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Relative Dirichlet Problems on Riemann Surfaces

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Introduction

M. Brelot [1] introduced relative Dirichlet problems on a metrizable compactification of a Green space and L. Naïm [4] obtained many results concerning this type of problems. Also, T. Ikegami [3] studied the problems on the Wiener compactification of a hyperbolic Riemann surface.

In this paper, we consider the relative Dirichlet problems on an arbitrary compactification of a hyperbolic Riemann surface R. We denote by $\mathfrak{A}_u$ the resolutivity of all finite continuous functions on the ideal boundary relative to a positive harmonic function u (§1, 7) and first give characterizations of $\mathfrak{A}_u$ for Q-compactifications (Theorem 1). Then we obtain that $\mathfrak{A}_u$ is satisfied for the Wiener compactification if and only if u is quasi-bounded (Theorem 2). As a corollary, we improve Ikegami's result as follows: There exists a unique pole of a minimal positive harmonic function on the Wiener boundary if and only if the function is bounded.

Next, in connection with Brelot's [1] and Naïm's works [4], we define the maximal compactification $R^*_{W_1}$ of R for which $\mathfrak{A}_u$ is satisfied for any $u>0$ (Theorem 3). As a corollary, we obtain Brelot's result ([1]): For the Martin compactification of R, $\mathfrak{A}_u$ is satisfied for any $u>0$. Finally, we prove that $R^*_{W_1}$ is not metrizable (Theorem 4) and we give an answer in the negative to a question in Naïm's remark (p. 268 in [4]).

§1 Preliminaries

Let R be a hyperbolic Riemann surface. For a subset A of R, we denote by ∂A and A^i the (relative) boundary and the interior of A respectively. We shall call a closed subset F of R *regular* if ∂F consists of at most a countable number of analytic arcs clustering nowhere in R. An *exhaustion* will mean an increasing sequence $\{R_n\}_{n=1}^{\infty}$ of relatively compact domains on R such that $\bigcup_{n=1}^{\infty} R_n = R$ and each ∂R_n consists of a finite number of closed Jordan curves. We denote by BC the family of all real valued bounded continuous functions on R and by C_0 the subfamily of BC consisting of functions with compact supports in R.

1. *Wiener functions* (cf. [2]).

For a finite continuous function f on R, we shall denote by $\overline{\mathcal{W}}_f$ (resp. $\underline{\mathcal{W}}_f$)

the family of all superharmonic (resp. subharmonic) functions s on R such that $s \geqq f$ (resp. $s \leqq f$) on $R-K_s$ for some compact set K_s in R. If $\overline{\mathfrak{W}}_f$ and $\underline{\mathfrak{W}}_f$ are not empty, then we define $\bar{h}_f(a)=\inf\{s(a);\ s \in \overline{\mathfrak{W}}_f\}$ and $\underline{h}_f(a)=\sup\{s(a);\ s \in \underline{\mathfrak{W}}_f\}$ $(a \in R)$. It is known that $\bar{h}_f$, $\underline{h}_f$ are harmonic and $\underline{h}_f \leqq \bar{h}_f$. If $\bar{h}_f=\underline{h}_f$, then f is said to be *harmonizable*. We write $h_f=\bar{h}_f=\underline{h}_f$ if f is harmonizable. If f_1 and f_2 are harmonizable, then min (f_1, f_2) is harmonizable and $h_{f_1} \wedge h_{f_2}$[1] $=h_{(min(f_1,f_2))}$. A finite continuous function f on R is called a *Wiener function* if $|f|$ has a superharmonic majorant and f is harmonizable. We denote by W the family of all finite continuous Wiener functions on R and set $BCW=BC \cap W$. We see that W is a vector lattice with respect to the maximum and minimum operations and also contains C_0 and constants.

2. *Compactifications.*

We follow C. Constantinescu and A. Cornea [2] for the definition of (Q-) compactifications. In particular, we denote by R_M^* (resp. R_W^*) the Martin compactification (resp. the Wiener compactification) of R. Let R^* be a compactification of R. We write $\Delta_M=R_M^*-R$, $\Delta_W=R_W^*-R$, $\Delta_Q=R_Q^*-R$ and $\Delta=R^*-R$. We denote by $C(R^*)$ the family of all real valued continuous functions on R^*. For any subset A of R, we shall denote by $\bar{A}^*$ (resp. $\bar{A}^M$, $\bar{A}^W$, $\bar{A}^Q$) the closure of A in R^* (resp. R_M^*, R_W^*, R_Q^*). Let R_1^* and R_2^* be two compactifications of R. If there exists a continuous mapping π of R_1^* onto R_2^* which is reduced to the identity on R, then we shall say that such a mapping is the *canonical mapping* of R_1^* onto R_2^* and that R_2^* is a *quotient space* of R_1^*. It is known ([2]) that if $Q_1 \subset Q_2$, then $R_{Q_1}^*$ is a quotient space of $R_{Q_2}^*$. Hence R_M^* is a quotient space of R_W^*.

