J. Sci. Hiroshima Univ. Ser. A-I
33 (1969) 73-83

Multiplication Rings Containing Only Finitely Many Minimal Prime Ideals

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(Received October 12, 1968)

1. Introduction

A commutative ring R is called an *AM-ring* if whenever A and B are ideals of R with A properly contained in B, then there is an ideal C of R such that $A=BC$. An AM-ring in which $RA=A$ for each ideal A of R is called a *multiplication ring.* This paper is principally concerned with the results of a paper by Gilmer and Mott [7] when the ring R is assumed to contain only a finite number of minimal prime ideals. One of the principal results of this paper is that a multiplication ring R is Noetherian if and only if R contains only finitely many minimal prime ideals. Unless otherwise stated, all rings considered in this paper are assumed to be commutative and to contain an identity. However, on some occasions it will be pointed out that the theorem proved can be proved when R does not necessarily contain an identity.

2. Preliminary results and definitions

Two very important properties to be considered are the properties that will be called $(*)$ and $(**)$ throughout this paper. A ring R satisfies $(*)$ if *an ideal of R with prime radical is primary*, and $(**)$ is the property that *an ideal of R with prime radical is a prime power.* Also important is the notion of the *kernel* of an ideal, which is defined as follows: if $\{P_\alpha\}$ is the collection of all minimal prime ideals of an ideal A of R, then by an isolated P_α-primary component of A we mean the intersection Q_α of all P_α-primary ideals which contain A. The kernel of A is the intersection of all Q_α'*s*.

The relationship between properties $(*)$ and $(**)$ and the kernel of an ideal were studied in [7]. We list here those results which are used most frequently in this paper.

Theorem 1. *A ring R satisfies $(*)$ if and only if R is one of the following:*

a) *a zero-dimensional ring,*

or

b) *a one-dimensional ring in which each non-maximal prime ideal P of R has the property that if M is a maximal ideal such that $P<M<R$ and if $p \in P$, then $p \in pM$.*

In Theorem 1, b) is equivalent to c) *R is one-dimensional and if P and M*

are prime ideals of R such that $P<M<R$, PR_M is the zero ideal of R_M.

For the proof of Theorem 1, see [5].

The next theorem does not appear in [7] but it follows easily from Theorem 1 and we omit the proof.

THEOREM 2. *Suppose R is a ring. The following are equivalent*:

a) *R satisfies* $(*)$.

b) *R_M satisfies* $(*)$ *for each maximal ideal M of R.*

c) *R_M is either a one-dimensional integral domain or a primary ring for each maximal ideal M of R.*

THEOREM 3. *A ring R satisfies* $(*)$ *if and only if each ideal of R is equal to its kernel.* (See [7; 43]).

THEOREM 4. *Suppose R is a ring. The following are equivalent.*

a) *R satisfies* $(**)$.

b) *R satisfies* $(*)$ *and primary ideals are prime powers.*

c) *R_M is a multiplication ring for each maximal ideal M of R.*

d) *R_M is a Dedekind domain or a special primary ring for each maximal ideal M of R.* (Compare [7; 46–49]).

THEOREM 5. *A multiplication ring satisfies* $(**)$.

All of the above theorems can be proved in a more general situation; namely, when R does not necessarily contain an identity, but is a u-ring [7; 41]. (A *u-ring* is a commutative ring R such that if A is a proper ideal of R, $\sqrt{A} \neq R$.)

This paper contains results obtained independently by Larsen and McCarthy [8]. One pertinent observation to make relative to this paper and to theirs is that a ring R which satisfies $(**)$ has only finitely minimal prime ideals if and only if R has few zero-divisors.

3. Principal results

The following theorem will be useful throughout the remainder of the paper.

