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On the Vector Bundles $m\xi_n$ over Real Projective Spaces

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§1. Introduction

Let ξ_n be the canonical line bundle over n-dimensional real projective space RP^n, and $m\xi_n$ the Whitney sum of m-copies of it.

The purpose of this note is to study the number *span* $m\xi_n$ of the linearly independent cross-sections of $m\xi_n$. These are related to the immersion problems of RP^n in the Euclidean space R^m by [2], and also to the submersion problems of $P_k^n = RP^n - RP^{k-1}$ in R^m by [7] and Theorem 2.4 below.

In §2, we study the simple properties of *span* $m\xi_n$. In order to make further calculations, we consider in §3 the Postnikov resolution of the universal sphere bundle and characterize the third k-invariant by the methods of [9], where the results obtained may be contained in [5]. These are applied to *span* $m\xi_n$ in §4, and we consider the submersion problems of P_k^n in §5. The author expresses his hearty thanks to Prof. M. Sugawara and Dr. T. Kobayashi for their valuable suggestions and discussions.

§2. Some properties of $m\xi_n$

If ξ is a real vector bundle, we denote by *span* ξ the maximum number of the linearly independent cross-sections of ξ. Especially, when M is a C^∞-manifold, we denote by *span* M the *span* $\tau(M)$, where $\tau(M)$ is the tangent vector bundle of M.

The following two lemmas are well known.

Lemma 2.1. *Let $f: X \to Y$ be a homotopy equivalence between CW-complexes X and Y, and ξ be a real vector bundle over Y. Then*

$$span\, f^{\#}\xi = span\, \xi,$$

where $f^{\#}\xi$ is the induced bundle of ξ by f.

Lemma 2.2. *Let ξ be a real vector bundle over a CW-complex X. If* $\dim \xi > \dim X$, *then* $span\, \xi \geqq \dim \xi - \dim X$, *and*

$$span(\xi \oplus 1) = 1 + span\, \xi,$$

where $\oplus$ is the Whitney sum and 1 in the left hand side is the 1-dimensional trivial bundle over X.

Now, let ξ_n be the canonical line bundle over the n-dimensional real projective space RP^n, and $m\xi_n$ be the Whitney sum of m-copies of ξ_n.

LEMMA 2.3. *If* $span(m+1)\xi_n \geqq p+1$ *and* $m-p+1 \leqq n$, *then* $\binom{m+1}{p} \equiv 0$ (mod 2).

PROOF. If $span(m+1)\xi_n \geqq p+1$, then there is a bundle η over RP^n such that $(m+1)\xi_n = (p+1) \oplus \eta$. Then the $(m-p+1)$-th Stiefel-Whitney class $w_{m-p+1}(\eta)$ of η is 0 because $\dim \eta = m-p$. On the other hand

$$w_{m-p+1}(\eta) = w_{m-p+1}((m+1)\xi_n) = \binom{m+1}{p} x^{m-p+1}$$

for the generator $x \in H^1(RP^n; Z_2)$. This shows the lemma. *q.e.d.*

THEOREM 2.4. *Let* $m \geqq n$, *then*

$$span(m+1)\xi_n = 1 + span(RP^m - RP^{m-n-1}),$$

where RP^{-1} *is the empty set.*

PROOF. Let the natural inclusion $RP^{m-n-1} \subset RP^m$ be defined by mapping $[x_0, \cdots, x_{m-n-1}] \in RP^{m-n-1}$ to $[x_0, \cdots, x_{m-n-1}, 0, \cdots, 0] \in RP^m$, and let $i: RP^n \rightarrow RP^m - RP^{m-n-1}$ be the into-homeomorphism defined by $i[x_0, \cdots, x_n] = [0, \cdots, 0, x_0, \cdots, x_n]$. Then, i is clearly a homotopy equivalence, and $i^!(\xi_m | RP^m - RP^{m-n-1}) = \xi_n$ where $\xi_m | RP^m - RP^{m-n-1}$ is the restriction of ξ_m. Hence we have

$$\begin{aligned} span(m+1)\xi_n &= span((m+1)\xi_m | RP^m - RP^{m-n-1}) \\ &= span(\tau^m \oplus 1 | RP^m - RP^{m-n-1}), \end{aligned}$$

by 2.1 and the well known facts $(m+1)\xi_m = \tau^m \oplus 1$, where $\tau^m = \tau(RP^m)$. Therefore, for the case $m > n$, this is equal to

$$1 + span(\tau^m | RP^m - RP^{m-n-1})$$

by 2.1 and 2.2, and the theorem is proved for this case.

