J. Sci. Hiroshima Univ. Ser. A-I
33 (1969), 29-40

Note on Outer Derivations of Lie Algebras

Shigeaki Tôgô

(Received March 5, 1969)

Introduction

Let $\mathfrak{O}$ be the set of Lie algebras L over a field Φ satisfying the conditions that $L \neq L^2$ and $Z(L) \neq (0)$, where $Z(L)$ denotes the center of L. Clearly every non-trivial nilpotent Lie algebra belongs to $\mathfrak{O}$. It is known ([4], [6], [8], [13]) that every $L \in \mathfrak{O}$ has an outer derivation. In [13] we have introduced the notion of Lie algebras of type (T) and shown that every Lie algebra L of type (T) such that $L^{(1)} \neq L^{(2)}$ admits an outer derivation belonging to $\mathfrak{R}$, the radical of the derivation algebra $\mathfrak{D}(L)$. It has been also shown that if $L \in \mathfrak{O}$ is not of type (T) there exists an abelian ideal of $\mathfrak{D}(L)$ containing an outer derivation. From these observations it seems to be interesting to study the case where L is of type (T) such that $L^{(1)} = L^{(2)}$. The main purpose of this note is to give a detailed consideration to the case just mentioned. Some additional remarks will be also given.

In Section 2 we shall show that a Lie algebra L of type (T) such that dim $Z(L) \neq 1$ or Φ is of characteristic 2 admits an outer derivation in $\mathfrak{R}$ and that a Lie algebra L of type (T) such that $L^{(1)} = L^{(2)}$, dim $Z(L) = 1$ and Φ is of characteristic $\neq 2$ admits an outer derivation in $\mathfrak{R}$ if and only if $L^{(1)}$ does (Theorem 2.2). In Section 3, we shall show that a Lie algebra L over a field of characteristic 0 admits a semisimple outer derivation in $\mathfrak{R}$ if the radical of L does (Proposition 3.1), and based on this result, for a Lie algebra L of type (T) such that $L^{(1)} = L^{(2)}$ and dim $Z(L) = 1$, we shall give several properties of the radical of $L^{(1)}$, each of which ensures the existence of a semisimple outer derivation in $\mathfrak{R}$ (Theorem 3.6).

In [12] we have studied the existence of the automorphisms of L, when Φ is of characteristic 0, outside the connected algebraic group such that the corresponding Lie algebra is the algebraic hull $\mathfrak{J}(L)^*$ of $\mathfrak{J}(L)$, the ideal of $\mathfrak{D}(L)$ consisting of all inner derivations of L. The final section 4 will be devoted to the discussions about the existence of the derivations of $L \in \mathfrak{O}$ which are contained in $\mathfrak{R}$ but not in $\mathfrak{J}(L)^*$.

§ 1. Preliminaries and notations

Throughout this note we shall consider a finite dimensional Lie algebra L over a field Φ. We denote by R the radical of L and by $Z(L)$ the center of L.

As in [13], we shall denote by $\mathfrak{O}$ the class of all the Lie algebras L over a field Φ such that $L \neq L^2$ and $Z(L) \neq (0)$. We shall consider a subclass of $\mathfrak{Q}$, the Lie algebras of type (T). L is called to be *of type* (T) [13] provided L has a non-zero subspace T satisfying the following conditions:

(1) $L = T + L^2, \qquad T \cap L^2 = (0)$.

(2) $[T, L^2] = (0)$.

(3) $[T, T] = (z)$ with $0 \neq z \in Z(L)$.

(4) The pairing θ which assigns to $(x, y) \in T \times T$ the coefficient of z in $[x, y]$ is a non-degenerate alternate form on T.

Then L is of type (T) if and only if $L \in \mathfrak{Q}$, $Z(L) \subsetneqq L^2$ and $Z(M) \nsubseteq L^2$ for every ideal M of L of codimension 1 [13; Theorem 2].

If H is a subalgebra of L stable under ad x, then ad x induces the derivation of H which we denote by $\mathrm{ad}_H x$. We furthermore employ the following notations.

$\mathfrak{D}(L)$: The derivation algebra of L, that is, the Lie algebra of all the derivations of L.

$\mathfrak{J}(L)$: The ideal of $\mathfrak{D}(L)$ consisting of all the inner derivations of L. The other notation of this is ad L.

$\mathfrak{C}(L)$: The ideal of $\mathfrak{D}(L)$ consisting of all the central derivations of L, that is, the derivations of L which map L into $Z(L)$.

For a Lie algebra L of endomorphisms of a finite dimensional vector space over a field of characteristic 0, we denote by L^* the algebraic hull of L, that is, the intersection of all the algebraic Lie algebras containing L. L is called to be splittable or almost algebraic if it contains the nilpotent and semisimple components of every $x \in L$. Then L is splittable if and only if L is decomposed in

$$L = S + R, \quad R = A + N, \quad S \cap R = (0), \quad A \cap N = (0), \quad [S, A] = (0),$$

where S is a semisimple subalgebra, A is a maximal toroidal subalgebra of R and N is the ideal of L consisting of all nilpotent elements of R [9; Theorem 1]. For brevity we shall call this a normal decomposition of L. A not-necessarily-linear Lie algebra L is called ad-splittable (resp. ad-algebraic) provided ad L is splittable (resp. algebraic).

§ 2.

This section is devoted to studying the existence of outer derivations in the radical of $\mathfrak{D}(L)$ for a Lie algebra L of type (T).

