\leqq|\overline{H}_\varphi^{q_1}(x)-\overline{H}_\varphi^{q^*}(x)|+|\overline{H}_\varphi^{q_2}(x)-\overline{H}_\varphi^{q^*}(x)|\\&\leqq\frac{||\varphi||}{c_d}\int G^{q^*}(x, y)\{2q^*(y)-q_1(y)-q_2(y)\}dy\\&=\frac{||\varphi||}{c_d}\int G^{q^*}(x, y)|q_1(y)-q_2(y)|dy.\end{aligned}$$

REMARK. The integrals in the above proof are finite, since

$$\frac{1}{c_d}\int G^{q^*}(x, y)|q_1(y)-q_2(y)|dy \leqq \frac{1}{c_d}\int G^{q^*}(x, y)q^*(y) \leqq 1$$

(cf. the corollary, (ii), to Theorem 2.1). Thus it also follows that $|\bar{H}_\varphi^{q_1}(x)-\bar{H}_\varphi^{q_2}(x)| \leqq \|\varphi\|$.

THEOREM 3.6. *Let* $q_n \to q$ $(n \to \infty)$, $q_n \leqq q^*$ $(n=1, 2, \dots)$ *and suppose* $\int G^q(x, y)q^*(y)dy < \infty$. *Then, for any bounded function* φ *on* Γ,

$$\bar{H}_\varphi^{q_n}(x) \to \bar{H}_\varphi^q(x) \quad \textit{and} \quad \underline{H}_\varphi^{q_n}(x) \to \underline{H}_\varphi^q(x) \qquad (n \to \infty).$$

PROOF. By the previous theorem,

$$|\bar{H}_\varphi^{q_n}(x)-\bar{H}_\varphi^q(x)| \leqq \frac{\|\varphi\|}{c_d}\int G^{\max(q_n, q)}(x, y)|q_n(y)-q(y)|dy$$

$$\leqq \frac{\|\varphi\|}{c_d}\int G^q(x, y)|q_n(y)-q(y)|dy.$$

Since $|q_n(y)-q(y)| \leqq q^*(y)$ and $\int G^q(x, y)q^*(y)dy < \infty$ by assumption, the Lebesgue convergence theorem implies that

$$\int G^q(x, y)|q_n(y)-q(y)|dy \to 0 \qquad (n \to \infty).$$

Hence $\bar{H}_\varphi^{q_n}(x) \to \bar{H}_\varphi^q(x)$ $(n \to \infty)$. By considering $-\varphi$, we also have $\underline{H}_\varphi^{q_n}(x) \to \underline{H}_\varphi^q(x)$ $(n \to \infty)$.

COROLLARY 1. *If* q_n *decreases to* q *and* $\int G^q(x, y)q_1(y)dy < \infty$, *then*

$$\bar{H}_\varphi^{q_n}(x) \to \bar{H}_\varphi^q(x) \quad \textit{and} \quad \underline{H}_\varphi^{q_n}(x) \to \underline{H}_\varphi^q(x) \qquad (n \to \infty)$$

for any bounded function φ *on* Γ.

COROLLARY 2. *If* $q_n \in Q_0$ $(n=1, 2, \dots)$ *and* q_n *decreases to* 0, *then* $H_\varphi^{q_n}(x) \to H_\varphi(x)$ *for any bounded resolutive function* φ.

REMARK. In Theorem 3.4 and in the above two corollaries, the convergence is locally uniform on X by Dini's theorem.

Appendix. We can define a topology in Q_0 by a family of metrics $\{d_Z\}_{Z\text{-compact}}$:

$$d_Z(q_1, q_2) = \sup_{x \in Z}\int G(x, y)|q_1(y)-q_2(y)|dy.$$

By the topology defined by $\{d_Z\}$, Q_0 is a metrizable space. Let $\boldsymbol{C}(X)$ be the space of continuous functions on X with the compact convergence topology and let $\boldsymbol{R}_B(\Gamma)$ be the space of all bounded resolutive functions on Γ with the sup topology. Then Theorem 3.5 implies that the mapping $(\varphi, q) \to H_\varphi^q$ is continuous from $\boldsymbol{R}_B(\Gamma) \times Q_0$ into $\boldsymbol{C}(X)$. (Also, we see that the mapping $q \to G^q(x, y)$ is continuous on Q_0 for any $x \neq y$; cf. Theorems 2.4 and 2.5.)

Let us say that X is a space of *bounded type* if $q(x) \equiv 1$ belongs to Q_0, i.e., if $\int G(x, y)\, dy < \infty$ for some (hence all) $x \in X$. Any bounded domain in R^d is of bounded type. Let $Q_b = \{q \in Q_0;\ q \text{ is bounded}\}$ and let $\|q_1 - q_2\| = \sup_{y \in X} |q_1(y) - q_2(y)|$. Then it is easy to see that if X is of bounded type, then, for each compact set Z, there exists $M_Z > 0$ such that

$$d_Z(q_1, q_2) \leqq M_Z \|q_1 - q_2\| \qquad \text{for all} \quad q_1, q_2 \in Q_b.$$

Thus, if X is of bounded type, then the mapping $(\varphi, q) \to H_\varphi^q$ is continuous from $\boldsymbol{R}_B(\Gamma) \times Q_b$ into $\boldsymbol{C}(X)$, where the topology in Q_b is given by the metric $\|q_1 - q_2\|$.

CHAPTER IV Normal Derivatives.

In what follows, we shall always assume that X is a Green space.

§4.1. The spaces $\mathcal{D}$ and $\widehat{\mathcal{D}}$.

Let $\mathcal{D}$ be the set of all locally summable functions g on X (with respect to dx) such that in each ball V in X, $\partial g/\partial x_i$, $i = 1, \cdots, d$, in the distribution sense are identified with square summable functions (on V) and such that $D[g] \equiv \int_X \sum_{i=1}^d (\partial g/\partial x_i)^2 dx < \infty$. Remark that this integral is determined coordinate-free. $\mathcal{D}$ is obviously a linear space. If $g_1, g_2 \in \mathcal{D}$, then their mutual Dirichlet integral $D[g_1, g_2] \equiv \int_X \{\sum_{i=1}^d (\partial g_1/\partial x_i)(\partial g_2/\partial x_i)\}\, dx$ is defined. We denote by $\| \ \|$ the corresponding norm, i.e., $\|g\|^2 = D[g]$.

The space $\mathcal{D}$ coincides with the space BL(X) in [10] in case X is a domain in R^d. Thus $\partial g/\partial x_i$ in the ordinary sense exists almost everywhere in a ball V and coincides with the one in the distribution sense ([10], [25]). Also, BLD-functions on X (see [5]; Dirichletsche Funktionen in [7]) belong to $\mathcal{D}$.

Let $\boldsymbol{H}_D$ be the set of all harmonic functions on X belonging to $\mathcal{D}$. It is also the space of all BLD-harmonic functions.

Next, we consider the subspace $\widehat{\mathcal{D}}$ of $\mathcal{D}$ consisting of all $g \in \mathcal{D}$ such that Δg in the distribution sense is identified with a (signed) Radon measure on X. It can be seen from Theorem 1.2 and a general theory in [10] that $g \in \mathcal{D}$ belongs to $\widehat{\mathcal{D}}$ if and only if $g = u + p$ a.e. on X, where $u \in \boldsymbol{H}_D$ and p is a difference of two Green potentials belonging to $\mathcal{D}$ (cf. Doob [11], §11).

We shall also consider the class $\widetilde{\mathscr{D}}$ of functions $\tilde{g}$ on X for each of which there exist a compact set Z in X and $g \in \widehat{\mathscr{D}}$ such that $\tilde{g} = g$ on $X - Z$.

§4.2. Definitions of normal derivatives

We consider a *resolutive* compactification $\hat{X}$ of X and define normal derivatives with respect to the *ideal boundary* $\Gamma = \hat{X} - X$, for functions in $\widetilde{\mathscr{D}}$ in general.

For any subset Λ of Γ, let

$$\boldsymbol{C}_D(\Lambda) = \{\varphi \in \boldsymbol{C}(\Gamma);\ H_\varphi \in \boldsymbol{H}_D,\ \varphi = 0 \text{ on } \Gamma - \Lambda\}.$$

$\boldsymbol{C}_D(\Lambda)$ is a linear subspace of $\boldsymbol{C}(\Gamma)$. In case Λ is a Borel set, by a measure on Λ, we shall mean the restriction of a signed Radon measure on Γ to Λ.

DEFINITION 1. Let Λ be a Borel set on Γ and ν be a measure on Λ. Given $g \in \widehat{\mathscr{D}}$, we say that ν is *a normal derivative of* g *on* Λ *in the weak sense*, or g *has a normal derivative* ν *on* Λ *in the weak sense*, if the following condition is satisfied:

For any $\varphi \in \boldsymbol{C}_D(\Lambda)$, $\int |H_\varphi|\, d|\Delta g| < \infty$ and

$$\int H_\varphi d(\Delta g) + D[H_\varphi, g] = -\int \varphi d\nu. \tag{4.1}$$

REMARK. This definition includes the definition given by Constantinescu-Cornea ([7], p. 218) as a special case.

Next, we consider the case where a normal derivative of $g \in \widehat{\mathscr{D}}$ on Λ in the weak sense is absolutely continuous with respect to the harmonic measure ω_x. In this case we fix $x_0 \in X$ and find an ω-measurable function γ on Λ such that $d\nu = \gamma d\omega_{x_0}$ (γ is determined only ω-a.e. for each ν). We shall again call γ a normal derivative of g on Λ in the weak sense (and with respect to x_0).

If we define normal derivatives as functions, then it is possible to make the following somewhat stronger definition, which is more suitable for our purpose: Let Λ be an ω-measurable subset of Γ and let

$$\boldsymbol{R}_{BD}(\Lambda) = \{\varphi;\ \text{bounded resolutive},\ H_\varphi \in \boldsymbol{H}_D,\ \varphi = 0\ \omega\text{-a.e. on } \Gamma - \Lambda\}.$$

DEFINITION 2. (Cf. definitions in [11] and [20].) Let γ be an ω-measurable function on an ω-measurable set $\Lambda \subset \Gamma$. We say that γ is *a normal derivative of* $g \in \widehat{\mathscr{D}}$ *on* Λ, or g *has a normal derivative* γ *on* Λ, if the following condition is satisfied:

For any $\varphi \in \boldsymbol{R}_{BD}(\Lambda)$, $\int |H_\varphi|\, d|\Delta g| < \infty$, $\int_\Lambda |\gamma\varphi|\, d\omega_{x_0} < \infty$ and

$$\int H_\varphi d(\Delta g) + D[H_\varphi, g] = -\int_\Lambda \gamma\varphi d\omega_{x_0}. \tag{4.1$'$}$$

Since $\boldsymbol{R}_{BD}(\Lambda)\subset \boldsymbol{C}_D(\Lambda)$, any normal derivative is that in the weak sense.

LEMMA 4.1. *Let $g_1, g_2 \in \widehat{\mathcal{D}}$ and $g_1=g_2$ a.e. outside a compact set in X. If $\varphi \in \boldsymbol{R}_{BD}(\Gamma)$ and if $\int |H_\varphi|\, d|\Delta g_1|<\infty$, then $\int |H_\varphi|\, d|\Delta g_2|<\infty$ and*

$$\int H_\varphi d(\Delta g_1)+D[H_\varphi, g_1]=\int H_\varphi d(\Delta g_2)+D[H_\varphi, g_2].$$

PROOF. Let f be a C^∞-function with compact support in X such that $f\equiv 1$ on a compact set outside of which $g_1=g_2$ a.e. Since $\int |fH_\varphi|\, d|\Delta g_i|<\infty$ for $i=1, 2$, $\int |H_\varphi|\, d|\Delta g_1|<\infty$ implies $\int |H_\varphi|\, d|\Delta g_2|<\infty$. Since fH_φ is a C^∞-function with compact support, we have

$$\begin{aligned}&\int H_\varphi d[\Delta(g_1-g_2)]+D[H_\varphi, g_1-g_2]\\ &\quad=\int fH_\varphi d[\Delta(g_1-g_2)]+D[fH_\varphi, g_1-g_2]=0.\end{aligned}$$

By this lemma, we can extend the definition of normal derivatives to functions in $\widetilde{\mathcal{D}}$: For $\tilde{g}\in\widetilde{\mathcal{D}}$, let $g\in\widehat{\mathcal{D}}$ be equal to $\tilde{g}$ outside a compact set in X. If g has a normal derivative γ (resp. in the weak sense) on Λ, then we define γ as a normal derivative of $\tilde{g}$ on Λ (resp. in the weak sense).

The following properties are immediate consequences of the definitions:

(a) Let $\Lambda_1\subset\Lambda_2$ and if γ is a normal derivative of $g\in\widetilde{\mathcal{D}}$ on Λ_2 (resp. in the weak sense), then the restriction of γ to Λ_1 is a normal derivative of g on Λ_1 (resp. in the weak sense).

(b) If γ_i is a normal derivative of $g_i\in\widetilde{\mathcal{D}}$ on Λ (resp. in the weak sense) for each $i=1, 2$, and if λ_i, $i=1, 2$, are real numbers, then $\lambda_1\gamma_1+\lambda_2\gamma_2$ is a normal derivative of $\lambda_1 g_1+\lambda_2 g_2$ on Λ (resp. in the weak sense). (Cf. the convention considered in Proposition 3.1, (i).)

(c) If γ is a normal derivative of $g\in\widetilde{\mathcal{D}}$ on Λ (resp. in the weak sense) and if $\gamma_1=\gamma$ ω-a.e. on Λ, then γ_1 is also a normal derivative of g on Λ (resp. in the weak sense).

EXAMPLE 1. Let X be a bounded domain in R^d such that its relative boundary Γ in R^d consists of a finite number of smooth closed hypersurfaces. Let dS be the surface element on Γ. If g is a C^2-function on X such that g and its first order partial derivatives are continuous on $\bar{X}=X\cup\Gamma$, then Green's formula implies that the measure $(\partial g/\partial n)dS$ is a normal derivative of g on Γ in the weak sense, where $\partial g/\partial n$ is the ordinary inner normal derivative on Γ. Since the harmonic measure is expressed as $d\omega_{x_0}=(\partial G_{x_0}/\partial n)dS$, we see that $(\partial g/\partial n)/(\partial G_{x_0}/\partial n)$ is a function valued normal derivative of g on Γ in the weak sense (cf. [11]). We can show that this is also a normal derivative of g on Γ in the sense given in Definition 2.

REMARK. In general, it can be seen that a normal derivative on Γ in the weak sense is also a normal derivative on Γ in the sense of Definition 2 if Γ is the Kuramochi boundary.

EXAMPLE 2. For any Green space X and any resolutive compactification $\hat{X}$, the Green function G_y has a normal derivative $c_d\chi_y$ on Γ (see [20], Proposition 1). More generally, we shall show (Proposition 4.4) that G_y^q has a normal derivative $c_d\chi_y^q$ on Γ for each $y \in X$. (See Corollary 3 to Theorem 3.1 for the function χ_y^q).

§4.3. Royden decomposition

We now consider the set $\boldsymbol{D}$ of all BLD-functions on X. It is a subspace of $\mathcal{D}$, so that $D[g_1, g_2]$ and $D[g]$ make sense for $g_1, g_2, g \in \boldsymbol{D}$. If $g \in \boldsymbol{D}$ and $\|g\|=0$, then $g\equiv$const. q.p., where "q.p." means "quasi-partout" or "except for a polar set" (see [3]). We know ([3], [10], [7])

PROPOSITION 4.1. *The quotient space of* $\boldsymbol{D}$ (resp. $\boldsymbol{H}_D$) *with respect to the equivalence relation* $\|g_1-g_2\|=0$ *is a Hilbert space with respect to the inner product* $D[g_1, g_2]$.

Since every $g \in \boldsymbol{D}$ is Lebesgue measurable, the integral $\int q(x)g^2(x)dx$ makes sense. We consider the spaces

$$\boldsymbol{D}^q=\{g \in \boldsymbol{D};\ \int q(x)g^2(x)dx<\infty\},$$

$$\boldsymbol{H}_D^q=\{u \in \boldsymbol{D}^q;\ u \text{ is } q\text{-harmonic on } X\}.$$

For $g_1, g_2, g \in \boldsymbol{D}^q$, let

$$D^q[g_1, g_2]=D[g_1, g_2]+\int q(x)g_1(x)g_2(x)dx,$$

$$D^q[g]=D^q[g, g] \quad \text{and} \quad \|g\|_q=\sqrt{D^q[g]}.$$

Obviously, if $q_1 \leqq q_2$, then $\boldsymbol{D}^{q_2} \subset \boldsymbol{D}^{q_1}$ and $D^{q_1}[g] \leqq D^{q_2}[g]$ for any $g \in \boldsymbol{D}^{q_2}$.

LEMMA 4.2. *If* $q \neq 0$, *then* $\|\ \|_q$ *is a norm in* $\boldsymbol{D}^q$ *and* $\boldsymbol{D}^q$ *is a Hilbert space with respect to the inner product* $D^q[g_1, g_2]$, *provided that we identify two functions which are equal q.p. on* X.

PROOF. Obviously, $\|\ \|_q$ is a semi-norm on $\boldsymbol{D}^q$. If $\|g\|_q=0$, then $\|g\|=0$ and $\int qg^2dx=0$. It follows that $g=0$ q.p. on X, since $q \neq 0$. Thus $\|\ \|_q$ is a norm on $\boldsymbol{D}^q$. $D^q[g_1, g_2]$ is the corresponding inner product. Hence it is enough to show that $\boldsymbol{D}^q$ is complete with respect to the norm $\|\ \|_q$. Suppose $\{g_n\}$ is a Cauchy sequence in $\boldsymbol{D}^q$. Since $\int q(g_m-g_n)^2dx \to 0$ $(n, m \to \infty)$, there exists a subsequence $\{g_{n_k}\}$ such that $\{g_{n_k}(x)\}$ is convergent almost everywhere on the set $\{x \in X; q(x)>0\}$. Since $\boldsymbol{D}$ is complete and $\|g_n - g_m\| \to 0$ $(n, m \to \infty)$, there exists $g \in \boldsymbol{D}$ such that $\|g_n - g\| \to 0$ $(n \to \infty)$. By choosing a subsequence again if necessary, we may assume that $\{g_{n_k}\}$ converges to g q.p. on X (cf. [5], [7] or [19]). Then we easily see that $g \in \boldsymbol{D}^q$ and $\|g_{n_k} - g\|_q \to 0$ $(k \to \infty)$, which implies $\|g_n - g\|_q \to 0$ $(n \to \infty)$. Hence $\boldsymbol{D}^q$ is complete.