Consider the case $m=n$, and set $span\ RP^n = d-1$, then $span(n+1)\xi_n = span(\tau^n \oplus 1) \geqq d$. Suppose $n \not\equiv 15$ (mod 16), then $n+1 = ud$, $d = 2^c$ (u: odd, $0 \leqq c \leqq 3$) by [1]. Also we have $\binom{n+1}{d} \equiv 1$ (mod 2), and so $span(n+1)\xi_n < d+1$ by 2.3. These show that $span(n+1)\xi_n = 1 + span\,RP^n$ for $n \not\equiv 15$ (mod 16).

If there is a bundle η over RP^n such that $(n+1)\xi_n = (d+1) \oplus \eta$, then $\tau^n \oplus 1 = (d \oplus \eta) \oplus 1$, and this implies that $\tau^n = d \oplus \eta$ for $n \equiv 1$ (mod 2) by [3, Cor. 1. 11]. This is impossible, because $d = 1 + span \tau^n$, and the above equality holds also for $n \equiv 1$ (mod 2). *q.e.d.*[1]

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[1] Our original proof for the case $n \equiv 15$ (mod 16) is based on K-theory (and [1], [2]) which is due to Dr. T. Kobayashi, and the above simple proof was suggested by Dr. B. Steer, to whom the author wishes to thank.

REMARKS. (2.5) *span* $m\xi_n=0$ if $0\leqq m\leqq n$, because the Stiefel-Whitney class $w_m(m\xi_n)$ is not zero.

(2.6) The case $m=n$ in 2.4 is equivalent to

$$span\ RP^n=n-\text{g. dim }(\tau^n-n),$$

where g. dim (τ^n-n) is the geometrical dimension of $\tau^n-n\in\widetilde{KO}(RP^n)$.

As an application of 2.4 for $m=n$, we have

THEOREM 2.7 *Let M be a C^∞-manifold, then*

$$span\ (M\times RP^n)\leqq\dim M+span\ RP^n.$$

Especially, if M is a π-manifold, and n is odd, then

$$span\ (M\times RP^n)=\dim M+span\ RP^n$$

PROOF. Let dim $M=m$ and $d-1=span\ RP^n$, and suppose $span(M\times RP^n)\geqq m+d$. Then there is a bundle ξ over $M\times RP^n$ such that $\tau(M\times RP^n)=(m+d)\oplus\xi$, and we have $m\oplus\tau(RP^n)=(m+d)\oplus j^\sharp\xi$ inducing by the inclusion map $j: RP^n=*\times RP^n\subset M\times RP^n$. Hence $span(1\oplus\tau^n)\geqq d+1$ by 2.2, which contradicts to 2.4 for $m=$n, and so the first relation is obtained.

If n is odd, there exists a vector bundle η over RP^n such that $\tau(RP^n)=1\oplus\eta$, as $span\ RP^n\geqq 1$. So, for π-manifold M,

$$\tau(\mathrm{M}\times RP^n)=p_1^\sharp\tau(M)\oplus p_2^\sharp(1\oplus\eta)=p_1^\sharp(\tau(M)\ \oplus 1)\oplus p_2^\sharp(\eta)$$
$$=(\dim M+1)\oplus p_2^\sharp(\eta)=\dim M\oplus p_2^\sharp\tau^n,$$

where p_i is the projection map onto the i-th factor. This shows that *span* $(M\times RP^n)\geqq\dim M+spanRP^n$, and the second equation. *q.e.d.*

Now, we consider the simple properties of $span(n\xi_k)$ for $n\geqq k+2$.

THEOREM 2.8. *Let k and n be integers such that $n\geqq k+2$.*

(a) *If* $\binom{n}{k}\equiv 1$ (mod 2), *then span* $(n\xi_k)=n-k$.

(b) *If k and n are even integers, then the inverse of* (a) *holds.*

PROOF. (a) is immediate from 2.2 and 2.3

(b): Let η be a vector bundle over RP^n such that $n\xi_k=(n-k)\oplus\eta$. For even k and n, $H^k(RP^k; Z)$ and $H^k(RP^k; Z_2)$ are isomorphic by the mod 2-reduction homomorphism, and η is orientable. Therefore, the fact that η has a non-zero cross-section is equivalent to $w_k(\eta)=0$, i.e., $\binom{n}{k}\equiv 0$ (mod 2) (cf. [6]).

q.e.d.