We begin with the following

LEMMA 2.1. *Let L be a Lie algebra over a field Φ of characteristic $\neq 2$ of type (T) such that* dim $Z(L) = 1$. *Then L admits an outer derivation in the radical of its derivation algebra if and only if $L^{(1)}$ does.*

PROOF. (i) Every derivation D of $L^{(1)}$ can be extended to a derivation $\tilde{D}$

of L and D is outer if and only if $\tilde{D}$ is outer. In fact, we denote $Z(L)=(z)$ and extend D to an endomorphism $\tilde{D}$ of L in such a way that for any $x \in T$

$$\tilde{D}x=\begin{cases} 0 & \text{if} \quad Dz=0 \\ \dfrac{\alpha}{2}x & \text{if} \quad Dz=\alpha z,\ \alpha\neq 0. \end{cases}$$

Then it is immediate that $\tilde{D}$ is a derivation of L. If D is inner, then $D=\mathrm{ad}_{L^{(1)}}\, y$ with $y \in L^{(1)}$. From $[T, L^{(1)}]=(0)$ it follows that $\tilde{D}=\mathrm{ad}_L\, y$. Conversely, if $\tilde{D}$ is inner, then $\tilde{D}=\mathrm{ad}_L\,(x+y)$ with $x \in T$ and $y \in L^{(1)}$. Hence it follows from $[T, L^{(1)}]=(0)$ that $D=\mathrm{ad}_{L^{(1)}}\, y$. Thus D is outer if and only if $\tilde{D}$ is outer.

(ii) We denote by $\tilde{\mathfrak{D}}(L^{(1)})$ the set of derivations of L obtained by extending the derivations of $L^{(1)}$ to L as in (i). Then $\mathfrak{D}(L)=\tilde{\mathfrak{D}}(L^{(1)})+\mathfrak{C}(L)+\mathfrak{S}$, where $\mathfrak{S}$ is a symplectic Lie algebra.

Let D be any derivation of L. Since $L^{(1)}$ is a characteristic ideal of L, the restriction of D to $L^{(1)}$ is in $\mathfrak{D}(L^{(1)})$. Extend it to a derivation $\tilde{D}$ of L as in (i). Then $D-\tilde{D}$ maps T into $T+Z(L)$ and $L^{(1)}$ into (0). In fact, for any $x \in T$ put $(D-\tilde{D})x=x'+y'$ with $x' \in T$ and $y' \in L^{(1)}$. For any $y \in L^{(1)}$,

$$[y', y]=[(D-\tilde{D})x, y]=(D-\tilde{D})[x, y]-[x, (D-\tilde{D})y]=0.$$

Hence $y' \in Z(L^{(1)})$. Thus $(D-\tilde{D})T\subseteqq T+Z(L)$. It is evident that $(D-\tilde{D})L^{(1)}=(0)$. Now denote by D_0 an endomorphism of L induced by $D-\tilde{D}$, which maps T into $Z(L)$ and $L^{(1)}$ into (0). Then $D_0 \in \mathfrak{C}(L)$. Furthermore $D-\tilde{D}-D_0$ is a derivation of L mapping T into T and $L^{(1)}$ into (0), and its restriction to T is skew symmetric relative to θ. The trivial extensions to L of endomorphisms of T which are skew symmetric relative to θ may be considered to form a symplectic Lie algebra $\mathfrak{S}$ contained in $\mathfrak{D}(L)$. Therefore $D \in \tilde{\mathfrak{D}}(L^{(1)})+\mathfrak{C}(L)+\mathfrak{S}$. It follows that $\mathfrak{D}(L)=\tilde{\mathfrak{D}}(L^{(1)})+\mathfrak{C}(L)+\mathfrak{S}$.

(iii) We denote by $\mathfrak{R}_1$ the radical of $\mathfrak{D}(L^{(1)})$ and by $\tilde{\mathfrak{R}}_1$ the subset of $\tilde{\mathfrak{D}}(L^{(1)})$ obtained from the elements of $\mathfrak{R}_1$. Then we show that the radical of $\mathfrak{D}(L)$ is $\tilde{\mathfrak{R}}_1+\mathfrak{C}(L)$.

First $\tilde{\mathfrak{D}}(L^{(1)})$ is a subalgebra of $\mathfrak{D}(L)$, because for any D_1, D_2 in $\mathfrak{D}(L^{(1)})$ we have $[\tilde{D}_1, \tilde{D}_2]=\widetilde{[D_1, D_2]}$ with the notations in (i). By the same reason, $\tilde{\mathfrak{R}}_1$ is a solvable subalgebra of $\mathfrak{D}(L)$. Since $Z(L)\subseteqq L^{(1)}$ and $Z(L)$ is a characteristic ideal of L, it follows that $\mathfrak{C}(L)$ is an abelian ideal of $\mathfrak{D}(L)$. Hence $\tilde{\mathfrak{R}}_1+\mathfrak{C}(L)$ is a solvable subalgebra of $\mathfrak{D}(L)$. It is easy to see that $[\tilde{\mathfrak{D}}(L^{(1)}), \mathfrak{S}]=(0)$, from which it follows that $\tilde{\mathfrak{R}}_1+\mathfrak{C}(L)$ is an ideal of $\mathfrak{D}(L)$. Putting $\mathfrak{R}=\mathfrak{R}_1+\mathfrak{C}(L)$, we shall now show that $\mathfrak{R}$ is the radical of $\mathfrak{D}(L)$. By virtue of (ii) we have

