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On the Homotopy Groups of Sphere Bundles over Spheres

Dedicated to Professor Atuo Komatu on his 60th birthday

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§ 0 Statement of Results.

Throughout this paper, p denotes always an odd prime. We consider a cell complex

(0.1) $$B_n(p)=S^{2n+1}\cup e^{2n+2p-1}\cup e^{4n+2p},$$

whose cohomology ring with Z_p-coefficient is

(0.2) $$H^*(B_n(p);Z_p)=\Lambda(u,\mathscr{P}^1u),\ u\in H^{2n+1}(B_n(p);Z_p).$$

We notice that the p-primary components $\pi_i(B_n(p);p)$ of the i-th homotopy groups of $B_n(p)$ appear in the following

THEOREM 0.1. *For the homotopy groups of the special unitary groups $SU(m)$ and the symplectic groups $Sp(m)$, we have the following direct sum decompositions:*

(0.3) $$\sum_{k=1}^{n}\pi_i(B_k(p);p)+\sum_{k=n+1}^{p-1}\pi_i(S^{2k+1};p)\approx\pi_i(SU(n+p);p),\ \textit{for } n<p,$$

(0.4) $$\sum_{k=1}^{n}\pi_i(B_{2k-1}(p);p)+\sum_{k=n+1}^{q}\pi_i(S^{4k-1};p)\approx\pi_i(Sp(n+q);p),\ \textit{for } n\leqq q=(p-1)/2.$$

These decompositions for $n=1$ are (1.4) and (1.5) of [6], and the similar direct sum decompositions for exceptional Lie groups are obtained recently by Mimura and Toda [4].

As a special kind of $B_n(p)$, we have the following

THEOREM 0.2. *There exist cell complexes $B_n(p)$ for $n\geqq 1$, satisfying* (0.1), (0.2) *and the following two conditions:*

(0.5) *$B_n(p)$ is an S^{2n+1}-bundle over $S^{2n+2p-1}$.*

(0.6) *There exists a map*

$$f:\ S^2B_n(p)\longrightarrow B_{n+1}(p)\quad \textit{for } n\geqq 1,$$

which induces isomorphisms of $H_i(\ ;Z)$ for $i<4n+2p+2$.

The purpose of this paper is to compute $\pi_i(B_n(p); p)$ of these $B_n(p)$ for $i<2n+1+2(p^2+p)(p-1)-5$.

From (0.5), we have the following exact sequence:

$$(0.7)\qquad \cdots \xrightarrow{j_*} \pi_{i+1}(S^{2n+2p-1}) \xrightarrow{\partial_n} \pi_i(S^{2n+1}) \xrightarrow{i_*} \pi_i(B_n(p)) \xrightarrow{j_*} \pi_i(S^{2n+2p-1}) \xrightarrow{\partial_n} \cdots,$$

and we consider the boundary homomorphisms ∂_n. We have easily

THEOREM 0.3. $\partial_n(S^2\gamma)=\alpha_1(2n+1)\circ S\gamma$ *for given* $\gamma \in \pi_{i-1}(S^{2n+2p-3})$, *where S denotes the suspension homomorphism and $\alpha_1(2n+1)$ is an element in $\pi_{2n+2p-2}(S^{2n+1})$ of order p.*

By means of the mapping cylinder construction of the map $B_n(p) \longrightarrow \Omega^2 B_{n+1}(p)$, induced by f of (0.6), we may regard as $B_n(p) \subset \Omega^2 B_{n+1}(p)$ and write

$$QB_n(p)=\Omega(\Omega^2 B_{n+1}(p),\ B_n(p)).$$

THEOREM 0.4. *We have the exact sequence*

$$(0.8)\qquad \cdots \xrightarrow{j_*} \pi_{j+1}(Q_2^{2n+2p-1}) \xrightarrow{\bar{\partial}_n} \pi_j(Q_2^{2n+1}) \xrightarrow{i_*} \pi_j(QB_n(p)) \xrightarrow{j_*} \pi_j(Q_2^{2n+2p-1}) \xrightarrow{\partial_n} \cdots,$$

where $Q_2^{2m-1}=\Omega(\Omega^2 S^{2m+1},\ S^{2m-1})$. Furthermore we obtain the following commutative diagram of exact sequences:

(0.9)

