J. Sci. Hiroshima Univ. Ser. A-I
33 (1969), 231-235

Comparison of the Classes of Wiener Functions

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(Received September 20, 1969)

Introduction

For a harmonic space satisfying the axioms of M. Brelot [1], one can define the notion of Wiener functions as a generalization of that for a Riemann surface or a Green space (see [2]). The class of Wiener functions may be used to see global properties of the harmonic space; in particular, in order to show that a compactification of the base space be resolutive with respect to the Dirichlet problem, it is enough to verify that every continuous function on the compactification is a Wiener function (see Theorem 4.4 in [2]). Thus, given two harmonic structures $\mathfrak{H}_1$ and $\mathfrak{H}_2$ on the same base space Ω, it may be useful to know when the inclusion $\boldsymbol{BW}^{(1)} \subset \boldsymbol{BW}^{(2)}$ holds, where $\boldsymbol{BW}^{(i)}$ $(i=1, 2)$ is the class of bounded Wiener functions with respect to $\mathfrak{H}_i$ $(i=1, 2)$. In this paper, we shall give a sufficient condition for the above inclusion, which includes the conditions given in [4] and [5] for special cases.

1. Harmonic spaces and Wiener functions

In this paper, we assume that a harmonic space $(\Omega, \mathfrak{H}) = \{\mathfrak{H}(G)\}_{G:\text{open}}$, satisfies Axioms 1, 2 and 3 of M. Brelot [1] and that Ω is non-compact. For an open set G in Ω, the set of all superharmonic functions on G with respect to $(\Omega, \mathfrak{H})$ is denoted by $\mathcal{S}_{\mathfrak{H}}(G)$. The set of all potentials with respect to $(\Omega, \mathfrak{H})$ is denoted by $\mathcal{P}_{\mathfrak{H}}$. In general, given a family $\boldsymbol{A}$ of (extended) real-valued functions, we use the notation $\boldsymbol{A}^+ = \{f \in \boldsymbol{A}; f \geqq 0\}$ and $\boldsymbol{BA} = \{f \in \boldsymbol{A}; f\text{: bounded}\}$.

We furthermore assume that $(\Omega, \mathfrak{H})$ satisfies

Axiom 4. $\quad 1 \in \mathcal{S}_{\mathfrak{H}}(\Omega)$ *and* $\mathcal{P}_{\mathfrak{H}} \neq \{0\}$.

Remark that under Axiom 4 the following minimum principle holds (see [1]):

If $v \in \mathcal{S}_{\mathfrak{H}}(\Omega)$ and if for any $\varepsilon > 0$ there exists a compact set K in Ω such that $v(x) > -\varepsilon$ on $\Omega - K$, then $v \geqq 0$.

Given an extended real-valued function f on Ω, we consider the classes

$$\overline{\mathcal{W}}_{\mathfrak{H}}(f) = \left\{ v \in \mathcal{S}_{\mathfrak{H}}(\Omega);\ \begin{array}{l}\text{there exists a compact set } K_v \text{ in } \Omega \\ \text{such that } v \geqq f \text{ on } \Omega - K_v\end{array} \right\}$$

and

$$\underline{\mathcal{W}}_{\mathfrak{H}}(f)=\{-v\,;\,v\in\overline{\mathcal{W}}_{\mathfrak{H}}(-f)\}.$$

In case $\overline{\mathcal{W}}_{\mathfrak{H}}(f)$ (resp. $\underline{\mathcal{W}}_{\mathfrak{H}}(f)$) is non-empty, we define

$$\bar{h}_f^{\mathfrak{H}}=\inf \overline{\mathcal{W}}_{\mathfrak{H}}(f) \quad (\text{resp. } \underline{h}_f^{\mathfrak{H}}=\sup \underline{\mathcal{W}}_{\mathfrak{H}}(f)).$$

It is known (cf. [2]) that $\bar{h}_f^{\mathfrak{H}}$ (resp. $\underline{h}_f^{\mathfrak{H}}$) $\in \mathfrak{H}(\Omega)$ if it exists. Remark that if f is bounded, then both $\overline{\mathcal{W}}_{\mathfrak{H}}(f)$ and $\underline{\mathcal{W}}_{\mathfrak{H}}(f)$ are non-empty and $\inf_{\Omega} f\leqq \underline{h}_f^{\mathfrak{H}}\leqq \bar{h}_f^{\mathfrak{H}}\leqq \sup_{\Omega} f$ (by Axiom 4 and the minimum principle).