$$\mathfrak{D}(L)/\mathfrak{R}=(\tilde{\mathfrak{D}}(L^{(1)})+\mathfrak{C}(L))/\mathfrak{R}+(\mathfrak{S}+\mathfrak{R})/\mathfrak{R},$$

where $(\tilde{\mathfrak{D}}(L^{(1)})+\mathfrak{C}(L))/\mathfrak{R}$ is isomorphic to $\mathfrak{D}(L^{(1)})/\mathfrak{R}_1$ and $(\mathfrak{S}+\mathfrak{R})/\mathfrak{R}$ is isomorphic to $\mathfrak{S}$. It follows that $\mathfrak{D}(L)/\mathfrak{R}$ is semisimple as a direct sum of semisimple ideals. Therefore $\mathfrak{R}$ is the radical of $\mathfrak{D}(L)$.

(iv) By the property (4) of Lie algebras of type (T), $\mathfrak{C}(L)$ consists of inner derivations. Hence the statement of the Lemma follows now from (i) and (iii).

Thus the proof is complete.

THEOREM 2.2. *Let L be a Lie algebra over a field Φ of type (T).*

(1) *If $L^{(1)}\neq L^{(2)}$ or* dim $Z(L)\neq 1$ *or Φ is of characteristic* 2, *then L admits an outer derivation in the radical of its derivation algebra.*

(2) *If $L^{(1)}=L^{(2)}$,* dim $Z(L)=1$ *and Φ is of characteristic* $\neq 2$, *then L admits an outer derivation in the radical of its derivation algebra if and only if $L^{(1)}$ does.*

PROOF. The statement (2) follows from Lemma 2.1. The statement (1) has been proved in the case where $L^{(1)}\neq L^{(2)}$ in [13; Theorem 3]. Therefore it remains to prove (1) in the cases where dim $Z(L)\neq 1$ and where Φ is of characteristic 2.

The case where dim $Z(L)\neq 1$: In this case, dim $Z(L)\geqq 2$. Choose $x\neq 0$ in T and let T_1 be a complementary subspace of (x) in T. Choose $z'\neq 0$ in $Z(L)\backslash(z)$. And define an endomorphism D of L in such a way that

$$Dx=z' \quad \text{and} \quad D(T_1+L^{(1)})=(0).$$

Then D belongs to $\mathfrak{C}(L)$ which is an abelian ideal of $\mathfrak{D}(L)$. D is not inner, since every inner derivation of L sends T into (z).

The case where Φ is of characteristic 2: Let D be an endomorphism of L which is identity on T and zero on $L^{(1)}$. Then D is an outer derivation of L. Moreover it is immediate that

$$[D, \mathfrak{D}(L)]\subseteqq \mathfrak{C}(L),$$

from which it follows that $(D)+\mathfrak{C}(L)$ is an ideal of $\mathfrak{D}(L)$. It is a solvable ideal of $\mathfrak{D}(L)$, since $\mathfrak{C}(L)$ is abelian.

The proof is complete.

REMARK 2.3. As an illustration of Theorem 2.2 (2) we shall consider the example of non-solvable Lie algebra of type (T) given in [13; p. 270]. Let L be the Lie algebra over a field Φ of characteristic $\neq 2$ described in terms of a basis $x_1, x_2, \cdots, x_8$ by the multiplication table:

$$[x_1, x_2]=x_8, \qquad [x_3, x_4]=2x_4, \qquad [x_3, x_5]=-2x_5,$$

$$[x_4, x_5]=x_3, \qquad [x_3, x_6]=-x_6, \qquad [x_3, x_7]=x_7,$$

$$[x_4, x_6]=-x_7, \quad [x_5, x_7]=-x_6, \qquad [x_6, x_7]=x_8.$$

Then $L^{(1)}=L^{(2)}=(x_3, x_4, x_5, x_6, x_7, x_8)$ and $Z(L)=(x_8)$. A decomposition of L as a Lie algebra of type (T) is given by

$$L=T+L^{(1)} \qquad \text{with} \qquad T=(x_1, x_2).$$

The radical $\mathfrak{R}$ of $\mathfrak{D}(L)$ is

$$(\tilde{D}, \mathrm{ad}_L\ x_6, \mathrm{ad}_L\ x_7)+(\mathrm{ad}_L\ x_1, \mathrm{ad}_L\ x_2),$$

where $\tilde{D}$ is a derivation such that

$$\tilde{D}x_i=\begin{cases} x_i & i=1, 2 \\ 0 & i=3, 4, 5 \\ x_i & i=6, 7 \\ 2x_i & i=8. \end{cases}$$

The radical $\mathfrak{R}_1$ of $\mathfrak{D}(L^{(1)})$ is

$$(D, \mathrm{ad}_L\ x_6, \mathrm{ad}_L\ x_7)$$

where D is the restriction of $\tilde{D}$ to $L^{(1)}$. Therefore

$$\mathfrak{R}=\tilde{\mathfrak{R}}_1+\mathfrak{C}(L).$$

This example shows that as for further study of Theorem 2.2 (2) it is a problem to find a semisimple outer derivation in the radical of the derivation algebra of a Lie algebra L over a field Φ of characteristic $\neq 2$ such that $L=L^2$ and dim $Z(L)=1$. This problem will be studied in the next section.

§ 3.