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \pi_{i+1}(S^{2n+2p-3}) \arrow[r, "\partial_{n-1}"] \arrow[d, "S^2"] & \pi_i(S^{2n-1}) \arrow[r, "i_*"] \arrow[d, "S^2"] & \pi_i(B_{n-1}(p)) \arrow[r, "j_*"] \arrow[d, "f_* S^2"] & \pi_i(S^{2n+2p-3}) \arrow[r] \arrow[d, "S^2"] & \cdots \\
\cdots \arrow[r] & \pi_{i+3}(S^{2n+2p-1}) \arrow[r, "\partial_n"] \arrow[d, "H^{(2)}"] & \pi_{i+2}(S^{2n+1}) \arrow[r, "i_*"] \arrow[d, "H^{(2)}"] & \pi_{i+2}(B_n(p)) \arrow[r, "j_*"] \arrow[d, "H^{(2)}"] & \pi_{i+2}(S^{2n+2p-1}) \arrow[r] \arrow[d, "H^{(2)}"] & \cdots \\
\cdots \arrow[r] & \pi_i(Q_2^{2n+2p-3}) \arrow[r, "\bar{\partial}_{n-1}"] \arrow[d, "p_*"] & \pi_{i-1}(Q_2^{2n-1}) \arrow[r, "i_*"] \arrow[d, "p_*"] & \pi_{i-1}(QB_{n-1}(p)) \arrow[r, "j_*"] \arrow[d, "p_*"] & \pi_{i-1}(Q_2^{2n+2p-3}) \arrow[r] \arrow[d, "p_*"] & \cdots \\
\cdots \arrow[r] & \pi_i(S^{2n+2p-3}) \arrow[r, "\partial_{n-1}"] \arrow[d] & \pi_{i-1}(S^{2n-1}) \arrow[r, "i_*"] \arrow[d] & \pi_{i-1}(B_{n-1}(p)) \arrow[r, "j_*"] \arrow[d] & \pi_{i-1}(S^{2n+2p-3}) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

By the above diagram, we can investigate ∂_n by using $\bar{\partial}_{n-1}$ and Theorem 0.3. Using the homomorphisms I' and I in the exact sequence

$$(0.10)\qquad \cdots \xrightarrow{\Delta} \pi_{i+2}(S^{2mp-1}; p) \xrightarrow{I'} \pi_i(Q_2^{2m-1}; p) \xrightarrow{I} \pi_{i+3}(S^{2mp+1}; p) \xrightarrow{\Delta} \pi_{i+1}(S^{2mp-1}; p) \xrightarrow{I'} \cdots$$

of [8; (2.5)], we have the following

THEOREM 0.5. *There is an integer* $x_n \not\equiv 0 \pmod p$ *such that*

$$I\bar{\partial}_n I'(S^3\gamma) = x_n \beta_1(2(n+1)p+1) \circ S^3\gamma \tag{0.11}$$

for any $\gamma \in \pi_{i-1}(S^{2(n+p)p-4}; p)$, *where* $\beta_1(2(n+1)p+1) \in \pi_{2(n+p)p-1}(S^{2(n+1)p+1}; p) \approx Z_p$ *is Toda's element in* [8].

Using these three theorems and the known results about the homotopy groups of spheres in [8], we can determine

$$\partial_n : \pi_{2n+2p-1+k}(S^{2n+2p-1}; p) \longrightarrow \pi_{2n+1+k+2(p-1)-1}(S^{2n+1}; p)$$

for $k < 2(p^2+p-1)(p-1)-4$ except the only one case

$$p=3,\ n=1,\ k=35. \tag{0.12}$$

For the determination of the extensions of groups in the exact sequence (0.7), we treat Lemmas 6.2 and 6.3 in §6. Consequently, the groups $\pi_{2n+1+k}(B_n(p); p)$, $k < 2(p^2+p)(p-1)-5$, are determined except the following two cases:

$$p=3,\ n=1,\ k=37,\ 38, \tag{0.13}$$

$$k = 2r(p-1)-2,\ r>p+3,\ 1<n<r-p-1 \tag{0.14}$$

$$\textit{and } r=2p+1,\ p^2+1,\ n=r-p-1.$$

The case (0.13) occurs from the indetermination of (0.12). In the case (0.14), we can determine the orders of groups.

Summarizing these facts, it is stated as follows:

THEOREM 0.6. *For* $n \geqq 1$ *and* $k < 2(p^2+p)(p-1)-5$, *we have the following direct sum decomposition*:

$$\pi_{2n+1+k}(B_n(p); p) = \bar{A}(n, k) + \bar{B}(n, k) + \bar{E}(n, k)$$
$$+ U_a(n, k) + U_b(n, k) + U_u(n, k),$$

where the definitions of direct factors are given in §6.

