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Commutative Rings for Which Each Proper Homomorphic Image is a Multiplication Ring[1]

Craig A. Wood

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In this paper, all rings considered are assumed to be commutative rings. A ring R is called an *AM-ring* (for allgemeine Multiplikationring) if whenever A and B are ideals of R with A properly contained in B, then there is an ideal C of R such that $A=BC$. An *AM*-ring R in which $RA=A$ for each ideal A of R is called a *multiplication ring*[2]. This paper considers a ring R *satisfying property* (*Hm*): Each proper homomorphic image of R is a multiplication ring. Numerous ring-theoretic properties (for example, Noetherian, or proper prime ideals are maximal) are inherited by a ring R if these properties hold in each proper homomorphic image of R. In Section 3 of this paper we show, however, that a ring satisfying (*Hm*) need not be a multiplication ring, and we give a characterization of rings with identity satisfying property (*Hm*). An outline is given for constructing examples of rings with identity satisfying (*Hm*) that are not multiplication rings.

Let R be a ring. We say that R *satisfies property* (*) if each ideal of R with prime radical is primary. Property (*) is considered by Gilmer in [3] and [4] and by Gilmer and Mott in [5]. Closely related to (*) is the *property* (**) which is also studied in [5] and in [1] by Butts and Phillips: *Each ideal of R with prime radical is a prime power.* If every proper homomorphic image of R satisfies property (*) (satisfies property (**)), we say that R *satisfies property* (H^*) (*satisfies property* (H^{**})). In [5] it is shown that an *AM*-ring satisfies (*) and (**) and that if S is a u-ring, S satisfies (**) if and only if S satisfies (*) and primary ideals of S are prime powers. It follows that if R contains an identity, then R a multiplication ring implies that R satisfies (**) and R satisfying (**) implies that R satisfies (*). Hence, in a ring with identity, (*Hm*) implies (H^{**}) and (H^{**}) implies (H^*). For this reason, we consider rings satisfying (H^*) in Section 1 and rings satisfying (H^{**}) in Section 2. In particular, rings with identity satisfying (H^{**}) are characterized in Section 2.

The notation and terminology is that of [9] with two exceptions: $\subseteq$ denotes containment and $\subset$ denotes proper containment, and we do not assume that a Noetherian ring contains an identity. If A is an ideal of a ring

1) This paper is a portion of the author's doctoral dissertation, written under the direction of Professor Robert W. Gilmer, Jr. of The Florida State University. This part of the dissertation was written under the direction of Professor Joe L. Mott, while Professor Gilmer was on leave of absence.

2) For a historical development of the theory of multiplication rings see [5, p. 40].

R, we say that A is a *proper ideal* of R if $(0)\subset A\subset R$ and that A is a *genuine ideal* of R if $A\subset R$.

1. Rings Satisfying Property (H^*)

We obtain in this section some results concerning rings satisfying (H^*) and give in Theorem 1.5 a partial characterization of such rings with identity.

We first, however, introduce some terminology which is used in this section. If A is an ideal of a ring R and $\{P_\alpha\}$ is the collection of minimal prime ideals of A, then by an *isolated primary component of A belonging to P_α* we mean the intersection Q_α of all P_α-primary ideals which contain A. The *kernel of A* is the intersection of all the Q_α's. Krull introduced the notion of the kernel of an ideal in [6]. The kernel of an ideal is also considered by Mori in [7] and by Mott in [8]. In this section we use the following fact: *A ring R satisfies* (*) *if and only if every ideal of R is equal to its kernel* [5, Theorem 4].

Lemma 1.1. *An integral domain satisfying* (H^*) *also satisfies* (*).

Proof. Let D be an integral domain satisfying (H^*). We show that D satisfies (*) by showing that each ideal of D is equal to its kernel. Let A be a nonzero ideal of D and consider D/A. In D/A, A/A is equal to its kernel since D/A satisfies (*). By the one-to-one correspondence between primary ideals of D/A and primary ideals of D containing A, it follows that A is equal to its kernel in D. Therefore, each ideal of D is equal to its kernel which implies that D satisfies (*).

Definition. *Let R be a ring. If there exists a chain $P_0\subset P_1\subset\cdots\subset P_n$ of $n+1$ prime ideals of R where $P_n\subset R$, but no such chain of $n+2$ prime ideals, then we say that R has dimension n and we write* $\dim R=n$.