Let $\boldsymbol{C}_0^\infty$ be the set of all infinitely differentiable functions with compact support on X. Obviously, $\boldsymbol{C}_0^\infty \subset \boldsymbol{D}^q$. In case $q=0$, $\boldsymbol{D}_0$ is the set of $g \in \boldsymbol{D}$ for which there exists a sequence $\{f_n\}$ of functions in $\boldsymbol{C}_0^\infty$ such that $\|f_n - g\| \to 0$ and $f_n \to g$ q.p. $(n \to \infty)$. If $q \neq 0$, then let $\boldsymbol{D}_0^q$ be the closure of $\boldsymbol{C}_0^\infty$ in $\boldsymbol{D}^q$ with respect to the norm $\| \ \|_q$. Thus if $g_0 \in \boldsymbol{D}_0^q$, then there exist $f_n \in \boldsymbol{C}_0^\infty$, $n=1, 2, \cdots$, such that $\|f_n - g_0\|_q \to 0$. An argument similar to the proof of the above lemma implies that there exists a subsequence $\{f_{n_k}\}$ which converges to g_0, q.p. on X. Therefore we have

LEMMA 4.3. $\boldsymbol{D}_0^q \subset \boldsymbol{D}_0$.

It is known ([5], [7]) that

PROPOSITION 4.2. *$\boldsymbol{H}_D$ and $\boldsymbol{D}_0$ are orthogonal to each other, i.e.,* $D[h, f]=0$ *for any $h \in \boldsymbol{H}_D$ and $f \in \boldsymbol{D}_0$. Any function $g \in \boldsymbol{D}$ is uniquely decomposed into $g=h+f$ with $h \in \boldsymbol{H}_D$ and $f \in \boldsymbol{D}_0$* (Royden decomposition).

Similarly we have (cf. [24], Theorem 3 for a special case)

LEMMA 4.4. *For any $q \neq 0$, $\boldsymbol{H}_D^q$ is complete with respect to the norm $\| \ \|_q$ and is orthogonal to the space $\boldsymbol{D}_0^q$,* i.e., $D^q[u, f]=0$ *for any $u \in \boldsymbol{H}_D^q$ and $f \in \boldsymbol{D}_0^q$. Any function $g \in \boldsymbol{D}^q$ is uniquely decomposed into $g=u+g_0$ with $u \in \boldsymbol{H}_D^q$ and $g_0 \in \boldsymbol{D}_0^q$.*

PROOF. Let $\tilde{\boldsymbol{H}}_D^q=\{g \in \boldsymbol{D}^q;\ g=u$ q.p. for some $u \in \boldsymbol{H}_D^q\}$. It is enough to show that $\tilde{\boldsymbol{H}}_D^q$ is the orthogonal complement of $\boldsymbol{C}_0^\infty$ in $\boldsymbol{D}^q$. If $f \in \boldsymbol{C}_0^\infty$ and $u \in \tilde{\boldsymbol{H}}_D^q$, then Green's formula implies

$$0=\int (\Delta u) f\, dx + D[u, f] = \int q u f\, dx + D[u, f] = D^q[u, f].$$

Conversely, suppose $g \in \boldsymbol{D}^q$ satisfies $D^q[g, f]=0$ for all $f \in \boldsymbol{C}_0^\infty$. Since $D[g, f] = -\int f\, d(\Delta g)$ in the distribution sense, it follows that $\int f\, d(\Delta g - q g)=0$ for all $f \in \boldsymbol{C}_0^\infty$, or $\Delta g - q g=0$ in the distribution sense. Therefore, by Proposition 1.1, g is equal to a q-harmonic function almost everywhere (hence q.p.) on X. Hence $g \in \tilde{\boldsymbol{H}}_D^q$.

§4.4. Properties of functions in $\boldsymbol{D}^q$

Let us recall that for any $g_0 \in \boldsymbol{D}_0$ there exists a potential p such that

$|g_0|\leqq p$ (see [7], Hilfssatz 7.7 and [19], Lemma 4). By Proposition 1.14, p is also a q-potential. Thus we have

LEMMA 4.5. *For any $g_0 \in \boldsymbol{D}_0^q$, there exists a q-potential p such that $|g_0|\leqq p$ on X.*

Using this lemma, we have

LEMMA 4.6. *If v is a q-superharmonic function on X and $v \in \boldsymbol{D}^q$, then v has a q-harmonic minorant. Furthermore, if $v=u+g$ is the decomposition into a q-harmonic function u and a q-potential g, then $u \in \boldsymbol{H}_D^q$ and $g \in \boldsymbol{D}_0^q$, so that $D^q[u]\leqq D^q[v]$ and $D^q[g]\leqq D^q[v]$. In particular, any q-potential which belongs to $\boldsymbol{D}^q$ belongs to $\boldsymbol{D}_0^q$.*

PROOF. Let $v=u_1+g_1$ be the decomposition into $u_1 \in \boldsymbol{H}_D^q$ and $g_1 \in \boldsymbol{D}_0^q$ (Proposition 4.2 and Lemma 4.4). By the previous lemma, there exists a q-potential p such that $|g_1|\leqq p$. Since v is q-superharmonic, so is g_1. Hence it follows that g_1 is a q-potential. Hence $u_1=u$ and $g_1=g$ and the lemma is proved.

LEMMA 4.7. *Let $u_1, u_2 \in \boldsymbol{H}_D^q$. Then the least q-harmonic majorant v of* $\max(u_1, u_2)$ *and the greatest q-harmonic minorant w of* $\min(u_1, u_2)$ *both belong to $\boldsymbol{H}_D^q$ and*

$$D^q[v]+D^q[w]\leqq D^q[u_1]+D^q[u_2].$$

PROOF. It is easy to see that $\max(u_1, u_2)$, $\min(u_1, u_2) \in \boldsymbol{D}^q$ and $D^q[\max(u_1, u_2)]+D^q[\min(u_1, u_2)]=D^q[u_1]+D^q[u_2]$ (cf. Satz 7.3 in [7]). Since $-\max(u_1, u_2)$ and $\min(u_1, u_2)$ are q-superharmonic, v and w belong to $\boldsymbol{H}_D^q$ by the previous lemma and $D^q[v]\leqq D^q[\max(u_1, u_2)]$ and $D^q[w]\leqq D^q[\min(u_1, u_2)]$.

LEMMA 4.8. (i) *If $u_n \in \boldsymbol{H}_D$, $n=1, 2, \cdots$, $D[u_n]\to 0$ and $u_n(x_0)\to 0$* (for a fixed $x_0 \in X$), *then $u_n\to 0$ locally uniformly on X.*

(ii) *If $q\neq 0$, $u_n \in \boldsymbol{H}_D^q$, $n=1, 2, \cdots$, and $D^q[u_n]\to 0$, then $u_n\to 0$ locally uniformly on X.*

PROOF. (i) is well-known (see [3], p. 11 and [5], Lemma 2 and n° 21).

(ii) Let $\{u_{n_j}\}$ be any subsequence of $\{u_n\}$. Since $D^q[u_{n_j}]\to 0$, $\int q(u_{n_j})^2 dx \to 0$ $(j\to\infty)$. Hence there exists a subsequence $\{n_j'\}$ of $\{n_j\}$ such that $u_{n_j'}\to 0$ $(j\to\infty)$ almost everywhere on $\{x \in X;\ q(x)>0\}$. Each $-|u_{n_j'}|$ is q-superharmonic, so that superharmonic (Proposition 1.9). Let $|u_{n_j'}|=h_j-p_j$, where h_j is harmonic and p_j is a potential, for each j. By the previous lemma, $D[h_j]$, $D[p_j]\leqq D[|u_{n_j'}|]\leqq D^q[u_{n_j'}]\to 0$ $(j\to\infty)$. Since $p_j \in \boldsymbol{D}_0$, there exists a subsequence $\{p_{j_k}\}$ such that $p_{j_k}\to 0$ q.p. (see [7], Hilfssatz 7.8). Then $h_{j_k}\to 0$ almost everywhere on $\{x \in X;\ q(x)>0\}$. Thus (i) implies that $h_{j_k}\to 0$ locally uniformly on X. Since $|u_{n_{j_k}}| \leqq h_{j_k}$, $u_{n_{j_k}} \to 0$ locally uniformly on X. Thus we have seen that any subsequence of $\{u_n\}$ contains another subsequence which converges to 0 locally uniformly. It then follows that $\{u_n\}$ itself converges to 0 locally uniformly on X.

§4.5. The space $\boldsymbol{R}_D^q(\Gamma)$

Given a resolutive compactification $\hat{X}$ and an ω-measurable subset Λ of $\Gamma = \hat{X} - X$, let

$$\boldsymbol{R}_D^q(\Lambda) = \left\{ \varphi\,; \begin{array}{l} q\text{-resolutive function on } \Gamma \text{ such that} \\ H_\varphi^q \in \boldsymbol{H}_D^q \text{ and } \varphi = 0\ \omega^q\text{-a.e. on } \Gamma - \Lambda \end{array} \right\}.$$

We shall study the properties of the space $\boldsymbol{R}_D^q(\Gamma)$ in this section.

LEMMA 4.9. *If* $\varphi_1, \varphi_2 \in \boldsymbol{R}_D^q(\Gamma)$, *then* $\max(\varphi_1, \varphi_2)$ *and* $\min(\varphi_1, \varphi_2)$ *both belong to* $\boldsymbol{R}_D^q(\Gamma)$ *and*

$$D^q[H^q_{\max(\varphi_1, \varphi_2)}] + D^q[H^q_{\min(\varphi_1, \varphi_2)}] \leqq D^q[H^q_{\varphi_1}] + D^q[H^q_{\varphi_2}].$$

PROOF. This is an immediate consequence of Proposition 3.1, (i) and Lemma 4.7.

By this lemma, we see that if $\varphi \in \boldsymbol{R}_D^q(\Gamma)$, then $|\varphi| \in \boldsymbol{R}_D^q(\Gamma)$ and that $D^q[H^q_{|\varphi|}] \leqq D^q[H^q_\varphi]$.

LEMMA 4.10. *If* $\varphi \in \boldsymbol{R}_D^q(\Gamma)$ *and* $\varphi \geqq 0$, *then* $\min(\varphi, \lambda) \in \boldsymbol{R}_D^q(\Gamma)$ *for any non-negative constant* λ *and* $D^q[H^q_{\min(\varphi,\lambda)}] \leqq D^q[H^q_\varphi]$.

PROOF. By Lemma 3.3, $\min(\varphi, \lambda)$ is q-resolutive. Since $H^q_\varphi \in \boldsymbol{D}$, $v = \min(H^q_\varphi, \lambda) \in \boldsymbol{D}$ and $D[v] \leqq D[H^q_\varphi]$. Since $0 \leqq v \leqq H^q_\varphi$, $\int qv^2\,dx \leqq \int q(H^q_\varphi)^2 dx < \infty$. Therefore, $v \in \boldsymbol{D}^q$ and $D^q[v] \leqq D^q[H^q_\varphi]$. Now, Lemma 4.6, together with Lemma 3.3, implies that $H^q_{\min(\varphi,\lambda)} \in H^q_D$ and $D^q[H^q_{\min(\varphi,\lambda)}] \leqq D^q[H^q_\varphi]$.

LEMMA 4.11. *Let* $\varphi \in \boldsymbol{R}_D^q(\Gamma)$ *and* $\varphi_n = \max(\min(\varphi, n), -n)$, $n = 1, 2, \ldots$. *Then* $D^q[H^q_{\varphi_n} - H^q_\varphi] \to 0$ $(n \to \infty)$.

PROOF. By virtue of Lemma 4.9, it is enough to show the case $\varphi \geqq 0$. Then $\varphi_n = \min(\varphi, n)$. Let $v_n = H^q_\varphi - \min(H^q_\varphi, n)$. As in the proof of the previous lemma, we see that $D^q[H^q_\varphi - H^q_{\varphi_n}] \leqq D^q[v_n]$. On the other hand, since $H^q_\varphi \in \boldsymbol{H}^q_D$, we have $D[v_n] \to 0$ and $\int q(v_n)^2 dx \to 0$, i.e., $D^q[v_n] \to 0$ $(n \to \infty)$. Hence we have the lemma.

For different q's, we have

PROPOSITION 4.3. *Let* $q_1 \leqq q_2$ *and suppose* (3.2) *is satisfied. Then* $\boldsymbol{R}_D^{q_2}(\Gamma) \subset$

$\boldsymbol{R}_D^{q_1}(\Gamma)$ and $D^{q_1}[H_\varphi^{q_1}] \leqq D^{q_2}[H_\varphi^{q_2}]$ for all $\varphi \in \boldsymbol{R}_D^{q_2}(\Gamma)$. In particular, if $q \in Q_0$, then $\boldsymbol{R}_D^q(\Gamma) \subset \boldsymbol{R}_D(\Gamma)$ and $D[H_\varphi] \leqq D^q[H_\varphi^q]$ for all $\varphi \in \boldsymbol{R}_D^q(\Gamma)$.

PROOF. If $\varphi \in \boldsymbol{R}_D^{q_2}(\Gamma)$ is bounded, then Theorem 3.2 implies that φ is q_1-resolutive and $H_\varphi^{q_1}$ is the least q_1-harmonic minorant of $H_\varphi^{q_2}$. Hence, using Lemma 4.6, we see that $\varphi \in \boldsymbol{R}_D^{q_1}(\Gamma)$ and $D^{q_1}[H_\varphi^{q_1}] \leqq D^{q_1}[H_\varphi^{q_2}] \leqq D^{q_2}[H_\varphi^{q_2}]$. If $\varphi \in \boldsymbol{R}_D^{q_2}(\Gamma)$ is not bounded, then we consider $\varphi_n = \max(\min(\varphi, n), -n)$. Then $\varphi_n \in \boldsymbol{R}_D^{q_2}(\Gamma)$ and $D^{q_2}[H_{\varphi_n}^{q_2} - H_{\varphi_m}^{q_2}] \to 0$ $(n, m \to \infty)$ by the previous lemma. Thus the above result for bounded φ implies $D^{q_1}[H_{\varphi_n}^{q_1} - H_{\varphi_m}^{q_1}] \leqq D^{q_2}[H_{\varphi_n}^{q_2} - H_{\varphi_m}^{q_2}] \to 0$ $(n, m \to \infty)$. By Lemmas 4.4 and 4.8, $H_{\varphi_n}^{q_1}$ tends to $u \in \boldsymbol{H}_D^{q_1}$ locally uniformly on X and also in the norm $\| \ \|_{q_1}$. It follows from Proposition 3.1, (iii) that $u = H_\varphi^{q_1}$. Hence $\varphi \in \boldsymbol{R}_D^{q_1}(\Gamma)$ and Lemma 4.11 and the above results imply

$$D^{q_1}[H_\varphi^{q_1}] = \lim_{n\to\infty} D^{q_1}[H_{\varphi_n}^{q_1}] \leqq \lim_{n\to\infty} D^{q_2}[H_{\varphi_n}^{q_2}] = D^{q_2}[H_\varphi^{q_2}].$$

§4.6. Normal derivatives of q-Green functions

First, we show

LEMMA 4.12. *For a sufficiently large λ_0, the set $V_\lambda = \{x \in X; G_y^q(x) \geqq \lambda\}$ is compact in X for all $\lambda \geqq \lambda_0$ and $g_\lambda = \min(G_y^q, \lambda)$ belongs to $\boldsymbol{D}_0^q$ ($y \in X$: fixed). Furthermore, $\mu_\lambda = -(\Delta g_\lambda - q g_\lambda)$ is a measure supported by V_λ and vaguely converges to $c_d \delta_y$ as $\lambda \to \infty$.*

PROOF. Since $G_y^q \leqq G_y$ and $\{x \in X; G_y(x) \geqq \lambda\}$ is compact for sufficiently large λ, V_λ is compact for such λ. Also, as in the case of $q=0$, we can show that $g_\lambda \in \boldsymbol{D}^q$. (For example, we may use the method in the proof of Hilfssatz 7.5 and Satz 7.2 in [7].) Since g_λ is a q-potential, it belongs to $\boldsymbol{D}_0^q$ by Lemma 4.6.

For any $f \in \boldsymbol{C}_0^\infty$, $\int f d\mu_\lambda = -\int (\Delta f - qf) g_\lambda dx \to -\int (\Delta f - qf) G_y^q dx = c_d f(y)$ as $\lambda \to \infty$, since $\Delta G_y^q - q G_y^q = -c_d \delta_y$. It follows that $\mu_\lambda \to c_d \delta_y$ vaguely as $\lambda \to \infty$.

By this lemma, we see that $G_y^q \in \widetilde{\mathscr{D}}$ for any $y \in X$. Now we have

PROPOSITION 4.4. *For any resolutive compactification $\hat{X}$, G_y^q has a normal derivative $c_d \chi_y^q$ on $\Gamma = \hat{X} - X$ for each $y \in X$.*

PROOF. For any $\varphi \in \boldsymbol{R}_{BD}(\Gamma)$, $D[H_\varphi, g_\lambda] = 0$ by the above lemma, together with Lemma 4.3 and Proposition 4.2, and $\int H_\varphi d(\Delta g_\lambda) = -\int H_\varphi d\mu_\lambda + \int q H_\varphi g_\lambda dx$. As $\lambda \to \infty$, $\int H_\varphi d\mu_\lambda \to -c_d H_\varphi(y)$ by the above lemma. Also we have $\int q H_\varphi g_\lambda dx \to \int q H_\varphi G_y^q dx$. Hence, using Theorem 3.1, we have

$$\int H_\varphi d(\Delta g_\lambda)+D[H_\varphi, g_\lambda]\to -c_d H_\varphi(y)+\int qH_\varphi G_y^q dx = -c_d H_\varphi^q(y) \qquad (\lambda\to\infty).$$

Since the left hand side is independent of λ by Lemma 4.1, we have

$$\int H_\varphi d(\Delta g_\lambda)+D[H_\varphi, g_\lambda] = -c_d H_\varphi^q(y) = -c_d\int \varphi d\omega_y^q$$
$$= -c_d\int \varphi \chi_y^q d\omega_{x_0},$$

so that $c_d\chi_y^q$ is a normal derivative of G_y^q.

§4.7. Uniqueness of normal derivatives

In [20], a resolutive compactification $\hat{X}$ was called regular if $\boldsymbol{C}_D(\Gamma)$ is dense in $\boldsymbol{C}(\Gamma)$ with respect to the uniform convergence topology. For examples of regular compactifications, see [20].

REMARK. By Stone's theorem, we see that $\hat{X}$ is regular if and only if $\boldsymbol{C}_D(\Gamma)$ separates points of Γ.