The subgroups $\bar{A}(n, k) + \bar{B}(n, k)$ $(k \neq 2(p^2+1)(p-1)-3) + \bar{E}(n, k)$ are mapped isomorphically into the stable groups $\pi_k^S(B; p) = \varinjlim_n \pi_{2n+1+k}(B_n(p); p)$.

In §1, Theorem 0.1 is proved. The bundles $B_n(p)$ of Theorem 0.2 are constructed in §2, and Theorems 0.3 and 0.4 are proved. Theorem 0.5 is proved in §3. Section 4 is used to quote the known results about the homotopy groups of spheres. The determination of ∂_n is in §5 and the proof of Theorem 0.6 is in §6.

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§ 1 Proof of Theorem 0.1.

Since $\pi_{2n}(S^{2k+1})$ is finite and has no p-torsion if $k<n<p$, it follows that $\pi_{2n}(SU(k+1))$ is finite and has no p-torsion. From the exactness of the sequence $\pi_{2n+1}(SU(n+1)) \xrightarrow{\pi_*} \pi_{2n+1}(S^{2n+1}) \longrightarrow \pi_{2n}(SU(n))$, there exists a map

$$f_n\colon S^{2n+1} \longrightarrow SU(n+1)$$

such that the mapping degree of the composition $\pi f_n\colon S^{2n+1} \longrightarrow S^{2n+1}$ is prime to p, for $n<p$. We put $f'_n = i_n f_n$, for the inclusion $i_n\colon SU(n+1) \longrightarrow SU(n+p)$. Since $\pi_{2n+2p-2}(SU(n+p))=0$ by Bott periodicity, the map f'_n is extended to a map

$$g_n\colon K_n = S^{2n+1} \cup e^{2n+2p-1} \longrightarrow SU(n+p),$$

where K_n is the $(2n+2p-1)$-skeleton of $B_n(p)$. And g_n^* are epimorphisms of $H^*(\ ;Z_p)$, since $\mathscr{P}^1 \neq 0$ holds in $SU(n+p)$ for $n<p$. According to Imanishi ([2] Theorem 1), the order a of $\pi_{4n+2p-1}(SU(n+p))$ is prime to p for $n \leqq 3(p-1)/2$. Replacing the attaching map β of the $(4n+2p)$-cell of $B_n(p)$ by $a\beta$, we obtain a cell complex $B'_n(p)$. Obviously we have

(1.1) $$\pi_i(B'_n(p);p) \approx \pi_i(B_n(p);p) \text{ for all } i.$$

The map g_n has an extension

$$h_n\colon B'_n(p) \longrightarrow SU(n+p) \quad \text{for } n<p$$

and h_n^* are epimorphisms of $H^*(\ ;Z_p)$. Using the maps h_i and f'_j and the multiplication of $SU(n+p)$, we obtain a map

$$F = h_1 \cdot h_2 \cdot \cdots \cdot h_n \cdot f'_{n+1} \cdot f'_{n+2} \cdot \cdots \cdot f'_{p-1}\colon$$
$$B'_1(p) \times B'_2(p) \times \cdots \times B'_n(p) \times S^{2n+3} \times S^{2n+5} \times \cdots \times S^{2p-1} \rightarrow SU(n+p)$$

which induces isomorphisms F^* of $H^*(\ ;Z_p)$. Thus, for $n<p$, the following isomorphisms hold:

(1.2) $$F_*\colon \sum_{k=1}^{n} \pi_i(B'_k(p);p) + \sum_{k=n+1}^{p-1} \pi_i(S^{2k+1};p) \xrightarrow{\approx} \pi_i(SU(n+p);p).$$

The decompositions (0.3) follow from (1.1) and (1.2).

Similarly we have a map

$$G\colon B'_1(p) \times \cdots \times B'_{2n-1}(p) \times S^{4n+3} \times \cdots \times S^{2p-3} \longrightarrow Sp(n+q)$$

and isomorphisms G_* of $\pi_i(\ ;p)$ for $n \leqq q = (p-1)/2$, and the decompositions (0.4) are obtained.

§ 2 Definition and Properties of $B_n(p)$.

Let $V_{m,k}$ denote the Stiefel manifold of orthonormal k-frames in R^m, the m-dimensional vector space over the reals. Then $V_{m,k}$ is a fibre bundle over $V_{m,k'}$ with fibre $V_{m-k',k-k'}$, for $1 \leqq k' \leqq k \leqq m$. Especially $V_{2n+3,2}$ is an S^{2n+1} bundle over S^{2n+2}. The characteristic class of this bundle $V_{2n+3,2}$ is an element $2\iota_{2n+1} \in \pi_{2n+1}(S^{2n+1}) \approx Z$, which is represented by a map of degree 2.