Lemma 1.2. *If a ring R satisfies* (H^*), *then* $\dim R\leq 1$.

Proof. If (0) is a prime ideal of R, R satisfies (*) by Lemma 1.1. Thus, $\dim R\leq 1$ [5, Theorem 1]. Assume that there exist prime ideals P_1, P_2, and P_3 of R such that $(0)\subset P_1\subset P_2\subseteq P_3\subset R$. Then R/P_1 is an integral domain satisfying (*) so that $\dim R/P_1\leq 1$ [5, Theorem 1]. Therefore, $P_2/P_1=P_3/P_1$ which implies that $P_2=P_3$. Thus, $\dim R\leq 1$.

Lemma 1.3. *Let R be a ring with identity satisfying* (H^*) *such that $\sqrt{(0)}=P$ is a genuine nonmaximal prime ideal of R. If $P=P^2$, R is a one-dimensional domain. Hence, R satisfies* (*).

Proof. We show that $P=(0)$. Assume that $P\neq(0)$ and let $a\in P\setminus\{0\}$. Since $\sqrt{(0)}=P$, $\sqrt{(a)/(a)}=(\sqrt{(a)})/(a)=P/(a)$, a prime ideal of $R/(a)$. Therefore,

$(a)/(a)$ is a $P/(a)$-primary ideal of $R/(a)$ since $R/(a)$ satisfies (*). Then [4, Theorem 2] implies that $(a)/(a)=P/(a)$ which shows that $P=(a)$. Thus, (0) contains a power of P and it follows that $P\neq P^2$. Hence, if $P=P^2$, $P=(0)$ and R is a one-dimensional domain by Lemma 1.2.

LEMMA 1.4. *Let R be a ring with identity satisfying (H^*). If P is a genuine nonmaximal prime ideal of R and $P^2\neq(0)$, then $P=P^2$.*

PROOF. Since $P^2\neq(0)$, R/P^2 satisfies (*) and $P/P^2=(P/P^2)^2=P^2/P^2$ [4, Corollary 2.2]. Therefore, $P=P^2$.

THEOREM 1.5. *Let R be a ring with identity. Then R satisfies (H^*) and each nonmaximal prime ideal of R is idempotent if and only if R satisfies* (*).

PROOF. $(\leftarrow)$ If R satisfies (*), R clearly satisfies (H^*) and each nonmaximal prime ideal of R is idempotent [4, Corollary 2.2].

$(\rightarrow)$ Assume that R satisfies (H^*) and that each nonmaximal prime ideal of R is idempotent. We consider three cases.

Case 1. $\sqrt{(0)}$ is not a prime ideal of R. Let A be an ideal of R such that $\sqrt{A}=P$ is a prime ideal of R. Then $A\neq(0)$ and R/A satisfies (*). Since $\sqrt{A/A}=(\sqrt{A})/A=P/A$ is a prime ideal of R/A, A/A is a P/A-primary ideal of R/A. Therefore, A is a P- primary ideal of R and R satisfies (*).

Case 2. $\sqrt{(0)}$ is a maximal ideal of R. It follows that R is a primary ring. Therefore, R clearly satisfies (*).

Case 3. $\sqrt{(0)}$ is a nonmaximal prime ideal of R. Since $P=P^2$ by hypothesis, Lemma 1.3 implies that R is a one-dimensional domain. Therefore, R satisfies (*).

EXAMPLE 1.6. We give an example here of a ring R with identity such that R satisfies (H^*) but R does not satisfy (*); in fact, R satisfies the following conditions: (i) Each nonzero ideal of R with prime radical is primary; (ii) R is a one-dimensional ring; (iii) R contains a unique minimal prime ideal $P\neq(0)$ such that $P=(a)$ for each $a\in P\setminus\{0\}$ and $P^2=(0)$; (iv) $\sqrt{(0)}=P$ but (0) is not a P-primary ideal of R. This example shows that the condition "each nonmaximal prime ideal of R is idempotent" is a necessary condition in Theorem 1.5.