LEMMA 4.13. *If $\hat{X}$ is a regular compactification, then $\boldsymbol{C}_D(\Lambda)$ is dense in $\boldsymbol{C}_0(\Lambda)$, where $\boldsymbol{C}_0(\Lambda)=\{\varphi\in \boldsymbol{C}(\Gamma);\ \varphi=0$ on $\Gamma-\Lambda\}$ with the uniform convergence topology.*

PROOF. Given $\varphi\in \boldsymbol{C}_0(\Lambda)$, $\varphi\geqq 0$ and $\varepsilon>0$, there exists $\varphi_1\in \boldsymbol{C}_D(\Gamma)$ such that $|\varphi-\varphi_1|<\varepsilon/2$ on Γ, since $\hat{X}$ is regular. Let $\varphi^*=\max(\varphi_1-\varepsilon/2, 0)$. Then $\varphi^*\in \boldsymbol{C}_D(\Gamma)$ by Lemma 4.9 and $\varphi^*=0$ on $\Gamma-\Lambda$, i.e., $\varphi^*\in \boldsymbol{C}_D(\Lambda)$. It is easy to see that $|\varphi-\varphi^*|<\varepsilon$ on Γ. If $\varphi\in \boldsymbol{C}_0(\Lambda)$ is not necessarily non-negative, then consider φ^+ and φ^- and we find $\varphi^*\in \boldsymbol{C}_D(\Lambda)$ such that $|\varphi-\varphi^*|<\varepsilon$.

PROPOSITION 4.5. *If $\hat{X}$ is a regular compactification and Λ is a relatively open subset of Γ, then a normal derivative of a given function $g\in\widetilde{\mathfrak{D}}$ on Λ* (in the weak sense or not) *is uniquely determined as a measure and ω-a.e. as a function.*

PROOF. Let ν_1 and ν_2 be two (measure valued) normal derivatives of g on Λ in the weak sense. Then, for any $\varphi\in \boldsymbol{C}_D(\Lambda)$, we have $\int\varphi d\nu_1=\int\varphi d\nu_2$. From the above lemma, it follows that this equality holds for any $\varphi\in \boldsymbol{C}_0(\Lambda)$. Since Λ is open in Γ, it follows that $\nu_1=\nu_2$ on Λ. The rest of the proposition now easily follows.

§4.8. Normal derivatives of $\boldsymbol{H}_D^q$-functions

Now we shall study normal derivatives of functions in $\boldsymbol{H}_D^q$. If $u\in \boldsymbol{H}_D^q$, then $\Delta u=qu$ and $u\in\widehat{\mathfrak{D}}$. Thus the left hand side of (4.1)′ in Definition 2 for

$g=u$ is reduced to $D^q[H_\varphi, u]$, provided that $H_\varphi \in \boldsymbol{D}^q$. Let

$$\boldsymbol{R}^q_{BD}(\Lambda) = \{\varphi \in \boldsymbol{R}^q_D(\Lambda);\ \varphi \text{ is bounded}\}$$

for any ω-measurable set Λ on Γ. Then we have

LEMMA 4.14. *Let q and q_1 satisfy $\int |q(x)-q_1(x)|\,dx<\infty$. Then $\boldsymbol{R}^q_{BD}(\Gamma) = \boldsymbol{R}^{q_1}_{BD}(\Gamma)$ and $D^q[u, H^q_\varphi]=D^q[u, H^{q_1}_\varphi]$ for any $u \in \boldsymbol{H}^q_D$ and for any $\varphi \in \boldsymbol{R}^q_{BD}(\Gamma)$.*

PROOF. Let $q_0=\min(q, q_1)$. Then $\int(q-q_0)dx<\infty$ and $\int(q_1-q_0)dx<\infty$. It follows that $\int G^{q_0}(x, y)[q(y)-q_0(y)]dy<\infty$ and $\int G^{q_0}(x, y)[q_1(y)-q_0(y)]dy <\infty$. Hence, by Theorems 3.1 and 3.2, we see that q-resolutivity, q_1-resolutivity and q_0-resolutivity all coincide. By Proposition 4.3, we see that $\boldsymbol{R}^q_{BD}(\Gamma) \subset \boldsymbol{R}^{q_0}_{BD}(\Gamma)$ and $\boldsymbol{R}^{q_1}_{BD}(\Gamma) \subset \boldsymbol{R}^{q_0}_{BD}(\Gamma)$. Let $\varphi \in \boldsymbol{R}^{q_0}_{BD}(\Gamma)$ and $\varphi \geqq 0$. Theorem 3.1 implies that $H^{q_0}_\varphi = H^q_\varphi + p$ with a bounded q-potential p. Since $D^q[H^{q_0}_\varphi] = D^{q_0}[H^{q_0}_\varphi]+\int(q-q_0)(H^{q_0}_\varphi)^2dx<\infty$, Lemma 4.6 implies $\varphi \in \boldsymbol{R}^q_{BD}(\Gamma)$ and $D^q[u, H^q_\varphi] = D^q[u, H^{q_0}_\varphi]$ for any $u \in \boldsymbol{H}^q_D$. Similarly, we see that $\varphi \in \boldsymbol{R}^{q_1}_{BD}(\Gamma)$. By Theorem 3.2, $H^{q_0}_\varphi = H^{q_1}_\varphi + p_1$ with a bounded q_0-potential p_1. Since $H^{q_1}_\varphi \in \boldsymbol{D}^{q_1} \subset \boldsymbol{D}^{q_0}$, $p_1 \in \boldsymbol{D}^{q_0}_0$ (Lemma 4.6). Since p_1 is bounded and $\int(q-q_0)dx<\infty$, we see that $p_1 \in \boldsymbol{D}^q$. By Proposition 1.14, p_1 is a q-potential. Hence $p_1 \in \boldsymbol{D}^q_0$ by Lemma 4.6. Therefore $D^q[u, H^{q_0}_\varphi]=D^q[u, H^{q_1}_\varphi]$ for any $u \in \boldsymbol{H}^q_D$. Thus we have seen that if $\varphi \in \boldsymbol{R}^{q_0}_{BD}(\Gamma)$ and $\varphi \geqq 0$, then $\varphi \in \boldsymbol{R}^q_{BD}(\Gamma)$, $\varphi \in \boldsymbol{R}^{q_1}_{BD}(\Gamma)$ and $D^q[u, H^q_\varphi]= D^q[u, H^{q_1}_\varphi]$ for any $u \in \boldsymbol{H}^q_D$. By considering φ^+ and φ^- in general, we can easily complete the proof.

THEOREM 4.1. *Suppose*

$$\int q(x)dx<\infty. \tag{4.2}$$

Let $u \in \boldsymbol{H}^q_D$, Λ be an ω-measurable subset of Γ and γ be an ω-measurable function on Λ. Then γ is a normal derivative of u on Λ if and only if, for any $\varphi \in \boldsymbol{R}_{BD}(\Lambda)$ $(=\boldsymbol{R}^q_{BD}(\Lambda))$, $\varphi\gamma$ is ω-summable on Λ and

$$D^q[H^q_\varphi, u] = -\int_\Lambda \varphi\gamma\, d\omega_{x_0}. \tag{4.3}$$

PROOF. By the above lemma, $\boldsymbol{R}_{BD}(\Lambda)=\boldsymbol{R}^q_{BD}(\Lambda)$ and $D^q[H_\varphi, u]=D^q[H^q_\varphi, u]$ for all $\varphi \in \boldsymbol{R}_{BD}(\Gamma)$. Therefore, as remarked at the beginning of this section, the left hand side of (4.1)$'$ in Definition 2 is reduced to $D^q[H^q_\varphi, u]$ and the theorem is proved.

REMARK. (4.2) implies $q \in Q_0$, so that "ω-a.e." and "ω^q-a.e." mean the same (Theorem 3.3) in this case.

Corollary. *Let q satisfy* (4.2). *If γ is a normal derivative of $u \in \boldsymbol{H}_D^q$ on Λ, then* (4.3) *holds for any $\varphi \in \boldsymbol{R}_D^q(\Lambda)$ such that $\int_\Lambda |\gamma\varphi|\, d\omega_{x_0} < \infty$.*

Proof. Let $\varphi \in \boldsymbol{R}_D^q(\Lambda)$ be such that $\int_\Lambda |\gamma\varphi|\, d\omega < \infty$. Then $\varphi_n \in \boldsymbol{R}_{BD}^q(\Lambda)$, where $\varphi_n = \max(\min(\varphi, n), -n)$, $n=1, 2, \dots$. Hence by the above theorem,

$$D^q[H_{\varphi_n}^q, u] = -\int_\Lambda \varphi_n \gamma\, d\omega_{x_0}, \qquad n=1, 2, \dots.$$

By Lemma 4.11, $D^q[H_{\varphi_n}^q, u] \to D^q[H_\varphi^q, u]$ as $n \to \infty$. On the other hand, since $\int_\Lambda |\varphi\gamma|\, d\omega_{x_0} < \infty$ and $\varphi_n \to \varphi$ ω-a.e., the Lebesgue convergence theorem implies $\int_\Lambda \varphi_n \gamma\, d\omega_{x_0} \to \int_\Lambda \varphi\gamma\, d\omega_{x_0}$ $(n \to \infty)$. Hence (4.3) holds for φ.

Remark 1. We shall show in the next chapter (Lemma 5.1) that any function in $\boldsymbol{R}_D(\Gamma)$ is ω-square summable. Hence the condition that $\int |\varphi\gamma|\, d\omega < \infty$ for all $\varphi \in \boldsymbol{R}_D^q(\Lambda) (\subset \boldsymbol{R}_D(\Lambda))$ is satisfied if γ is ω-square summable on Λ.

Remark 2. Even when q does not satisfy (4.2), we may formally define a "q-normal derivative" of $u \in \boldsymbol{H}_D^q$ on an ω^q-measurable subset Λ on Γ as an ω^q-measurable function γ on Λ such that (4.3) holds for all $\varphi \in \boldsymbol{R}_{BD}^q(\Lambda)$ (or for all $\varphi \in \boldsymbol{R}_D^q(\Lambda)$). With this definition, it is possible to obtain a theory analogous to that given in the next two chapters, replacing normal derivatives by q-normal derivatives.

CHAPTER V Boundary Value Problems.

Throughout this and the next chapters let $\hat{X}$ be a resolutive compactification of X and let $\Gamma = \hat{X} - X$. We also fix $x_0 \in X$ and denote $\omega = \omega_{x_0}$. Moreover we assume that the condition

$$(4.2) \qquad \int q(x)\, dx < \infty$$

is satisfied (cf. Remark 2 at the end of the previous chapter).

§5.1. Problem setting

We now formulate a general boundary value problem, which includes the Neumann problem, the third boundary value problem and the mixed problem.

Suppose an ω-measurable subset Λ of Γ and an ω-measurable *non-negative* function β on Λ are given. We shall call the pair $[\Lambda, \beta]$ a *boundary condition.* As boundary data, we consider two ω-measurable functions τ on $\Gamma - \Lambda$ and γ on Λ. With such boundary condition and data, we set

Problem P$[\Lambda, \beta; \tau, \gamma; q]$: To find $u=H_{\psi}^{q} \in \boldsymbol{H}_{D}^{q}$ with $\psi \in \boldsymbol{R}_{D}^{q}(\Gamma)$ such that

(i) $\psi=\tau$ ω-a.e. on $\Gamma-\Lambda$,

(ii) u has a normal derivative $\beta\psi+\gamma$ on Λ.

REMARK 1. The above problem includes the following as special cases:

(a) The case $\omega(\Lambda)=0$. In this case the problem is regarded as the Dirichlet problem. Since we assumed that $\hat{X}$ is resolutive (hence q-resolutive), the existence and the uniqueness of the problem in this case are trivial.

(b) The case $\omega(\Gamma-\Lambda)=0$ and $\beta\equiv 0$ ω-a.e. on Λ. In this case condition (i) is not in force and condition (ii) is reduced to

(ii)$'$ u has a normal derivative γ on Λ.

Thus in this case the problem is regarded as the Neumann problem. By Theorem 4.1, condition (ii)$'$ is rewritten as

$$D^{q}[H_{\varphi}^{q}, u]=-\int\varphi\gamma d\omega \qquad \text{for all} \quad \varphi \in \boldsymbol{R}_{BD}(\Gamma).$$

Hence, if $q=0$, then it is necessary that $\int\gamma d\omega=0$.

(c) The case $\omega(\Gamma-\Lambda)=0$ and $\beta\not\equiv 0$ ω-a.e. on Λ. The problem in this case may be regarded as the third boundary value problem.

(d) The case $\omega(\Lambda)>0$, $\omega(\Gamma-\Lambda)>0$ and $\beta\equiv 0$ ω-a.e. on Λ. The problem in this case is regarded as the mixed problem.

REMARK 2. We can formulate similar problems with respect to normal derivatives in the weak sense. For example, as a Neumann problem, we may consider a problem to find $u \in \boldsymbol{H}_{D}^{q}$ having a given signed measure ν on Γ as a normal derivative in the weak sense. In fact, Constantinescu-Cornea [7] treated this form of problem for $q=0$ with respect to the Kuramochi boundary. However, we shall restrict ourselves to normal derivatives given in Definition 2 in treating a general form of problem on a general ideal boundary.

From the formulation of the problem, we immediately see the following:

PROPOSITION 5.1. *The problem* P$[\Lambda, \beta; \tau, \gamma; q]$ *is linear in* τ *and* γ, *i.e., if* u_i *is a solution of* P$[\Lambda, \beta; \tau_i, \gamma_i; q]$ *for each* $i=1, 2$ *and if* λ_1, λ_2 *are real numbers, then* $\lambda_1 u_1+\lambda_2 u_2$ *is a solution of* P$[\Lambda, \beta; \lambda_1\tau_1+\lambda_2\tau_2, \lambda_1\gamma_1+\lambda_2\gamma_2; q]$ (cf. the convention given in Proposition 3.1, (i)).

§5.2. Uniqueness of solutions

THEOREM 5.1. *If the problem* P$[\Lambda, \beta; \tau, \gamma; q]$ *has a solution of the form* $u=H_{\psi}^{q}$ *with* $\psi \in \boldsymbol{R}_{D}^{q}(\Gamma)$, *then it is uniquely determined* (*up to an additive con-*

stant in case $q=0$, $\omega(\Gamma-\Lambda)=0$ *and* $\beta\equiv 0$ ω*-a.e. on* Γ).

PROOF. Let $u_i=H^q_{\psi_i}$, $i=1, 2$, be two solutions of the problem. Since $\psi_1=\psi_2=\tau$ ω-a.e. on $\Gamma-\Lambda$, $\psi_1-\psi_2 \in \boldsymbol{R}^q_D(\Lambda)$. Let $\varphi_n=\max(\min(\psi_1-\psi_2, n), -n)$. Then $\varphi_n \in \boldsymbol{R}_{BD}(\Lambda)$. Hence, by condition (ii) of the problem and by Theorem 4.1, we have

$$D^q[u_1-u_2, H^q_{\varphi_n}] = -\int_\Lambda \{(\beta\psi_1+\gamma)-(\beta\psi_2+\gamma)\}\varphi_n d\omega \tag{5.1}$$

$$= -\int_\Lambda \beta(\psi_1-\psi_2)\varphi_n d\omega \leqq 0.$$

By Lemma 4.11, we have $0\leqq D^q[u_1-u_2]=\lim_{n\to\infty}D^q[u_1-u_2, H^q_{\varphi_n}]\leqq 0$. Therefore, $u_1=u_2$ if $q\neq 0$ and $u_1=u_2+\text{const.}$ if $q=0$. In the latter case, if $\omega(\Gamma-\Lambda)>0$, then $\psi_1=\psi_2=\tau$ ω-a.e. on $\Gamma-\Lambda$ implies $u_1=u_2$; if $\beta\not\equiv 0$ ω-a.e., then (5,1) implies that $\psi_1=\psi_2$ ω-a.e. on the set $\{\xi \in \Lambda;\ \beta(\xi)>0\}$, since $\psi_1=\psi_2+c$ ω-a.e. on Γ implies $\varphi_n=\psi_1-\psi_2$ ω-a.e. for $n\geqq |c|$. It then follows that $u_1=u_2$ in this case, too.

§5.3. More properties of the space $\boldsymbol{R}^q_D(\boldsymbol{\Gamma})$

In addition to those given in §4.5, we shall need the following properties of $\boldsymbol{R}^q_D(\Gamma)$ to obtain the existence theorem for our problem. In this and the next section, the assumption (4.2) can be weakened to the assumption $q \in Q_0$.

In case $q=0$, we also consider the space

$$\boldsymbol{R}_{D,1}=\{\varphi \in \boldsymbol{R}_D(\Gamma);\ \int\varphi d\omega=0\}.$$

The following lemma was obtained by Doob [11] in case $q=0$:

LEMMA 5.1. *There exists a constant* $M_q>0$ *such that*

$$\int\varphi^2 d\omega \leqq M_q D^q[H^q_\varphi] \tag{5.2}$$

for all $\varphi \in \boldsymbol{R}^q_D(\Gamma)$ *in case* $q\neq 0$; *for all* $\varphi \in \boldsymbol{R}_{D,1}$ *in case* $q=0$. *In particular,* $\boldsymbol{R}^q_D(\Gamma)\subset \boldsymbol{L}^2(\omega)$ *for any* q.

PROOF. We prove the case $q\neq 0$. First consider $\varphi \in \boldsymbol{R}^q_D(\Gamma)$ which is non-negative and bounded. Let $u=H^q_\varphi$ and consider the measure $\mu=(1/c_d)\Delta u^2$ associated with the superharmonic function $-u^2$. Then

$$\int d\mu = \frac{1}{c_d}\{2D[u]+2\int u(\Delta u)dx\} \tag{5.3}$$

$$= \frac{1}{c_d}\{2D[u]+2\int qu^2dx\} = \frac{2}{c_d}D^q[u]<\infty.$$

It then follows that the potential G_μ exists (i.e., $\not\equiv\infty$), so that $u^2=h-G_\mu$ with a harmonic function h. By Lemma 3.4, there exists a positive superharmonic function s such that $u+\varepsilon s \in \bar{\mathfrak{S}}_\varphi^q$ for all $\varepsilon>0$. By the same method as in the proof of Lemma 2 of [20], we see that $h+\varepsilon s \in \bar{\mathfrak{S}}_{\varphi^2}$. Hence $h\geqq \bar{H}_{\varphi^2}$. On the other hand, $u^2(x)=\left(\int\varphi d\omega_x^q\right)^2\leqq\int\varphi^2 d\omega_x\leqq \bar{H}_{\varphi^2}(x)$ (Lemma 3.5). Since h is the least harmonic majorant of u^2, it follows that $h\leqq\bar{H}_{\varphi^2}$. Hence $h=\bar{H}_{\varphi^2}$ or $u^2=\bar{H}_{\varphi^2}-G_\mu$.