In this section we shall study the semisimple outer derivation of a Lie algebra L over a field of characteristic 0 such that $L=L^2$ and dim $Z(L)=1$. We begin with the improvement of [13; Theorem 5].

Proposition 3.1. *Let L be a non-solvable Lie algebra over a field of characteristic 0. Then L admits a semisimple outer derivation in the radical of its derivation algebra if the radical of L does.*

Proof. Let R be the radical of L and assume that R admits a semisimple outer derivation in the radical $\mathfrak{R}$ of $\mathfrak{D}(R)$. Let $L=S+R$ be a Levi decomposition and $\mathfrak{A}(S)$ be the subalgebra of $\mathfrak{D}(L)$ consisting of all the derivations of L which map S into (0). Then by Lemma 3 in [11], there exists a maximal toroidal subalgebra $\mathfrak{T}$ of $\mathfrak{R}$ which can be imbedded in $\mathfrak{A}(S)$. As in the proof of Theorem 5 in [13], we see that $\mathfrak{T}$ contains a semisimple outer derivation.

Denote by $\mathfrak{R}_1$ the set of derivations in $\mathfrak{R}$ which can be trivially extended to the derivations of L. Then $\mathfrak{T}\subseteqq\mathfrak{R}_1$. Let $\tilde{\mathfrak{R}}_1$ (resp. $\tilde{\mathfrak{T}}$) be the set of trivial extensions of elements of $\mathfrak{R}_1$ (resp. $\mathfrak{T}$) to L. Then we assert that $\tilde{\mathfrak{R}}_1+\mathrm{ad}_L\ R$ is a solvable ideal of $\mathfrak{D}(L)$. In fact, $\tilde{\mathfrak{R}}_1$ is a solvable subalgebra and $\mathrm{ad}_L\ R$ is a solvable ideal of $\mathfrak{D}(L)$. Hence $\tilde{\mathfrak{R}}_1+\mathrm{ad}_L\ R$ is a solvable subalgebra of $\mathfrak{D}(L)$.

Now it is easy to see that

$$[\tilde{\mathfrak{R}}_1, \mathrm{ad}_L S]=(0) \quad \text{and} \quad [\tilde{\mathfrak{R}}_1, \mathfrak{A}(S)]\subseteqq\tilde{\mathfrak{R}}_1.$$

Since $\mathfrak{D}(L)=\mathfrak{A}(S)+\mathfrak{J}(L)$[5], it follows that $\tilde{\mathfrak{R}}_1+\mathrm{ad}_L R$ is an ideal of $\mathfrak{D}(L)$ and therefore a solvable ideal of $\mathfrak{D}(L)$. Thus the radical of $\mathfrak{D}(L)$ contains $\tilde{\mathfrak{A}}$ and therefore a semisimple outer derivation.

The proof is complete.

LEMMA 3.2. *Let L be a non-solvable Lie algebra over a field of characteristic* 0. *Then L admits a semisimple derivation with trace $\neq 0$ if and only if the radical of L does.*

PROOF. Assume that L has a semisimple derivation D with trace $\neq 0$. Let R be the radical of L and let $L=S+R$ be a Levi decomposition of L. Let $\mathfrak{A}(S)$ be the subalgebra of $\mathfrak{D}(L)$ as in the proof of Proposition 3.1. Then $\mathfrak{A}(S)$ is splittable. Let $\mathfrak{A}(S)=\mathfrak{S}+\mathfrak{A}+\mathfrak{N}$ be a normal decomposition of $\mathfrak{A}(S)$. Then

$$\begin{aligned}\mathfrak{D}(L)&=\mathfrak{J}(L)+\mathfrak{A}(S)\\ &=(\mathrm{ad}_L\, S+\mathfrak{S})+\mathfrak{A}+\mathrm{ad}_L\, R+\mathfrak{N}.\end{aligned}$$

By considering the components of D in the above factors, D may be supposed to belong to $\mathfrak{A}+\mathrm{ad}_L R$. Then the restriction D' of D to R is a derivation of R and $\mathrm{Tr}\ D'=\mathrm{Tr}\ D\neq 0$. Let $\mathfrak{D}(R)=\mathfrak{S}'+\mathfrak{A}'+\mathfrak{N}'$ be a normal decomposition of $\mathfrak{D}(R)$. The component D_1 of D' in $\mathfrak{A}'$ is a semisimple derivation with trace $\neq 0$.

Conversely, assume that R admits a semisimple derivation D_1 with trace $\neq 0$. Then D_1 is contained in a maximal toroidal subalgebra of the radical $\mathfrak{R}_1$ of $\mathfrak{D}(R)$. It is known [11] that there exists a maximal toroidal subalgebra $\mathfrak{A}_1$ of $\mathfrak{R}_1$ which can be imbedded in $\mathfrak{A}(S)$. By the conjugacy of maximal toroidal subalgebras of $\mathfrak{R}_1$, $\mathfrak{A}_1$ contains a semisimple derivation with trace $\neq 0$ and therefore $\mathfrak{D}(L)$ does. By considering a Levi decomposition of $\mathfrak{D}(L)$, we see that the radical of $\mathfrak{D}(L)$ contains a semisimple derivation with trace $\neq 0$, completing the proof.