Let $D=\pi_2[X, Y]$ where X and Y are indeterminates over π_2, the Galois field of two elements. Let $M=(X, Y)$, let $P=(X)$, and let $B=(X^2, XY)$. If $f(X, Y)\in P$,

$$f(X,\ Y)=f_{10}X+\sum_{\substack{j=0,\ i\geq 2\\ j\geq 1,\ i\geq 1}} f_{ij}X^iY^j$$

where each $f_{ij}\in\pi_2$. Therefore, $f(X, Y)\equiv f_{10}X(B)$. If $f(X, Y)\in P\setminus B$, $f_{10}=1$

so that $B+(f(X, Y))=P$ for each $f(X, Y) \in P\setminus B$. Note that B is not a P-primary ideal of D since $XY \in B$, $X \notin B$, and $Y \notin P$.

Let $R=D/B=\pi_2[X, Y]/(X^2, XY)$. Since D is a two-dimensional domain, it is clear that R is a one-dimensional ring. Also, P/B is the unique minimal prime ideal of R, P/B is principal and is generated by any nonzero element in P/B, and $(P/B)^2=B/B$. If A is a nonzero ideal of R such that $\sqrt{A}=P/B$, $A=P/B$. If A is a nonzero ideal of R and if $\sqrt{A}$ is a maximal ideal of R or is equal to R, A is clearly a primary ideal. Therefore, each nonzero ideal of R with prime radical is primary. This implies that R satisfies (H^*). But $\sqrt{B/B}=P/B$ and B/B is not a P/B-primary ideal of R since B is not a P-primary ideal of D. Thus, R does not satisfy (*).

2. Rings Satisfying Propetry $(\boldsymbol{H}^{**})$

In this section Theorems 2.2, 2.3, and 2.5 and Lemma 2.4 give a characterization of rings with identity satisfying (H^{**}). In Example 3.13 an example will be given of a ring with identity satisfying (Hm), but not (**).

Definitions. *Let R be a ring. If A is an ideal of R, we say that A is simple if there exist no ideals properly between A and A^2. If R contains an identity, R is called a primary ring if R contains a unique genuine prime ideal. Finally, a primary ring R with genuine prime ideal M is called a special primary ring (special P. I. R.) if R is a principal ideal ring such that $M^k=(0)$ for some $k \in w$.*

Lemma 2.1. *Let R be a ring satisfying (H^{**}). If P is a prime ideal of R such that $P^2 \neq (0)$, then P is simple.*

Proof. If $P=P^2$, P is clearly simple. Assume that $P^2 \subset P$ and let A be an ideal of R such that $P^2 \subset A \subseteq P$. Then R/P^2 satisfies (**) and $\sqrt{A/P^2}=(\sqrt{A})/P^2=P/P^2$, a prime ideal of R/P^2. Thus, $A/P^2=(P/P^2)^n$ for some $n \in w$. Since $P^2 \subset A \subseteq P$, it follows that $n=1$. Therefore, $A/P^2=P/P^2$ which implies that $A=P$. This shows that P is a simple ideal of R.

Theorem 2.2. *Let R be a ring with identity. If R is not a primary ring, R satisfies (H^{**}) if and only if R satisfies (**).*

Proof. $(\leftarrow)$ If R satisfies (**), R clearly satisfies (H^{**}).

$(\rightarrow)$ Assume that R satisfies (H^{**}). Let A be a genuine ideal of R with prime radical. We show that A is a power of its radical by considering three cases.

Case 1. $\sqrt{A}=M$ is a maximal ideal of R. Since R is not a primary ring, $A \neq (0)$. Therefore, R/A satisfies (**) and $\sqrt{A/A}=M/A$ which implies that

$A/A=(M/A)^n$ for some $n \in w$. This shows that $A=M^n+A$. Hence, $M^n \subseteq A \subseteq M$. Since R is not a primary ring, $M^2 \neq (0)$ and it follows from Lemma 2.1 that M is simple. Thus, $A=M^k$ for some $k \in w$ [5, Lemma 3].