3. *Reduced functions.*

Let R^* be a compactification of R and denote by Δ the ideal boundary R^*-R. Let u be a positive harmonic function on R. For a compact subset A of Δ, we consider the following class:

$$\mathcal{S}_{A,R^*}^u=\left\{s;\ \begin{array}{l}\text{superharmonic} \geqq 0 \text{ on } R,\ s \geqq u \text{ on } U \cap R \\ \text{for some neighborhood } U \text{ of } A \text{ in } R^*\end{array}\right\}.$$

Then the function

$$u_A(a)=\inf\{s(a);\ s \in \mathcal{S}_{A,R^*}^u\} \quad (a \in R)$$

is harmonic on R and $0 \leqq u_A \leqq u$.

We can easily show

LEMMA 1. *Let u and A be as above. Let $\{U_n\}_{n=1}^\infty$ be any sequence of neighborhoods of A in R^*. Then there exists a sequence $\{F_n\}_{n=1}^\infty$ of regular*

1) $h_{f_1} \wedge h_{f_2}$ is the greatest harmonic minorant of min (h_{f_1}, h_{f_2}).

closed sets in R such that

(a) *The closure $\bar{F}_n^*$ of each F_n is a neighborhood of A,*

(b) $U_n \cap R \supset F_n (n=1, 2, \dots)$ *and* $\bigcap_{n=1}^{\infty} F_n = \phi$,

(c) $\overline{R - F_n^{i*}} \cap \bar{F}_{n+1}^* = \phi$ $(n=1, 2, \dots)$,

(d) u_{F_n}[2] *decreases to* u_A *as* $n \to \infty$.

By the aid of the above lemma we can prove the following properties:

(A1) If $A_1 \subset A_2$ and $u_1 \leqq u_2$, then $(u_1)_{A_1} \leqq (u_2)_{A_2}$.

(A2) $(u_1 + u_2)_A = (u_1)_A + (u_2)_A$.

(A3) If $c \geqq 0$ is a constant, then $(cu)_A = cu_A$.

(A4) If $A_1 \subset A_2$, then $u_{A_1} = (u_{A_1})_{A_2} = (u_{A_2})_{A_1}$.

(A5) If u_k increases to u as $k \to \infty$, then $(u_k)_A$ increases to u_A as $k \to \infty$.

LEMMA 2. *Let u be a positive harmonic function on R. If F is a regular closed set in R, then $u_F \geqq u_{\bar{F}^W \cap \Delta_W}$.*

PROOF. Since $v = u - u_F \geqq 0$ is a continuous Wiener function on R, it can be continuously extended over R_W^*. We denote by v^* the continuous extension of v over R_W^*. For each $\varepsilon > 0$, we set $U_\varepsilon = \{z \in R_W^*;\ v^*(z) < \varepsilon\}$. Since $v^* = 0$ on $\bar{F}^W$, U_ε is an open neighborhood of $\bar{F}^W \cap \Delta_W$ and $u_F + \varepsilon > u$ on $U_\varepsilon \cap R$. Hence $u_F + \varepsilon \geqq u_{\bar{F}^W \cap \Delta_W}$. Since $\varepsilon > 0$ is arbitrary, we complete the proof.

COROLLARY 1. *If $\{F_n\}_{n=1}^{\infty}$ is a sequence of regular closed sets in R such that $F_n \supset F_{n+1}$ $(n=1, 2, \dots)$ and $\bigcap_{n=1}^{\infty} F_n = \phi$, then u_{F_n} decreases to u_A, where $A = \bigcap_{n=1}^{\infty} \bar{F}_n^W$.*

COROLLARY 2. *If F is a regular closed set in R, then $\lim_{n \to \infty} u_{F - R_n} = u_{\bar{F}^W \cap \Delta_W}$, where $\{R_n\}_{n=1}^{\infty}$ is an exhaustion of R.*

4. *Singular harmonic functions.*

Let u be a non-negative harmonic function on R. If u is the limit function of an increasing sequence of non-negative bounded harmonic functions, then u is said to be *quasi-bounded.* If any non-negative bounded harmonic function dominated by u is always zero, then u is said to be *singular.* Hence an unbounded positive minimal harmonic function is singular. It is known (Parreau) that any positive harmonic function is uniquely represented as the sum of a quasi-bounded harmonic function and a singular harmonic function.