Now we prove that there exists $M_q>0$ such that (5.2) holds for all $\varphi\in \boldsymbol{R}_D^q(\Gamma)$ which are bounded non-negative. Suppose this is not true. Then there would exist a sequence $\{\varphi_n\}$ of bounded non-negative functions in $\boldsymbol{R}_D^q(\Gamma)$ such that $\int\varphi_n^2 d\omega=1$ and $D^q[H_{\varphi_n}^q]<1/2^n$, $n=1, 2, \cdots$. Let $u_n=H_{\varphi_n}^q$ and $\mu_n=(1/c_d)\Delta(u_n)^2$. Then, by (5.3), $c_d\int d\mu_n=2D^q[u_n]<1/2^{n-1}$. Hence $\sum_{n=1}^\infty\mu_n$ defines a measure with finite total mass. It follows that G_{μ_n} tends to zero almost everywhere on X. Since $u_n\to0$ locally uniformly on X by Lemma 4.8 and $u_n^2=\bar{H}_{\varphi_n^2}-G_{\mu_n}$, it follows that $\{\bar{H}_{\varphi_n^2}\}$ converges to 0 a.e. on X, and hence locally uniformly on X (cf. Proposition 1.7). This contradicts the assumption that $\bar{H}_{\varphi_n^2}(x_0)\geqq\int\varphi_n^2 d\omega=1$. Therefore, there exists $M_q>0$ such that (5.2) holds for all $\varphi\in\boldsymbol{R}_D^q(\Gamma)$ which are bounded non-negative. Then we see from Lemmas 4.9 and 4.10 that (5.2) holds for all $\varphi\in\boldsymbol{R}_D^q(\Gamma)$.

The above proof is a modification of the proof for $q=0$ given in [11] and [20].

Let A be an ω-measurable subset of Γ and let τ be an ω-measurable function on $\Gamma-A$. We define

$$\boldsymbol{R}_D^q(A;\tau)=\{\varphi\in\boldsymbol{R}_D^q(\Gamma);\ \varphi=\tau\ \omega\text{-a.e. on } \Gamma-A\}.$$

Obviously, $\boldsymbol{R}_D^q(A;0)=\boldsymbol{R}_D^q(A)$ and if $\omega(\Gamma-A)=0$, then $\boldsymbol{R}_D^q(A;\tau)=\boldsymbol{R}_D^q(\Gamma)$ for any τ. We shall denote $\boldsymbol{H}_D^q(A;\tau)=\{H_\varphi^q;\ \varphi\in\boldsymbol{R}_D^q(A;\tau)\}$, $\boldsymbol{H}_D^q(A)=\boldsymbol{H}_D^q(A;0)$ and $\boldsymbol{H}_{D,1}=\{H_\varphi;\ \varphi\in\boldsymbol{R}_{D,1}\}$.

Lemma 5.2. *In case $q\not\equiv0$ or in case $q=0$ and $\omega(\Gamma-A)>0$, $\boldsymbol{H}_D^q(A;\tau)$ is complete with respect to the norm $\|\ \|_q$, if it is non-empty. In particular, $\boldsymbol{H}_D^q(A)$ is a Hilbert space in this case. In case $q=0$ and $\omega(\Gamma-A)=0$, $\boldsymbol{H}_{D,1}$ is a Hilbert space with respect to $D[u_1, u_2]$.*

Proof. First suppose $q\not\equiv0$. Let $u_n=H_{\varphi_n}^q\in\boldsymbol{H}_D^q(A;\tau)$, $\varphi_n\in\boldsymbol{R}_D^q(A;\tau)$ and let $\{u_n\}$ be a Cauchy sequence with respect to $\|\ \|_q$. By Lemma 4.4, there exists $u\in\boldsymbol{H}_D^q$ such that $\|u_n-u\|_q\to0$. On the other hand, Lemma 5.1 implies that $\{\varphi_n\}$ is a Cauchy sequence in $\boldsymbol{L}^2(\omega)$, so that there exists $\varphi\in\boldsymbol{L}^2(\omega)$ such that $\int(\varphi_n-\varphi)^2 d\omega\to0$ $(n\to\infty)$. Then Proposition 1.7 implies that $\int(\varphi_n-\varphi)^2 d\omega_x\to0$ for any $x\in X$. By Lemma 3.7, (ii), we may assume that φ is resolutive (hence q-resolutive). Thus, for each $x \in X$, we have

$$|H_\varphi^q(x)-u_n(x)|^2=\left|\int(\varphi_n-\varphi)d\omega_x^q\right|^2$$

$$\leqq\int(\varphi_n-\varphi)^2 d\omega_x\to 0 \qquad (n\to\infty).$$

It then follows from Lemma 4.8, (ii) that $u=H_\varphi^q$. Hence $\varphi\in \boldsymbol{R}_D^q(\Gamma)$. We can choose a subsequence $\{\varphi_{n_j}\}$ of $\{\varphi_n\}$ such that φ_{n_j} tends to φ ω-a.e. on Γ. Since $\varphi_{n_j}=\tau$ ω-a.e. on $\Gamma-\Lambda$, $\varphi=\tau$ ω-a.e. on $\Gamma-\Lambda$. Hence $\varphi\in \boldsymbol{R}_D^q(\Lambda;\tau)$, so that $u\in \boldsymbol{H}_D^q(\Lambda;\tau)$.

Quite similarly, we can show that $\boldsymbol{H}_{D,1}$ is complete (cf. [20]).

Finally suppose $q=0$ and $\omega(\Gamma-\Lambda)>0$. It is easy to see that if $\|u_1-u_2\|=0$ for $u_1, u_2\in \boldsymbol{H}_D(\Lambda;\tau)$ then $u_1=u_2$. Let $u_n=H_{\varphi_n}\in \boldsymbol{H}_D(\Lambda;\tau)$ with $\varphi_n\in \boldsymbol{R}_D(\Lambda;\tau)$ and suppose $\{u_n\}$ is a Cauchy sequence with respect to $\|\ \|$. Let $c_n=u_n(x_0)$ for each n. Then $u_n-c_n\in \boldsymbol{H}_{D,1}$ for each n. Since $\boldsymbol{H}_{D,1}$ is complete, there exists $u^*\in \boldsymbol{H}_{D,1}$ such that $u_n-c_n\to u^*$ in $\boldsymbol{H}_{D,1}$. Let $u^*=H_{\varphi^*}$ with $\varphi^*\in \boldsymbol{R}_{D,1}$. By Lemma 5.1, $\int(\varphi_n-c_n-\varphi^*)^2 d\omega\to 0$. Since $\varphi_n=\tau$ ω-a.e. on $\Gamma-\Lambda$, we have $\int_{\Gamma-\Lambda}(\tau-c_n-\varphi^*)^2 d\omega\to 0$. Therefore, $\varphi^*=\tau+c$ (const.) ω-a.e. on $\Gamma-\Lambda$. Hence $u=u^*-c\in \boldsymbol{H}_D(\Lambda;\tau)$ and $\|u_n-u\|\to 0$.

LEMMA 5.3. *Let Λ be an ω-measurable subset of Γ such that $\omega(\Gamma-\Lambda)>0$. Then there exists a constant $M'>0$ such that*

$$\int\varphi^2 d\omega\leqq M'D[H_\varphi] \tag{5.2$'$}$$

for all $\varphi\in \boldsymbol{R}_D(\Lambda)$.

PROOF. By Lemma 5.1, we have $\int(\varphi-H_\varphi(x_0))^2 d\omega\leqq MD[H_\varphi]$ for all $\varphi\in \boldsymbol{R}_D(\Gamma)$. Hence it is enough to show that there exists $M_1>0$ such that $|H_\varphi(x_0)|^2\leqq M_1 D[H_\varphi]$ for all $\varphi\in \boldsymbol{R}_D(\Lambda)$. Suppose the contrary. Then there would exist $\varphi_n\in \boldsymbol{R}_D(\Lambda)$, $n=1, 2, \dots$, such that $H_{\varphi_n}(x_0)=1$ for all n and $D[H_{\varphi_n}]\to 0$ $(n\to\infty)$. By Lemma 4.8, H_{φ_n} tends to constant 1, while the previous lemma implies that it tends to constant 0, a contradiction.

§5.4. Condition $(\mathrm{B})_q$ for a boundary condition

We shall consider the following condition:

$$(\mathrm{B})_q: \qquad \int_\Lambda \beta\varphi^2 d\omega<\infty \qquad \textit{for all} \quad \varphi\in \boldsymbol{R}_D^q(\Lambda)$$

for a boundary condition $[\Lambda;\beta]$.

If $q_1\leqq q_2$, then $(B)_{q_1}$ implies $(B)_{q_2}$ (Proposition 4.3). If β is bounded (ω-essentially) on A, then $(B)\equiv(B)_0$ (hence $(B)_q$ for any q) is satisfied by virtue of Lemma 5.1.

LEMMA 5.4. *Let $[A, \beta]$ be a boundary condition satisfying* $(B)_q$ *and suppose* $\int_A \beta d\omega>0$. *Then there exists a constant $M_q(\beta)>0$ such that*

$$\int_A \beta\varphi^2 d\omega\leqq M_q(\beta)D^q[H_\varphi^q] \tag{5.4}$$

for all $\varphi\in \boldsymbol{R}_D^q(A)$ in case $q\neq 0$ or in case $q=0$ and $\omega(\Gamma-A)>0$; for all $\varphi\in \boldsymbol{R}_{D1}$ in case $q=0$ and $\omega(\Gamma-A)=0$.

PROOF. Let $q\neq 0$ or $q=0$ and $\omega(\Gamma-A)>0$ (resp. $q=0$ and $\omega(\Gamma-A)=0$). Suppose there is no $M_q(\beta)>0$ satisfying (5.4) for all $\varphi\in \boldsymbol{R}_D^q(A)$ (resp. $\in \boldsymbol{R}_{D1}$). Then we would find $\varphi_n\in \boldsymbol{R}_D^q(A)$ (resp. $\in \boldsymbol{R}_{D1}$) such that $\int_A\beta\varphi_n^2 d\omega=1$ and $D^q[H_{\varphi_n}^q]<1/2^n$, $n=1, 2, \ldots$ Let $\varphi_n^*=|\varphi_1|+\cdots+|\varphi_n|$ for each n. Then $\varphi_n^*\in \boldsymbol{R}_D^q(A)$ (resp. $\in \boldsymbol{R}_D(\Gamma)$) and $\int_A \beta(\varphi_n^*)^2 d\omega\geqq n$. Using Lemma 4.9, we see that $D^q[H_{\varphi_n^*}^q-H_{\varphi_m^*}^q]\to 0$ $(n, m\to\infty)$. Furthermore, in case $q=0$, $0\leqq H_{\varphi_m^*}(x_0)-H_{\varphi_n^*}(x_0)=\sum_{k=n+1}^m\int|\varphi_k| d\omega\leqq\sum_{k=n+1}^m\left(\int\varphi_k^2 d\omega\right)^{1/2}\leqq M_q^{1/2}\sum_{k=n+1}^m D[H_{\varphi_k}]^{1/2}$ for $m>n$, by Lemmas 5.1 and 5.3. Therefore $\{H_{\varphi_n^*}(x_0)\}$ is convergent in this case. Hence, by Lemma 5.2, there exists $\varphi^*\in \boldsymbol{R}_D^q(A)$ (resp. $\in \boldsymbol{R}_D(\Gamma)$) such that $D^q[H_{\varphi_n^*}^q-H_{\varphi^*}^q]\to 0$ $(n\to\infty)$. Since φ_n^* is monotone increasing, Lemmas 5.1 and 5.3 imply that $\varphi^*=\lim_{n\to\infty}\varphi_n^*$ ω-a.e. on Γ (resp. we may assume that $\varphi^*=\lim_{n\to\infty}\varphi_n^*$). By assumption, $\int_A\beta\varphi^{*2}d\omega<\infty$, while $n\leqq\int_A\beta(\varphi_n^*)^2 d\omega\leqq\int_A\beta\varphi^{*2}d\omega$ for all n, which is a contradiction.

LEMMA 5.5. *Let $[A, \beta]$ be a boundary condition satisfying* $(B)_q$ *and suppose* $\int_A\beta d\omega>0$. *If we define an inner product $(u_1, u_2)_{\beta,q}$ on $\boldsymbol{H}_D^q(A)$ by*

$$(u_1, u_2)_{\beta,q}=D^q[u_1, u_2]+\int_A\beta\varphi_1\varphi_2 d\omega$$

for $u_1=H_{\varphi_i}^q$, $\varphi_i\in \boldsymbol{R}_D^q(A)$, $i=1, 2$, then $\boldsymbol{H}_D^q(A)$ is a Hilbert space with respect to this inner product.

PROOF. If $q\neq 0$ or if $q=0$ and $\omega(\Gamma-A)>0$, then the above lemma asserts that $\|\ \|_q$ and the norm corresponding to $(\ ,\ )_{\beta,q}$ are equivalent norms. Hence $\boldsymbol{H}_D^q(A)$ is a Hilbert space with respect to $(\ ,\ )_{\beta,q}$ by Lemma 5.2.

Next let $q=0$ and $\omega(\Gamma-A)=0$. Since $\int_A\beta d\omega>0$, $(u, u)_\beta=0$ implies $u=0$. If $\{u_n\}$ is a Cauchy sequence in $\boldsymbol{H}_D(A)$ with respect to $(\ ,)_\beta$, i.e., $(u_n-u_m, u_n-u_m)_\beta$

$\to 0$ $(n, m\to\infty)$, then $D[u_n-u_m]\to 0$ and $\int_A \beta(\varphi_n-\varphi_m)^2 d\omega\to 0$ $(n, m\to\infty)$, where $u_n=H_{\varphi_n}$, $\varphi_n \in \boldsymbol{R}_D(A)$, $n=1, 2, \cdots$. By Lemma 5.2, there exists $u=H_\varphi$, $\varphi\in \boldsymbol{R}_{D,1}$ such that $D[u_n-u]\to 0$ $(n\to\infty)$. On the other hand, the previous lemma implies that $\int_A \beta\{(\varphi_n-c_n)-(\varphi_m-c_m)\}^2 d\omega\to 0$ $(n, m\to\infty)$, where $c_n=\int \varphi_n d\omega$ $(n=1, 2, \cdots)$. It follows that $\{c_n\}$ is convergent. Let $c=\lim_{n\to\infty}c_n$. By the previous lemma again, $\int_A \beta(\varphi_n-c_n-\varphi)^2 d\omega\to 0$, so that $\int_A \beta\{\varphi_n-(\varphi+c)\}^2 d\omega\to 0$. Hence $(u_n-(u+c), u_n-(u+c))_\beta\to 0$. Therefore $\boldsymbol{H}_D(A)$ is complete with respect to the norm induced by $(\ ,\ )_\beta$.

§5.5. The existence theorem

Let $[A, \beta]$ be a boundary condition satisfying $(\mathrm{B})_q$. We shall denote by $\|\ \|_{\beta,q}$ the norm induced by $(\ ,\ )_{\beta,q}$ on $\boldsymbol{H}_D^q(A)$, i.e.,

$$\|u\|_{\beta,q}^2=\|u\|_q^2+\int_A \beta\varphi^2 d\omega$$

for $u=H_\varphi^q \in \boldsymbol{H}_D^q(A)$. Now, for functions τ and γ given on $\Gamma-A$ and on A respectively, we consider the following conditions, which depend on $[A, \beta]$ and q:

$(\mathrm{T})_q$: $\int_A \beta\varphi_1^2 d\omega<\infty$ *for some* $\varphi_1\in \boldsymbol{R}_D^q(A;\tau)$.

$(\Gamma)_q$: *For any* $\varphi\in \boldsymbol{R}_{BD}(A)$, $\varphi\gamma$ *is* ω*-summable and there exists a constant* $K_\gamma>0$ *such that*

$$\left|\int_A \varphi\gamma d\omega\right|\leqq K_\gamma\|H_\varphi^q\|_{\beta,q}$$

for all $\varphi\in \boldsymbol{R}_{BD}(A)$.

REMARK. (a) If τ satisfies condition $(\mathrm{T})_q$, then $\int_A \beta\varphi^2 d\omega<\infty$ for *any* $\varphi\in \boldsymbol{R}_D^q(A;\tau)$ by virtue of condition $(\mathrm{B})_q$.

(b) $\tau\equiv 0$ satisfies $(\mathrm{T})_q$ because of $(\mathrm{B})_q$.

(c) If β is bounded on A, then any τ on $\Gamma-A$ for which $\boldsymbol{R}_D^q(A;\tau)$ is non-empty satisfies $(T)_q$. (Cf. the similar remark for $(\mathrm{B})_q$ in the previous section; also cf. Doob [11].)

(d) If $\mathrm{P}[A, \beta; \tau, \gamma; q]$ has a solution $u=H_\psi^q$, then, by Theorem 4.1, $D^q[u, H_\varphi^q]+\int_A (\beta\psi+\gamma)\varphi d\omega=0$ for all $\varphi\in \boldsymbol{R}_{BD}(A)$. It follows that

$$\left|\int_A \gamma\varphi d\omega\right|^2\leqq D^q[u]D^q[H_\varphi^q]+\left(\int\beta\psi^2 d\omega\right)\left(\int\beta\varphi^2 d\omega\right)$$
$$\leqq \max\left(D^q[u], \int\beta\psi^2 d\omega\right)\|H_\varphi^q\|_{\beta,q}^2$$

for all $\varphi \in \boldsymbol{R}_{BD}(\Lambda)$. Thus condition $(\Gamma)_q$ is a necessary condition for $P[\Lambda, \beta; \tau, \gamma; q]$ to have a solution, under conditions $(B)_q$ and $(T)_q$ (cf. (a) above).

(e) By virtue of Lemma 5.4, condition $(\Gamma)_q$ can be replaced by the following (under condition $(B)_q$):

$(\Gamma)_q'$: For any $\varphi \in \boldsymbol{R}_{BD}(\Lambda)$, $\varphi\gamma$ is ω-summable and there exists a constant $K_\gamma' > 0$ such that

$$\left|\int_\Lambda \varphi\gamma\, d\omega\right| \leqq K_\gamma' \|H_\varphi^q\|_q$$

for all $\varphi \in \boldsymbol{R}_{BD}(\Lambda)$ in case $q \neq 0$ or in case $q=0$ and $\omega(\Gamma - \Lambda) > 0$; for all $\varphi \in \boldsymbol{R}_{D,1}$ in case $q=0$ and $\omega(\Gamma-\Lambda)=0$. In addition $\int \gamma\, d\omega = 0$ in case $q=0$, $\omega(\Gamma-\Lambda)=0$ and $\beta \equiv 0$ ω-a.e.