Case 2. $\sqrt{A}=P$ is a nonmaximal prime ideal of R and $\sqrt{(0)}$ is not a prime ideal. It follows that $A \neq (0)$ and that $P^2 \neq (0)$. Since R satisfies (H^{**}), R satisfies (H^*) [5, Theorem 15]. Thus, $P=P^2$ by Lemma 1.4. Since R/A satisfies (**) and $\sqrt{A/A}=P/A$, $A/A=(P/A)^n$ for some $n \in w$. But $(P/A)^n=(P^n+A)/A=(P+A)/A=P/A$ so that $A/A=P/A$. Therefore, $A=P$.

Case 3. $\sqrt{A}=P$ is a nonmaximal prime ideal of R and $\sqrt{(0)}$ is a prime ideal. Since R satisfies (H^*), $\sqrt{(0)}=P$ by Lemma 1.2. If $P=(0)$, $A=P$. Therefore, assume that $P \neq (0)$. We want to show that $P^2=(0)$. Assume that $P^2 \neq (0)$. Then Lemma 1.4 implies that $P=P^2$ and it follows from Lemma 1.3 that R is a one-dimensional domain. Thus, $P=(0)$ which yields a contradiction. Therefore, $P^2=(0)$. If $x \in P\setminus\{0\}$, $R/(x)$ satisfies (**) and $P/(x)=(P/(x))^2=(P^2+(x))/(x)=(x)/(x)$ [4, Corollary 2.2]. It follows that $P=(x)$ and that P is a simple ideal of R. Thus, either $A=P$ or $A=P^2$.

Theorem 2.3. *Let R be a primary ring satisfying (H^{**}) with maximal ideal M. Then*

(a) *Each proper homomorphic image of R is a special primary ring.*

(b) *If $M^2 \neq (0)$, R is a special primary ring. Thus, R satisfies* (**).

Proof. Let $\bar{R}$ be a proper homomorphic image of R. Then $\bar{R}$ satisfies (**) and if $\bar{R} \neq (0)$, $\bar{M}$ is the unique genuine prime ideal of $\bar{R}$, where $\bar{M}$ is the image of M in $\bar{R}$. Now $\sqrt{\overline{(0)}}=\bar{M}$ so that $\overline{(0)}=\bar{M}^n$ for some $n \in w$. Since $\bar{R}$ satisfies (**), $\bar{M}$ is clearly a simple ideal of $\bar{R}$. Therefore, $\bar{R}, \bar{M}, \bar{M}^2, \cdots, \bar{M}^n=\overline{(0)}$ is the collection of ideals of $\bar{R}$. If $x \in \bar{M}\setminus\bar{M}^2$, then $(x)=\bar{M}$. Hence, $\bar{R}$ is a special primary ring.

It follows that each proper homomorphic image of R is Noetherian, which implies that R is Noetherian. Since $\sqrt{(0)}=M$, there exists a positive integer k such that $M^k=(0)$. Let k be minimal with the property that $M^k=(0)$. Then if $M^2 \neq (0)$, $R, M, M^2, \cdots, M^k=(0)$ is the collection of ideals of R by Lemma 2.1. If $y \in M\setminus M^2$, then $(y)=M$ and R is a special primary ring.

Lemma 2.4. *Let R be a primary ring such that $M^2=(0)$ where M is the maximal ideal of R. Then M is simple if and only if R is a special primary ring.*

Proof. If $M=(0)$, the lemma is clear. Therefore, we assume that $M \neq (0)$.

$(\rightarrow)$ If M is a simple ideal of R, $M=(x)$ for each $x \in M\setminus\{0\}$. Hence, R is a special primary ring.

$(\leftarrow)$ This is a consequence of [9, Corollary 1, p. 237].

THEOREM 2.5. *Let R be a primary ring satisfying (H^{**}) such that R is not a special primary ring and $M^2=(0)$ where M is the maximal ideal of R. Then the following conditions hold in R:*

(i) *M is generated by any two elements in M which do not compare; that is, if $x, y \in M$ are such that $(x) \nsubseteq (y)$ and $(y) \nsubseteq (x)$, then $(x, y)=M$.*

(ii) *M is not simple.*

(iii) *If A is an ideal of R such that $(0) \subset A \subset M$, A is a principal ideal of R.*

(iv) *R does not contain a chain of five ideals.*

(v) *R is Noetherian.*

Conversely, if conditions (i) *and* (ii) *hold in R, then each nontrivial proper homomorphic image of R is a special primary ring. Thus, R satisfies (Hm).*