In case $\overline{\mathcal{W}}_{\mathfrak{H}}(f)$ and $\underline{\mathcal{W}}_{\mathfrak{H}}(f)$ are both non-empty and $\underline{h}_f^{\mathfrak{H}}=\bar{h}_f^{\mathfrak{H}}$, we say that f is $\mathfrak{H}$-harmonizable (cf. [2]) and denote $\underline{h}_f^{\mathfrak{H}}=\bar{h}_f^{\mathfrak{H}}$ by $h_f^{\mathfrak{H}}$. Obviously, any function in $\mathcal{S}_{\mathfrak{H}}^{+}(\Omega)$ is $\mathfrak{H}$-harmonizable and if $p\in\mathcal{P}_{\mathfrak{H}}$, then $h_p^{\mathfrak{H}}=0$.

The set of all continuous $\mathfrak{H}$-harmonizable functions (called $\mathfrak{H}$-Wiener functions) will be denoted by $\boldsymbol{W}^{\mathfrak{H}}$. We define (the class of $\mathfrak{H}$-Wiener potentials)

$$\boldsymbol{W}_0^{\mathfrak{H}}=\{f\in \boldsymbol{W}^{\mathfrak{H}}\,;\,h_f^{\mathfrak{H}}=0\}.$$

It is known ([2]) that $\boldsymbol{W}^{\mathfrak{H}}$ and $\boldsymbol{W}_0^{\mathfrak{H}}$ are real linear spaces; if $f,\ g\in \boldsymbol{W}^{\mathfrak{H}}$ and λ, μ are reals, then $h_{\lambda f+\mu g}^{\mathfrak{H}}=\lambda h_f^{\mathfrak{H}}+\mu h_g^{\mathfrak{H}}$. Also, constant functions belong to $\boldsymbol{W}^{\mathfrak{H}}$ and $0\leqq h_1^{\mathfrak{H}}\leqq 1$.

We can easily prove the following lemma:

LEMMA 1. *If $f\in \boldsymbol{W}^{\mathfrak{H}}$ and g is a bounded function on Ω, then*

$$\bar{h}_{f+g}^{\mathfrak{H}}=h_f^{\mathfrak{H}}+\bar{h}_g^{\mathfrak{H}} \quad \textit{and} \quad \underline{h}_{f+g}^{\mathfrak{H}}=h_f^{\mathfrak{H}}+\underline{h}_g^{\mathfrak{H}}.$$

2. Comparison of the classes of Wiener functions

Now we consider two harmonic structures $\mathfrak{H}_1$ and $\mathfrak{H}_2$ on the same space Ω (non-compact). We assume that both $(\Omega, \mathfrak{H}_1)$ and $(\Omega, \mathfrak{H}_2)$ satisfy Axioms 1~4. For simplicity, we replace the index $\mathfrak{H}_i$ by (i), e.g., we write $\mathcal{S}_{(1)}(G)$ for $\mathcal{S}_{\mathfrak{H}_1}(G)$, $\overline{\mathcal{W}}_{(2)}(f)$ for $\overline{\mathcal{W}}_{\mathfrak{H}_2}(f)$, $h_f^{(1)}$ for $h_f^{\mathfrak{H}_1}$, etc.

LEMMA 2. *Suppose $\boldsymbol{BW}^{(1)}\subset \boldsymbol{BW}^{(2)}$. Then, $\boldsymbol{BW}_0^{(1)}\subset \boldsymbol{BW}_0^{(2)}$ if and only if $h_f^{(2)}=h_{f_1}^{(2)}$ for any $f\in \boldsymbol{BW}^{(1)}$, where $f_1=h_f^{(1)}$.*

PROOF. The "if" part is obvious. Suppose now that $\boldsymbol{BW}_0^{(1)}\subset \boldsymbol{BW}_0^{(2)}$ and let $f\in \boldsymbol{BW}^{(1)}$. Then $f-f_1\in \boldsymbol{BW}_0^{(1)}\subset \boldsymbol{BW}_0^{(2)}$. Hence $h_{f-f_1}^{(2)}=0$, so that $h_f^{(2)}=h_{f_1}^{(2)}$.

THEOREM 1. *Suppose the following condition* (C) *is satisfied*:

(C) *There exists $p\in\mathcal{P}_{(2)}$ such that for any $v\in \mathcal{S}_{(1)}^{+}(\Omega)$ with $0\leqq v\leqq 1$ there is $w\in \mathcal{S}_{(2)}^{+}(\Omega)$ with the property that $|v-w|\leqq p$ on Ω.*

Then $\boldsymbol{BW}^{(1)}\subset \boldsymbol{BW}^{(2)}$ and $\boldsymbol{BW}_0^{(1)}\subset \boldsymbol{BW}_0^{(2)}$.