Thus, under condition $(B)_q$, $(\Gamma)_q$ can be stated in a form independent of β.

(f) In case $q=0$, $\omega(\Gamma-\Lambda)=0$ and $\beta\equiv 0$ ω-a.e., $(\Gamma)_q$ implies $\int \gamma\, d\omega = 0$, since $\varphi=1$ belongs to $\boldsymbol{R}_{BD}(\Gamma)$. Cf. Remark 1, (b) in §5.1.

(g) If γ is ω-square summable on Λ (and in addition $\int \gamma\, d\omega = 0$ in the case of (f)), then it satisfies $(\Gamma)_q$ by Lemma 5.1. Cf. [11] and [20].

Now we have the following existence theorem for the problem $P[\Lambda, \beta; \tau, \gamma; q]$:

THEOREM 5.2. (The existence theorem) *Let $[\Lambda, \beta]$ be a boundary condition satisfying $(B)_q$ and let τ and γ be ω-measurable functions on $\Gamma - \Lambda$ and Λ respectively satisfying conditions $(T)_q$ and $(\Gamma)_q$. Then the problem $P[\Lambda, \beta; \tau, \gamma; q]$ has a solution $u = H_\psi^q$ with $\psi \in \boldsymbol{R}_D^q(\Gamma)$.*

This theorem for $q=0$ includes the existence theorems in Doob [11] (for the Martin boundary) and in Maeda [20]; cf. the above remark, (c) and (g).

§5.6. A proof of the existence theorem

(I) *The Neumann problem for $q=0$, i.e.,* $P[\Gamma, 0; 0, \gamma; 0]$.

For each $u = H_\varphi$ with $\varphi \in \boldsymbol{R}_{D,1} \cap \boldsymbol{R}_{BD}(\Gamma)$, let

$$l(u) = -\int \varphi\gamma\, d\omega.$$

By condition (Γ), l is a continuous linear form on $\boldsymbol{H}_{D,1} \cap \boldsymbol{H}_{BD}$, where $\boldsymbol{H}_{BD} = \{H_\varphi;\ \varphi \in \boldsymbol{R}_{BD}(\Gamma)\}$. By Lemma 4.11, we see that $\boldsymbol{H}_{D,1} \cap \boldsymbol{H}_{BD}$ is dense in the Hilbert space $\boldsymbol{H}_{D,1}$. Therefore there exists $u_0 = H_\psi \in \boldsymbol{H}_{D,1}$ such that

$$l(u) = D[u, u_0]$$

for all $u \in \boldsymbol{H}_{D,1} \cap \boldsymbol{H}_{BD}$, i.e.,

$$D[H_\varphi, u_0] = -\int \varphi\gamma\, d\omega$$

for all $\varphi \in \boldsymbol{R}_{D,1} \cap \boldsymbol{R}_{BD}(\Gamma)$. Since $\int \gamma\, d\omega = 0$ by assumption, this equality holds for all $\varphi \in \boldsymbol{R}_{BD}(\Gamma)$. Hence γ is a normal derivative of u_0 on Γ by Theorem 4.1, so that u_0 is a solution of P$[\Gamma, 0; 0, \gamma; 0]$.

(II) *Dirichlet principle*: P$[A, 0; \tau, 0; q]$ *with* $\omega(\Gamma - A) > 0$.

For this problem, condition $(\mathrm{T})_q$ is reduced to $\boldsymbol{R}_D^q(A; \tau) \neq \emptyset$ (see Remark, (c) in the previous section). Then $\boldsymbol{H}_D^q(A; \tau)$ is a non-empty convex set and is complete by Lemma 5.2. Therefore, there exists $u_0 = H_\psi^q \in \boldsymbol{H}_D^q(A; \tau)$ such that

$$\|u_0\|_q = \min\{\|u\|_q;\ u \in \boldsymbol{H}_D^q(A; \tau)\}.$$

For any $\varphi \in \boldsymbol{R}_D^q(A)$ and for any real number λ, $u_0 + \lambda H_\varphi^q \in \boldsymbol{H}_D^q(A; \tau)$. Hence $\|u_0 + \lambda H_\varphi^q\|_q \geqq \|u_0\|_q$ for all λ, i.e.,

$$2\lambda D^q[u_0, H_\varphi^q] + \lambda^2 D^q[H_\varphi^q] \geqq 0$$

for all λ. It follows that $D^q[u_0, H_\varphi^q] = 0$. Therefore u_0 has a normal derivative zero on A, and hence it is a solution of P$[A, 0; \tau, 0; q]$.

(III) *General case*: P$[A, \beta; \tau, \gamma; q]$, *excluding the cases* (I) *and* (II).

Since τ satisfies $(\mathrm{T})_q$ with respect to $[A, 0]$, there exists a solution $u_1 = H_{\psi_1}^q$ of the problem P$[A, 0; \tau, 0; q]$ (the case (II); if $\omega(\Gamma - A) = 0$, then $u_1 = 0$). We consider the linear mapping

$$l(u) = -\int_A (\beta\varphi\psi_1 + \varphi\gamma)\, d\omega$$

defined for $u = H_\varphi^q \in \boldsymbol{H}_{BD}^q(A) = \{H_\varphi^q;\ \varphi \in \boldsymbol{R}_{BD}(A)\}$. By conditions $(\mathrm{B})_q$ and $(\mathrm{T})_q$, we see that

$$\left|\int_A \beta\varphi\psi_1\, d\omega\right|^2 \leqq \left(\int_A \beta\psi_1^2\, d\omega\right)\left(\int_A \beta\varphi^2\, d\omega\right)$$

$$\leqq \left(\int_A \beta\psi_1^2\, d\omega\right)\|H_\varphi^q\|_{\beta,q}^2$$

and by condition $(\Gamma)_q$, we have

$$\left|\int_A \varphi\gamma\, d\omega\right| \leqq K_\gamma \|\varphi\|_{\beta,q}$$

for all $\varphi \in \boldsymbol{R}_{BD}(A)$. Hence l is a continuous linear form on $\boldsymbol{H}_{BD}^q(A)$ with the norm $\|\ \|_{\beta,q}$. By Lemma 4.11, we see that $\boldsymbol{H}_{BD}^q(A)$ is dense in the Hilbert space $\boldsymbol{H}_D^q(A)$ (Lemma 5.5). Hence there exists $u_2 = H_{\psi_2}^q \in \boldsymbol{H}_D^q(A)$ such that

$$l(u) = (u, u_2)_{\beta,q}$$

for all $u \in \boldsymbol{H}_{BD}^q(A)$, i.e.,

$$D^q[H_\varphi^q, u_2]+\int_A \beta\varphi\psi_2 d\omega = -\int_A (\beta\varphi\psi_1+\varphi\gamma)d\omega$$

for all $\varphi \in \boldsymbol{R}_{BD}(A)$. Therefore, by Theorem 4.1, $u_0=u_1+u_2$ has a normal derivative $\beta\psi_0+\gamma$ on A, where $\psi_0=\psi_1+\psi_2$. Obviously $\psi_0=\tau$ ω-a.e. on $\Gamma-A$. Hence u_0 is a solution of $\mathrm{P}[A, \beta; \tau, \gamma; q]$.

§5.7. General properties of solutions

In this and the next sections, we shall always assume that, for a given problem $\mathrm{P}[A, \beta; \tau, \gamma; q]$, $[A, \beta]$ satisfies $(\mathrm{B})_q$, τ satisfies $(\mathrm{T})_q$ and $|\gamma|$ satisfies $(\Gamma)_q$ (cf. Lemma 5.6 below). Also we *exclude the case* $q=0$, $\omega(\Gamma-A)=0$ *and* $\beta\equiv 0$ ω-a.e.. Thus the problem has a unique solution.

First we remark

LEMMA 5.6. *If* $|\gamma|$ *satisfies* $(\Gamma)_q$ *with respect to* $[A, \beta]$, *then* γ^+, γ^- *and* γ *also satisfy* $(\Gamma)_q$ *and if* $u=H_\psi^q$ *is the solution of* $\mathrm{P}[A, \beta; \tau, \gamma; q]$ *in this case, then*

$$(u, H_\varphi^q)_{\beta,q} = -\int_A \gamma\varphi d\omega \tag{5.5}$$

holds for all $\varphi \in \boldsymbol{R}_D^q(A)$.

PROOF. The first assertions are obvious if we remark that $\varphi \in \boldsymbol{R}_{BD}(A)$ implies $|\varphi| \in \boldsymbol{R}_{BD}(A)$ (Lemma 4.9).

For any $\varphi \in \boldsymbol{R}_D^q(A)$, let $\varphi_n=\max(\min(\varphi, n), -n)$. By condition $(\Gamma)_q$ for $|\gamma|$ and condition $(\mathrm{B})_q$, we see that

$$\int |\gamma\varphi_n|\, d\omega \leqq K\|H_{|\varphi_n|}^q\|_{\beta,q} \leqq K_\gamma \|H_\varphi^q\|_{\beta,q} < \infty.$$

Since $|\gamma\varphi_n|$ increases to $|\gamma\varphi|$, it follows that $\int|\gamma\varphi|\, d\omega<\infty$. Hence, by the corollary to Theorem 4.1, we conclude that (5.5) holds for any $\varphi \in \boldsymbol{R}_D^q(A)$.

THEOREM 5.3. *If* $\tau\geqq 0$ *ω-a.e. on* $\Gamma-A$ *and* $\gamma\leqq 0$ *ω-a.e. on* A, *then the solution* u *of* $\mathrm{P}[A, \beta; \tau, \gamma; q]$ *is non-negative.*

PROOF. Let $u=H_\psi^q$ and let $\psi^+=\max(\psi, 0)$. Since $\psi=\tau\geqq 0$ ω-a.e. on $\Gamma-A$, $\psi^+-\psi \in \boldsymbol{R}_D^q(A)$. Hence, by the above lemma,

$$(u, H_{\psi^+}^q)_{\beta,q}+\int_A \gamma\psi^+ d\omega = (u, u)_{\beta,q}+\int_A \gamma\psi d\omega.$$

Since $\gamma\leqq 0$ ω-a.e. on A and $\psi^+\geqq\psi$, we have $\int_A \gamma\psi^+ d\omega \leqq \int_A \gamma\psi d\omega$. Therefore

$$(u, H_{\psi^+}^q)_{\beta,q} \geqq (u, u)_{\beta,q} = \|u\|_{\beta,q}^2.$$

Hence

$$0 \leqq \|u-H^q_{\psi^+}\|^2_{\beta,q} \leqq \|H^q_{\psi^+}\|^2_{\beta,q}-\|u\|^2_{\beta,q}.$$

On the other hand, $D^q[H^q_{\psi^+}] \leqq D^q[u]$ by Lemma 4.9. Also, $\int \beta\psi^{+2}d\omega \leqq \int \beta\psi^2 d\omega$. Hence $\|H^q_{\psi^+}\|_{\beta,q} \leqq \|u\|_{\beta,q}$. It then follows that $\|u-H^q_{\psi^+}\|_{\beta,q}=0$ or $u=H^q_{\psi^+} \geqq 0$.

COROLLARY. *Let u_i be the solution of* $\mathrm{P}[\Lambda, \beta; \tau_i, \gamma_i; q]$ *for each* $i=1, 2$. *If $\tau_1 \geqq \tau_2$ ω-a.e. on $\Gamma-\Lambda$ and $\gamma_1 \leqq \gamma_2$ ω-a.e. on Λ, then $u_1 \geqq u_2$.*

THEOREM 5.4. *Let u_i be the solution of* $\mathrm{P}[\Lambda_i, \beta_i; \tau_i, \gamma_i; q]$ *for each* $i=1, 2$. *If* (a) $\Lambda_1 \supset \Lambda_2$, (b) $\beta_1 \leqq \beta_2$ *ω-a.e. on Λ_2,* (c) $\tau_1 \geqq \tau_2$ *ω-a.e. on $\Gamma-\Lambda_1$,* (d) $\gamma_1 \leqq \gamma_2$ *ω-a.e. on Λ_2,* (e) $\tau_2 \leqq 0$ *ω-a.e. on $\Lambda_1-\Lambda_2$,* (f) $\tau_1 \geqq 0$ *ω-a.e. on $\Gamma-\Lambda_1$ and* (g) $\gamma_1 \leqq 0$ *ω-a.e. on Λ_1, then $u_1 \geqq u_2$.*

PROOF. Let $u_i=H^q_{\psi_i}$, $i=1, 2$ and let $\psi^*=\max(\psi_1, \psi_2)$ and $\psi_*=\min(\psi_1, \psi_2)$. By (c), $\psi_1 \geqq \psi_2$ ω-a.e. on $\Gamma-\Lambda_1$. By (f) and (g), $\psi_1 \geqq 0$ ω-a.e. (the previous theorem). Hence, together with (e), we see that $\psi_1 \geqq \psi_2$ ω-a.e. on $\Lambda_1-\Lambda_2$. It follows that $\psi^*-\psi_1=\psi_2-\psi_* \in \boldsymbol{R}^q_D(\Lambda_2) \subset \boldsymbol{R}^q_D(\Lambda_1)$. Therefore, by Lemma 5.6, we have

$$(u_1, H^q_{\psi^*}-u_1)_{\beta_1,q} = -\int_{\Lambda_1} \gamma_1(\psi^*-\psi_1)d\omega \tag{5.6}$$

and

$$(u_2, H^q_{\psi_*}-u_2)_{\beta_2,q} = -\int_{\Lambda_2} \gamma_2(\psi_*-\psi_2)d\omega. \tag{5.7}$$

Now

$$(u_1, H^q_{\psi^*}-u_1)_{\beta_1,q} = (u_1, H^q_{\psi^*}-u_1)_{\beta_2,q} + \int_{\Lambda_2} (\beta_1-\beta_2)\psi_1(\psi^*-\psi_1)d\omega.$$

Since $\psi_1 \geqq 0$ and $\psi^*-\psi_1 \geqq 0$, condition (b) implies that the last term is non-positive. Hence, by (5.6), we have

$$(u_1, H^q_{\psi^*}-u_1)_{\beta_2,q} \geqq -\int_{\Lambda_1} \gamma_1(\psi^*-\psi_1)d\omega. \tag{5.6$'$}$$

Combining (5.6)$'$ and (5.7) and using the relation $\psi^*-\psi_1=\psi_2-\psi_*$, we obtain

$$(u_1, H^q_{\psi^*}-u_1)_{\beta_2,q}+(u_2, H^q_{\psi^*}-u_2)_{\beta_2,q} \geqq -\int_{\Lambda_2} (\gamma_1-\gamma_2)(\psi^*-\psi_1)d\omega.$$

By condition (d), the right hand side is non-negative. Hence

$$(u_1, H^q_{\psi^*}-u_1)_{\beta_2,q}+(u_2, H^q_{\psi_*}-u_2)_{\beta_2,q} \geqq 0.$$

Thus we have

$$
\begin{aligned}
0 &\leqq \|u_1-H^q_{\psi^*}\|^2_{\beta_2,q}+\|u_2-H^q_{\psi_*}\|^2_{\beta_2,q} \\
&=\|H^q_{\psi^*}\|^2_{\beta_2,q}+\|H^q_{\psi_*}\|^2_{\beta_2,q}-\|u_1\|^2_{\beta_2,q}-\|u_2\|^2_{\beta_2,q} \\
&\quad -2\{(u_1, H^q_{\psi^*}-u_1)_{\beta_2,q}+(u_2, H^q_{\psi_*}-u_2)_{\beta_2,q}\} \\
&\leqq \|H^q_{\psi^*}\|^2_{\beta_2,q}+\|H^q_{\psi_*}\|^2_{\beta_2,q}-\|u_1\|^2_{\beta_2,q}-\|u_2\|^2_{\beta_2,q}.
\end{aligned}
$$

By virtue of Lemma 4.9 and the relation $(\psi^*)^2+(\psi_*)^2=\psi_1^2+\psi_2^2$, we see that the last expression is non-positive. Hence we have $u_1=H^q_{\psi^*}\geqq H^q_{\psi_*}=u_2$.

§5.8. Dependence of the solution on τ and γ

THEOREM 5.5. *Let u be the solution of* $\mathrm{P}[\Lambda, \beta; \tau, \gamma; q]$ *and let M be a positive constant. If $|\tau|\leqq M$ ω-a.e. on $\Gamma-\Lambda$ and if $|\gamma|\leqq M\beta$ ω-a.e. on Λ, then $|u|\leqq M$.*

PROOF. First let $\tau\geqq 0$ and $\gamma\geqq 0$. Then $u\geqq 0$ by Theorem 5.3. Let $u=H^q_\psi$ and $\psi_1=\min(\psi, M)$. By Lemma 4.10, $\psi_1 \in \boldsymbol{R}^q_D(\Gamma)$ and $D^q[H^q_{\psi_1}]\leqq D^q[u]$. By the assumption that $\tau\leqq M$ ω-a.e. on $\Gamma-\Lambda$, we see that $\psi-\psi_1 \in \boldsymbol{R}^q_D(\Lambda)$. Therefore, Lemma 5.6 implies

$$
D^q[u, u-H^q_{\psi_1}]=-\int_\Lambda(\beta\psi+\gamma)(\psi-\psi_1)d\omega.
$$

$\psi-\psi_1\geqq 0$ everywhere on Γ and if $\psi(\xi)-\psi_1(\xi)>0$, then $\psi(\xi)>M$, so that $\beta(\xi)\psi(\xi)\geqq M\beta(\xi)\geqq-\gamma(\xi)$. Hence $D^q[u, u-H^q_{\psi_1}]\leqq 0$. Hence

$$
0\leqq D^q[u-H^q_{\psi_1}]\leqq D^q[H^q_{\psi_1}]-D^q[u]\leqq 0.
$$

If $q\neq 0$ or if $\omega(\Gamma-\Lambda)>0$, then it follows that $u=H^q_{\psi_1}$. If $q=0$ and $\omega(\Gamma-\Lambda)=0$, then it follows that either $u=H_{\psi_1}$ or $u=c$ (const. $>M$). Suppose the latter case occurs. Since $\int\beta d\omega>0$ in this case, Lemma 5.6 implies

$$
0=D[u]=-\int_\Lambda(\beta c+\gamma)c\,d\omega<-\int_\Lambda(\beta M+\gamma)c\,d\omega\leqq 0,
$$

a contradiction. Therefore $u=H^q_{\psi_1}$ in any case. By Proposition 3.1, (ii), $H^q_{\psi_1}\leqq M$. Hence $u\leqq M$.