THEOREM 2.9. *Let n be even, k be odd such that $n\geqq k+2$, then*

$$span(n\xi_k) \geqq n-k+1.$$

Moreover, if $\binom{n}{k-1} \equiv 1$ (mod 2), *then span* $(n\xi_k) = n-k+1$.

PROOF. Put $n\xi_k = (n-k)\otimes\eta$. Since k is odd, the obstruction for η to have a non-zero cross-section is $\delta w_{k-1}(\eta)$, where $\delta\colon H^{k-1}(RP^k;\, Z_2) \to H^k(RP^k;\, Z)$ is the Bockstein operator. Since this δ is zero, we have $\delta w_{k-1}(\eta) = 0$, and the first is obtained. The rest is easy from 2.3. *q.e.d.*

THEOREM 2.10. *Let l, m and n be integers, and $d=2$, 4 or 8. Then,*

$$span(dl+m)\xi_n = dl \qquad \textit{for } 0 \leqq m \leqq n \leqq d-1.$$

PROOF. $span\,(dl+m)\xi_n \geqq span\,(dl+m)\xi_{d-1} \geqq span\,(dl\xi_{d-1}) = dl$ because $span(RP^{d-1}) = d-1$. Also, $span(dl+m)\xi_n < dl+1$ by 2.3, because $\binom{dl+m}{dl} \equiv 1$ (mod 2). *q.e.d.*

§3. Postnikov resolution of the universal sphere bundle for the third stage

Let (E, p, B, F) be a fiber space over a CW-complex B with $(n-1)$-connected fiber F, and assume that the fundamental group $\pi_1(B)$ acts trivially on the homology group $H_*(F;\, G)$ with coefficient group G. Let $w\colon B \to C$ be a map into the Eilenberg-MacLane space $C = K(\Pi,\, n+1)$, and $(E_1,\, P_1,\, B,\, \Omega C)$ be the principal fiber space with classifying map w [9], where $\Omega C = K(\Pi,\, n)$ is the loop space of C. As is well known, the homotopy set $[B,\, C]$ is naturally isomorphic to $H^{n+1}(B;\, \Pi)$, and so we identify these. Also, assume that $p^*w = 0$, which is equivalent to the existence of the map $q\colon E \to E_1$ such that $p_1 \circ q = p$.

Consider the following commutative diagram in [9]

$$\begin{tikzcd}
\Omega C \arrow[rr, phantom, "="] \arrow[d] & & \Omega C \arrow[d] \\
\Omega C \times E \arrow[r, "1\times q"] \arrow[d, "\pi"] & \Omega C \times E_1 \arrow[r, "\mu"] & E_1 \arrow[d, "p_1"] \\
E \arrow[rr, "p"] & & B
\end{tikzcd}$$

where μ is the action map and π is the projection map.

Put $\nu = \mu \circ (1 \times q)$. If $s\colon E = * \times E \subset \Omega C \times E$ is the inclusion map, then $\nu \circ s$ is homotopic to q [9].

Under the above notations, it follows:

THEOREM 3.1. [9, Cor. 1] *For any abelian group G, the sequence*

$$\cdots \to H^i(\Omega C \times E;\, G) \xrightarrow{\tau_0} H^{i+1}(B,\, E;\, G) \xrightarrow{l^*} H^{i+1}(E_1 \colon G)$$
$$\xrightarrow{\nu^*} H^{i+1}(\Omega C \times E;\, G) \to \cdots \to H^{2n}(\Omega C \times E;\, G)$$

is exact, where (B, E) *should be considered as* (M_p, E) $(M_p$ *is the mapping cylinder of* $p)$, $l=j\circ p_1(j\colon B\to(B, E)$ *is the inclusion map), and* τ_0 *is the relative transgression homomorphism.*

COROLLARY 3.2. *Assume that the following two conditions* (a) *and* (b) *hold for a positive integer* i $(\leqq 2n-1)$ *and for a coefficient group* G:

(a) $\operatorname{Ker} p_1^* \supset \operatorname{Ker} p^*$ *in dimension* i,

(b) p^* *is surjective in dimension* i.