PROOF. By Lemma 2, it is enough to show that if $f \in \boldsymbol{W}^{(1)}$ and $0 \leqq f \leqq 1$, then $f \in \boldsymbol{W}^{(2)}$ and $h_f^{(2)} = h_{f_1}^{(2)}$, where $f_1 = h_f^{(1)}$. Given such an f, let $v \in \overline{\mathfrak{W}}_{(1)}(f)$ and $0 \leqq v \leqq 1$. By condition (C), there exists $w \in \mathcal{S}^+_{(2)}(\Omega)$ such that $|v-w| \leqq p$. Since $w+p \in \mathcal{S}_{(2)}(\Omega)$ and $w+p \geqq v \geqq f$ outside a compact set in Ω, we have $w+p \in \overline{\mathfrak{W}}_{(2)}(f)$. Hence $w+p \geqq \bar{h}_f^{(2)}$, and hence $v+2p \geqq w+p \geqq \bar{h}_f^{(2)}$. Taking the infimum of $v \in \overline{\mathfrak{W}}_{(1)}(f)$, we have

$$h_f^{(1)} + 2p \geqq \bar{h}_f^{(2)}. \tag{1}$$

By applying the above result to the function $1-f$, we have $h_{1-f}^{(1)} + 2p \geqq \bar{h}_{1-f}^{(2)}$. By virtue of Lemma 1, this inequality can be written as

$$h_1^{(1)} - h_f^{(1)} + 2p \geqq h_1^{(2)} - \underline{h}_f^{(2)}. \tag{2}$$

(1) and (2) imply

$$\bar{h}_f^{(2)} - 2p \leqq h_f^{(1)} = f_1 \leqq \underline{h}_f^{(2)} + h_1^{(1)} - h_1^{(2)} + 2p \leqq \underline{h}_f^{(2)} + (1 - h_1^{(2)}) + 2p.$$

Since $1 - h_1^{(2)} \in \mathcal{P}_{(2)}$, it follows that f is $\mathfrak{H}_2$-harmonizable and $h_f^{(2)} = h_{f_1}^{(2)}$. Hence we have the theorem.

The following theorem is an easy consequence of the above theorem:

THEOREM 2. *If there exists a compact set K (may be empty) such that $\boldsymbol{B}\mathcal{S}^+_{(1)}(\Omega - K) \subset \boldsymbol{B}\mathcal{S}^+_{(2)}(\Omega - K)$, then $\boldsymbol{BW}^{(1)} \subset \boldsymbol{BW}^{(2)}$ and $\boldsymbol{BW}_0^{(1)} \subset \boldsymbol{BW}_0^{(2)}$.*

PROOF. Since $\mathcal{P}_{(2)} \neq \{0\}$ by assumption, there exists $p \in \mathcal{P}_{(2)}$ such that $p \geqq 1$ on a neighborhood of K. Given $v \in \mathcal{S}^+_{(1)}(\Omega)$ such that $0 \leqq v \leqq 1$, let $w = \inf(1, v+p)$. Since $v \,|\, \Omega - K \in B\mathcal{S}^+_{(1)}(\Omega - K) \subset B\mathcal{S}^+_{(2)}(\Omega - K)$, $w \,|\, \Omega - K \in \mathcal{S}^+_{(2)}(\Omega - K)$. Also, $w(x) \equiv 1$ on a neighborhood of K. Hence $w \in \mathcal{S}^+_{(2)}(\Omega)$. On the other hand, we see $0 \leqq w - v = \inf(1-v, p) \leqq p$. Thus condition (C) of Theorem 1 is satisfied, and hence our conclusion holds.

P. A. Loeb [3] defined that $\mathfrak{H}_1 \geqq \mathfrak{H}_2$ if there exists a compact set K in Ω such that $\mathfrak{H}_1^+(G) \subset \mathcal{S}^+_{(2)}(G)$ for any open set G contained in $\Omega - K$. In this case, we have $\mathcal{S}^+_{(1)}(\Omega - K) \subset \mathcal{S}^+_{(2)}(\Omega - K)$ (cf. [3]). Hence we have

COROLLARY. *If $\mathfrak{H}_1 \geqq \mathfrak{H}_2$ in Loeb's sense, then $\boldsymbol{BW}^{(1)} \subset \boldsymbol{BW}^{(2)}$ and $\boldsymbol{BW}_0^{(1)} \subset \boldsymbol{BW}_0^{(2)}$.*

3. Applications to the solutions of $\Delta u - qu = 0$

Now let Ω be a locally Euclidean space having a Green function (or a hyperbolic Riemann surface) and consider the differential equation $\Delta u - qu = 0$ on Ω, where q is a locally Hölder continuous non-negative function on Ω. Then the solutions of this equation form a harmonic space $(\Omega, \mathfrak{H}_q)$ satisfying

Axioms 1~4 (see [4]). We denote by $G^q(x, y)$ the Green function on Ω for this equation.