THEOREM 6. *If R is a ring with only a finite number of minimal prime ideals $P_1, \cdots, P_n$, the following are equivalent*:

a) *R is a finite direct sum of integral domains and primary rings.*

b) $(0)=Q_1 \cap Q_2 \cdots \cap Q_k \cap Q_{k+1} \cap \cdots \cap Q_n$

where i) $Q_i+Q_j=R$ for $i \neq j$

ii) *Q_i is a prime ideal for* $1 \leq i \leq k$, and

iii) *Q_i is a primary ideal belonging to a maximal ideal for*

$k+1 \leq i \leq n$.

c) *R_M is either an integral domain or a primary ring for each maximal ideal M of R.*

PROOF. It is well known that (a) and (b) are equivalent [11; thm. 32]. We show that (b) implies (c). Since R is a direct sum of integral domains D_i and primary rings R_j and since $(0)=Q_1 \cap \cdots \cap Q_k \cap \cdots \cap Q_n$ where $D_i \simeq R/Q_i$ for $1 \leq i \leq k$ and $R_j = R/Q_j$ for $k+1 \leq j \leq n$, it follows that the minimal prime ideals of R are precisely $Q_1, \cdots, Q_k$, $M_{k+1}, \cdots, M_n$ where $M_j = \sqrt{Q_j}$ for $j \geq k+1$. Furthermore, if M is a maximal ideal of R, then M contains exactly one minimal prime ideal P_{i_0}. Let $A_M = \{a \in R \mid as=0 \text{ for some } s \in RM\}$ be contraction of the extension of (0) relative to R_M [11; 218–224]. If M properly contains P_{i_0}, then $A_M = P_{i_0}$ and R_M is an integral domain. If $M = P_{i_0}$, then M is a minimal prime ideal of R and A_M is an isolated M-primary component of (0) [8; thm. 6]. In this case, R_M is a primary ring.

Next we show that (c) implies (b). For each maximal ideal M_α of R, let A_{M_α} be the extension of the contraction of (0) relative to R_{M_α} and let $A = \bigcap_\alpha A_{M_\alpha}$. If $a \in A$ and $C = \{c \in R \mid ca=0\}$, then C is an ideal of R which is not contained in any maximal ideal of R. Hence $C=R$ and $A=(0)$.

Let $P_1, \cdots, P_k$ be the minimal prime ideals of R which are not maximal and $P_{k+1}, \cdots, P_n$ be those which are also maximal.

Let M be a maximal ideal of R. If M is a minimal prime ideal then $M = P_i$ for some i between $k+1$ and n, and A_M is an isolated M-primary component of (0). In this case, we shall write $A_M = Q_i$. If M is not a minimal prime ideal of R, then R_M is an integral domain and A_M is a minimal prime ideal of R contained in M. In this case, $A_M = P_j$ for some j between 1 and k. Thus $P_1 \cap \cdots P_k \cap Q_{k+1} \cdots \cap Q_n = \bigcap_\alpha A_{M_\alpha} = (0)$ and (b) follows.

It should be observed that either (a) or (b) implies that R has only finitely many minimal prime ideals.

LEMMA 7. *If $R = R_1 \oplus R_2 \cdots \oplus R_n$ and if M is a maximal ideal of R such that $M = R_1 \oplus \cdots \oplus M_i \oplus \cdots \oplus R_n$ where M_i is a maximal ideal of R_i, then $R_M \simeq (R_i)_{M_i}$.*

This is an immediate consequence of properties of direct sums and quotient ring formation [11; 221–227] and the proof is omitted.

We use the above lemma and Theorem 6 to obtain the following corollary.

COROLLARY 8. *Suppose R is a ring. Then R is a finite direct sum of integral domains if and only if R contains only finitely many minimal prime ideals and R_M is an integral domain for each maximal ideal M of R.*

PROOF. If R is a finite direct sum of integral domains D_k, then $(0) = P_1 \cap \cdots \cap P_n$ where P_i is a prime ideal of R for each i and $P_i + P_j = R$ for $i \neq j$.

Furthermore $D_k \simeq R/P_k$ for each k. Clearly, $\{P_1,\cdots,P_n\}$ is the set of minimal prime ideals of R. If M is a maximal ideal of R, then for some i between 1 and n, and for some maximal ideal M_i of D_i, $M=D_1\oplus\cdots\oplus M_i\oplus\cdots\oplus D_n$ [11; 175]. Lemma 7 then shows that $R_M \simeq (D_i)_{M_i}$. And since D_i is an integral domain it follows then that R_M is also an integral domain.