In the general case, we consider solutions u_1 and u_2 of $\mathrm{P}[\Lambda, \beta; \tau^+, -\gamma^-; q]$ and $\mathrm{P}[\Lambda, \beta; \tau^-; -\gamma^+; q]$ respectively. Then $u=u_1-u_2$, $0\leqq u_1\leqq M$ and $0\leqq u_2\leqq M$. Hence $|u|\leqq M$.

COROLLARY. *Let u be the solution of* $\mathrm{P}[\Lambda, \beta; \tau, \gamma; q]$ *and let M be a non-negative constant. If $\tau\leqq M$ ω-a.e. on $\Gamma-\Lambda$ and if $\gamma\geqq 0$ ω-a.e. on Λ, then $u\leqq M$.*

PROOF. Let u_1 be the solution of $\mathrm{P}[\Lambda, \beta; \tau^+, 0; q]$. Then $u_1\leqq M$ by the above theorem, while $u \leqq u_1$ by Theorem 5.4 (or the corollary to Theorem 5.3). Hence $u \leqq M$.

THEOREM 5.6. *Let a boundary condition* $[\Lambda, \beta]$ *and* q *be given.*

(i) *There exists* $M>0$ *such that if* u *is the solution of* $\mathrm{P}[\Lambda, \beta; 0, \gamma; q]$, *then*

$$D^q[u] \leqq M \int_\Lambda \gamma^2 d\omega.$$

(ii) *Given a compact set* Z *in* X, *there exists* $M_Z>0$ *such that if* u *is the solution of* $\mathrm{P}[\Lambda, \beta; 0, \gamma; q]$, *then*

$$|u(x)|^2 \leqq M_Z \int_\Lambda \gamma^2 d\omega$$

for all $x \in Z$.

PROOF. (i) First, we consider the case $q \neq 0$ or the case $q=0$ and $\omega(\Gamma-\Lambda)>0$. Let $u=H_\psi^q$. By Lemmas 5.6, 5.1 and 5.3,

$$\begin{aligned} D^q[u] &= -\int_\Lambda (\beta\psi+\gamma)\psi d\omega \leqq -\int_\Lambda \gamma\psi d\omega \\ &\leqq \left(\int_\Lambda \psi^2 d\omega\right)^{1/2}\left(\int_\Lambda \gamma^2 d\omega\right)^{1/2} \\ &\leqq M^{1/2} D^q[u]^{1/2}\left(\int_\Lambda \gamma^2 d\omega\right)^{1/2}, \end{aligned}$$

where M depends only on Λ and q. Hence $D^q[u] \leqq M \int_\Lambda \gamma^2 d\omega$.

Next, consider the case $q=0$, $\omega(\Gamma-\Lambda)=0$ and $\beta \neq 0$ ω-a.e. In this case $\int (\beta\psi+\gamma) d\omega=0$, since $\psi \equiv 1$ belongs to $\boldsymbol{R}_{BD}(\Lambda)$. Let $c=\left(\int \gamma d\omega\right)/\left(\int \beta d\omega\right)$. Then $\int \beta(\psi-c) d\omega=0$. Let $\boldsymbol{R}_{D,\beta}=\{\varphi \in \boldsymbol{R}_D(\Gamma);\ \int \beta\varphi d\omega=0\}$. We shall show that there exists $M_\beta>0$ such that $\int \varphi^2 d\omega \leqq M_\beta D[H_\varphi]$ for all $\varphi \in \boldsymbol{R}_{D,\beta}$. By Lemma 5.1,

$$\int \varphi^2 d\omega \leqq M_0\{D[H_\varphi]+|H_\varphi(x_0)|^2\}.$$

Therefore, it is enough to show that there exists $M'>0$ such that $|H_\varphi(x_0)|^2 \leqq M' D[H_\varphi]$ for all $\varphi \in \boldsymbol{R}_{D,\beta}$. Suppose this is not true. Then we would find $\varphi_n \in \boldsymbol{R}_{D,\beta}$, $n=1, 2, \cdots$, such that $H_{\varphi_n}(x_0)=1$ for each n and $D[H_{\varphi_n}] \to 0$ $(n \to \infty)$. Since $1-\varphi_n \in \boldsymbol{R}_{D,1}$ and $\int \beta\varphi_n d\omega=0$, we have, using Lemma 5.4,

$$\begin{aligned} 0<\int \beta d\omega &= \int \beta(1-\varphi_n) d\omega \leqq \left(\int \beta d\omega\right)^{1/2}\left(\int \beta(1-\varphi_n)^2 d\omega\right)^{1/2} \\ &\leqq (M(\beta)\int \beta d\omega)^{1/2} D[H_{\varphi_n}]^{1/2} \to 0 \qquad (n \to \infty), \end{aligned}$$

a contradiction.

Now we have

$$\begin{aligned}D[u]&=-\int(\beta\psi+\gamma)(\psi-c)d\omega\\&\leqq c\int\gamma d\omega-\int\gamma(\psi-c)d\omega\\&\leqq\left(\int\gamma d\omega\right)^2\Big/\left(\int\beta d\omega\right)+\left(\int\gamma^2 d\omega\right)^{1/2}\left(\int(\psi-c)^2 d\omega\right)^{1/2}\\&\leqq\left(\int\gamma^2 d\omega\right)\Big/\left(\int\beta d\omega\right)+M_\beta^{1/2}\left(\int\gamma^2 d\omega\right)^{1/2}D[u]^{1/2}.\end{aligned}$$

It follows that

$$D[u]\leqq\left(M_\beta+\frac{2}{\int\beta d\omega}\right)\int\gamma^2 d\omega.$$

(ii) By Proposition 1.7, there exists $K_Z>0$, depending only on Z and q, such that

$$|u(x)|\leqq H^q_{|\psi|}(x)\leqq K_Z\int|\psi|\,d\omega$$

for all $x\in Z$. Hence

$$|u(x)|^2\leqq K_Z^2\int\psi^2 d\omega$$

for all $x\in Z$.

If $q\neq 0$ or if $q=0$ and $\omega(\Gamma-\Lambda)>0$, then Lemmas 5.1 and 5.3 imply $\int\psi^2 d\omega\leqq M_q D^q[u]$. Hence, by the above result, we see that

$$|u(x)|^2\leqq K_Z^2 M_q D^q[u]\leqq K_Z^2 M_q M\int\gamma^2 d\omega.$$

If $q=0$, $w(\Gamma-\Lambda)=0$ and $\beta\not\equiv 0$ ω-a.e., then consider $\psi-c\in\boldsymbol{R}_{D,\beta}$. Using the result in (i), we have

$$\begin{aligned}\int\psi^2 d\omega&\leqq 2\left\{\int(\psi-c)^2 d\omega+c^2\right\}\\&\leqq 2\left\{M_\beta D[u]+\frac{\int\gamma^2 d\omega}{\left(\int\beta d\omega\right)^2}\right\}\\&\leqq 2\left\{M_\beta M+\frac{1}{\left(\int\beta d\omega\right)^2}\right\}\int\gamma^2 d\omega.\end{aligned}$$

Hence

$$|u(x)|^2 \leqq 2K_Z^2\left\{M_\beta M+\frac{1}{\left(\int\beta d\omega\right)^2}\right\}\int\gamma^2 d\omega$$

for all $x \in Z$.

THEOREM 5.7. *Let $\omega(\Gamma-\Lambda)>0$ and $\{\tau_n\}$ be a monotone sequence of ω-measurable functions on $\Gamma-\Lambda$. Suppose all τ_n, $n=1, 2, \cdots$ and $\tau_0=\lim_{n\to\infty}\tau_n$ satisfy* $(\mathrm{T})_q$ *for* $[\Lambda, \beta]$. *Suppose furthermore there exist $\varphi_n \in \boldsymbol{R}_D^q(\Lambda; \tau_n)$, $n=1, 2, \cdots$, such that $D^q[H_{\varphi_n}^q-H_{\varphi_m}^q]\to 0$ $(n, m\to\infty)$. Then the solution u_n of* $\mathrm{P}[\Lambda, \beta; \tau_n, 0; q]$ *converges to the solution u_0 of* $\mathrm{P}[\Lambda, \beta; \tau_0, 0; q]$ *locally uniformly on X and $D^q[u_n-u_0]\to 0$ $(n\to\infty)$.*

PROOF. We assume that $\{\tau_n\}$ is monotone increasing. Let $u_n=H_{\psi_n}^q$, $n=0, 1, \cdots$. By the corollary to Theorem 5.3, $\{u_n\}$ is monotone increasing and $u_n \leqq u_0$ for all n. Next let $v_n=H_{\rho_n}^q$ be the solution of $\mathrm{P}[\Lambda, 0; \tau_n, 0; q]$ for each $n=0, 1, \cdots$. Then it is easy to see that $D^q[v_n-v_m]\leqq D^q[H_{\varphi_n}^q-H_{\varphi_m}^q]$ $\to 0$ $(n, m\to\infty)$. Also, again by the corollary to Theorem 5.3, $\{v_n\}$ is monotone increasing and $v_n \leqq v_0$. Since $\int\beta(\psi_0-\psi_1)^2 d\omega<\infty$ and $\int\beta(\rho_0-\rho_1)^2 d\omega<\infty$ by condition $(\mathrm{T})_q$ for τ_0 and τ_1, we have $\int\beta(\psi_n-\psi_m)(\rho_n-\rho_m)d\omega\to 0$ $(n, m\to\infty)$. Since $\psi_n-\rho_n \in \boldsymbol{R}_D^q(\Lambda)$ for each n, Lemma 5.6 implies

$$D^q[u_n-u_m, (u_n-u_m)-(v_n-v_m)]=-\int\beta(\psi_n-\psi_m)[(\psi_n-\psi_m)-(\rho_n-\rho_m)]d\omega$$
$$\leqq\int\beta(\psi_n-\psi_m)(\rho_n-\rho_m)d\omega.$$

Thus we have

$$0\leqq D^q[(u_n-u_m)-(v_n-v_m)]$$
$$\leqq D^q[v_n-v_m]-D^q[u_n-u_m]+\int\beta(\psi_n-\psi_m)(\rho_n-\rho_m)d\omega.$$

Therefore

$$D^q[u_n-u_m]\leqq D^q[v_n-v_m]+\int\beta(\psi_n-\psi_m)(\rho_n-\rho_m)d\omega\to 0 \qquad (n, m\to\infty).$$

Hence $\{u_n\}$ is a Cauchy sequence in $\boldsymbol{H}_D^q$.

Next let $\tilde{u}=\lim_{n\to\infty}u_n$ and $\tilde{\psi}=\lim_{n\to\infty}\psi_n$. Then Proposition 3.1, (iii) implies that $\tilde{u}=H_{\tilde{\psi}}^q$. By Lemma 4.8, we see that $\tilde{u}\in\boldsymbol{H}_D^q$ and $D^q[u_n-\tilde{u}]\to 0$ $(n\to\infty)$. Obviously $\tilde{\psi}=\tau_0$ ω-a.e. on $\Gamma-\Lambda$. By Theorem 4.1,

$$D^q[u_n, H_\varphi^q]=-\int\beta\psi_n\varphi d\omega$$

for all $\varphi \in \boldsymbol{R}_{BD}(A)$. Letting $n\to\infty$, we obtain

$$D^q[\tilde{u}, H_\varphi^q] = -\int \beta\tilde{\psi}\varphi d\omega$$

for all $\varphi \in \boldsymbol{R}_{BD}(A)$. Hence $\tilde{u}$ is the solution of $\mathrm{P}[A, \beta; \tau_0, 0; q]$, i.e., $\tilde{u}=u_0$.

COROLLARY. *Let $\omega(\Gamma-A)>0$ and let τ be an ω-measurable function on $\Gamma-A$ satisfying $(\mathrm{T})_q$. Let $\tau_n=\max(\min(\tau, n), -n)$ on $\Gamma-A$. Then the solution u_n of $\mathrm{P}[A, \beta; \tau_n, 0; q]$ converges to the solution u_0 of $\mathrm{P}[A, \beta; \tau, 0; q]$ locally uniformly and $D^q[u_n-u_0]\to 0$ $(n\to\infty)$.*

PROOF. First suppose $\tau\geqq 0$. It is easy to verify that each τ_n satisfies $(\mathrm{T})_q$. Let $u_0=H_{\psi_0}^q$ and let $\varphi_n=\min(\psi_0, n)$. Then $\varphi_n \in \boldsymbol{R}_D^q(A; \tau_n)$ for each n and $D^q[H_{\varphi_n}^q-H_{\varphi_m}^q]\to 0$ $(n, m\to\infty)$ by Lemma 4.11. Hence our corollary follows from the theorem. If τ is arbitrary, then it is enough to consider τ^+ and

§5.9. Dependence of the solution on boundary condition

Theorem 5.4 gives one result on the dependence of the solution of our problem on boundary condition $[A, \beta]$. We shall give two more results in this direction.

THEOREM 5.8. *Let $\{A_n\}$ be a monotone decreasing sequence of ω-measurable subsets of Γ and let $A_0=\bigcap_{n=1}^\infty A_n$. In case $q=0$, we further assume that $\omega(\Gamma-A_1)>0$. Let γ_1 be an ω-measurable function on A_1 such that $|\gamma_1|$ satisfies $(\Gamma)_q$ with respect to $[A_1, 0]$ and let γ_n be the restriction of γ_1 to A_n $(n=0, 1, \dots)$. Then the solution u_n of $\mathrm{P}[A_n, 0; 0, \gamma_n; q]$ converges to the solution u_0 of $\mathrm{P}[A_0, 0; 0, \gamma_0; q]$ locally uniformly on X and $D^q[u_n-u_0]\to 0$ $(n\to\infty)$.*

PROOF. Obviously, each γ_n satisfies $(\Gamma)_q$ with respect to $[A_n, 0]$, $n=0, 1, \dots$. Let $u_n=H_{\psi_n}^q$, $n=0, 1, \dots$. First suppose $\gamma_1\leqq 0$ on A_1. By Theorems 5.3 and 5.4, $\{u_n\}$ is a monotone decreasing sequence of non-negative functions. Thus $\tilde{\psi}=\lim\psi_n$ and $\tilde{u}=\lim u_n$ exist and $\tilde{u}=H_{\tilde{\psi}}^q$ by Proposition 3.1, (iii). Obviously $\tilde{\psi}=0$ ω-a.e. on $\Gamma-A_0$. If $n<m$, then $\psi_n\geqq\psi_m$ (ω-a.e.) and $\psi_m \in \boldsymbol{R}_D^q(A_m)\subset \boldsymbol{R}_D^q(A_n)$. Since $\gamma_1\leqq 0$, we have, by Lemma 5.6,

$$D^q[u_n] = -\int_{A_n}\gamma_1\psi_n d\omega \geqq -\int_{A_m}\gamma_1\psi_m d\omega = D^q[u_m].$$

It follows that $\{D^q[u_n]\}$ is convergent. Also we have

$$D^q[u_n, u_m] = -\int_{A_n}\gamma_1\psi_m d\omega = -\int_{A_m}\gamma_1\psi_m d\omega = D^q[u_m].$$

Hence

$$0 \leqq D^q[u_n - u_m] = D^q[u_n] - D^q[u_m] \to 0 \qquad (n, m \to \infty).$$

Thus $\{u_n\}$ is a Cauchy sequence in $\boldsymbol{H}_D^q$. It follows that $\tilde{u} \in \boldsymbol{H}_D^q$, i.e., $\tilde{\psi} \in \boldsymbol{R}_D^q(\Gamma)$ and that $D^q[u_n - \tilde{u}] \to 0$ $(n \to \infty)$. Since $\boldsymbol{R}_D^q(\Lambda_0) \subset \boldsymbol{R}_D^q(\Lambda_n)$ for any n, Theorem 4.1 implies

$$D^q[u_n, H_\varphi^q] = -\int_{\Lambda_n} \gamma_1 \varphi d\omega = -\int_{\Lambda_1} \gamma_1 \varphi d\omega$$

for any $\varphi \in \boldsymbol{R}_{BD}(\Lambda_0)$ and for any n. Letting $n \to \infty$, we obtain

$$D^q[\tilde{u}, H_\varphi^q] = -\int_{\Lambda_1} \gamma_1 \varphi d\omega = -\int_{\Lambda_0} \gamma_0 \varphi d\omega$$

for any $\varphi \in \boldsymbol{R}_{BD}(\Lambda_0)$. Therefore $\tilde{u}$ is a solution of $\mathrm{P}[\Lambda_0, 0; 0, \gamma_0; q]$, i.e., $\tilde{u} = u_0$.

If γ_1 is not necessarily non-positive, then we consider $-\gamma_1^+$ and $-\gamma_1^-$ and obtain the required result.