Then, the sequence

$$0\longrightarrow H^i(E_1;\, G)\xrightarrow{\nu^*} H^i(\Omega C\times E;\, G)\xrightarrow{\tau_1} H^{i+1}(B;\, G)$$

is exact, where $\tau_1=j^*\circ\tau_0$.

PROOF. (a) implies $\operatorname{Im} l^*=p_1^*(\operatorname{Im} j^*)=p_1^*(\operatorname{Ker} p^*)=0$, and so ν^* in the above sequence is monomorphic by 3.1.

By the exact sequence of (B, E) and (b), $j^*\colon H^{i+1}(B, E;\, G)\to H^{i+1}(B;\, G)$ is a monomorphism, and so $\operatorname{Ker}\tau_0=\operatorname{Ker}\tau_1$. These and 3.1. show the exactness. *q.e.d.*

Let $n\geqq 4$, and $S^{n-1}\xrightarrow{i} BSO(n-1)\xrightarrow{\pi} BSO(n)$ be the universal oriented $(n-1)$-sphere bundle. π is homotopically equivalent to the natural inclusion $BSO(n-1)\subset BSO(n)$.

The Postnikov resolution of π for the third stage is as follows:

$$(*)\qquad \begin{tikzcd} BSO(n-1) \arrow[rr, "q'"'] \arrow[dr, "q"'] \arrow[dd, "\pi"'] & & E' \arrow[r, "k'"'] \arrow[dl, "p'"] & K(Z_2,\, n+2) \\ & E \arrow[r, "k"'] \arrow[dl, "p"] & K(Z_2,\, n+1) & \\ BSO(n) \arrow[r, "X_n"'] & K(Z,\, n) & & \end{tikzcd}$$

where $X_n \in H^n(BSO(n);\, Z)$ is the Euler class, $(E, p, BSO(n))$ is the principal fiber space with classifying map X_n, q is the map such that $p\circ q=\pi$, k is the second k-invariant, (E', p', E) is the principal fiber space with classifying map k, q' is the map such that $p'\circ q'=q$, and k' is the third k-invariant.

The two conditions of 3.2 for the bundle $(BSO(n-1),\, \pi,\, BSO(n),\, S^{n-1})$ hold for $0<i\leqq 2n-3$ and $G=Z_2$ by [9, p. 20]. So,

$$(**)\quad 0\to H^i(E;\, Z_2)\xrightarrow{\nu^*} H^i(K(Z,\, n-1)\times BSO(n-1);\, Z_2)\xrightarrow{\tau_1} H^{i+1}(BSO(n);\, Z_2)$$

is exact for $0<i\leqq 2n-3$ by 3.2, where $\nu=\mu\circ(1\times q)$, $\mu\colon K(Z,\, n-1)\times E\to E$ is the action map.

Also, the invariant k is characterized uniquely by the equation [9, p. 21]:

$$\nu^*k = Sq^2\iota\otimes 1+\iota\otimes w_2$$

where ι is the generator of $H^{n-1}(K(Z,\, n-1);\, Z_2)=Z_2$, $w_i \in H^i(BSO(n-1);\, Z_2)$ is the i-th Stiefel-Whitney class, and Sq is the Steenrod square operation.

Now, to consider the characterization of k', we consider the bundle $(BSO(n-1), q, E)$. For the conditions of 3.2 of this bundle, we have:

LEMMA 3.3 *For* $n \geqq 5$, *and for coefficient group* Z_2, *we have*

(a) $\operatorname{Ker} p'^* \supset \operatorname{Ker} q^*$ *in* $\dim n+2$,

(b) q^* *is surjective in* $\dim n+2$.

PROOF. (a): Since $\nu \circ s$ is homotopic to q and $n \geqq 5$,

$$\nu^*: H^{n+2}(E; Z_2) \frown \operatorname{Ker} q^* \cong \operatorname{Ker} \tau_1 \frown \operatorname{Ker} s^* \frown H^{n+2}(K(Z, n-1) \times BSO(n-1); Z_2)$$

is isomorphic by the exact sequence (**) for $i=n+2$, where $s: BSO(n-1) \to K(Z, n-1) \times BSO(n-1)$ is the inclusion map.