We shall prove

LEMMA 3. *Suppose u is singular. For each integer $n > 0$, we set $F_n = \{z \in R;\ u(z) \geqq n\}$. Then $u_{F_n} = u$ on R for each n.*

PROOF. $v = u - u_{F_n}$ is a bounded continuous Wiener function on $R - F_n$.

2) See p. 43 in [2] for this notation.

By Lemma 1.3 in [3], we see that $u=0$ on Γ_W, where Γ_W is the harmonic boundary of R_W^* (cf. [2]). Since $u_{F_n}\leqq u$ on R, $u_{F_n}=0$ on Γ_W. Hence we have $v=0$ on $(\Gamma_W-\bar{F}_n^W)\cup\partial\bar{F}_n^W$. By the minimum principle (Satz 8.4 in [2]), we obtain that $v=0$ on $R-F_n$. This completes the proof.

REMARK: We can furthermore show the following: *Let u be a positive harmonic function. For each integer $n>0$, we set $F_n=\{z\in R;\ u(z)\geqq n\}$. Then $\lim_{n\to\infty} u_{F_n}$ is equal to the singular part of u.*

PROOF. (*i*) Let u be quasi-bounded and A be a compact subset of Δ_W such that $1_A=0$. Suppose u is the limit function of an increasing sequence $\{u_k\}_{k=1}^\infty$ of positive bounded harmonic functions. Then, by (A5), we have $u_A=\lim_{k\to\infty}(u_k)_A$. Since $(u_k)_A\leqq(\sup u_k)1_A=0\,(k=1, 2, \cdots)$, it follows that $u_A=0$.

(ii) Let u be an arbitrary positive harmonic function. We set $A=\bigcap_{n=1}^\infty \bar{F}_n^W$. Since $1_{F_n}\leqq(1/n)(\min(u, n))\leqq u/n\,(n=1, 2, \cdots)$, it follows from Corollary 1 to Lemma 2 that $1_A=0$. Hence, if u is quasi-bounded, then $u_A=0$ by (i). Now suppose u is not quasi-bounded. Let w be the singular part of u and $\Omega_n=\{z\in R;\ w(z)\geqq n\}$ for each integer $n>0$. Since $\Omega_n\subset F_n$ for each n, it follows from Lemma 3 and Corollary 1 to Lemma 2 that $w_A=w$. By (i), we see that $(u-w)_A=0$, so that $u_A=w_A$ by (A2). This completes the proof.

As a corollary, we obtain:

a) *u is quasi-bounded if and only if* $\lim_{n\to\infty} u_{F_n}=0$ (M. Nakai: Proc. Japan Acad., 41 (1965), 215–217).

b) *u is singular if and only if $u_{F_n}=u$ on R for each n* (cf. Lemma 3).

5. *Poles on the ideal boundary.*

For $b\in\Delta_M=R_M^*-R$, let k_b be the Martin kernel (cf. p. 135 in [2]). Let Δ_1 be the set of all minimal points of Δ_M. It is known ([4]) that if $b\in\Delta_1$ and if F is a closed set in R, then $(k_b)_F$ is either equal to k_b or a Green potential; in fact $(k_b)_F$ is a Green potential if and only if F is thin[3] at b.

Let b be a point in Δ_1 and R^* be a compactification of R. Then we know that there exists at least one point z on Δ such that $(k_b)_{\{z\}}=k_b$ (Lemma 2.2 in [3]). We call such a point z a *pole* of b on Δ. If $(k_b)_F=k_b$ for some closed set F in R, then there exists at least one pole of b on Δ which is contained in $\bar{F}^*\cap\Delta$. The set of all poles of b on Δ_W is denoted by $\Phi(b)$. It is known (Theorem 2.1 in [3]) that $\Phi(b)=\bigcap_{E\in\mathfrak{F}_b}\bar{E}^W$ where $\mathfrak{F}_b=\{E\subset R;\ R-E \text{ is thin at } b\}$. If U is a neighborhood of b in R_M^*, then it follows form Hilfssatz 13.2 in [2] that $U\cap R\in\mathfrak{F}_b$.

LEMMA 4. *Let b be a point in Δ_1 and F be a regular closed set in R. Then F is thin at b if and only if $\bar{F}^W\cap\Phi(b)=\phi$.*

3) See p. 201 of [4]; this is called effilé.