Let A and B be spaces, and $I=[0, 1]$ the unit interval. We denote by $A*B$ the join of A with B, and $d\colon A \times B \times I \longrightarrow A*B$ the canonical map. Then the homeomorphism

$$h\colon S^m * S^1 \longrightarrow S^{m+2}$$

is given by $hd(x, \theta, t)=(\lambda x, \mu\cos\theta, \mu\sin\theta)$, $\lambda=\cos(\pi t/2)$, $\mu=\sin(\pi t/2)$, $0 \leqq \theta \leqq 2\pi$. We define a map

$$g\colon V_{2n+3,2} * S^1 \longrightarrow V_{2n+5,2}$$

by $gd((x, y), \theta, t)=((\lambda x, \mu\cos\theta, \mu\sin\theta), (\lambda y, -\mu\sin\theta, \mu\cos\theta))$, then we obtain the following diagram in which the left square is homotopy commutative and the right one is commutative.

$$\begin{array}{ccccc}
S^{2n+1} * S^1 & \xrightarrow{i*1} & V_{2n+3,2} * S^1 & \xrightarrow{\pi*1} & S^{2n+2} * S^1 \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
S^{2n+3} & \xrightarrow{i} & V_{2n+5,2} & \xrightarrow{\pi} & S^{2n+4}
\end{array}$$

Since $V_{2n+3,2} * S^1$ has the same homotopy type as $S^2 V_{2n+3,2}$, we get the following

PROPOSITION 2.1. *There exists a map*

$$\bar{g}\colon S^2 V_{2n+3,2} \longrightarrow V_{2n+5,2}$$

such that, in the following diagram (2.1), $\pi \bar{g} = S^2\pi$ *and* $\bar{g} S^2 i \simeq i$ *hold:*

$$\begin{tikzcd}
 & S^2 V_{2n+3,2} \arrow[dr, "S^2\pi"] \arrow[dd, "\bar{g}"] & \\
S^{2n+3} \arrow[ur, "S^2 i"] \arrow[dr, "i"'] & & S^{2n+4} \\
 & V_{2n+5,2} \arrow[ur, "\pi"'] &
\end{tikzcd} \tag{2.1}$$

where $\simeq$ *means homotopic.*

Let $\alpha_1(3)$ be the generator of $\pi_{2p}(S^3; p) \approx Z_p$ with mod p Hopf invariant one. Then $S^\infty \alpha_1(3) = \alpha_1$ is the first non-trivial element of the p-component of the stable homotopy groups of spheres. We put

(2.2) $$\alpha_1(m)=S^{m-3}\alpha_1(3)\in\pi_{m+2p-3}(S^m;p)\quad \textit{for } m\geqq 3.$$

DEFINITION 2.2. *We denote by $B_n(p)$, the induced bundle of the bundle $V_{2n+3,2}$ by the map which represents the element $\frac{1}{2}\alpha_1(2n+2)\in\pi_{2n+2p-1}(S^{2n+2};p)$, for $n\geqq 1$.*

PROOF OF THEOREM 0.2. By definition, the conditions (0.1), (0.2) and (0.5) are satisfied obviously. The space $B_n(p)$ consists of pairs (x, y) in $S^{2n+2p-1}\times V_{2n+3,2}$ satisfying $a(x)=\pi(y)$, where a denotes a representative of $\frac{1}{2}\alpha_1(2n+2)$. We define a map f: $S^2B_n(p)=B_n(p)\wedge S^2\longrightarrow B_{n+1}(p)$ by

$$f((x, y)\wedge z)=(x\wedge z,\ \bar{g}(y\wedge z)),$$

for any elements $(x, y)\in B_n(p)\subset S^{2n+2p-1}\times V_{2n+3,2}$ and $z\in S^2$, where $\wedge$ denotes the smash product. Then the map f is well defined, since $(S^2a)(x\wedge z)=\pi(\bar{g}(y\wedge z))$ by Proposition 2.1. We can verify easily that this map f satisfies the condition (0.6). q.e.d.

Let X and Y be spaces and let A be a subspace of X. We denote by $[X, Y]$ the set of homotopy classes of base-point preserving maps $X\rightarrow Y$, and $\Omega(X, A)$ the space of paths $(I, 0, 1)\rightarrow(X, *, A)$ with compact-open topology. $\Omega X=\Omega(X, *)$ is the loop space of X. Let p: $E\rightarrow B$ be a fibering and $F=p^{-1}(*)$ the fibre over $*$, then the boundary map Δ: $[SX, B]\rightarrow[X, F]$ is defined as usual, and the following lemma is verified easily.