The right side is Z_2 generated by $\iota \otimes w_3 + Sq^3 \iota \otimes 1$, because

$$\tau_1(\iota \otimes w_3) = w_n w_3,$$

$$\tau_1(Sq^3 \iota \otimes 1) = Sq^3 \tau_1(\iota \otimes 1) = Sq^3 w_n = w_n w_3$$

by [8], [9] and a formula of Wu [11]. On the other hand,

$$\begin{aligned} \nu^* Sq^1 k &= Sq^1 \nu^* k = Sq^1(Sq^2 \iota \otimes 1 + \iota \otimes w_2) \\ &= Sq^1 Sq^2 \iota \otimes 1 + \iota \otimes Sq^1 w_2 + Sq^1 \iota \otimes w_2 = Sq^3 \iota \otimes 1 + \iota \otimes w_3, \end{aligned}$$

and so, $H^{n+2}(E; Z_2) \frown \operatorname{Ker} q^*$ is equal to Z_2 generated by $Sq^1 k$. Also, $p'^* Sq^1 k = Sq^1 p'^* k = Sq^1 0 = 0$, and we have (a).

(b): This follows from the fact that π^* is an epimorphism for coefficient group Z_2 in all dimensions. *q.e.d.*

By 3.2 and 3.3, we have

COROLLARY 3.4. *For* $n \geqq 5$,

$$0 \to H^{n+2}(E'; Z_2) \xrightarrow{\nu'^*} H^{n+2}(K(Z_2, n) \times BSO(n-1); Z_2) \xrightarrow{\tau_1'} H^{n+3}(E; Z_2)$$

is an exact sequence, where ν', τ_1', *are defined similarly as before.*

The following characterization of k' is obtained [5]:

THEOREM 3.5. *For* $n \geqq 6$, $k' \in H^{n+2}(E'; Z_2)$ *is characterized uniquely by the equation*:

$$\nu'^* k' = \iota' \otimes w_2 + Sq^2 \iota' \otimes 1$$

where ι' *is the generator of* $H^n(K(Z_2, n); Z_2) = Z_2$.

PROOF. By [9, Property 2, p. 14], we have

$$\tau_1'(\iota' \otimes w_2) = k p^* w_2,$$

$$\tau_1'(Sq^2 \iota' \otimes 1) = Sq^2 \tau_1'(\iota' \otimes 1) = Sq^2 k.$$

These are not zero and mapped to the same element by ν^*, because

$$\begin{aligned}\nu^*(kp^*w_2) &= \nu^*k\cup\nu^*p^*w_2\\ &= (Sq^2\iota\otimes 1+\iota\otimes w_2)(1\otimes w_2) = Sq^2\iota\otimes w_2+\iota\otimes w_2^2,\\ \nu^*Sq^2k &= Sq^2\nu^*k = Sq^2(Sq^2\iota\otimes 1+\iota\otimes w_2)\\ &= Sq^2Sq^2\iota\otimes 1+Sq^2\iota\otimes w_2+Sq^1\iota\otimes Sq^1w_2+\iota\otimes Sq^2w_2\\ &= Sq^3Sq^1\iota\otimes 1+Sq^2\iota\otimes w_2+\iota\otimes w_2^2 = Sq^2\iota\otimes w_2+\iota\otimes w_2^2.\end{aligned}$$

As $n\geqq 6$, these and the exactness of (**) show that

$$\tau_1'(\iota'\otimes w_2)=\tau_1'(Sq^2\iota'\otimes 1).$$

And so, $\iota'\otimes w_2+Sq^2\iota'\otimes 1$ is the only non-zero element of $\nu'^*(\operatorname{Ker} q'^*)$. Therefore, we have 3.5. *q.e.d.*

§4. Obstructions for cross-sections of vector bundles

Now, let X be a CW-complex and ξ be an orientable real vector bundle of dimension n over X. The equivalence class of ξ corresponds bijectively to a homotopy class of a map ξ: $X\to BSO(n)$.

Consider the diagram (*) and suppose that $\xi^*X_n=0$. Then there is a map η: $X\to E$ such that $p\circ\eta=\xi$. We define, as in [10],

$$k(\xi)=\bigcup_{\eta}\eta^*k\subset H^{n+1}(X;\ Z_2),$$

where the union is taken over all maps η: $X\to E$ such that $p\circ\eta=\xi$. As is well known, $k(\xi)\ni 0$ if and only if ξ has a non-zero cross-section over the $(n+1)$-skeleton of X.

We obtain the following theorem as a special case of [10].