PROPOSITION 3.3. *Let L be a Lie algebra over a field of characteristic* 0. *Let R be the radical* (*resp. the radical which is nilpotent*) *of L. Then the following statements are equivalent*:

(1) *L admits a derivation with trace $\neq 0$.*

(2) *L admits a semisimple derivation with trace $\neq 0$.*

(3) *L admits a semisimple* (*resp. semisimple outer*) *derivation with trace $\neq 0$ in the radical of $\mathfrak{D}(L)$.*

(4) *R admits a derivation with trace $\neq 0$.*

(5) *R admits a semisimple derivation with trace $\neq 0$.*

(6) *R admits a semisimple* (*resp. semisimple outer*) *derivation with trace*

$\neq 0$ *in the radical of* $\mathfrak{D}(N)$.

PROOF. Since $\mathfrak{D}(L)$ is splittable, (1) obviously implies (2). Now assume (2). Let D be a semisimple derivation with trace $\neq 0$. We consider a Levi decomposition of $\mathfrak{D}(L)$: $\mathfrak{D}(L)=\mathfrak{S}+\mathfrak{R}$. Then the component D_1 of D in $\mathfrak{R}$ is a derivation with trace $\neq 0$. Since $\mathfrak{R}$ is splittable, the semisimple component of D_1 belongs to $\mathfrak{R}$ and has the trace $\neq 0$. In the special case where R is nilpotent, $\mathrm{ad}_L R$ consists of nilpotent elements, and therefore the semisimple component of D_1 is obviously outer. Thus we have (3). Consequently (1), (2) and (3) are equivalent.

The equivalence of (4), (5) and (6) is a consequence of the proof stated above. By virtue of Lemma 3.2, (2) and (5) are equivalent. Therefore all the six statements are equivalent and the proof is complete.

DEFINITION 3.4. *Let N be a nilpotent Lie algebra over a field Φ. Let $n=\dim N/N^2$. We define the following properties.*

(Δ_0): *There exist $\alpha_1, \alpha_2, \cdots, \alpha_m (m \leqq n)$ in Φ which are not all zero and the subspaces $U_{1,\alpha_1}, U_{2,\alpha_2}, \cdots, U_{m,\alpha_m}$ such that*

$$N=U_{1,\alpha_1}+\cdots+U_{m,\alpha_m}+N^2 \qquad (\textit{direct sum})$$

and, putting for any $\alpha \in \Phi$

$$V_\alpha=\sum_{\alpha_j=\alpha} U_{j,\alpha_j}+\sum_{\alpha_j+\alpha_k=\alpha}[U_{j,\alpha_j}, U_{k,\alpha_k}]$$
$$+\sum_{\alpha_j+\alpha_k+\alpha_l=\alpha}([[U_{j,\alpha_j}, U_{k,\alpha_k}], U_{l,\alpha_l}]+[U_{j,\alpha_j}, [U_{k,\alpha_k}, U_{l,\alpha_l}]])$$
$$+\cdots,$$
$$N=\sum_\alpha V_\alpha \qquad (\textit{direct sum}).$$

(Δ): *The property (Δ_0) with the further condition*

$$\sum_\alpha \alpha \dim V_\alpha \neq 0.$$

The quasi-cyclicity of N is the property (Δ_0) with $m=1$ and $\alpha_1=1$ and is the property (Δ) with $m=1$ and $\alpha_1=1$ if Φ is of characteristic 0. It is easy to find the further examples of nilpotent Lie algebras with the property (Δ_0) or (Δ) in [3] and [7].

LEMMA 3.5. *Let N be a nilpotent Lie algebra over a field Φ. Then N has the property (Δ_0) (resp. (Δ)) if and only if N has a non-zero derivation (resp. a derivation with trace $\neq 0$) which is diagonal for a suitable choice of basis of N.*

PROOF. If N has the property (Δ_0), define an endomorphism of L in such a way that for any $\alpha \in \Phi$

$$Dx=\alpha x \qquad \text{if} \qquad x \in V_\alpha.$$

Then D is a non-zero derivation of L and is diagonal if we choose a basis of L consisting of elements of V_α.

Conversely, if N has a non-zero diagonal derivation D, then we can choose $x_1, \cdots, x_n$ in $N \setminus N^2$ with $n = \dim N/N^2$ so that

$$N = (x_1, x_2, \cdots, x_n) + N^2,$$
$$Dx_j = \alpha_j x_j \qquad \text{for} \quad j = 1, 2, \cdots, n.$$

Now it is immediate that $\alpha_1, \cdots, \alpha_n$ and $U_{1,\alpha_1} = (x_1), \cdots, U_{n,\alpha_n} = (x_n)$ satisfy the condition of the property (Δ_0).

The proof is similar for the property (Δ). Therefore we omit it.

THEOREM 3.6. *Let L be a Lie algebra over a field Φ of characteristic* 0 *of type (T) such that $L^{(1)} = L^{(2)}$ and* $\dim Z(L) = 1$. *Then L admits a semisimple outer derivation in the radical of $\mathfrak{D}(L)$ in each of the following cases:*

(1) *The radical N of $L^{(1)}$ admits a non-zero semisimple derivation in the radical of $\mathfrak{D}(N)$.*

(2) *N admits a derivation with trace* $\neq 0$.

(3) *N has the property* (Δ).

PROOF. Let $\mathfrak{R}$ and $\mathfrak{R}_1$ be the radicals of $\mathfrak{D}(L)$ and $\mathfrak{D}(L^{(1)})$ respectively. With the notations in the proof of Lemma 2.1, we know that $\mathfrak{R} = \tilde{\mathfrak{R}}_1 + \mathfrak{C}(L)$, where $\tilde{\mathfrak{R}}_1$ is the set of trivial extensions of elements of $\mathfrak{R}_1$ for some Levi decomposition of $L^{(1)}$. Since $\mathfrak{C}(L)$ consists of inner derivations, it follows that $\mathfrak{R}$ contains a semisimple outer derivation if and only if $\mathfrak{R}_1$ does.