LEMMA 2.3. $\Delta(\alpha\circ S\beta)=\Delta(\alpha)\circ\beta$ *for any $\alpha\in[SY, B]$ and $\beta\in[X, Y]$.*

Let ∂ be the boundary homomorphism in the homotopy exact sequence of the bundle $V_{2n+3,2}$. The characteristic class of this bundle is

(2.3) $$\partial(\iota_{2n+2})=2\iota_{2n+1},$$

and the following diagram is commutative:

(2.4) $$\begin{tikzcd} \pi_{i+1}(S^{2n+2p-1}) \arrow[r, "\partial_n"] \arrow[d, "a_*"] & \pi_i(S^{2n+1}) \\ \pi_{i+1}(S^{2n+2}) \arrow[ur, "\partial"'] & \end{tikzcd},$$

where $\{a\}=\frac{1}{2}\alpha_1(2n+2)$.

PROOF OF THEOREM 0.3. By Lemma 2.3 for $\Delta=\partial$, (2.3) and (2.4),

$$\partial_n(S^2\gamma)=\partial\Big(\frac{1}{2}\alpha_1(2n+2)\circ S^2\gamma\Big)$$
$$=\partial\Big(\iota_{2n+2}\circ S\Big(\frac{1}{2}\alpha_1(2n+1)\circ S\gamma\Big)\Big)$$

$$=2\iota_{2n+1}\circ\frac{1}{2}\alpha_1(2n+1)\circ S\gamma=\alpha_1(2n+1)\circ S\gamma. \qquad \text{q.e.d.}$$

Let $p: E\rightarrow B$ be a fibering with the fibre $F=p^{-1}(*)$, and assume that E, B and F have the same homotopy types as CW-complexes. Since Ωp: $\Omega(E, F)\rightarrow\Omega B$ induces isomorphisms of homotopy groups, it is a homotopy equivalence. The projection p_0: $\Omega(E, F)\rightarrow F$ induces homomorphisms of homotopy groups equivalent to the boundary homomorphism in the homotopy exact sequence of the pair (E, F). Replacing $\Omega(E, F)$ and p_0 by ΩB and the composition with a homotopy inverse of Ωp respectively, we get the following

Lemma 2.4. *$p: E\longrightarrow B$, F and p_0 are as above. There is a map*

$$\rho: \Omega B\longrightarrow F \tag{2.5}$$

such that p_0 is homotopic to $\rho\Omega p$, and the following diagram is commutative:

$$\begin{tikzcd} \pi_{i+1}(B) \arrow[r, "\partial"] \arrow[d, "\approx"', "\Omega"] & \pi_i(F) \\ \pi_i(\Omega B) \arrow[ur, "\rho_*"'] & \end{tikzcd}.$$

The following proposition is proved easily.

Proposition 2.5. *Let $p: E\longrightarrow B$ and $p': E'\longrightarrow B'$ be fiberings with fibres F and F', and assume that the following two conditions hold:*

(i) *E, B, E', B', F and F' have the same homotopy types as CW-complexes.*

(ii) *E', B' and F' are subspaces of E, B and F respectively, and the following diagram is homotopy commutative:*

$$\begin{tikzcd} F' \arrow[r, "i'"] \arrow[d] & E' \arrow[r, "p'"] \arrow[d] & B' \arrow[d] \\ F \arrow[r, "i"] & E \arrow[r, "p"] & B, \end{tikzcd}$$

where vertical arrows are inclusions.

Then we obtain the following commutative diagram of exact sequences:

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & \pi_{i+1}(B') \arrow[r] \arrow[d] & \pi_i(F') \arrow[r] \arrow[d] & \pi_i(E') \arrow[r] \arrow[d] & \pi_i(B') \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \pi_{i+1}(B) \arrow[r] \arrow[d] & \pi_i(F) \arrow[r] \arrow[d] & \pi_i(E) \arrow[r] \arrow[d] & \pi_i(B) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \pi_{i+1}(B, B') \arrow[r] \arrow[d] & \pi_i(F, F') \arrow[r] \arrow[d] & \pi_i(E, E') \arrow[r] \arrow[d] & \pi_i(B, B') \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \pi_i(B') \arrow[r] \arrow[d] & \pi_{i-1}(F') \arrow[r] \arrow[d] & \pi_{i-1}(E') \arrow[r] \arrow[d] & \pi_{i-1}(B') \arrow[r] \arrow[d] & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{tikzcd}$$