Conversely, suppose R contains only finitely many minimal prime ideals and R_M is an integral domain for each maximal ideal M of R. By Theorem 6, $R=D_1\oplus\cdots\oplus D_n\oplus R_1\oplus\cdots\oplus R_k$ where D_i is an integral domain and R_i is a primary ring for each i.

We show that each R_i is a field in this case. For let M_i be a maximal ideal of R_i and M the corresponding maximal ideal of R. Lemma 7 implies that $R_M \simeq (R_i)_{M_i}$. Since R_M is an integral domain and R_i is a primary ring, it follows that M_i is the zero ideal of R_i and that R_i is a field. Hence each summand in the direct sum representation of R is an integral domain.

THEOREM 9. *In the ring R, these conditions are equivalent*:

a) *R satisfies (*) and contains only finitely many minimal prime ideals.*

b) *R is a finite direct sum of one-dimensional integral domains and primary rings.*

PROOF. Suppose R satisfies (*) and contains only finitely many minimal prime ideals. By Theorem 2, R_M is a one-dimensional domain or a primary ring for each maximal ideal M of R. Therefore, by Theorem 6, R is a finite direct sum of integral domains D_i and primary rings. Since each D_i is a homomorphic image of R, each D_i satisfies (*). Hence each D_i is one-dimensional by Theorem 1. (Note that if D_i is zero-dimensional, then D_i is a field and, hence, is a primary ring.)

Clearly, (b) implies (a).

Part (c) of Theorem 1 shows that if M is a maximal ideal of a ring R which satisfies (*) and if M properly contains a prime ideal P, then P is the unique prime ideal properly contained in P. The next corollary follows immediately from this fact and from Theorem 9.

COROLLARY 10. *In the ring R, these conditions are equivalent*:

a) *R satisfies (*) and contains only finitely many maximal ideals.*

b) *R is a finite direct sum of semi-quasi-local one-dimensional integral domains and primary rings.*

THEOREM 11. *Suppose R is a ring. The following are equivalent*:

a) *R is a multiplication ring containing only finitely many minimal prime ideals.*

b) *R is a multiplication ring and the zero ideal of R is a finite intersection of primary ideals.*

c) *R is a Noetherian multiplication ring.*

d) *R is a finite direct sum of Dedekind domains and special primary rings.*

Proof. It is well-known that (c) and (d) are equivalent [1; thm. 5]. That (d) implies (a) is clear. We show (a) implies (b).

The kernel of (0) is the intersection of all isolated P_α-primary components of (0) where P_α is a minimal prime ideal of (0). Since a minimal prime ideal of (0) is a minimal prime ideal of R we see that the kernel of (0) is a finite intersection of primary ideals. In a multiplication ring each ideal is equal to its kernel by Theorems 5, 4, and 3. Thus (b) follows.

We show (b) implies (d). If a prime ideal P of R is such that $P<M<R$ where M is a maximal ideal of R, then P is the unique prime ideal properly contained in M and, in fact, P is contained in every M-primary ideal of R since R satisfies (*). By hypothesis (0) is a finite intersection of primary ideals and since primary ideals of a multiplication ring are prime ideals powers, it follows that $(0)=P_1\cap\cdots\cap P_k\cap M_{k+1}^{s_{k+1}}\cdots\cap M_n^{s_n}$ where each P_i is a non-maximal prime and each M_j is both a maximal ideal and a minimal prime ideal. It is clear that $P_1, \cdots, P_k, M_{k+1}, \cdots, M_n$ are pairwise comaximal. Thus $R=D_1\oplus\cdots\oplus D_k\oplus R_{k+1}\oplus\cdots\oplus R_n$ where $D_i\simeq R/P_i$ and $R_j\simeq R/M_j^{s_j}$. Since each D_i is a multiplication ring and an integral domain, each D_i is a Dedekind domain. Similarly, each R_j is a multiplication ring and a primary ring, and is therefore a special primary ring.

Definition. An integral domain D is an *almost Dedekind domain* if D_M is a Dedekind domain for each maximal ideal M of D. A Noetherian almost Dedekind domain is Dedekind [3; thm. 8].