Combining this fact with Proposition 3.1, we see that if (1) is satisfied then L admits a semisimple outer derivation in $\mathfrak{R}$.

By Proposition 3.3, we see that (2) implies (1).

Finally assume that (3) is satisfied. Then by Lemma 3.5, N has a semisimple derivation with trace $\neq 0$. Let $\mathfrak{D}(N) = \mathfrak{S} + \mathfrak{R}_2$, $\mathfrak{R}_2 = \mathfrak{A} + \mathfrak{N}$ be a normal decomposition of $\mathfrak{D}(N)$. Then $\mathfrak{A} \neq (0)$, for if $\mathfrak{A} = (0)$ every D in $\mathfrak{D}(N)$ has the trace 0. Thus we see that (3) implies (1).

The proof is complete.

REMARK 3.7. As an illustration of Theorem 3.6, we continue to consider a Lie algebra L in Remark 2.3. The radical N of $L^{(1)}$ is (x_6, x_7, x_8). The radical of $\mathfrak{D}(N)$ is

$$(D) + (\mathrm{ad}_N\, x_6, \mathrm{ad}_N\, x_7),$$

where D is a derivation of N such that

$$Dx_i = \begin{cases} x_i & i = 6, 7 \\ 2x_i & i = 8. \end{cases}$$

Putting $S = (x_3, x_4, x_5)$, $L^{(1)} = S + N$ is a Levi decomposition. The set $\mathfrak{R}_1$ of elements of the radical of $\mathfrak{D}(N)$ which can be trivially extended to derivations of $L^{(1)}$ is (D). Let $\tilde{D}$ be the trivial extension of D to $L^{(1)}$. Then the radical of $\mathfrak{D}(L^{(1)})$ is identical to $(\tilde{D})+\mathrm{ad}_{L^{(1)}}N$. Furthermore N has obviously the property (Δ).

§ 4.

At the end of the paper [12], we have tried to study the outer automorphism of a Lie algebra L over an arbitrary field of characteristic 0. It has been defined as an automorphism which is not in the connected algebraic group corresponding to $\mathfrak{J}(L)^*$. This section is devoted to the study of the outer derivations which do not belong to $\mathfrak{J}(L)^*$.

In [11], we have studied among other things the properties of the Lie algebra L such that $\mathfrak{D}(L)=\mathfrak{J}(L)^*$. Therefore we are now only concerned with the properties corresponding to Theorem 3 in [13].

In the proof of the following theorems we shall use the results on algebraic Lie algebras in [1], [2] without references. As in [13], we denote $\mathfrak{N}_0$ (resp. $\mathfrak{C}_0$) the abelian ideal of $\mathfrak{D}(L)$ consisting of all derivations which map L into L^2 (resp. $Z(L)$) and L^2 (resp. $Z(L)$) into (0).

THEOREM 4.1. *Let L be a Lie algebra in $\mathfrak{O}$ over a field Φ of characteristic 0 and not of type (T). Assume that L is ad-splittable. Then there exists an abelian ideal $\mathfrak{A}$ of $\mathfrak{D}(L)$ such that $\mathfrak{A}\backslash\mathfrak{J}(L)^*\neq\emptyset$.*

PROOF. (i) The case where L has no non-zero abelian direct summands: By Proposition 10 in [9], we may assume that L is a splittable linear Lie algebra. Hence we have a normal decomposition of L as follows:

$$L=S+R, \quad R=A+N, \quad [S, A]=(0),$$

where S is a maximal semisimple subalgebra of L, A is a maximal toroidal subalgebra and N is the ideal of nilpotent elements of R. Since S and N are algebraic, we have

$$L^*=S+R^*, \quad R^*=A^*+N, \quad [S, A^*]=(0).$$

If $A=(0)$, $L=L^*$ and therefore by Lemma 1 in [10]

$$(\mathrm{ad}\ L)^*=\mathrm{ad}_L\ L^*=\mathrm{ad}\ L.$$

As in the proof of Theorem 3 in [13], take an ideal M of L of codimension 1 such that $Z(M)\subseteqq L^2$. Let $L=(x_1)+M$. The endomorphism D of L defined in such a way that

$$Dx_1=z\in Z(M)\backslash[L, Z(M)] \quad \text{and} \quad DM=(0)$$

is an outer derivation in $\mathfrak{N}_0\backslash(\mathrm{ad}\ L)=\mathfrak{N}_0\backslash(\mathrm{ad}\ L)^*$.