PROOF OF THEOREM 0.4. The double suspension S^2: $\pi_i(S^{2m-1})\longrightarrow\pi_{i+2}(S^{2m+1})$ is equivalent to the map induced by the inclusion $S^{2m-1}\longrightarrow\Omega^2 S^{2m+1}$, and the homomorphism $H^{(2)}$ is defined as follows:

$$H^{(2)}=k_*\Omega^3\colon \pi_{i+2}(S^{2m+1})\xrightarrow{\approx}\pi_{i-1}(\Omega^3 S^{2m+1})\longrightarrow\pi_{i-1}(Q_2^{2m-1}),$$

for the inclusion k: $\Omega^3 S^{2m+1}\longrightarrow Q_2^{2m-1}$. The projection p: $Q_2^{2m-1}=\Omega(\Omega^2 S^{2m+1}, S^{2m-1})\longrightarrow S^{2m-1}$ is a fibering with fibre $\Omega^3 S^{2m+1}$. First and second columns of the diagram (0.9) are obtained from the homotopy exact sequence of this fibering for $m=n+p-1$ and n respectively. Similarly, third column of (0.9) is obtained from the fibering p: $QB_{n-1}(p)\longrightarrow B_{n-1}(p)$. Then in Proposition 2.5, putting $F=\Omega^2 S^{2n+1}$, $E=\Omega^2 B_n(p)$, $B=\Omega^2 S^{2n+2p-1}$, $F'=S^{2n-1}$, $E'=B_{n-1}(p)$ and $B'=S^{2n+2p-3}$, we obtain (0.8) and (0.9). q.e.d.

Now we consider the cohomology spectral sequence associated with the fibering

(2.6) $$\Omega S^{2n+1}\xrightarrow{\Omega i}\Omega B_n(p)\xrightarrow{\Omega j}\Omega S^{2n+2p-1}.$$

Then $E_\infty^{**}\approx E_2^{**}\approx H^*(\Omega S^{2n+1};Z_p)\otimes H^*(\Omega S^{2n+2p-1};Z_p)$ holds, since both ΩS^{2n+1} and $\Omega S^{2n+2p-1}$ have vanishing cohomology of odd degrees. In more detail:

(2.7) *$H^*(\Omega S^{2n+1};Z_p)$ has the following Z_p-basis*

$$\{x_{i_1}^{r_1}\cdots x_{i_s}^{r_s};\ 0\leqq i_1<\cdots<i_s,\ 0\leqq r_1,\cdots,r_s<p\},\ \deg x_i=2np^i.$$

And $H^*(\Omega S^{2n+2p-1};Z_p)$ has the following Z_p-basis

$$\{y_{i_1}^{r_1}\cdots y_{i_s}^{r_s};\ 0\leqq i_1<\cdots<i_s,\ 0\leqq r_1,\cdots,r_s<p\},\ \deg y_i=2(n+p-1)p^i.$$

PROPOSITION 2.6. *$H^*(\Omega B_n(p);Z_p)$ has the following Z_p-basis*

$$\{a_{i_1}^{r_1}\cdots a_{i_q}^{r_q}b_{j_1}^{t_1}\cdots b_{j_s}^{t_s};\ 0\leqq i_1<\cdots<i_q,\ 0\leqq j_1<\cdots<j_s,\ 0\leqq r_1,\cdots,r_q<p,$$
$$0\leqq t_1,\cdots,t_s<p,\quad q,s\geqq 1\},\ \textit{and}\ \deg a_k=2np^k,\ \deg b_k=2(n+p-1)p^k.$$

Furthermore the elements a_k and b_k satisfy the following conditions:

(i)$_k$ *$(\Omega i)^*a_k=x_k$ and $b_k=(\Omega j)^*y_k$, up to non-zero coefficients.*

(ii)$_k$ *$\mathcal{P}^{p^k}a_k=b_k$ and $\mathcal{P}^i a_k=0$ for $i>p^k$.*

PROOF. Put $a_0=\sigma u$ and $b_0=\sigma\mathcal{P}^1 u$, for the cohomology suspension σ and $u\in H^{2n+1}(B_n(p);Z_p)$. Since $\mathcal{P}^1$ commutes with σ, (i)$_0$ and (ii)$_0$ hold.

We have the sequence:

(2.8) $$0\longrightarrow H^*(\Omega S^{2n+2p-1};Z_p)\xrightarrow{\Omega j^*}H^*(\Omega B_n(p);Z_p)\xrightarrow{\Omega i^*}H^*(\Omega S^{2n+1};Z_p)\longrightarrow 0$$

which is exact as Hopf algebras, with respect to the diagonal map μ^* induced by the loop-multiplication μ. The sets of primitive elements in $H^*(\Omega S^{2n+2p-1}; Z_p)$ and $H^*(\Omega S^{2n+1}; Z_p)$ are spanned by y_0 and x_0 respectively. It follows from Proposition 3.12 of [3] that

(2.9) *The primitive elements of* $H^*(\Omega B_n(p); Z_p)$ *are spanned by* a_0 *and* b_0.