Theorem 12. *Suppose R is a ring containing only finitely many minimal prime ideals. The following are equivalent:*

a) *R is a finite direct sum of almost Dedekind domains and special primary rings.*

b) *R satisfies (**).*

Proof. Suppose (a) holds and let $R=R_1\oplus\cdots\oplus R_k\oplus R_{k+1}\oplus\cdots\oplus R_n$ where R_i is an almost Dedekind domain for $1\leq i\leq k$ and R_i is a special primary ring for $k+1\leq j\leq n$. Let M be a maximal ideal of R and M_i be the corresponding maximal ideal of R_i. By Lemma 7, $R_M\simeq(R_i)_{M_i}$. If $i\leq k$, $(R_i)_{M_i}$ is a Dedekind domain, and if $k+1\leq i$, then $(R_i)_{M_i}=R_i$, a special primary ring. Thus by Theorem 4, R satisfies (**).

Conversely suppose R satisfies (**). Since R also satisfies (*), Theorem 9 implies that R is a finite direct sum of one-dimensional integral domains $\{R_i\}_{i=1}^k$ and primary rings $\{R_j\}_{j=k+1}^n$. Let M_i be a maximal ideal of some R_i and M the corresponding maximal ideal of R. If $i\leq k$, $(R_i)_{M_i}$ is a one-dimensional integral domain. Since $R_M\simeq(R_i)_{M_i}$ and R_M is either a Dedekind domain or special primary ring, it follows that $(R_i)_{M_i}$ is a Dedekind domain. If $k+1$

$\leq i$, $(R_i)_{M_i}=R_i$ since R_i is a primary ring. In this case it follows that R_i is a special primary ring. Therefore (b) implies (a).

THEOREM 13. *Suppose R is a ring. The following are equivalent:*

a) *R is a Noetherian multiplication ring.*
b) *Every ideal is a finite product of prime ideals.*
c) *Every ideal of R is a finite intersection of powers of prime ideals.*
d) *R satisfies (**) and every ideal of R is a finite intersection of primary ideals.*
e) *R is Noetherian and satisfies (**).*
f) *R is Noetherian and R_M is a multiplication ring for each maximal ideal M or R.*

PROOF. It is well known that (a) and (b) are equivalent [1, thm. 5]. Recently, Butts and Gilmer have shown that (b) and (c) are equivalent [2; cor. 6]. Clearly (a) implies (d). It is clear from the definition (**), however, that (d) implies (c). It is also clear that (a) implies (e). Theorem 4 shows that (e) and (f) are equivalent. Finally, we show that (e) implies (a).

Since a Noetherian ring contains only finitely many minimal prime ideals, Theorem 12 shows that (e) implies that R is a finite direct sum of almost Dedekind domains and special primary rings. However, since R is Noetherian, each almost Dedekind summand is in fact a Dedekind domain. Thus R is a finite direct sum of Dedekind domains and special primary rings and by Theorem 11, R is a Noetherian multiplication ring.

THEOREM 14. *Suppose R is a ring containing only finitely many maximal ideals. Then R satisfies (**) if and only if R is a principal ideal ring.*

PROOF. A principal ideal ring R is a multiplication ring, and hence, (**) holds in any principal ideal ring.

On the other hand, since R contains only finitely many maximal ideals and since a maximal ideal of a ring satisfying (*) or (**) contains a unique minimal prime ideal, it follows that R contains only finitely many minimal prime ideals. Thus, R is a finite direct sum of almost Dedekind domains and special primary rings by Theorem 12. By [6, thm. 3], an almost Dedekind domain with only a finite number of maximal ideals is a principal ideal domain. Therefore, R is a finite direct sum of principal ideal domains and special primary rings. By [11; thm. 33], R is a principal ideal ring.

The above theorem can be proved when R need not contain an identity but is assumed to be a u-ring. The more general form of Theorem 14 will be apparent from the following theorem and the fact that in a u-ring which satisfies (*), proper ideals are contained in maximal ideals [7; 41].

THEOREM 15. *If R is a u-ring containing only finitely many maximal ideals and if each proper ideal of R is contained in a maximal ideal, then R*

contains an identity.

PROOF. Since R is a u-ring, a) $R=R^2$ and b) maximal ideals are prime [7; thm. 2]. The union of all maximal ideals of R is a proper subset of R. Otherwise, R is contained in the finite union of prime ideals and hence is contained in one of them [11; 215]. This, of course, violates the fact that a maximal ideal is a proper ideal. By hypothesis, if x is not in any maximal ideal of R, then $R=(x)$. Therefore, R is a finitely generated idempotent ideal, and by [4; 185] or [1; 86], R is generated by a single idempotent element e. This element e is the identity of R.