We now assume that $A\neq(0)$. Take x_1 in A and the subspace A_1 of A

complementary to (x_1). Put $M=S+A_1+N$. Then M is a maximal ideal of L of codimension 1. Denoting by $C_{L^*}(M)$ the centralizer of M in L^* we assert that

$$C_{L^*}(M)\subseteqq Z(M)+A^*.$$

In fact, let $x\in C_{L^*}(M)$. Then

$$x=s+a+n \quad \text{with} \quad s\in S,\quad a\in A^*,\quad n\in N.$$

From $[x, S]=(0)$ it follows that $s=0$. Since $[a, S+A_1^*]\subseteqq[A^*, S]+[A^*, A^*]=(0)$, it follows that $[n, S+A_1]=(0)$. Take a subspace U of N in such a way that

$$N=U+N^2,\quad [a, U]\subseteqq U,\quad U\cap N^2=(0).$$

Since $[a+n, N]=0$, we have $[a, U]\subseteqq[n, U]\subseteqq N^2$ and therefore $[a, U]=(0)$. U generating N, we have $[a, N]=(0)$. It follows that $[n, N]=(0)$. Thus $[n, M]=(0)$ and therefore $n\in Z(M)$. Hence $x=a+n\in A^*+Z(M)$, as was asserted. We now have

$$[x_1, C_{L^*}(M)]\subseteqq[x_1, A^*+Z(M)]=[x_1, Z(M)]$$

and therefore $[x_1, C_{L^*}(M)]=[x_1, Z(M)]$. Take an endomorphism D of L so that

$$Dx_1=z\in Z(M)\backslash[L, Z(M)] \quad \text{and} \quad DM=(0).$$

Then D is a derivation of L. Moreover $D\notin \mathfrak{J}(L)^*$. In fact, assume that D is in $(\mathrm{ad}\, L)^*$. Then $D=\mathrm{ad}_L x$ with $x\in L^*$ by Lemma 1 in [10]. Since $(\mathrm{ad}\, x)\, M=0$, we have $x\in C_{L^*}(M)$. It follows that

$$z=Dx_1=[x, x_1]\in[x_1, C_{L^*}(M)]=[x_1, Z(M)],$$

contradicting the choice of z. Thus we conclude that $D\in \mathfrak{N}_0\backslash\mathfrak{J}(L)^*$.

(ii) The case where L has a non-zero abelian direct summand L_1: In this case $L=L_1\oplus L_2$ where L_2 is an ideal such that $Z(L_2)\subset L_2^2$.

If $Z(L)$ is not a direct summand, the endomorphism $D\neq 0$ of L such that

$$DL_1\subseteqq Z(L_2) \quad \text{and} \quad DL_2=(0)$$

is in $\mathfrak{N}_0\cap\mathfrak{C}(L)$. Moreover $D\notin(\mathrm{ad}\, L)^*$ since $(\mathrm{ad}\, L)^*L_1=[L^*, L_1]=(0)$.

If $Z(L)$ is a direct summand and $L/Z(L)\neq(L/Z(L))^2$, every non-zero endomorphism D of L such that

$$DL_2\subseteqq Z(L)=L_1 \quad \text{and} \quad D(L_1+L_2^2)=(0)$$

belongs to $\mathfrak{C}_0$. Moreover $D\notin\mathfrak{J}(L)^*$, since $(\mathrm{ad}\, L)^*L_2\subseteqq L_2$.

If $Z(L)$ is a direct summand and $L/Z(L)=(L/Z(L))^2$, the endomorphism D of L such that D is the identity on L_1 and $DL_2=(0)$ is in $Z(\mathfrak{D}(L))$. Moreover $D\notin\mathfrak{J}(L)^*$, since $(\mathrm{ad}\, L)^*\, L_1=(0)$.

Thus the proof is complete.

THEOREM 4.2. *Let L be a Lie algebra of type (T) over a field Φ of characteristic* 0. *If $L^{(1)} \neq L^{(2)}$ or* dim $Z(L) \neq 1$, *then $\mathfrak{R} \setminus \mathfrak{J}(L)^* \neq \emptyset$, where $\mathfrak{R}$ is the radical of $\mathfrak{D}(L)$. If $L^{(1)} = L^{(2)}$ and* dim $Z(L)=1$, *then $\mathfrak{R} \setminus \mathfrak{J}(L)^* \neq \emptyset$ if and only if $\mathfrak{R}_1 \setminus \mathfrak{J}(L^{(1)})^* \neq \emptyset$, where $\mathfrak{R}_1$ is the radical of $\mathfrak{D}(L^{(1)})$.*

PROOF. By Ado's theorem, we may assume that L is a linear Lie algebra. Since L is of type (T), there exists a subspace T such that

$$L = T + L^2, \quad T \cap L^2 = (0), \quad [T, L^2] = (0), \quad [T, T] = (z_0) \subseteqq Z(L).$$

(i) The case where $L^{(1)} \neq L^{(2)}$: We have obviously $L^{(1)} = (z_0) + L^{(2)}$, from which it follows that L is the direct sum of the ideals L_1 and L_2, where $L_1 = T + (z_0)$ and $L_2 = L^{(2)}$. The endomorphism D of L such that

$$Dx = x \quad \text{for any } x \in T, \quad Dz_0 = 2z_0, \quad DL_2 = (0)$$

is a derivation in $\mathfrak{R}$. Since $(\text{ad } L)^* z_0 = [L^*, z_0] = (0)$, it follows that $D \in \mathfrak{R} \setminus \mathfrak{J}(L)^*$.

(ii) The case where dim $Z(L) \geqq 2$: The endomorphism D of L such that

$$DT \subseteqq Z(L) \setminus (z_0) \quad \text{and} \quad DL^{(1)} = (0)$$

is in $\mathfrak{D}(L)$. Since

$$(\text{ad } L)^* T = [L^*, T] = [T^* + L^{(1)}, T] = [T^*, T] = [T, T] = (z_0),$$

it follows that $D \in \mathfrak{C}(L) \setminus \mathfrak{J}(L)^*$.