Now we consider the case $n>1$. Assume that there are elements a_k and b_k satisfying $(\mathrm{i})_k$, $(\mathrm{ii})_k$ and the following conditions $(\mathrm{iii})_k$ for $k=0, 1, \cdots, r$.

$(\mathrm{iii})_k$ *There exist elements* $a_k \in H^{2np^k}(\Omega B_n(p); Z)$ *and* $b_k \in H^{2(n+p-1)p^k}(\Omega B_n(p); Z)$ *whose* mod p *reductions are the elements* a_k *and* b_k *of above. Such elements satisfy* $a_{k-1}^p = pa_k$ *and* $b_{k-1}^p = pb_k$.

From $(\mathrm{ii})_r$, $a_r^p = \mathscr{P}^{np^r} a_r = 0$. This means that $a_r^p = pa_{r+1}$ for some $a_{r+1} \in H^{2np^{r+1}}(\Omega B_n(p); Z)$. From $(\mathrm{iii})_k$, $k \leqq r$, $p^{1+p+\cdots+p^r} a_{r+1} = a_0^{p^{r+1}}$ holds in Z-coefficient. Then we have

$$
\begin{aligned}
p^{1+p+\cdots+p^r} \mu^*(a_{r+1}) &= \mu^*(a_0^{p^{r+1}}) \\
&= \sum_{i=1}^{p^{r+1}-1} \binom{p^{r+1}}{i} a_0^i \otimes a_0^{p^{r+1}-i} + a_0^{p^{r+1}} \otimes 1 + 1 \otimes a_0^{p^{r+1}} \\
&= p^{1+p+\cdots+p^r} \left(\sum_{i=1}^{p^{r+1}-1} \frac{1}{p^{r+1-\nu}} \binom{p^{r+1}}{i} a_r^{\alpha_r} \cdots a_\nu^{\alpha_\nu} \otimes a_r^{\beta_r} \cdots a_\nu^{\beta_\nu} + a_{r+1} \otimes 1 + 1 \otimes a_{r+1} \right),
\end{aligned}
$$

where $i = \sum_{t=\nu}^{r} \alpha_t p^t$ and $p^{r+1} - i = \sum_{t=\nu}^{r} \beta_t p^t$ are p-adic expansions, $\alpha_\nu, \beta_\nu \neq 0$.

Remark that $c_i = \frac{1}{p^{r+1-\nu}} \binom{p^{r+1}}{i}$ is an integer prime to p, for $0 < i < p^{r+1}$. So, we have

$$
(2.10) \qquad \mu^*(a_{r+1}) = a_{r+1} \otimes 1 + 1 \otimes a_{r+1} + \sum_{i=1}^{p^{r+1}-1} c_i a_r^{\alpha_r} \cdots a_\nu^{\alpha_\nu} \otimes a_r^{\beta_r} \cdots a_\nu^{\beta_\nu},
$$

in Z_p-coefficient. Using the Cartan formula and $(\mathrm{ii})_k$, $k \leqq r$, (2.10) implies the following

$$
\begin{aligned}
(2.11) \qquad & \mu^*(\mathscr{P}^{p^{r+1}} a_{r+1}) \\
&= \mathscr{P}^{p^{r+1}} a_{r+1} \otimes 1 + 1 \otimes \mathscr{P}^{p^{r+1}} a_{r+1} + \sum_{i=1}^{p^{r+1}-1} c_i b_r^{\alpha_r} \cdots b_\nu^{\alpha_\nu} \otimes b_r^{\beta_r} \cdots b_\nu^{\beta_\nu}, \\
& \mu^*(\mathscr{P}^i a_{r+1}) = \mathscr{P}^i a_{r+1} \otimes 1 + 1 \otimes \mathscr{P}^i a_{r+1} \quad \textit{for } i > p^{r+1}.
\end{aligned}
$$

On the other hand, we have similarly

$$
(2.12) \qquad \mu^*(b_{r+1}) = b_{r+1} \otimes 1 + 1 \otimes b_{r+1} + \sum_{i=1}^{p^{r+1}-1} c_i b_r^{\alpha_r} \cdots b_\nu^{\alpha_\nu} \otimes b_r^{\beta_r} \cdots b_\nu^{\beta_\nu}.
$$

The elements $\mathscr{P}^{p^{r+1}} a_{r+1} - b_{r+1}$ and $\mathscr{P}^i a_{r+1} (i > p^{r+1})$ are primitive by (2.11) and (2.12), and so vanished by (2.9), that is, $(\mathrm{ii})_{r+1}$ holds. Therefore the proof can be done by the induction on r.