PROOF. Let $x \in M \setminus \{0\}$. Then Lemma 2.4 shows that $(x) \subset M$. This shows that (ii) holds. If $y \in M \setminus (x)$, $(0) \subset (x) \subseteq (x, y) \subset M$. Since $R/(x)$ satisfies $(**)$, $M/(x)$ is a simple ideal of $R/(x)$; there do not exist any ideals properly between $M/(x)$ and $(M/(x))^2=(M^2+(x))/(x)=(x)/(x)$. Therefore, $M=(x, y)$ which proves condition (i). Let A be an ideal of R such that $(0) \subset A \subset M$. If $a \in A \setminus \{0\}$, $(0) \subset (a) \subseteq A \subset M$. Since there are no ideals properly between M and (a), $A=(a)$; that is, condition (iii) holds. This also shows that conditions (iv) and (v) hold in R.

Conversely, if conditions (i) and (ii) hold in R, it is clear that R/A is a special primary ring for each proper ideal A of R. For if $A=M$, R/M is a field and if $(0) \subset A \subset M$, M/A is the only proper ideal of R/A. Thus, R satisfies (Hm).

3. Rings Satisfying Property (Hm)

We are now in a position to give a characterization of rings with identity satisfying (Hm). In particular, Theorem 3.8 shows that in a nonprimary ring R with identity, R satisfying (Hm) is equivalent to R being a multiplication ring. This section concludes with an outline for constructing examples of rings with identity satisfying (Hm) that are not multiplication rings.

THEOREM 3.1. *Let A be an ideal of a ring R with identity satisfying (Hm) such that $A \nsubseteq \sqrt{(0)}$. If B is an ideal of R containing A, there exists an ideal C of R such that $A=BC$. Therefore, if $\sqrt{(0)}=(0)$, R is a multiplication ring.*

PROOF. Let $a \in A \setminus \sqrt{(0)}$. Then there exists an ideal N of R such that $A/(a)=(B/(a))(N/(a))=(BN+(a))/(a)$ and an ideal N' of R such that $(a)/(a^2)=(B/(a^2))(N'/(a^2))=(BN'+(a^2))/(a^2)$. Hence, $A=BN+(a)=BN+(BN'+(a^2))=B(N+N')$ since $a^2 \in BN$. The ideal $N+N'$ is the desired ideal C in our theorem.

THEOREM 3.2. *If R is a ring with identity satisfying (Hm), then R satisfies (*). Therefore, each ideal of R is equal to its kernel.*

PROOF. If R is a primary ring, each ideal of R with prime radical is primary. If R is not a primary ring, R satisfies (**) by Theorem 2.2. Thus, R satisfies (*) [5, Theorem 15].

REMARK 3.3. Let R be a ring with identity satisfying (*Hm*). Assume that $\sqrt{(0)}=P$ is a prime ideal of R. If P is a maximal prime ideal of R, R is either a special primary ring by Theorem 2.3 and Lemma 2.4 or R is a primary ring satisfying conditions (i) through (v) in Theorem 2.5. If P is a nonmaximal prime ideal of R, $\sqrt{(0)}=P$ is the unique minimal prime ideal of R and R is not a primary ring. Therefore, R satisfies (**) by Theorem 2.2. Since R also satisfies (*), (0) is a P-primary ideal of R. Thus, $P=(0)$ [4, Theorem 2] and R is a Dedekind domain by Theorem 3.1 and [8, p. 429].

THEOREM 3.4. *Let R be a ring with identity satisfying (Hm). If $\sqrt{(0)}=P$ is a genuine nonmaximal prime ideal of R, then $P=(0)$ and R is a Dedekind domain.*

PROOF. This is an immediate consequence of Remark 3.3.

COROLLARY 3.5. *Let R be a ring with identity satisfying (Hm). If P is a nonmaximal prime ideal of R, $P=P^2$.*

PROOF. If $P^2=(0)$, then $\sqrt{(0)}=P$ and Theorem 3.4 shows that $P=(0)$. If $P^2\neq(0)$, $P/P^2=(P/P^2)^2=P^2/P^2$ [4, Corollary 2.2]. In both cases we have that $P=P^2$.