Theorem 4.1. *For $n\geqq 4$,*

$$k(\xi)\in H^{n+1}(X;\ Z_2)/(w_2\otimes 1+1\otimes Sq^2)\cdot H^{n-1}(X;\ Z_2)$$

where the dot operates by ξ [10].[2]

Proof. Put $\tilde{\mu}^*=\mu^*-p_0^*$, where μ: $K(Z,\ n-1)\times E\to E$ is the action map and p_0: $K(Z,\ n-1)\times E\to E$ is the projection. Then

$$\begin{aligned}(1\times q)^*\tilde{\mu}^*k &= (\nu^*-(1\times q)^*p_0^*)(k)\\ &= \nu^*k-(1\times q)^*(1\otimes k)=\nu^*k = Sq^2\iota\otimes 1+\iota\otimes w_2.\end{aligned}$$

Because $\operatorname{Im}\tilde{\mu}^*\subset\sum_{0\leqq i\leqq 2}H^{n+1-i}(K(Z,\ n-1);\ Z_2)\otimes H^i(E;\ Z_2)$ and q^*: $H^i(E;\ Z_2)$

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[2] This means that $(w_2\otimes 1+1\otimes Sq^2)\cdot H^{n-1}(X;\ Z_2)=\{w_2(\xi)x+Sq^2x \mid x\in H^{n-1}(X;\ Z_2)\}$

$\to H^i(BSO(n-1); Z_2)$ is injective for $i \leqq 2$, we have

$$\tilde{\mu}^*k = Sq^2\iota \otimes 1 + \iota \otimes p^*w_2 = (w_2 \otimes 1 + 1 \otimes Sq^2) \cdot (\iota \otimes 1),$$

where the dot operates by $p \circ p_0$, and the proof is completed by [10]. *q.e.d.*

Suppose η be a map such that $p \circ \eta = \xi$. Define similarly

$$k'(\eta) = \bigcup_{\zeta} \zeta^* k' \subset H^{n+2}(X; Z_2)$$

where the union is taken over all maps ζ: $X \to E'$ such that $p' \circ \zeta = \eta$. Then, $k'(\eta) \ni 0$ if and only if ξ has a non-zero cross-section over the $(n+2)$-skeleton of X.

Using 3.5, we have

THEOREM 4.2. *For* $n \geqq 6$,

$$k'(\eta) \in H^{n+2}(X; Z_2)/(w_2 \otimes 1 + 1 \otimes Sq^2) \cdot H^n(X; Z_2),$$

where the dot operates by ξ.

PROOF. By 3.5 and the same technique as in the proof of 4.1, we see that

$$k'(\eta) \in H^{n+2}(X; Z_2)/(p^*w_2 \otimes 1 + 1 \otimes Sq^2) \cdot H^n(X; Z_2),$$

where the dot operates by η. But, we have

$$(p^*w_2 \otimes 1 + 1 \otimes Sq^2) \cdot H^n(X; Z_2) = (w_2 \otimes 1 + 1 \otimes Sq^2) \cdot H^n(X; Z_2)$$

by the definition of the operations, and 4.2 is obtained. *q.e.d.*

Now, we shall apply these two results to the bundles over RP^n.

THEOREM 4.3. *Let k and n be integers such that $n \geqq k+2 \geqq 7$. Suppose one of the following two conditions* (a) *and* (b) holds:

(a) $n \equiv 0 \pmod 4$, $k \equiv 0 \pmod 4$, $\binom{n}{k} \equiv 0 \pmod 2$,

(b) $n \equiv 2 \pmod 4$, $k \equiv 2 \pmod 4$, $\binom{n}{k} \equiv 0 \pmod 2$.

Then,

$$span(n\xi_k) \geqq n-k+2.$$

PROOF. By 2.8(b), there is a $(k-1)$-dimensional vector bundle η over RP^k such that $n\xi_k = (n-k+1) \oplus \eta$. As $H^{k-1}(RP^k; Z) = 0$, η has a non-zero cross-section over the $(k-1)$-skeleton of RP^k, and the final obstructions of the non-zero cross-section of η extending to RP^k form a coset of

$$(w_2 \otimes 1 + 1 \otimes Sq^2) \cdot H^{k-2}(RP^k; Z_2),$$

where the dot operates by η, by 4.1. But we have easily

$$(w_2\otimes 1+1\otimes Sq^2)\cdot H^{k-2}(RP^k;\ Z_2)=H^k(RP^k;\ Z_2)$$

by the assumption. So, η has a non-zero cross-section and the proof is completed. *q.e.d.*

THEOREM 4.4. *Let k and n be integers such that $n\geqq k+2\geqq 7$. Suppose one of the following conditions* (a) *and* (b) *holds*:

(a) $n\equiv 0 \pmod 4$, $k\equiv 1 \pmod 4$, $\binom{n}{k-1}\equiv 0 \pmod 2$,

(b) $n\equiv 2 \pmod 4$, $k\equiv 3 \pmod 4$, $\binom{n}{k-1}\equiv 0 \pmod 2$.