THEOREM 5.9. *Let Λ be an ω-measurable subset of Γ and let $\{\beta_n\}$ be a monotone sequence (increasing or decreasing) of non-negative ω-measurable functions on Λ such that $[\Lambda, \beta_n]$ satisfies condition $(\mathrm{B})_q$ for each $n = 0, 1, \cdots$, where $\beta_0 = \lim_{n\to\infty} \beta_n$. In case $q = 0$ and $\omega(\Gamma - \Lambda) = 0$, we further assume that $\beta_n \not\equiv 0$ ω-a.e. for any $n = 0, 1, \cdots$. Let τ be a function on $\Gamma - \Lambda$ satisfying $(\mathrm{T})_q$ with respect to all $[\Lambda, \beta_n]$, $n = 0, 1, \cdots$ and γ be a function on Λ such that $|\gamma|$ satisfies $(\Gamma)_q$ with respect to $[\Lambda, \beta_0]$. If u_n is the solution of $\mathrm{P}[\Lambda, \beta_n; \tau, \gamma; q]$ for each $n = 0, 1, \cdots$, then u_n tends to u_0 locally uniformly on X and $D^q[u_n - u_0] \to 0$ as $n \to \infty$.*

PROOF. Let $u_n = H_{\psi_n}^q$, $n = 0, 1, \cdots$. First suppose $\tau \geqq 0$ on $\Gamma - \Lambda$ and $\gamma \leqq 0$ on Λ. If β_n increases to β_0 (resp. decreases to β_0), then $\{u_n\}$ is a monotone decreasing sequence (resp. monotone increasing sequence, dominated by u_0) of non-negative functions by Theorems 5.3 and 5.4. Hence $\tilde{\psi} = \lim \psi_n$ and $\tilde{u} = \lim u_n$ exist and $\tilde{u} = H_{\tilde{\psi}}^q$. Obviously, $\tilde{\psi} = \tau$ ω-a.e. on $\Gamma - \Lambda$. Since $\psi_n - \psi_m \in \boldsymbol{R}_D^q(\Lambda)$, Lemma 5.6 implies

$$D^q[u_n - u_m, u_n] = -\int_\Lambda (\beta_n \psi_n + \gamma)(\psi_n - \psi_m) d\omega$$

for any n, m. Hence

$$D^q[u_n - u_m] = -\int_\Lambda (\beta_n \psi_n - \beta_m \psi_m)(\psi_n - \psi_m) d\omega.$$

Let $\beta^* = \beta_0$ (resp. $= \beta_1$) and $\psi^* = \psi_1$ (resp. $= \tilde{\psi}$). Then $|\beta_n \psi_n - \beta_m \psi_m| \leqq \beta^* \psi^*$ (ω-a.e.). By conditions $(\mathrm{B})_q$ and $(\mathrm{T})_q$, we see that $J = \int_\Lambda \beta^* (\psi^*)^2 d\omega < \infty$. Hence

$$\begin{aligned} D^q[u_n - u_m] &\leqq \int_\Lambda \beta^* \psi^* |\psi_n - \psi_m| d\omega \\ &\leqq J^{1/2} \left[\int_\Lambda \beta^* (\psi_n - \psi_m)^2 d\omega \right]^{1/2}. \end{aligned}$$

Now ψ_n converges to $\tilde{\psi}$, $|\psi_n-\psi_m|\leqq|\psi_1-\tilde{\psi}|$ and $\int_A \beta^*(\psi_1-\tilde{\psi})^2 d\omega<\infty$ by $(\mathrm{B})_q$. Hence $\int_A \beta^*(\psi_n-\psi_m)^2 d\omega\to 0$ $(n, m\to\infty)$. Therefore $D^q[u_n-u_m]\to 0$ $(n, m\to\infty)$, i.e., $\{u_n\}$ is a Cauchy sequence in $\boldsymbol{H}_D^q$. It then follows that $\tilde{u}\in\boldsymbol{H}_D^q$ and $D^q[u_n-\tilde{u}]\to 0$ $(n\to\infty)$. For any $\varphi\in\boldsymbol{R}_{BD}(A)$.

$$D^q[u_n, H_\varphi^q]=-\int_A(\beta_n\psi_n+\gamma)\varphi d\omega.$$

By the above result, $D^q[u_n, H_\varphi^q]\to D^q[\tilde{u}, H_\varphi^q]$ $(n\to\infty)$. On the other hand, since $\int\beta^*(\varphi^*)^2 d\omega<\infty$, $\int\beta^*\varphi^2 d\omega<\infty$ by $(\mathrm{B})_q$, $0\leqq\beta_n\psi_n\leqq\beta^*\psi^*$ and $\beta_n\psi_n\to\beta_0\tilde{\psi}$, the Lebesgue convergence theorem implies

$$\int_A\beta_n\psi_n\varphi d\omega\to\int_A\beta_0\tilde{\psi}\varphi d\omega \qquad (n\to\infty).$$

Therefore

$$D^q[\tilde{u}, H_\varphi^q]=-\int_A(\beta_0\tilde{\psi}+\gamma)\varphi d\omega$$

for all $\varphi\in\boldsymbol{R}_{BD}(A)$. Hence we have $\tilde{u}=u_0$ by Theorem 4.1 and the present theorem is proved in case $\tau\geqq 0$ and $\gamma\leqq 0$.

In the general case, we consider problems $\mathrm{P}[A, \beta_n; \tau^+, -\gamma^-; q]$ and $\mathrm{P}[A, \beta_n; \tau^-, -\gamma^+; q]$ and obtain the theorem.

§5.10. Dependence of the solution on q

Finally we investigate the dependence of the solution of our problem on q.

THEOREM 5.10. *Let $q_1\leqq q_2$ and suppose $[A, \beta]$ is given to satisfy* $(\mathrm{B})_{q_1}$; *in case $q_1=0$ and $\omega(\Gamma-A)=0$, suppose further that $\beta\not\equiv 0$ ω-a.e. Let τ* (resp. γ) *be a non-negative* (resp. *non-positive*) *function on $\Gamma-A$* (resp. *on A*) *satisfying* $(\mathrm{T})_{q_1}$ (resp. $(\Gamma)_{q_1}$) *with respect to $[A, \beta]$. If u_i is the solution of* $\mathrm{P}[A, \beta; \tau, \gamma; q_i]$ *for each $i=1$, 2 and if u_i are both bounded, then $u_1\geqq u_2$.*

PROOF. By Proposition 4.3, $\boldsymbol{R}_D^{q_2}(\Gamma)\subset\boldsymbol{R}_D^{q_1}(\Gamma)$. Hence $[A, \beta]$ satisfies also $(\mathrm{B})_{q_2}$. Since u_1 is bounded, τ is (ω-essentially) bounded. Hence Lemma 4.14 implies that τ satisfies $(\mathrm{T})_{q_2}$ and Lemma 4.14 and Proposition 4.3 imply that γ satisfies $(\Gamma)_{q_2}$. By Theorem 5.3, u_i are non-negative. Let $u_i=H_{\psi_i}^{q_i}$, $i=1$, 2. Since $\psi_1=\psi_2=\tau$ ω-a.e. on $\Gamma-A$, $\psi_1-\psi_2$ and $\psi^*=\max(\psi_1-\psi_2, 0)$ both belong to $\boldsymbol{R}_{BD}(A)$. Hence

$$D^{q_1}[u_1, H_{\psi^*-(\psi_1-\psi_2)}^{q_1}]=-\int_A(\beta\psi_1+\gamma)[\psi^*-(\psi_1-\psi_2)]d\omega$$

by Theorem 4.1. On the other hand, $D^{q_1}[H^{q_1}_{\psi^*}] \leqq D^{q_1}[H^{q_1}_{\psi_1-\psi_2}]$ by Lemma 4.9. Therefore

$$(5.8)\quad D^{q_1}[H^{q_1}_{\psi^*-(\psi_1-\psi_2)}] \leqq 2D^{q_1}[H^{q_1}_{\psi_1-\psi_2}] - 2D^{q_1}[H^{q_1}_{\psi^*}, u_1] + 2D^{q_1}[H^{q_1}_{\psi^*}, H^{q_1}_{\psi_2}]$$
$$= 2\{D^{q_1}[u_1, H^{q_1}_{\psi_1-\psi_2}] - D^{q_1}[u_1, H^{q_1}_{\psi^*}] + 2D^{q_1}[H^{q_1}_{\psi^*-(\psi_1-\psi_2)}, H^{q_1}_{\psi_2}]\}$$
$$= 2\int_A (\beta\psi_1+\gamma)[\psi^*-(\psi_1-\psi_2)]d\omega + 2D^{q_1}[H^{q_1}_{\psi^*-(\psi_1-\psi_2)}, H^{q_1}_{\psi_2}].$$

By Lemma 4.14, we have

$$D^{q_1}[H^{q_1}_{\psi^*-(\psi_1-\psi_2)}, H^{q_1}_{\psi_2}] = D^{q_1}[H^{q_1}_{\psi^*-(\psi_1-\psi_2)}, u_2].$$

Since $H^{q_1}_{\psi^*-(\psi_1-\psi_2)} \geqq 0$, $u_2 \geqq 0$ and $q_1 \leqq q_2$, we have

$$D^{q_1}[H^{q_1}_{\psi^*-(\psi_1-\psi_2)}, u_2] \leqq D^{q_2}[H^{q_1}_{\psi^*-(\psi_1-\psi_2)}, u_2].$$

Using Lemma 4.14 again, we have

$$D^{q_2}[H^{q_1}_{\psi^*-(\psi_1-\psi_2)}, u_2] = D^{q_2}[H^{q_2}_{\psi^*-(\psi_1-\psi_2)}, u_2]$$
$$= -\int_A (\beta\psi_2+\gamma)[\psi^*-(\psi_1-\psi_2)]d\omega.$$

Hence, together with (5.8), we obtain

$$0 \leqq \|H^{q_1}_{\psi^*-(\psi_1-\psi_2)}\|^2_{\beta,q_1}$$
$$\leqq 2\int_A \beta(\psi_1-\psi_2)[\psi^*-(\psi_1-\psi_2)]d\omega + \int_A \beta[\psi^*-(\psi_1-\psi_2)]^2 d\omega$$
$$= \int_A \beta[\psi^*+(\psi_1-\psi_2)][\psi^*-(\psi_1-\psi_2)]d\omega \leqq 0.$$

Therefore we have $H^{q_1}_{\psi^*-(\psi_1-\psi_2)} = 0$. Thus, using Lemma 3.2, we obtain

$$u_1 = H^{q_1}_{\psi_1} = H^{q_1}_{\psi_2} + H^{q_1}_{\psi^*} \geqq H^{q_1}_{\psi_2} \geqq H^{q_2}_{\psi_2} = u_2.$$

The assumption that u_i are bounded can be eliminated; we shall prove this in the next chapter (§6.4). We also postpone to the next chapter the discussion on the convergence of solutions according to a monotone convergence of q's.

CHAPTER VI Green Functions for General Mixed Problems.

§6.1. α-q-Green function

We consider an ω-measurable function $\alpha(\xi)$ on Γ such that $0 \leqq \alpha \leqq 1$ and regard it as a boundary condition equivalent to $[\Lambda_\alpha, \beta_\alpha]$, where $\Lambda_\alpha = \{\xi \in \Gamma;\ \alpha(\xi) > 0\}$ and $\beta_\alpha(\xi) = 1/\alpha(\xi) - 1$ on Λ_α. The condition $(B)_q$ for $[\Lambda_\alpha, \beta_\alpha]$ becomes

$$(\mathrm{A})_q: \qquad \int_{\Lambda_\alpha}\left(\frac{1}{\alpha}-1\right)\varphi^2 d\omega < \infty \qquad \text{for all} \quad \varphi \in \boldsymbol{R}_D^q(\Lambda_\alpha).$$

In this chapter, we shall always assume that α satisfies $(\mathrm{A})_q$ for a given q. In case $q=0$, we also assume that $\alpha \neq 1$ ω-a.e. on Γ.

For each $y \in X$, consider the non-negative bounded ω-measurable function χ_y^q on Γ satisfying $d\omega_y^q = \chi_y^q d\omega$ (cf. Corollary 3 to Theorem 3.1). By Theorems 5.1 and 5.2, there exists a unique solution $U_{\alpha,y}^q$ of the problem $\mathrm{P}[\Lambda_\alpha, \beta_\alpha; 0, -c_d\chi_y^q; q]$ for each $y \in X$. Here, we remark that $-c_d\chi_y^q$ is considered only on Λ_α and that it satisfies $(\Gamma)_q$ with respect to $[\Lambda_\alpha, \beta_\alpha]$, since it is bounded (cf. Remark, (g) in §5.5). Let $U_{\alpha,y}^q = H_{\Phi_{\alpha,y}^q}^q$. $\Phi_{\alpha,y}^q$ is determined ω-a.e. and belongs to $\boldsymbol{R}_D^q(\Lambda_\alpha)$.

The function

$$G_\alpha^q(x, y) \equiv G_{\alpha,y}^q(x) = G_y^q(x) + U_{\alpha,y}^q(x)$$

is called the *α-q-Green function*. This is the Green function for the boundary condition α. In fact $\Phi_{\alpha,y}^q$ can be regarded as the boundary value of $G_{\alpha,y}^q$, which vanishes ω-a.e. on $\Gamma - \Lambda_\alpha$ and Proposition 4.4 implies

Proposition 6.1. *For each $y \in X$, $G_{\alpha,y}^q$ has a normal derivative $\beta_\alpha \Phi_{\alpha,y}^q$ on Λ_α.*

In case $\alpha=0$ (i.e., the case of the Dirichlet problem), $\Lambda_\alpha = \Gamma$, and hence $U_{0,y}^q = 0$, i.e., $G_{0,y}^q = G_y^q$, for each $y \in X$.

The following lemma, which is an immediate consequence of the definition of $U_{\alpha,y}^q$, is fundamental in the subsequent discussions:

Lemma 6.1. *Let $y \in X$. Then $U_{\alpha,y}^q$ satisfies*

$$c_d H_\varphi^q(y) = D^q[U_{\alpha,y}^q, H_\varphi^q] + \int \beta_\alpha \Phi_{\alpha,y}^q \varphi\, d\omega$$

for all $\varphi \in \boldsymbol{R}_D^q(\Lambda_\alpha)$. Conversely, if $u = H_\psi^q$, $\psi \in \boldsymbol{R}_D^q(\Lambda_\alpha)$, satisfies

$$c_d H_\varphi^q(y) = D^q[u, H_\varphi^q] + \int \beta_\alpha \psi\varphi\, d\omega$$

for all $\varphi \in \boldsymbol{R}_{BD}(\Lambda_\alpha)$, then $u = U_{\alpha,y}^q$.

Proof. By the definition of $U_{\alpha,y}^q$ and Theorem 4.1, $u = U_{\alpha,y}^q$ $(u = H_\psi^q)$ if and only if

$$D^q[u, H_\varphi^q] = -\int (\beta_\alpha\psi - c_d\chi_y^q)\varphi\, d\omega$$

for all $\varphi \in \boldsymbol{R}_{BD}(\Lambda_\alpha)$. Since χ_y^q is bounded, $\boldsymbol{R}_{BD}(\Lambda_\alpha)$ can be replaced by $\boldsymbol{R}_D^q(\Lambda_\alpha)$ by Lemma 4.11 (cf. Lemma 5.6). On the other hand

$$-\int(\beta_\alpha\psi - c_d\chi_y^q)\varphi\, d\omega = -\int \beta_\alpha\psi\varphi\, d\omega + c_d H_\varphi^q(y).$$

Hence we have the lemma.

§6.2. Expression of the solution in terms of $U^q_{\alpha,x}$

THEOREM 6.1. *Let τ and γ be functions on $\Gamma - A_\alpha$ and on A_α respectively, satisfying* $(\mathrm{T})_q$ *and* $(\Gamma)_q$ *with respect to* $[A_\alpha, \beta_\alpha]$. *Then the solution u_0 of* $\mathrm{P}[A_\alpha, \beta_\alpha; \tau, \gamma; q]$ *is given by*

$$(6.1)\quad u_0(x) = H^q_\psi(x) - \frac{1}{c_d} D^q[U^q_{\alpha,x}, H^q_\psi] - \frac{1}{c_d}\int_{A_\alpha} \beta_\alpha \psi \Phi^q_{\alpha,x} d\omega - \frac{1}{c_d}\int_{A_\alpha} \gamma \Phi^q_{\alpha,x} d\omega$$

for any $x \in X$, where ψ is any function in $\boldsymbol{R}^q_D(A_\alpha; \tau)$. In particular, $u_0 = H^q_{\psi_0}$ satisfies

$$(6.2)\quad D^q[U^q_{\alpha,x}, u_0] + \int_{A_\alpha} \beta_\alpha \psi_0 \Phi^q_{\alpha,x} d\omega + \int_{A_\alpha} \gamma \Phi^q_{\alpha,x} d\omega = 0$$

for any $x \in X$.

REMARK. The right hand side of (6.1) does not depend on the choice of $\psi \in \boldsymbol{R}^q_D(A_\alpha; \tau)$, by virtue of Lemma 6.1.

PROOF of the THEOREM. Since $\Phi^q_{\alpha,x} \in \boldsymbol{R}^q_D(A_\alpha)$, (6.2) is an immediate consequence of the fact that u_0 is the solution of $\mathrm{P}[A_\alpha, \beta_\alpha; \tau, \gamma; q]$ and Lemma 5.6. On the other hand, $\psi_0 - \psi \in \boldsymbol{R}^q_D(A_\alpha)$ for any $\psi \in \boldsymbol{R}^q_D(A_\alpha; \tau)$. Hence Lemma 6.1 implies

$$c_d\{u_0(x) - H^q_\psi(x)\} = D^q[U^q_{\alpha,x}, u_0] - D^q[U^q_{\alpha,x}, H^q_\psi] + \int \beta_\alpha \psi_0 \Phi^q_{\alpha,x} d\omega - \int \beta_\alpha \psi \Phi^q_{\alpha,x} d\omega.$$

Hence, using (6.2), we obtain (6.1).

The converse of the last half of this theorem immediately follows from (6.1) and (6.2):

COROLLARY. *Let τ and γ be as in the above theorem. If $\psi_0 \in \boldsymbol{R}^q_D(A_\alpha; \tau)$ and $u_0 = H^q_{\psi_0}$ satisfies* (6.2) *for all $x \in X$, then u_0 is the solution of* $\mathrm{P}[A_\alpha, \beta_\alpha; \tau, \gamma; q]$.

THEOREM 6.2. *Let τ and γ be as in the previous theorem. Suppose, for each $x \in X$, $G^q_{\alpha,x}$ has a normal derivative γ_x on Γ such that it coincides with $\beta \Phi^q_{\alpha,x}$ on A_α and $\gamma_x \tau$ is ω-summable on $\Gamma - A_\alpha$. Then the solution u_0 of* $\mathrm{P}[A_\alpha, \beta_\alpha; \tau, \gamma; q]$ *is given by*

$$u_0(x) = \frac{1}{c_d}\left\{\int_{\Gamma - A_\alpha} \tau \gamma_x d\omega - \int_{A_\alpha} \gamma \Phi^q_{\alpha,x} d\omega\right\}.$$

PROOF. Since G^q_x has a normal derivative $c_d \chi^q_x$ on Γ (Proposition 4.4), $U^q_{\alpha,x}$ has a normal derivative $\gamma_x - c_d \chi^q_x$ on Γ. By assumption, $(\gamma_x - c_d \chi^q_x)\psi$ is

ω-summable on Γ for any $\psi \in \boldsymbol{R}_D^q(\Lambda_\alpha; \tau)$. Hence, by the corollary to Theorem 4.1,

$$D^q[U_{\alpha,x}^q, H_\psi^q] = -\int (\gamma_x - c_d \chi_x^q)\psi d\omega$$

$$= -\int_{\Gamma - \Lambda_\alpha} \gamma_x \tau d\omega - \int_{\Lambda_\alpha} \beta_\alpha \Phi_{\alpha,x}^q \psi d\omega + c_d H_\psi^q(x)$$

for any $\psi \in \boldsymbol{R}_D^q(\Lambda_\alpha; \tau)$. Then we obtain the theorem by virtue of (6.1) in the previous theorem.

Remark. If $\hat{X}$ is a regular compactification and if Λ_α is relatively open, then a normal derivative γ_x of $G_{\alpha,x}^q$, if it exists, is equal to $\beta \Phi_{\alpha,x}^q$ ω-a.e. on Λ_α (cf. Propositions 6.1 and 4.5).