The assumption that each proper ideal is contained in a maximal ideal is a necessary one as is shown by example 4 of [4].

We prove the next theorem in its most general form also.

THEOREM 16. *An AM-ring R containing only finitely many maximal ideals is a principal ideal ring.*

PROOF. If $R \neq R^2$, then $R=(r)$ where $r \in R \backslash R^2$ and each ideal of R is a power of R [17; thm. 10]. If $R=R^2$, then R is a u-ring in which each proper ideal is contained in a maximal ideal [7; 45]. Hence by Theorem 16, R contains an identity. Since a multiplication ring satisfies (**), R is a principal ideal ring by Theorem 15.

We return now to our original convention of assuming that the ring R contains an identity.

The following theorem is an immediate consequence of Corollary 8 and Theorem 12.

THEOREM 17. *A ring R containing only finitely many minimal prime ideals is a finite direct sum of almost Dedekind domains if and only if R_M is a Dedekind domain for each maximal ideal M of R.*

COROLLARY 18. *A Noetherian ring R is a finite direct sum of Dedekind domains if and only if R_M is a Dedekind domain for each maximal ideal M* of *R.*

4. Related results

We consider in this section conditions other than those given in Corollary 18 in order that a ring R be a finite direct sum of Dedekind domains. Our main result in this vein is the following theorem.

THEOREM 19. *If R is a Noetherian ring and R is not a field, then the following are equivalent*:

a) *R is a finite direct sum of proper Dedekind domains.*

b) *Every maximal ideal of R is invertible.*

c) *There are no ideals between a maximal ideal M and its square, and M properly contains a prime ideal P such that the only primary ideal for P is P itself.*

d) *R is integrally closed, and one of the following holds:*

  i) *each maximal ideal M of R properly contains exactly one other prime ideal P, and $MP=P$.*

  ii) *each maximal ideal M of P properly contains exactly one other prime ideal P, and P is contained in every primary ideal contained in M.*

Before proceeding with the proof of this theorem some definitions and basic results will be required.

If A is an ideal of the ring R and F is the total quotient ring of R, let $A^{-1}=\{x \in F \mid xA \subset R\}$. We say A is *invertible* if $AA^{-1}=R$. Equivalently A is invertible if $AC=(d)$ for some regular element $d \in A$ and some ideal C of R. An ideal A of R is a *reqular* ideal if A contains a regular element.

An integral domain D is said to be *proper* if D is not a field. This terminology is convenient since some of the following theorems are false for fields.

It is clear that if A is an ideal of R contained in the invertible ideal B, then there is an ideal C such that $A=BC$. Other properties of invertible ideals can be found in [11; 272]; of particular importance is the fact that an invertible ideal has a finite basis. The following general fact will be useful.

LEMMA 20. *If Q is a primary ideal for the prime ideal P and A is an invertible ideal such that $A \not\subset P$, then $A \cap Q=AQ$.*

PROOF. Since $A \cap Q \subset A$ and A is invertible, there is an ideal C such that $A \cap Q=AC$. Furthermore, since $AQ \subset A \cap Q=AC$, on multiplying by A^{-1}, we see that $Q \subset C$.

On the other hand, $AC \subset Q$ and $A \not\subset P$ imply that $C \subset Q$ since Q is primary for P. Hence $C=Q$ and the lemma is proved.

The following is a list of generally known properties of invertible prime ideals but the author is unaware of a reference in the literature.