PROOF. We set $\alpha=\bar{F}^W\cap\Delta_W$. It suffices to prove that F is thin at b if and only if $\alpha\cap\Phi(b)=\phi$. Let $\{R_n\}_{n=1}^{\infty}$ be an exhaustion of R. First suppose F is thin at b. Then $F-R_n$ is thin at b for each n. Hence $(k_b)_{F-R_n}$ is a Green potential. Thus, by Corollary 2 to Lemma 2, we obtain that $(k_b)_\alpha=0$. This shows that $\alpha\cap\Phi(b)=\phi$. Conversely, suppose $\alpha\cap\Phi(b)=\phi$. Since $\Phi(b)=\bigcap_{E\in\mathcal{F}_b}\bar{E}^W$, for each $z\in\alpha$, we can find a regular closed set F_z in R such that $\bar{F}_z^W$ is a neighborhood of z in R_W^* and F_z is thin at b. Since α is compact, we can choose a finite number of points $\{z_k\}_{k=1}^n$ in α such that $\bigcup_{k=1}^n\bar{F}_{z_k}^W$ is a neighborhood of α. If we set $F_0=\bigcup_{k=1}^n F_{z_k}$, then F_0 is thin at b. Since $F-R_m\subset F_0$ for sufficiently large m, we see that F is thin at b.

COROLLARY. Let $\tilde{\mathcal{G}}_b=\{G\subset R;\ R-G$ is a regular closed set in R and thin at $b\}$.

(i) *For any $G\in\tilde{\mathcal{G}}_b$, there exists a neighborhood U of $\Phi(b)$ in R_W^* such that $U\cap R\subset G$.*

(ii) *For any neighborhood U of $\Phi(b)$ in R_W^*, there exists a $G\in\tilde{\mathcal{G}}_b$ such that $G\subset U\cap R$.*

(iii) $\Phi(\mathrm{b})=\bigcap_{G\in\tilde{\mathcal{G}}_b}\bar{G}^W$.

For each $b\in\Delta_1$, we set $\mathcal{G}_b=\{G\subset R;\ R-G$ is a closed set in R and thin at $b\}$. Then $\tilde{\mathcal{G}}_b\subset\mathcal{G}_b$ for each $b\in\Delta_1$. For a function f in BC, we define $\mathcal{F}(f)=\{b\in\Delta_1;\ \bigcap_{G\in\mathcal{G}_b}\overline{f(G)}$ is one point$\}$, where $\overline{f(G)}$ means the closure of $f(G)$ in the real numbers (see p. 147 in [2]). It is known ([2]) that $\mathcal{F}(f)$ is a Borel set.

The following properties are easy to prove:

(B1) Let f be a function in BCW. Then $b\in\mathcal{F}(f)$ if and only if f can be continuously extended over $\Phi(b)$ by a constant.

(B2) If a function f in BC can be continuously extended over R_M^*, then $\mathcal{F}(f)=\Delta_1$.

6. *Relative Dirichlet problems.*

Let R^* be an arbitrary compactification of R and u be a positive harmonic function on R. Given a function f (extended real valued) on Δ, we consider the following classes:

$$\bar{\mathcal{S}}_{f,R^*}^u=\left\{s;\ \begin{array}{l}\text{superharmonic on } R,\ s/u \text{ is bounded below,}\\ \varliminf_{a\to z}[s(a)/u(a)]\geq f(z) \text{ for any } z\in\Delta\end{array}\right\}\cup\{\infty\}$$

and

$$\underline{\mathcal{S}}_{f,R^*}^u=\{-s;\ s\in\bar{\mathcal{S}}_{-f,R^*}^u\}.$$

We define $\bar{\mathcal{D}}_{f,u}(a)=\inf\{s(a);\ s\in\bar{\mathcal{S}}_{f,R^*}^u\}$ and $\underline{\mathcal{D}}_{f,u}(a)=\sup\{s(a);\ s\in\underline{\mathcal{S}}_{f,R^*}^u\}(a\in R)$.

It is known (Perron-Brelot) that $\bar{\mathcal{D}}_{f,u}$ (resp. $\underline{\mathcal{D}}_{f,u}$) is either harmonic, $\equiv +\infty$ or $\equiv -\infty$. If $\bar{\mathcal{D}}_{f,u}=\underline{\mathcal{D}}_{f,u}$ and are harmonic, then we say that f is *u-resolutive* and $\mathcal{D}_{f,u}=\bar{\mathcal{D}}_{f,u}=\underline{\mathcal{D}}_{f,u}$ is called the u-Dirichlet solution of f (with respect to R^*). In case $u=1$, a u-resolutive function is called resolutive. If any finite continuous function on Δ is resolutive, then we shall say that R^* is *resolutive.*

The following properties are easy to see:

(C1) If f is the characteristic function of a compact subset A of Δ, then $u_A=\bar{\mathcal{D}}_{f,u}$.

(C2) If f is a finite continuous function, then $(-\max|f|)u \leqq \underline{\mathcal{D}}_{f,u} \leqq \bar{\mathcal{D}}_{f,u} \leqq (\max|f|)u$.