Then,

$$span(n\xi_k)\geqq n-k+2.$$

Moreover, if $k\geqq 8$,

$$span(n\xi_k)\geqq n-k+3.$$

PROOF. By 2.9, we can write $n\xi_k=(n-k+1)\oplus\eta_1$, where η_1 is the $(k-1)$-dimensional vector bundle over RP^k.

As $H^{k-1}(RP^k;\ Z)$ is isomorphic to $H^{k-1}(RP^k;\ Z_2)$ by the mod 2-reduction homomorphism, it follows by the assumption that the Euler class $X(\eta_1)$ of η_1 is zero. By 4.1, the obstructions of the non-zero cross-section of η_1 extending to RP^k form a coset of

$$(w_2\otimes 1+1\otimes Sq^2)\cdot H^{k-2}(RP^k;\ Z_2),$$

which is equal to $H^k(RP^k;\ Z_2)$ by the assumption.

So, η_1 has a non-zero cross-section and we can write $n\xi_k=(n-k+2)\oplus\eta_2$ where η_2 is the $(k-2)$-dimensional vector bundle over RP^k.

Now, the Euler class $X(\eta_2)$ of η_2 is zero, because $H^{k-2}(RP^k;\ Z)=0$. So, η_2 has a non-zero cross-section over the $(k-2)$-skeleton of RP^k, and the obstructions extending to the $(k-1)$-skeleton of RP^k form a coset

$$(w_2\otimes 1+1\otimes Sq^2)\cdot H^{k-3}(RP^k;\ Z_2),$$

by 4.1, where the dot operates by η_2, and this group is equal to $H^{k-1}(RP^k;\ Z_2)$.

So, η_2 has a non-zero cross-section over the $(k-1)$-skeleton of RP^k, and the obstructions extending to RP^k form a coset of

$$(w_2\otimes 1+1\otimes Sq^2)\cdot H^{k-2}(RP^k;\ Z_2)=H^k(RP^k;\ Z_2)$$

by 4.2.

So, η_2 has a non-zero cross-section over RP^k and the proof is completed. *q.e.d.*

§5. Applications to the submersions of P_k^n

Let M^n be an open C^∞-manifold of dimension n, and W^p be a C^∞-manifold of dimension p. Then by [7], we say a differentiable map f: $M^n \to W^p$ $(n \geqq p)$ is a submersion if f has rank p at each point of M^n. In this case, we say that M^n submerges in W^p. R^p denotes the p-dimensional Euclidean space.

Now, we consider the problem of submersions in R^p. Our results are based on the following theorem of [7]:

THEOREM 5.1. *M^n submerges in R^p if and only if span $M^n \geqq p$.*

By $RP^n \subseteq R^{n+k}$, we mean that RP^n is immersible in R^{n+k}.

THEOREM 5.2. *$RP^{n+k}-RP^{k-1}$ submerges in R^n if and only if $RP^n \subseteq R^{n+k}$.*

PROOF. By [2, Theorem 1.1], *span*$(n+k+1)\xi_n \geqq n+1$ if and only if $RP^n \subseteq R^{n+k}$. So, the proof follows from 2.4 and 5.1. *q.e.d.*

By 2.4 and 5.1, we have also

LEMMA 5.3. *If $RP^{n+k}-RP^{k-1}$ submerges in R^n, then $RP^{n+k}-RP^k$ submerges in R^n and $RP^{n+k+1}-RP^k$ submerges in R^n.*

We denote by $s(n, k)$ the number s such that $P_k^n = RP^n - RP^{k-1}$ submerges in R^s and not in R^{s+1}. Then, we have the following results, using 2.8, 2.9, 2.10, 4.3 and 4.4:

(5.4) Let k and n be integers such that $n \geqq k+2$.

(a) If $\binom{n}{k} \equiv 1 \pmod 2$, then $s(n-1, n-k-1) = n-k-1$.

(b) If k and n are even integers, then the inverse of (a) holds.