LEMMA 21. *Suppose P is a proper invertible prime ideal of R.*

a) *If $P=AB$ where A and B are ideals of R, then either $A=R$ or $B=R$.*

b) *If A is an invertible ideal of R properly containing P, then $A=R$.*

c) *If $P'=\bigcap_{n=1}^{\infty} P^n$, then P' is a prime ideal, and if P'' is a prime ideal properly contained in P, then $P'' \subset P'$. If P' has a finite basis and H is a primary ideal contained in P, then $P' \subset H$; in fact, $P'=H$ or the radical of H is P. In particular if P' has a finite basis, then P' is the only prime ideal properly contained in P.*

d) *An ideal Q is P-primary if and only if Q is a power of P.*

e) *The only invertible ideals between P and P^n, where n is a positive in-*

teger, are powers of P.

f) *If P is a maximal ideal, the only ideals between P and P^n are powers of P.*

PROOF. The proofs of (a) and (b) are quite straightforward and so we proceed to prove (c). The proof that P' is a prime ideal is identical with a proof given by Nakano in [10; thm. 4]. Furthermore, Lemma 20 implies that $P''=P''P$ if P'' is a prime ideal properly contained in P. From this if follows that $P''=PP''=P^2P''=\cdots$, etc. so that $P''\subset P'$.

Now suppose P' has a finite basis and H is a primary ideal contained in P. Lemma 2 of [11; 215] implies there is an element $z \in P$ such that $(1-z)P'=(0)$ since $PP'=P'$. Therefore, $(1-z)$ is not contained in the radical of H since $(1-z)\not\subset P$. Thus, $P'\subset H$ since $(1-z)P'\subset H$. In particular, the first part of (c) implies that P' is the only prime ideal properly contained in P. Thus, if H is a primary ideal properly containing P', then the radical of H must be P.

To show (d) we first show that P^n is P-primary for each positive integer n. Suppose A and B are ideals such that $AB\subset P^n$ and suppose further that $A\not\subset P^n$. To show that P^n is primary, it will be sufficient to show that $B\subset P$. Since $AB\subset P^n$ and P^n is invertible, there is an ideal C such that $AB=P^nC$. Furthermore, since $A\not\subset P^n$, there is a non-negative integer $k<n$ such that $A\subset P^k$ but $A\not\subset P^{k+1}$ (here we mean that $P^0=R$). Therefore, there is an ideal C' such that $A=P^kC'$ where $C'\not\subset P$ (if $k=0$, then $P^k=R$ and $C'=A$). Consequently, $P^nC=AB=P^kC'B$ and $P^{n-k}C=BC'\subset P$ since $n-k>0$. From this it follows that $B\subset P$ since $C'\not\subset P$.

Next suppose Q is P-primary. Since P is invertible, P has finite basis, and therefore it is clear that Q contains a power of P. Consequently, there is an integer n such that $Q\subset P^n$ but $Q\not\subset P^{n+1}$. Then $Q=P^nQ'$ where $Q'\not\subset P$. But since Q is P-primary, $P^n\subset 0$. Hence $Q=P^n$.

We will prove (e) by induction on n. We first prove there is no invertible ideals between P and P^2. Suppose $P^2<A\subset P$ where A is an invertible ideal. There is an ideal A' such that $A=PA'$ and $A'\not\subset P$. But then $P=P^{-1}P^2\subset P^{-1}A=A'$. Since A is invertible, A' is also, and according to (b), $A'=R$ or $A'=P$. However, $A'\not\subset P$ so that $A'=R$ and $A=P$.

Suppose for $n\geq 2$, there are no invertible ideals between P and P^n except powers of P. Then if $P^{n+1}<A\subset P$ and if A is invertible, we will show that A is a power of P. Since $A\subset P$, there is an ideal A' such that $A=PA'$ and $A'\not\subset P^n$. Furthermore, there is an ideal B such that $P^{n+1}=AB$. Therefore, $P^n=A'B$ so that $P^n\subset A'$. If $A'\subset P$, then the inductive hypothesis implies that A' is a power of P and hence that A is also. Suppose $A'\not\subset P$. Since P^n is P-primary, it follows that $B\subset P^n$ and hence that $B=P^n$. It follows that $A'=R$ and, as a result, $A=P$.

To show (f) it is sufficient to show there are no ideals between the maximal ideal P and its square [1; 83]. If $P^2\subset A\subset P$, there is an ideal A' such that $A=PA'$ and $P=P^{-1}P^2\subset P^{-1}A=A'$. Therefore since P is a maximal ideal,

$A'=R$, or $A'=R$. Hence $A=P$ or $A=P^2$.

LEMMA 22. *A Noetherial domain D is a Dedekind domain if and only if every non-zero maximal ideal of D is invertible.*

PROOF. This result is well-known, but the proof follows immediately from part (iii) of Lemma 21.