(C3) If f and g are finite continuous functions, then $\underline{\mathcal{D}}_{(f-g),u} \leqq \underline{\mathcal{D}}_{f,u}-\underline{\mathcal{D}}_{g,u}$ and $\bar{\mathcal{D}}_{f,u}-\bar{\mathcal{D}}_{g,u} \leqq \bar{\mathcal{D}}_{(f-g),u}$.

We shall prove

PROPOSITION 1. *Let u be a positive harmonic function on R and R^* be a compactification of R. Then a continuous function f on R^* is u-resolutive if and only if fu is a Wiener function. Furthermore, in this case, $\mathcal{D}_{f,u}=h_{fu}$.*

PROOF. Since $\bar{\mathfrak{W}}_{fu} \subset \bar{\mathfrak{S}}^u_{f,R^*}$, we obtain that $\bar{h}_{fu} \geqq \bar{\mathcal{D}}_{f,u}$. Let s be any function in $\bar{\mathfrak{S}}^u_{f,R^*}$. For $\varepsilon>0$, there exists a neighborhood U of Δ in R^* such that $s/u \geqq f-\varepsilon$ on $U \cap R$. Hence we have $s+\varepsilon u \in \bar{\mathfrak{W}}_{fu}$. Thus $s+\varepsilon u \geqq \bar{h}_{fu}$ for any $\varepsilon>0$, so that $s \geqq \bar{h}_{fu}$. It follows that $\bar{\mathcal{D}}_{f,u} \geqq \bar{h}_{fu}$, and hence $\bar{\mathcal{D}}_{f,u}=\bar{h}_{fu}$. Similarily, we can show that $\underline{h}_{fu}=\underline{\mathcal{D}}_{f,u}$. Hence f is u-resolutive if and only if fu is harmonizable. Since fu has a superharmonic majorant $(\sup|f|)u$, we complete the proof.

COROLLARY (Hilfssatz 8.2 in [2]). f is resolutive if and only if it is a Wiener function.

7. *Brelot's axioms.*

Let R^* be a compactification of R and u be a positive harmonic function on R.

Brelot [1] considered the following axioms:

AXIOM $\mathfrak{A}_u$: *Any finite continuous function on Δ is u-resolutive.*

AXIOM $\mathfrak{A}_u'''$: $(u_{A_1})_{A_2}=0$ *for any mutually disjoint compact subsets A_1 and A_2 of Δ.*

The following lemma is due to Brelot [1]:

LEMMA 5. *In case R^* is metrizable, $\mathfrak{A}_u$ is equivalent to $\mathfrak{A}_u'''$.*

We can easily obtain

LEMMA 6. *Let R_1^* and R_2^* be two compactifications of R. Suppose R_2^* is a quotient space of R_1^*. If $\mathfrak{A}_u'''$ is satisfied for R_1^*, then so is for R_2^*.*

§ 2 Main results

8. *W^u-compactifications.*

For a positive harmonic function u on R we set

$$W^u=\{f \in BC;\ fu \in W\}.$$

We see that W^u is a vector lattice with respect to the maximum and minimum operations and also contains C_0 and constants. If u is bounded, then $BCW \subset W^u$.

We can easily prove

LEMMA 7. *If $b \in \Delta_1$ is a singular point,* i.e., *k_b is bounded, then $BCW \subset W^{k_b}$.*

LEMMA 8 (Satz 14.2 in [2]). *Let f be a function in BC and $u=\int_{\Delta_1} k_b d\mu(b)$ be a positive harmonic function. Then fu is a Wiener function if and only if $\mu(\Delta_1-\mathfrak{J}(f))=0$.*

PROPOSITION 2. *Let b be any point in Δ_1. Then $W^{k_b}=\{f \in BC;\ b \in \mathfrak{J}(f)\}$.*

We shall prove

THEOREM 1. *Let u be a positive harmonic function on R and Q be a nonempty subfamily of BC. Then the following conditions are mutually equivalent.*

a) *$Q \subset W^u$.*

b) *$\mathfrak{A}_u$ is satisfied for R_Q^*.*

c) *$\mathfrak{A}_u'''$ is satisfied for R_Q^*.*

PROOF. a)$\Rightarrow$b): We set $Q'=C(R_Q^*)\cap W^u$. Then Q' is a vector lattice with respect to the maximum and minimum operations and contains C_0 and constants. Since $Q\subset Q'$, we see that Q' separates points of R_Q^*. By Proposition 1, (C2) and (C3), we can show that Q' is closed with respect to the uniform convergence topology on R_Q^*. Hence, by Stone-Weierstrass' theorem (cf. [2]), we obtain that $Q'=C(R_Q^*)$. Therefore $C(R_Q^*)\subset W^u$. It follows from Proposition 1 that $\mathfrak{A}_u$ is satisfied for R_Q^*.