(iii) The case where $L^{(1)} = L^{(2)}$ and dim $Z(L) = 1$: In this case, with the notation in the proof of Lemma 1.1.,

$$\mathfrak{R} = \mathfrak{C}(L) + \tilde{\mathfrak{R}}_1 \quad \text{and} \quad \text{ad } L = \mathfrak{C}(L) + \text{ad}_L L^{(1)}.$$

$\mathfrak{C}(L)$ consists of nilpotent elements and $\text{ad}_L L^{(1)} = (\text{ad } L)^2$. Hence both of them are algebraic and therefore ad L is algebraic. Hence $\mathfrak{R} \setminus \mathfrak{J}(L)^* \neq \emptyset$ if and only if $\tilde{\mathfrak{R}}_1 \setminus (\text{ad}_L L^{(1)})^* \neq \emptyset$ and therefore if and only if $\mathfrak{R}_1 \setminus \mathfrak{J}(L^{(1)})^* \neq \emptyset$.

The following corollaries are immediate from Theorems 4.1 and 4.2.

COROLLARY 4.3. *Let L be a Lie algebra over a field of characteristic* 0. *In each of the following cases, $\mathfrak{R} \setminus \mathfrak{J}(L)^* \neq \emptyset$.*

(1) *$L \neq L^{(1)}$, $L^{(1)} \neq L^{(2)}$, $Z(L) \neq (0)$ and L is ad-splittable.*

(2) *L is solvable and ad-splittable, and $Z(L) \neq (0)$.*

(3) *L is nilpotent.*

COROLLARY 4.4. *Let L be an ad-algebraic Lie algebra in $\mathfrak{O}$ over a field of characteristic* 0. *If L is not of type (T), then there exists an abelian ideal $\mathfrak{A}$* of *$\mathfrak{D}(L)$ such that $\mathfrak{A} \setminus \mathfrak{J}(L)^* \neq \emptyset$.*

REMARK 4.5. In Theorem 4.1, we have assumed that L is ad-splittable.

The assumption was only used to prove the statement in the case (i) where L has no non-zero abelian direct summands. We shall here give an example which shows that, if L is not ad-splittable, the case (i) cannot be proved by our method used in the proof of Theorem 3 in [13].

Let L be a Lie algebra over a field of characteristic 0 described in terms of basis $x_1, x_2, \cdots, x_5$ by the following multiplication table:

$$[x_1, x_2]=x_5, \quad [x_1, x_3]=x_3, \quad [x_1, x_4]=-x_4, \quad [x_3, x_4]=x_5.$$

Then L is not ad-splittable and $\mathfrak{D}(L)$ is solvable. Every outer derivation constructed by choosing, as in the proof of Theorem 3 in [13], a maximal ideal M of L of codimension 1 such that $Z(M)\subseteqq L^2$ belongs to (ad L)*. In fact, such a maximal ideal of L of codimension 1 is

$$M=(\alpha x_1+\beta x_2, x_3, x_4, x_5), \quad \alpha\neq 0$$

and every derivation obtained as stated above is a linear combination of D and ad x_2, where D is the derivation such that

$$Dx_2=x_5 \quad \text{and} \quad Dx_i=0 \qquad \text{for} \quad i=1, 3, 4, 5.$$

Here D is the nilpotent component of ad x_1, whence $D \in \mathfrak{J}(L)^*\backslash\mathfrak{J}(L)$. Therefore we have the assertion.

References

[1] C. Chevalley, Théorie des Groupes de Lie, Tome II, Groupes algébriques, Paris, 1951.

[2] C. Chevalley, Théorie des Groupes de Lie, Tome III, Théorèmes généraux sur les algèbres de Lie, Paris, 1955.

[3] J. Dixmier, *Sur les représentations unitaires des groupes de Lie nilpotents III*, Canad. J. Math., **10** (1958), pp. 321-348.

[4] J. Dozias, *Sur les dérivations des algèbres de Lie*, C. R. Acad. Sci. Paris, **259** (1964), pp. 2748-2750.

[5] G. Hochschild, *Semi-simple algebras and generalized derivations*, Amer. J. Math., **64** (1942), pp. 677-694.

[6] N. Jacobson, *A note on automorphisms and derivations of Lie algebras*, Proc. Amer. Math. Soc., **6** (1955), pp. 281-283.

[7] V. V. Morozov, *Classification of nilpotent Lie algebras of six order*. (Russian) Izv. Vysš. Učebn. Zaved. Matematika 1958, no. 4 (5), pp. 161-171.

[8] T. Satô, *On derivations of nilpotent Lie algebras*, Tôhoku Math. J., **17** (1965), pp. 244-249.

[9] S. Tôgô, *On splittable linear Lie algebras*, J. Sci. Hiroshima Univ. Ser. A, **18** (1955), pp. 289-306.

[10] S. Tôgô, *Derivations of Lie algebras*, J. Sci. Hiroshima Univ. Ser. A-I, **28** (1964), pp. 133-158.

[11] S. Tôgô, *Lie algebras which have few derivations*, J. Sci. Hiroshima Univ. Ser. A-I, **29** (1965), pp. 29-41.

[12] S. Tôgô, *Derivations of Lie algebras*, Bull. Amer. Math. Soc., **72** (1966), pp. 690-692.

[13] S. Tôgô, *Outer derivations of Lie algebras*, Trans. Amer. Math. Soc., **128** (1967), pp. 264-276.

Department of Mathematics,
Faculty of Science,
Hiroshima University