(5.5) Let n be an even integer, k be an odd integer such that $n \geqq k+2$, then $s(n-1, n-k-1) \geqq n-k$. Moreover, if $\binom{n}{k-1} \equiv 1 \pmod 2$, then $s(n-1, n-k-1) = n-k$.

(5.6) Let l, m and n be integers, and $d=2$, 4 or 8. Then $s(dl+m-1, dl+m-n-1) = dl-1$ for $0 \leqq m \leqq n \leqq d-1$.

(5.7) Under the assumptions of 4.3, $s(n-1, n-k-1) \geqq n-k+1$.

(5.8) Under the assumptions of 4.4, $s(n-1, n-k-1) \geqq n-k+1$ for $k \geqq 5$ and $s(n-1, n-k-1) \geqq n-k+2$ for $k \geqq 8$.

By 5.3 and (5.4)-(5.6), $s(n, k)$ are determined partially as follows:

(5.9) $s(n+8, k+8) = 8+s(n, k)$ for $n-7 \leqq k \leqq n$,

$s(n+8, k) = s(n, k)$ for $0 < k \leqq l \leqq 6$ where $n = 8m+l$.

Moreover, we have the following table of $s(n, k)$ for $n \leqq 30 = 2^5 - 2$, which is a partial improvement of the table of [7, p. 201]. The symbols in the table are used in the following sense:

⬚ is determined by (5.9).
∗ comes from 5.2 and the known results concerning the immersion of RP^n.
○ is a consequence of (5.8).
† comes from [4, Th. 1, (vi) and Prop. 3].
△ comes from *K*-theory as in [7].

k \ n		15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
(0)		8	0	1	0	3	0	1	0	7	0	1	0	3	0	1	0
1	▨	8,10	8,12	1	1	3	3	1	1	7	7	▨	▨	▨	▨	▨	▨
2	▨	8,10	8,12	8,12	2	3	3	3	2	7	7	7	▨	▨	▨	▨	▨
3	▨	8,10	8,12	8,12	8,13	3	3	3	3	7	7	7	7	▨	▨	▨	▨
4	▨	8,10	△ 8,12	8,12	8,13	8,13	4	5	4	7	7	7	7	7	▨	▨	▨
5	▨	∗ 8,10	∗ 11,13	∗ 8,12	8,13	8,13	8,13	5	5	7	7	7	7	7	7	▨	▨
6	▨	○ △ 9,14	△ 11,13	11,14	∗ △ 12,14	○ 9,13	† 9,13	○ 9,13	6	7	7	7	7	7	7	7	▨
7	▨	9,16	△ 11,15	△ 11,14	12,15	12,16	9,13	9,13	∗ 15	7	7	7	7	7	7	7	7
8	▨	15	11,16	11,16	△ 12,15	12,16	12,16	∗ 9,13	15	15	8	9	8	9,12	8	9	8
9	▨	15	15	11,16	12,16	12,16	12,16	12,16	15	15	15	9	9	10,12	9,12	9	9
10	▨	15	15	15	12,16	○ 13,16	12,16	○ 13,16	15	15	15	15	10	11	10,12	11	10
11	▨	15	15	15	15	13,16	13,16	13,16	15	15	15	15	15	11	11	11	11
12	▨	15	15	15	15	15	13,16	13,16	15	15	15	15	15	15	12	13	12
13	▨	15	15	15	15	15	15	14,16	15	15	15	15	15	15	15	13	13
14	▨	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	14
15		15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15
16			16	17	16	19	16	17	16	▨	16	17	16	17,20	16	17	16
17				17	17	19	19	17	17	▨	▨	17	17	18,20	17,20	17	17
18					18	19	19	19	18	▨	▨	▨	18	19	18,20	19	18
19						19	19	19	19	▨	▨	▨	▨	19	19	19	19
20							20	21	20	▨	▨	▨	▨	▨	20	21	20
21								21	21	▨	▨	▨	▨	▨	▨	21	21
22									22	▨	▨	▨	▨	▨	▨	▨	22
23										▨	▨	▨	▨	▨	▨	▨	▨

$$\begin{array}{c} n \\ \downarrow \\ k \rightarrow \boxed{\;s\;} \end{array} \qquad \begin{array}{c} n \\ \downarrow \\ k \rightarrow \boxed{\;s,\ t\;} \end{array}$$

$s(n, k) = s$ $\qquad$ $s \leqq s(n,k) < t$

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