b)$\Rightarrow$c): Let A_1 and A_2 be mutually disjoint compact subsets of Δ_Q. Then there exist two open neighborhoods U_1 and U_2 of A_1 and A_2 respectively such that $\overline{U_1\cap R}^Q\cap\overline{U_2\cap R}^Q=\phi$ in R_Q^*. We can choose $f_k \in C(R_Q^*)$ $(k=1, 2)$ such that $0\leqq f_k\leqq 1$, $f_k=1$ on U_k $(k=1, 2)$ and $\min(f_1, f_2)=0$. It is easy to see that $u_{A_k}\leqq h_{f_k u}$ $(k=1, 2)$. Hence we obtain that

$$(u_{A_1})_{A_2}\leqq h_{f_1 u}\wedge h_{f_2 u}=h_{(\min(f_1,f_2))u}=0.$$

c)$\Rightarrow$a): Let f_0 be any function in Q and set $Q_0=\{f_0\}$. Then $\mathfrak{A}_u'''$ is satisfied for $R_{Q_0}^*$ by Lemma 6 and $R_{Q_0}^*$ is metrizable. It follows from Lemma 5 that $\mathfrak{A}_u$ is satisfied for $R_{Q_0}^*$. Hence, by Proposition 1, we see that f_0 belongs to W^u.. Therefore $Q \subset W^u$.

COROLLARY 1. *If* $u = k_b (b \in \Delta_1)$, *then one of the above conditions* a), b) *and* c) *is equivalent to the following condition*:

b) *There exists a unique pole of* b *on* Δ_Q.

PROOF. It suffices to prove the equivalence between c) and d).

c)$\Rightarrow$d): Suppose there exist two distinct poles z_1, z_2 of b on Δ_Q. Then $((k_b)_{\{z_1\}})_{\{z_2\}} = k_b$. This is a contradiction. Hence d) is valid.

d)$\Rightarrow$c): Suppose $((k_b)_{A_1})_{A_2} = k_b$ for mutually disjoint compact subsets A_1 and A_2 of Δ_Q. Since $(k_b)_{A_i} = k_b (i=1, 2)$, there exists a pole $z_i (i=1, 2)$ of b on $A_i (i=1, 2)$. $A_1 \cap A_2 = \phi$ implies $z_1 \neq z_2$. This is a contradiction. Hence c) is valid.

COROLLARY 2. *Let* b *be any point of* Δ_1. *Then there exists a unique pole of* b *on* Δ_W *if and only if* $BCW \subset W^{k_b}$. *In particular, if* k_b *is bounded, then there exists a unique pole of* b *on* Δ_W.

COROLLARY 3. *A compactification* R^* *of* R *is resolutive if and only if* $(1_{A_1})_{A_2} = 0$ *for any mutually disjoint compact subsets* A_1 *and* A_2 *of* $\Delta = R^* - R$.

REMARK. (i) Corollary 1 is a generalization of a part of Théorème 21 in [1].

(ii) The last half of Corollary 2 was obtained by Ikegami [3].

9. *A characterization of* $\mathfrak{A}_u$ *for* R^*_W.

THEOREM 2. $\mathfrak{A}_u$ *is satisfied for* R^*_W *if and only if* u *is quasi-bounded.*

PROOF. (i) Suppose u is quasi-bounded and is the limit function of an increasing sequence $\{u_k\}_{k=1}^{\infty}$ of positive bounded harmonic functions. Let A_1 and A_2 be compact subsets of Δ_W such that $A_1 \cap A_2 = \phi$. Then, by (A5), we see that $((u_k)_{A_1})_{A_2}$ increases to $(u_{A_1})_{A_2}$ as $k \to \infty$. Since $((u_k)_{A_1})_{A_2} \leqq (\sup u_k)(1_{A_1})_{A_2} = 0$ by Corollary 3 to Theorem 1, we have $(u_{A_1})_{A_2} = 0$. Hence $\mathfrak{A}_u'''$ is satisfied for R^*_W. Thus, by Theorem 1, we see that $\mathfrak{A}_u$ is satisfied for R^*_W and $BCW \subset W^u$.