LEMMA 3.6. *Let R be a ring and let x be a nonzero nilpotent element of R such that $R/(x)$ contains a nonzero, nonidentity, idempotent element. Then R contains a nonzero, nonidentity, idempotent element.*

PROOF[3]. Let $\bar{e}=e+(x)$ be a nonzero, nonidentity, idempotent element of $R/(x)$ where $e\in R$. Then $e\notin(x)$ and $e+(x)=(e+(x))^2=e^2+(x)$ which implies that $e^2-e\in(x)$. By an inductive argument, it is clear that $e^k-e\in(x)$ for each $k\in w$. Since $e\notin(x)$, $e^k\neq 0$ for each $k\in w$. Now $e^2-e=rx$ for some $r\in R$. There exists an odd positive integer n such that $x^n=0$. Therefore, $(e^2-e)^n=(rx)^n=0$. It follows that $se^{n+1}-e^n=0$ where s is a combination of powers of e. This shows that $(e^{n+1})=(e^n)$. Thus, $(e^n)^2=(e^n)\neq(0)$. Since $(0)\subset(e^n)\subset R$, (e^n) is generated by a nonzero, nonidentity, idempotent element [2, Corollary 2].

LEMMA 3.7. *Let e be an idempotent element of a ring R, let $A=Re=\{re: r\in R\}$, and let $B=\{x-xe: x\in R\}$. Then the following conditions hold:*

3) The method of proving Lemma 3. 6 was suggested to me by Professor Gilmer.

(a) *B is an ideal of R.*
(b) *A is a ring with identity, namely e.*
(c) *$R=A\oplus B$.*

Proof. The proofs of (a) and (b) are clear. Thus, we prove only part (c). If $x\in R$, $x=xe+(x-xe)\in A+B$ which implies that $R=A+B$. Therefore, to show that R is a direct sum of A and B we show that $A\cap B=(0)$. Let $a\in A\cap B$. Then $ae=a$ since $a\in A$, and $a=x-xe$ for some $x\in R$ since $a\in B$. It follows that $a=ae=(x-xe)e=xe-xe^2=xe-xe=0$. Thus, $A\cap B=(0)$ which shows that $R=A\oplus B$.

Theorem 3.8. *Let R be a ring with identity. If R is not a primary ring, then R satisfies (Hm) if and only if R is a multiplication ring.*

Proof. $(\leftarrow)$ This half of the proof is clear.

$(\rightarrow)$ Assume that R satisfies (Hm). If $\sqrt{(0)}=(0)$, R is a multiplication ring by Theorem 3.1. If $(0)\subset\sqrt{(0)}$, let $x\in\sqrt{(0)}\setminus\{0\}$. Then we consider the multiplication ring $R/(x)$. If $R/(x)$ is indecomposable, $R/(x)$ is a Dedekind domain [5, Theorem 16]. Thus, there exists a prime ideal P of R such that $P=(x)$. Since x is a nilpotent element of R, $\sqrt{(0)}=P$. It follows from Theorem 3.4 that R is a Dedekind domain. If $R/(x)$ is decomposable, $R/(x)$ contains a nonzero, nonidentity, idempotent element. Therefore, R contains a nonzero, nonidentity, idempotent element which implies that R is decomposable. It follows that R can be written as the direct sum of two multiplication rings. Thus, R is a multiplication ring.

Remark 3.9. Let R be a ring with identity satisfying (Hm). If R is not a primary ring and $(0)\subset\sqrt{(0)}$, the proof of Theorem 3.8 shows that $R/(x)$ is a decomposable ring for each $x\in\sqrt{(0)}\setminus\{0\}$.

Corollary 3.10. *Let R be a ring with identity satisfying (Hm) such that $(0)\subset\sqrt{(0)}$ and let $x\in\sqrt{(0)}\setminus\{0\}$. If $R/(x)$ contains no proper idempotent ideal, R is a primary ring.*

Proof. Since a decomposable ring with identity contains proper idempotent ideals, $R/(x)$ is indecomposable. But $R/(x)$ is not a Dedekind domain since $(0)\subset\sqrt{(0)}$. Therefore, $R/(x)$ is a special primary ring [5, Theorem 16]. This implies that R is a primary ring.