§6.3. Properties of $U_{\alpha,x}^q$

Theorem 6.3. *$U_{\alpha,x}^q(y) = U_{\alpha,y}^q(x)$ and $G_\alpha^q(x, y) = G_\alpha^q(y, x)$ for any $x, y \in X$.*

Proof. By Lemma 6.1,

$$c_d U_{\alpha,x}^q(y) = D^q[U_{\alpha,y}^q, U_{\alpha,x}^q] + \int \beta_\alpha \Phi_{\alpha,x}^q \Phi_{\alpha,y}^q d\omega.$$

Hence $U_{\alpha,x}^q(y) = U_{\alpha,y}^q(x)$. Then, by the corollary to Theorem 2.1, $G_\alpha^q(x, y) = G_\alpha^q(y, x)$.

Theorem 6.4. *If $\alpha_1 \leqq \alpha_2$ on Γ, then*

$$0 \leqq U_{\alpha_1,x}^q \leqq U_{\alpha_2,x}^q$$

for any $x \in X$.

Proof. By Theorem 5.3, $U_{\alpha,x}^q \geqq 0$ for any α. If $\alpha_1 \leqq \alpha_2$, then $\Lambda_{\alpha_1} \subset \Lambda_{\alpha_2}$ and $\beta_{\alpha_1} \geqq \beta_{\alpha_2}$ on Λ_{α_1}. Hence, by Theorem 5.4, $U_{\alpha_1,x}^q \leqq U_{\alpha_2,x}^q$.

Theorem 6.5. *$U_{\alpha,x}^q$ is a bounded function for each $x \in X$.*

Proof. Fix q, α and x and let $U = U_{\alpha,x}^q$. Let λ_0 be a positive number such that $V_0 = \{y \in X; G_x^q(y) \geqq \lambda_0\}$ is compact and let $g_\lambda = \min(G_x^q, \lambda)$ for $\lambda \geqq \lambda_0$. We shall write g_0 for g_{λ_0}. Let $\lambda_1 = \sup_{y \in V_0} U(y)$. Then $0 \leqq \lambda_1 < \infty$. We consider two functions $v = U + g_0$ and $v_1 = \min(\lambda_0 + \lambda_1, v)$. Both v and v_1 are q-superharmonic, $v = G_{\alpha,x}^q$ on $X - V_0$ and $v = v_1$ on V_0. Let $v_1 = u_1 + g^*$, where u_1 is q-harmonic and g^* is a q-potential. Then it is easy to see that $u_1 = H_{\phi_1}^q$ with $\phi_1 = \min(\lambda_0 + \lambda_1, \Phi_{\alpha,x}^q)$ (cf. Lemma 3.5). By Lemma 4.12, $v \in \boldsymbol{D}^q$, and it follows that $v_1 \in \boldsymbol{D}^q$ and $D^q[v_1] \leqq D^q[v]$ (cf. Lemma 4.10). Therefore, $u_1 \in \boldsymbol{H}_D^q$ i.e., $\phi_1 \in \boldsymbol{R}_D^q(\Gamma)$, and $g^* \in \boldsymbol{D}_0^q$ by Lemma 4.6. Now we compute

$$D^q[v, v-v_1] = D^q[U+g_0, U-u_1+g_0-g^*]$$
$$= D^q[U, U-u_1]+D^q[g_0, v-v_1].$$

Since $\Phi^q_{\alpha,x}-\psi_1 \in \boldsymbol{R}^q_D(\Lambda_\alpha)$, Lemma 6.1 implies

$$(6.3) \qquad D^q[U, U-u_1] = c_d\{U(x)-u_1(x)\} - \int \beta_\alpha \Phi^q_{\alpha,x}(\Phi^q_{\alpha,x}-\psi_1)d\omega$$
$$\leqq c_d\{U(x)-u_1(x)\}.$$

Since $v-v_1=0$ on V_0, $D^q[g_0, v-v_1]=D^q[g_\lambda, v-v_1]$ for any $\lambda \geqq \lambda_0$. We have

$$D^q[g_\lambda, v-v_1] = D^q[g_\lambda, g_0-g^*] = D^q[g_\lambda, g_0]-D^q[g_\lambda, g^*].$$

Since g_0 and g^* are continuous q-potentials, we can show, by using Lemma 4.12 (cf. the methods in the proofs of Hilfssatz 7.5 and Satz 7.2 of [7]), that $D^q[g_\lambda, g_0] \to c_d g_0(x)$ and $D^q[g_\lambda, g^*] \to c_d g^*(x)$ as $\lambda \to \infty$. Hence

$$(6.4) \qquad D^q[g_0, v-v_1] = c_d\{g_0(x)-g^*(x)\}.$$

By (6.3) and (6.4), we have

$$D^q[v, v-v_1] \leqq c_d\{U(x)-u_1(x)+g_0(x)-g^*(x)\}$$
$$= c_d\{v(x)-v_1(x)\} = 0.$$

Hence, $D^q[v_1] \leqq D^q[v]$ implies

$$0 \leqq D^q[v-v_1] \leqq D^q[v_1]-D^q[v] \leqq 0.$$

It follows that $v=v_1$. Since v_1 is bounded, so is v. Since $0 \leqq U \leqq v$, U is also bounded.

COROLLARY. *For any compact set Z in X, there exists a constant $K_Z>0$ such that $U^q_{\alpha,y} \leqq K_Z$ for all $y \in Z$.*

PROOF. This follows from Proposition 1.7, Theorem 6.3 and the above theorem.

The next theorem follows immediately from Theorems 5.8 and 5.9:

THEOREM 6.6. *Let $\{\alpha_n\}$ be a monotone sequence of boundary conditions and let $\alpha=\lim_{n\to\infty}\alpha_n$. If $q=0$, then we assume $\alpha_n \not\equiv 0$ ω-a.e. for all $n=0, 1, \dots$. In each of the following two cases, $U^q_{\alpha_n,x}$ tends to $U^q_{\alpha,x}$ locally uniformly on X and $D^q[U^q_{\alpha_n,x}-U^q_{\alpha,x}] \to 0$ $(n\to\infty)$ for any $x \in X$;*

(a) *$\{\alpha_n\}$ is decreasing and each α_n is a characteristic function of a subset Λ_n of Γ.*

(b) *$\Lambda_{\alpha_n} \equiv \Lambda_\alpha$ for all n.*

§6.4. Dependence of $U^q_{\alpha,x}$ on q

THEOREM 6.7. *Let $q_1 \leqq q_2$; if $q_1 = 0$, then assume that $\alpha \neq 1$ ω-a.e. Then*

$$U^{q_1}_{\alpha,x} \geqq U^{q_2}_{\alpha,x} \quad \textit{and} \quad G^{q_1}_{\alpha,x} \geqq G^{q_2}_{\alpha,x}$$

for any $x \in X$.

PROOF. By Lemma 3.2, we have $\chi^{q_1}_x \geqq \chi^{q_2}_x$. Let v be the solution of $P[\Lambda_\alpha, \beta_\alpha; 0, -c_d\chi^{q_2}_x; q_1]$. Theorems 5.3 and 5.4 imply $0 \leqq v \leqq U^{q_1}_{\alpha,x}$. Thus, by Theorem 6.5, v and $U^{q_2}_{\alpha,x}$ are both bounded. Hence Theorem 5.10 implies $U^{q_2}_{\alpha,x} \leqq v$, and hence $U^{q_2}_{\alpha,x} \leqq U^{q_1}_{\alpha,x}$. The inequality $G^{q_1}_{\alpha,x} \leqq G^{q_2}_{\alpha,x}$ then follows from Theorem 2.3.

Next we give an improvement of Theorem 5.10:

THEOREM 5.10$'$. *Under the same assumptions as in Theorem* 5.10, *we have $u_1 \geqq u_2$ even if u_1, u_2 are not bounded.*

PROOF. Let $\tau_n = \min(\tau, n)$, $n = 1, 2, \cdots$ and let $u^{(i)}_n$ be the solution of $P[\Lambda, \beta; \tau_n, 0; q_i]$ for $i = 1, 2$. Then, by Theorem 5.5, $u^{(i)}_n$ are bounded. Hence Theorem 5.10 implies $u^{(1)}_n \geqq u^{(2)}_n$ for each n. On the other hand, the corollary to Theorem 5.7 implies that $\{u^{(i)}_n\}$ tends to the solution v_i of $P[\Lambda, \beta; \tau, 0; q_i]$ as $n \to \infty$ for each $i = 1, 2$. Hence $v_1 \geqq v_2$.

Next let w_i be the solution of $P[\Lambda, \beta; 0, \gamma; q_i]$, $i = 1, 2$. Let $\alpha = 0$ on $\Gamma - \Lambda$ and $\alpha = 1/(1+\beta)$ on Λ. Then $\Lambda = \Lambda_\alpha$ and $\beta = \beta_\alpha$. Therefore, by Theorem 6.1, we have

$$w_i(x) = -\frac{1}{c_d}\int_\Lambda \gamma \Phi^{q_i}_{\alpha,x} d\omega \qquad (i = 1, 2)$$

for all $x \in X$. Since $\Phi^{q_1}_{\alpha,x} \geqq \Phi^{q_2}_{\alpha,x}$ ω-a.e. by the above theorem and since $\gamma \leqq 0$, we obtain $w_1(x) \geqq w_2(x)$ for all $x \in X$. Since $u_i = v_i + w_i$ $(i = 1, 2)$, we have the theorem.

THEOREM 6.8. *Let $\{q_n\}$ be a monotone increasing* (resp. *decreasing*) *sequence such that $q_n \to q$ $(n \to \infty)$. If $q_1 = 0$* (resp. $q = 0$), *then we assume that $\alpha \neq 1$ ω-a.e. on Γ. Then $U^{q_n}_{\alpha,x}$ decreases* (resp. *increases*) *to $U^q_{\alpha,x}$ and $D[U^{q_n}_{\alpha,x} - U^q_{\alpha,x}] \to 0$ $(n \to \infty)$ for any $x \in X$.*

PROOF. Fix $x \in X$ and let $\psi_n = \Phi^{q_n}_{\alpha,x}$, $U_n = U^{q_n}_{\alpha,x}$. By Theorem 6.7, $\{\psi_n\}$ and $\{U_n\}$ are monotone decreasing (resp. increasing) and are uniformly bounded by Theorem 6.5. Hence $\psi_0 = \lim_{n\to\infty}\psi_n$ and $u_0 = \lim_{n\to\infty} U_n$ exist and $\psi_0 = 0$ ω-a.e. on $\Gamma - \Lambda_\alpha$. Using Lemma 3.2, Proposition 3.1 and Theorem 3.6, we have

$$|U_n - H^q_{\psi_0}| = |H^{q_n}_{\psi_n} - H^{q_n}_{\psi_0}| + |H^{q_n}_{\psi_0} - H^q_{\psi_0}|$$
$$\leqq H_{|\psi_n - \psi_0|} + |H^{q_n}_{\psi_0} - H^q_{\psi_0}| \to 0$$

as $n\to\infty$. Hence $u_0=H^q_{\psi_0}$.

If $q_n\leqq q_m$, then $D^{q_n}[U_m]\leqq D^{q_m}[U_m]$. Hence, using Lemmas 6.1 and 3.2, we obtain

$$\begin{aligned}
D[U_n-U_m]&\leqq D^{q_n}[U_n-U_m]\\
&\leqq D^{q_n}[U_n]+D^{q_m}[U_m]-2D^{q_n}[U_n, U_m]\\
&= c_dU_n(x)+c_dU_m(x)-2c_dH^{q_n}_{\psi_m}(x)-\int\beta_\alpha(\psi_n^2+\psi_m^2-2\psi_n\psi_m)d\omega\\
&= c_dH^{q_n}_{\psi_n-\psi_m}(x)+c_d\{H^{q_m}_{\psi_m}(x)-H^{q_n}_{\psi_m}(x)\}-\int\beta_\alpha(\psi_n-\psi_m)^2d\omega\\
&\leqq c_dH^{q_n}_{\psi_n-\psi_m}(x)\leqq c_dH_{\psi_n-\psi_m}(x)\to 0 \qquad (n, m\to\infty).
\end{aligned}$$

Therefore, $\{U_n\}$ is a Cauchy sequence in $\boldsymbol{D}$. Since $U_n\to u_0$, it follows that $u_0\in\boldsymbol{D}$ and $D[U_n-u_0]\to 0$ $(n\to\infty)$ (cf. [5] or [19], Lemma 2). Hence $\psi_0\in\boldsymbol{R}_{BD}(\Lambda_\alpha)$.

On the other hand, by Lemmas 6.1 and 4.14, we have

$$\begin{aligned}
c_dH^{q_n}_\varphi(x)&=D^{q_n}[U_n, H^{q_n}_\varphi]+\int\beta_\alpha\varphi\psi_nd\omega\\
&=D^{q_n}[U_n, H^q_\varphi]+\int\beta_\alpha\varphi\psi_nd\omega\\
&=D[U_n, H^q_\varphi]+\int q_nU_nH^q_\varphi dx+\int\beta_\alpha\varphi\psi_nd\omega
\end{aligned}$$

for any $\varphi\in\boldsymbol{R}_{BD}(\Lambda_\alpha)$. Since $\{U_n\}$ and $\{\psi_n\}$ are uniformly bounded, $\int q(x)dx<\infty$ (resp. $\int q_1(x)dx<\infty$), $0\leqq\psi_n\leqq\psi^*$ and $\int\beta_\alpha(\psi^*)^2d\omega<\infty$, where $\psi^*=\psi_1$ (resp. $=\psi_0$), the Lebesgue convergence theorem and the above result, together with Theorem 3.6, imply

$$\begin{aligned}
c_dH^q_\varphi(x)&=D[u_0, H^q_\varphi]+\int qu_0H^q_\varphi dx+\int\beta_\alpha\varphi\psi_0d\omega\\
&=D^q[u_0, H^q_\varphi]+\int\beta_\alpha\varphi\psi_0d\omega
\end{aligned}$$

for all $\varphi\in\boldsymbol{R}_{BD}(\Lambda_\alpha)$. Then, by virtue of Lemma 6.1, $u_0=U^q_{\alpha,x}$ and the theorem is completely proved.

Combining this theorem with Corollary 1 to Theorem 2.4 and the corollary to Theorem 2.5, we obtain

COROLLARY. *Under the same assumptions as in the above theorem on* $\{q_n\}$ *and* q,

$$G^{q_n}_{\alpha,x}(y) \to G^q_{\alpha,x}(y) \qquad (n \to \infty)$$

for each $x, y \in X$ $(x \neq y)$.

Finally we give

THEOREM 6.9. *Let* $\{q_n\}$ *be a monotone increasing* (resp. *decreasing*) *sequence such that* $q_n \to q$ $(n \to \infty)$. *Let* $[A, \beta]$ *be a boundary condition satisfying* $(\mathrm{B})_{q_1}$, (resp. $(\mathrm{B})_q$), τ *be a bounded function on* $\Gamma - A$ *satisfying* $(\mathrm{T})_{q_1}$ (resp. $(\mathrm{T})_q$) *and* γ *be a function on* A *such that* $|\gamma|$ *satisfies* $(\Gamma)_{q_1}$ (resp. $(\Gamma)_q$). *Then the solution* u_n *of* $\mathrm{P}[A, \beta; \tau, \gamma; q_n]$ *converges to the solution* u *of* $\mathrm{P}[A, \beta; \tau, \gamma; q]$.

PROOF. Let $\alpha = 0$ on $\Gamma - A$ and $\alpha = 1/(1+\beta)$ on A. Choose $\varphi \in \boldsymbol{R}^q_D(A; \tau)$ which is bounded. By Theorem 6.1,

$$u_n(x) = H^{q_n}_\varphi(x) - \frac{1}{c_d} D^{q_n}[U^{q_n}_{\alpha,x}, H^{q_n}_\varphi] - \frac{1}{c_d}\int_A \beta\varphi\Phi^{q_n}_{\alpha,x} d\omega - \frac{1}{c_d}\int_A \gamma\Phi^{q_n}_{\alpha,x} d\omega$$

for all $x \in X$. By Theorem 3.6, $H^{q_n}_\varphi(x) \to H^q_\varphi(x)$ $(n \to \infty)$ for each $x \in X$. By Lemma 4.14, we have

$$\begin{aligned} D^{q_n}[U^{q_n}_{\alpha,x}, H^{q_n}_\varphi] &= D^{q_n}[U^{q_n}_{\alpha,x}, H_\varphi] \\ &= D[U^{q_n}_{\alpha,x}, H_\varphi] + \int q_n U^{q_n}_{\alpha,x} H_\varphi dx. \end{aligned}$$

By the previous theorem, we see that $D[U^{q_n}_{\alpha,x}, H_\varphi] \to D[U^q_{\alpha,x}, H_\varphi]$. Also, as in the proof of the previous theorem, we have $\int q_n U^{q_n}_{\alpha,x} H_\varphi dx \to \int q U^q_{\alpha,x} H_\varphi dx$. Hence, using Lemma 4.14 again, we have

$$\begin{aligned} D^{q_n}[U^{q_n}_{\alpha,x}, H^{q_n}_\varphi] \to D[U^q_{\alpha,x}, H_\varphi] + \int q U^q_{\alpha,x} H_\varphi dx &= D^q[U^q_{\alpha,x}, H_\varphi] \\ &= D^q[U^q_{\alpha,x}, H^q_\varphi] \qquad (n \to \infty). \end{aligned}$$

Finally, since $\Phi^{q_n}_{\alpha,x} \to \Phi^q_{\alpha,x}$ by the previous theorem, conditions $(\mathrm{B})_{q_1}$ and $(\mathrm{T})_{q_1}$ (resp. $(\mathrm{B})_q$ and $(\mathrm{T})_q$) imply

$$\int \beta\varphi\Phi^{q_n}_{\alpha,x} d\omega \to \int \beta\varphi\Phi^q_{\alpha,x} d\omega$$

and condition $(\Gamma)_{q_1}$ (resp. $(\Gamma)_q$) implies

$$\int \gamma\Phi^{q_n}_{\alpha,x} d\omega \to \int \gamma\Phi^q_{\alpha,x} d\omega$$

$(n \to \infty)$. Hence, $u_n(x)$ tends to

$$H^q_\varphi(x) - \frac{1}{c_d} D^q[U^q_{\alpha,x}, H^q_\varphi] - \frac{1}{c_d}\int_A \beta\varphi\Phi^q_{\alpha,x} d\omega - \frac{1}{c_d}\int_A \gamma\Phi^q_{\alpha,x} d\omega,$$

which is equal to $u(x)$ by Theorem 6.1. Therefore, $u_n(x) \to u(x)$ for each $x \in X$.

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