(ii) Next suppose u is singular. For each integer $n > 0$, we set $F_n = \{z \in R; u(z) \geqq n\}$. Since u is a continuous Wiener function, for each n, there exists a function ϕ_n in BCW such that $0 \leqq \phi_n \leqq 1$, $\phi_n = 0$ on $(R - F^i_{2n-1}) \cup F_{2n+1}$, $=1$ on ∂F_{2n} and ϕ_n is harmonic in $F^i_{2n-1} - F_{2n+1} - \partial F_{2n}$. If we set $f_n = \sum_{k=1}^{n} \phi_k$, then f_n is a function in BCW and tends to a function f in BC on R as $n \to \infty$. We shall prove that f is contained in BCW. Since $f_n \leqq f \leqq f_n + u/(2n+1)$ on $R (n=1, 2, \cdots)$, we obtain that

$$0 \leqq \bar{h}_f - \underline{h}_f \leqq u/(2n+1) \text{ on } R (n=1, 2, \cdots).$$

By letting $n \to \infty$, we have $\bar{h}_f = \underline{h}_f$. Since $|f|$ is bounded, it follows that f is a function in BCW. For each $\alpha(0<\alpha<1)$, we set

$$\Omega_{\alpha,n}=\{z\in F_{2n-1};f(z)\geqq\alpha\}\cup F_{2n}$$

and

$$C_\alpha=\{z\in R;f(z)=\alpha\}.$$

Then $\Omega_{\alpha,n}$ and C_α are regular closed and $\partial\Omega_{\alpha,n}\subset C_\alpha$. Since $u_{\Omega_{\alpha,n}}=u$ on R by Lemma 3, $u_{\partial\Omega_{\alpha,n}}=u$ on $R-\Omega_{\alpha,n}$. Hence $u_{C_\alpha}=u$ on $R-\Omega_{\alpha,n}$ for each α and n. This shows that $u_{C_\alpha}=u$ on R for each α. We set $A_\alpha=\bar{C}_\alpha^W\cap\Delta_W$. By Corollary 2 to Lemma 2, we see that $u_{A_\alpha}=u$ on R for each α. Since f is a continuous Wiener function, $A_{\alpha_1}\cap A_{\alpha_2}=\phi$ if $\alpha_1\neq\alpha_2$. Since $(u_{A_{\alpha_1}})_{A_{\alpha_2}}=u$ on R, it follows that $\mathfrak{A}_u'''$ is not satisfied for R_W^*. Hence, by Theorem 1, we see that $\mathfrak{A}_u$ is not satisfied for R_W^* and $BCW\not\subset W^u$.

(iii) Let u be an arbitrary positive harmonic function which is not quasi-bounded. Then u is uniquely decomposed into a quasi-bounded part u_1 and a singular part u_2. Since $u_2>0$, it follows from (ii) that there exists a function f in BCW such that $fu_2\notin W$. Since $fu_1\in W$ by (i), we see that $fu\notin W$. Hence $BCW\not\subset W^u$ and $\mathfrak{A}_u$ is not satisfied for R_W^* by Theorem 1. Therefore we complete the proof.

Corollary 1 (cf. Corollary 2 to Theorem 1). *Let b be a point in Δ_1. Then there exists a unique pole of b on Δ_W if and only if k_b is bounded.*

Corollary 2. *For each $b\in\Delta_1$, either $\Phi(b)$ consists of only one point or contains an uncountable number of points according as b is a singular point or not.*

Proof. Let $u=k_b(b\in\Delta_1)$ be unbounded. Then u is a singular harmonic function. In the proof of the theorem we see that there exists a pole $z(\alpha)$ of b on A_α for each $\alpha\in(0,1)$. If $\alpha_1\neq\alpha_2$, then $A_{\alpha_1}\cap A_{\alpha_2}=\phi$, so that $z(\alpha_1)\neq z(\alpha_2)$. Hence $\Phi(b)$ contains an uncountable number of points. By the above corollary, we complete the proof.

Corollary 3. *If R^* is a resolutive compactification of R, then $\mathfrak{A}_u$ is satisfied for R^* for any positive quasi-bounded harmonic function u.*

Proof. By the aid of (A5) and Corollary 3 to Theorem 1, we have the corollary.

10. *W_1-compactifications.*

We define a class

$$W_1=\bigcap_{u>0}W^u=\{f\in BC;fu\in W\text{ for any positive harmonic function }u\}.$$

By definition, we see that $W_1\subset BCW$. By Lemma 7, $f\in W_1$ if and only if $\mathfrak{J}(f)=\Delta_1$. Hence, by Proposition 2 and (B1), we have

PROPOSITION 3. $W_1=\bigcap_{b\in\Delta_1}W^{k_b}=\{f\in BC;\, \mathfrak{I}(f)=\Delta_1\}=\{f\in BC; f$ can be continuously extended over each $\Phi(b)$ by a constant for any $b\in\Delta_1\}$.