Corollary 3.11. *Let R be a ring with identity. If R is an indecomposable ring satisfying (Hm), R is either a primary ring or a Dedekind domain.*

Proof. If R is not a primary ring, R is a multiplication ring by Theorem 3.8. Then [5, Theorem 16] implies that R is a Dedekind domain.

THEOREM 3.12. *Let R be a ring with identity. If R is a primary ring with maximal ideal M, R satisfies (Hm) if and only if R is either a special primary ring or R satisfies the following three conditions*:

(i) $M^2=(0)$.

(ii) *M is generated by any two elements in M that do not compare*; *that is, if $x, y \in M$ such that $(x) \nsubseteq (y)$ and $(y) \nsubseteq (x)$, then $M=(x, y)$.*

(iii) *M is not simple.*

PROOF. $(\rightarrow)$ Assume that R satisfies (Hm). Then R also satisfies (H^{**}). If $M^2 \neq (0)$, R is a special primary ring by Theorem 2.3. If $M^2=(0)$ and M is simple, R is a special primary ring by Lemma 2.4. Finally, if $M^2=(0)$ and M is not simple, R satisfies conditions (i), (ii), and (iii) by Theorem 2.5.

$(\leftarrow)$ If R is a special primary ring, R clearly satisfies (Hm). If R satisfies conditions (i), (ii), and (iii), each nontrivial proper homomorphic image of R is a special primary ring. Thus, R satisfies (Hm).

EXAMPLE 3.13. If R is a ring with identity satisfying (Hm), R need not be a multiplication ring. We give here an outline for constructing such examples in general.

Let M be a maximal ideal of the ring S with identity such that M/M^2, considered as a vector space over S/M, has finite dimension $n>1$. (Thus, if S is Noetherian, M is any maximal ideal which is not simple.) Now, M/M^2 contains an n-2 dimensional subspace A/M^2 where A is an ideal of S. Therefore, if $x \in M \setminus A$, $A+(x)$ is an ideal of S such that $A \subset A+(x) \subset M$. If $y \in M \setminus A$ such that $A+(x) \nsubseteq A+(y)$ and $A+(y) \nsubseteq A+(x)$, then $A+(x) \subset A+(x, y) \subseteq M$. Since the dimension of M/M^2 over S/M is equal to n and the dimension of A/M^2 over S/M is equal to n-2, $A+(x, y)=M$.

Let $R=S/A$. Then R is a primary ring with maximal ideal M/A such that R satisfies (Hm) by Theorem 3.12. But R is not a multiplication ring since M/A is not a simple ideal of R; in fact, R does not satisfy $(**)$.

For a particular example, let $S=K[X_1, \cdots, X_m]$ where K is a field, $X_1, \cdots, X_m$ are indeterminates over K, and $m>1$. Then any maximal ideal of S satisfies the required conditions since any maximal ideal of S has a basis of m, but no basis of $m-1$, elements.

References

[1] H. S. Butts and R. C. Phillips, *Almost multiplication rings*, Can. J. Math., **17** (1965), 267-277.

[2] R. W. Gilmer, *Eleven nonequivalent conditions on a commutative ring*, Nagoya Math. J., **26** (1966), 183-194.

[3] R. W. Gilmer, *Extension of results concerning rings in which semi-primary ideals are primary*, Duke Math. J., **31** (1964), 73-78.

[4] R. W. Gilmer, *Rings in which semi-primary ideals are primary*, Pacific J. Math., **12** (1962), 1273-1276.

[5] R. W. Gilmer and J. L. Mott, *Multiplication rings as rings in which ideals with prime radical are primary*, Trans. Am. Math. Soc., **114** (1965), 40-52.

[6] W. Krull, *Idealtheorie in Ringen ohne Endlichkeitsbedingung*, Math. Annalen, **101** (1929), 729-744.
[7] S. Mori, *Uber allgemeine Multiplikationsringe* I, J. Sci. Hiroshima Univ., Ser. A, **4** (1934), 1-26.
[8] J. L. Mott, *Equivalent conditions for a ring to be a multiplication ring*, Can. J. Math., **16** (1964), 429-434.
[9] O. Zariski and P. Samuel, *Commutative algebra*, Vol. I, Van Nostrand, Princeton, 1958.

The Florida State University
Tallahassee, Florida

Oklahoma State University
Stillwater, Oklahoma