The proof of the following lemma is straightforward and will be omitted. See [11; 256] for the definition of integral closure.

LEMMA 23. *If $R=R_1\oplus R_2\oplus\cdots\oplus R_n$, then R is integrally closed if and only if each R_i is integrally closed.*

We are now prepared to give the proof of Theorem 19.

Assume, first, that R is a finite direct sum of proper Dedekind domains, $R=R_1\oplus R_2\oplus\cdots\oplus R_n$. If M is a maximal ideal of R, M is of the form.

$$M=R_1\oplus R_2\oplus\cdots\oplus R_{k-1}\oplus M_k\oplus R_{k+1}\oplus\cdots\oplus R_n,$$

where M_k is a maximal ideal of R_k. Since R_k is a proper Dedekind domain, M_k is invertible. Therefore, there is an ideal C_k of R_k such that $M_kC_k=(d_k)$ where $d_k\neq 0$ in R_k. Hence, the ideal

$$C=R_1\oplus R_2\oplus\cdots\oplus R_{k-1}\oplus C_k\oplus R_{k+1}\cdots\oplus R_n$$

is such that MC is a principal ideal generated by the regular element

$$e_1+e_2+\cdots+e_{k-1}+d_k+e_{k+1}\cdots+e_n,$$

where e_i is the identity element of R_i. It follows, then, that M is invertible.

If every maximal ideal of the Noetherian ring R is invertible and $(0)=Q_1\cap Q_2\cap\cdots\cap Q_n$ is an irredundant representation of (0) by primary components, where P_i is the radical of Q_i, then it is clear that no P_i is a maximal ideal since each P_i consists entirely of zero-divisors [11; 214]. Therefore each P_i is properly contained in a maximal ideal M_i, and by Lemma 21, P_i is the only prime properly contained in M_i and $Q_i=P_i$. Then (c) follows from (b).

Assuming (c), Lemma 4 of [1; 86] implies that in the Noetherian ring R, if M is a maximal ideal properly containing a prime ideal P, then, since there are no ideals between M and M^2, P is the only prime ideal properly contained in P and $P=\bigcap_{n=1}^{\infty}M^n$. Furthermore, by hypothesis, the only ideal primary for P is P itself. Thus, (0) can be represented as an irredundant intersection of non-maximal prime ideals. These prime ideals are pairwise comaximal since each maximal ideal properly contains only one prime ideal. Consequently, R is a finite direct sum of proper Noetherian domains. Further more, since there are no ideals between a maximal and its square, Theorem 8 of [3; 33] implies that each summand is a Dedekind domain. Thus (a) follows from (c).

If R is a finite direct sum of proper Dedekind domains, then Lemma 23 implies that R is integrally closed. Furthermore, it is clear that every maximal ideal M properly contains exactly one prime ideal P and $P=MP$.

From Lemma 2 of [11; 215] it follows that (di) implies (dii) in any Noetherian ring.

Assume that R is integrally closed and (dii) is valid in R. We will show that (a) follows. Let $(0)=Q_1 \cap Q_2 \cdots \cap Q_n$ be an irredundant representation of (0) by primary components. Since every maximal ideal properly contains only one prime ideal it follows that each $Q_i \subset M_i$ where M_i is a maximal ideal and $M_i \neq M_j$ for $i \neq j$. Let P_i be the prime ideal properly contained in M_i. Then since $P_i \subset Q_i$ it follows that $(0)=P_1 \cap P_2 \cdots \cap P_n$. Refine this representation to an irredundant one, say $(0)=P_1 \cap P_2 \cdots \cap P_k$. These prime ideals are pairwise comaximal since each maximal ideal properly contains only one prime ideal. Thus R is a direct sum of proper Noetherian integrally closed domains in which proper prime ideals are maximal. Consequently, each summand is a Dedekind domain [11; thm. 13], and (a) follows. This completes the proof of the theorem.

We also wish to remark that if R is a Noetherian ring in which every maximal ideal is a regular ideal and any one of the conditions 1–8 of Theorem 5 of [1; 89] is valid in R, then R is a finite direct sum of proper Dedekind domains, and conversely.

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