If q_1 and q_2 are two locally Hölder continuous non-negative functions on Ω, then we obtain the following result as a consequence of Theorem 1:

PROPOSITION. *If*

$$\int G^{q_2}(x, y) \max (q_1(y)-q_2(y), 0)\ dy<+\infty$$

for some $x \in \Omega$, *then* $\boldsymbol{BW}^{(1)} \subset \boldsymbol{BW}^{(2)}$ *and* $\boldsymbol{BW}_0^{(1)} \subset \boldsymbol{BW}_0^{(2)}$, *where we put* $\mathfrak{H}_1=\mathfrak{H}_{q_1}$ *and* $\mathfrak{H}_2=\mathfrak{H}_{q_2}$.

PROOF. Under the condition of the proposition,

$$p(x)=\frac{1}{c_d}\int G^{q_2}(x, y) \max (q_1(y)-q_2(y), 0)\ dy$$

(see [4] for the constant c_d) is an $\mathfrak{H}_2$-potential, i.e., $p \in \mathcal{P}_{(2)}$. Given $v \in \mathcal{S}_{(1)}^{+}(\Omega)$ such that $0 \leqq v \leqq 1$, let $w=v+p$. Then, in the distribution sense, we have (cf. [4]) $\Delta v-q_1 v \leqq 0$ and $\Delta p-q_2 p=-\max(q_1-q_2, 0)$. Hence

$$\begin{aligned} \Delta w-q_2 w &= \Delta v-q_2 v+\Delta p-q_2 p \\ &\leqq (q_1-q_2) v-\max(q_1-q_2, 0) \leqq 0. \end{aligned}$$

Thus $w \in \mathcal{S}_{(2)}^{+}(\Omega)$ (see [4]) and $0 \leqq w-v=p$. Therefore condition (C) of Theorem 1 is satisfied and the proposition is proved.

COROLLARY 1. *If there exists* $\alpha>0$ *such that* $q_1 \leqq \alpha q_2$ *outside a compact set in* Ω, *then* $\boldsymbol{BW}^{(1)} \subset \boldsymbol{BW}^{(2)}$ *and* $\boldsymbol{BW}_0^{(1)} \subset \boldsymbol{BW}_0^{(2)}$.

PROOF. We may prove only the case $\alpha \geqq 1$. In this case $q_1-q_2 \leqq (\alpha-1) q_2$ outside a compact set. Since

$$\int G^{q_2}(x, y) q_2(y)\, dy \leqq c_d$$

for all $x \in \Omega$ (cf. [4]), the condition in the above proposition is easily verified.

COROLLARY 2. (a) *For any* $q(\geqq 0)$, $\boldsymbol{BW} \subset \boldsymbol{BW}^{(q)}$ *and* $\boldsymbol{BW}_0 \subset \boldsymbol{BW}_0^{(q)}$; (b) *If* $\int G(x, y) q(y) dy<+\infty$, *then* $\boldsymbol{BW}^{(q)}=\boldsymbol{BW}$ *and* $\boldsymbol{BW}_0^{(q)}=\boldsymbol{BW}_0$. Here, $\boldsymbol{W}^{(q)}$ (resp. $\boldsymbol{W}_0^{(q)}$) is the class of $\mathfrak{H}_q$-Wiener functions (resp. $\mathfrak{H}_q$-Wiener potentials) and $\boldsymbol{W}$ (resp. $\boldsymbol{W}_0$) is the class of ordinary Wiener functions (resp. Wiener potentials).

REMARK. The above proposition and Corollary 1 show that our results contain the results given by Hidematu Tanaka [5]. Also, Theorem 3. 1, (i) and Corollary 2 to Theorem 3.2 in [4] are immediate consequences of the above corollaries.

Added in proof: We can improve Lemma 2 as follows: If $\boldsymbol{BW}^{(1)} \subset \boldsymbol{BW}^{(2)}$, then $\boldsymbol{BW}_0^{(1)} \subset \boldsymbol{BW}_0^{(2)}$ and $h_f^{(2)} = h_{f_1}^{(2)}$ for any $f \in \boldsymbol{BW}^{(1)}$, where $f_1 = h_f^{(1)}$.

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