COROLLARY. (i) *$R^*_{W_1}$ is a quotient space of R^*_W.*
(ii) *R^*_M is a quotient space of $R^*_{W_1}$.*

PROOF. Since $W_1\subset BCW$, we have (i). By (B2), we see that (ii) is valid.

The following theorem is an immediate consequence of Theorem 1, Corollary 1 to Theorem 1 and Proposition 3.

THEOREM 3. *Let Q be a non-empty subfamily of BC. Then the following conditions are mutually equivalent.*

a) $Q\subset W_1$.
b) *$\mathfrak{a}_u$ is satisfied for R^*_Q for any $u>0$.*
c) *$\mathfrak{a}'''_u$ is satisfied for R^*_Q for any $u>0$.*
d) *For any $b\in\Delta_1$, there exists a unique pole of b on Δ_Q.*

COROLLARY 1 (Brelot [1]). *For the Martin compactification of R, $\mathfrak{a}_u$ is satisfied for any $u>0$.*

COROLLARY 2. *Let R^* be a compactification of R. Suppose R^* is a quotient space of $R^*_{W_1}$ and R^*_M is a quotient space of R^*. For each $b\in\Delta_1$, we denote by z_b the unique pole of b on Δ_Q. Then $b\to z_b$ is a one to one mapping of Δ_1 into Δ_Q.*

REMARK. The equivalence between b) and d) in the theorem is a generalization of Théorème 24 in [1].

We shall prove

THEOREM 4. *$R^*_{W_1}$ is not metrizable.*

PROOF. We shall prove that any point z of Δ_{W_1} never has a countable system of basis for neighborhoods. Let π be the canonical mapping of $R^*_{W_1}$ onto R^*_M. Suppose z has a countable system $\{U_n\}_{n=1}^\infty$ of basis for open neighborhoods and set $\pi(z)=b$. We may assume that $\pi(U_n)\subset\{a\in R^*_M;\ d(a,b)<1/n\}(n=1, 2, \dots)$, where d is a Martin's metric on R^*_M. Furthermore, we may assume that $U_n\supset\overline{U_{n+1}\cap R}^{W_1}(n=1, 2, \dots)$. For each n, we take a compact disk K_n in $(U_n-\overline{U_{n+1}\cap R}^{W_1})\cap R$ with center at a_n. Let f_n be a function in BC such that $0\leqq f_n\leqq 1$, $f_n(a_n)=1$ and $f_n=0$ on $R-K_n$. If we set $f=\sum_{n=1}^\infty f_n$, then f is a function in BC.

First we assume that $b\in\Delta_1$. Then we can choose $\{K_n\}_{n=1}^\infty$ in such a way that $p=\sum_{n=1}^\infty(k_b)_{K_n}$ is a potential. If we set $F=\bigcup_{n=1}^\infty K_n$, then F is a regular closed set in R and $(k_b)_F\leqq p$. Hence F is thin at b. It follows that $b\in\mathfrak{I}(f)$. Obviously, $b'\in\mathfrak{I}(f)$ for $b'\in\Delta_1-\{b\}$. Thus $\mathfrak{I}(f)=\Delta_1$ and hence $f\in W_1$. Next if

$b \in \varDelta_M - \varDelta_1$, then obviously $\mathfrak{I}(f) = \varDelta_1$. Hence $f \in W_1$. It follows that $\bigcap_{n=1}^{\infty} U_n$ contains an uncountable number of points. This is a contradiction. Therefore we complete the proof.

COROLLARY 1. *If π is the canonical mapping of $R^*_{W_1}$ onto R^*_M, then, for each $b \in \varDelta_M$, $\pi^{-1}(b)$ contains an uncountable number of points.*

COROLLARY 2. *$R^*_{W_1}$ is not homeomorphic to R^*_M.*

11. *On Naïm's remark.*

By the aid of Corollary 2 to Theorem 4, we shall give an answer in the negative to a question in Naïm's remark ([4], p. 268): *Suppose a metrizable compactification R^* of R satisfies*

α) $\mathfrak{a}_u$ is satisfied for R^* for any $u > 0$

and

β) For each $b \in \varDelta_1$, we denote by z_b the unique pole of b on $\varDelta = R^* - R$. Then $b \rightarrow z_b$ is a one to one mapping of $\varDelta_1$ into $\varDelta$.

Then is R^* homeomorphic to R^*_M?

By Corollary 2 to Theorem 4, we see that there exists a function f in W_1 which can not be continuously extended over R^*_M. If we set $Q = M \cup \{f\}$[4], then R^*_Q is metrizable and satisfies α) and β) by Corollary 2 to Theorem 3. However, it is not homeomorphic to R^*_M.

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4) For the definition of the class M, see p. 134 in [2].