For the case $n=1$, we choose an element a_{r+1} such as $(\Omega i)^* a_{r+1}=x_{r+1}$. Then $\mu^*(a_{r+1})$ has a form of (2.10) for some $c_i \in Z_p$. Applying Hopf algebra homomorphism $(\Omega i)^*$ and comparing with $\mu^*(x_{r+1})$, we have $c_i=\frac{1}{p^{r+1-\nu}}\binom{p^{r+1}}{i}$. The rest of the proof is similar to the case $n>1$ and omitted. q.e.d.

REMARK (i) In the case $n=1$, the relations $a_k^p=b_k (k=0, 1, \cdots)$ hold.

REMARK (ii) The following relation in $(\mathrm{ii})_1$ is essential for the proof of Theorem 0.5 (§3):

$$\mathscr{P}^p(a_1)=b_1. \tag{2.13}$$

REMARK (iii) Using Dyer-Lashof's operations ([1]) and Nishida's formula ([5]), we can determine the reduced power operations in $H^*(\Omega B_n(p); Z_p)$. Let $L=S^{2n}\cup e^{2n+2p-2}$ be the mapping cone of $\alpha_1(2n)$, $n>1$, and $Q(L)$ be the limit space $\varinjlim_N \Omega^N S^N L$. Then S^3L is a subcomplex of $B_{n+1}(p)$. Using f in (0.6), we obtain a map $\Omega B_n(p) \longrightarrow \Omega^3 S^3 L \subset Q(L)$ which induces a monomorphism of $H_*(\ ; Z_p)$. $H_*(\Omega B_n(p); Z_p) \approx Z_p[a, b]$, deg $a=2n$, deg $b=2n+2p-2$ and $a=\mathscr{P}_*^1 b$ hold, where $\mathscr{P}_*^i$ denotes the dual operation of $\mathscr{P}^i$ in the sense of [5]. Then a_k(resp. b_k) is the dual element of a^{p^k}(resp. b^{p^k}), which can be written by iterated Dyer-Lashof operations on a(resp. b). And so, applying Nishida's formula, the relations $(\mathrm{ii})_k$ are obtained.

§3 Proof of Theorem 0.5.

We shall quote the following four propositions from [8], with respect to the homomorphisms I and I' in (0.10) and (0.11).

We denote by Y^n the Moore space of type $(Z_p, n-1)$, i.e., the mapping cone of a map $S^{n-1} \longrightarrow S^{n-1}$ of degree p.

PROPOSITION 3.1 (Lemma 2.5 of [8]). *Assume that $2mp-h \geqq 6$. Then there exists a map G: $Y^{2mp-h-2} \longrightarrow \Omega^h Q_2^{2m-1}$, uniquely up to homotopy equivalence, such that G^* are isomorphisms of $H^i(\ ; Z_p)$ for $i \leqq 2mp-h-2$. For such a map the following diagram is commutative*:

$$\begin{array}{ccccc}
\pi_i(S^{2mp-h-3}; p) & \xrightarrow{i_*} & \pi_i(Y^{2mp-h-2}; p) & \xrightarrow{\pi_*} & \pi_i(S^{2mp-h-2}; p) \\
 & & \downarrow G_* & & \\
\downarrow x S^{h+2} & & \pi_i(\Omega^h Q_2^{2m-1}; p) & & \downarrow y S^{h+3} \\
 & & \approx \uparrow \Omega^h & & \\
\pi_{i+h+2}(S^{2mp-1}; p) & \xrightarrow{I'} & \pi_{i+h}(Q_2^{2m-1}; p) & \xrightarrow{I} & \pi_{i+h+3}(S^{2mp+1}; p),
\end{array} \tag{3.1}$$

for some integers x, $y \not\equiv 0 \pmod p$.

PROPOSITION 3.2 ((2.12) (ii) of [8]). *There exists a map h_p: $\Omega S^{2m+1} \longrightarrow$*

ΩS^{2mp+1} *such that h_p^* is an isomorphism of $H^{2mp}(\ ;Z_p)$ and that the following diagram is commutative:*

(3.2)
$$\begin{array}{ccccc} \pi_i(\Omega S^{2m+1};p) & & \xrightarrow{\quad h_{p*} \quad} & & \pi_i(\Omega S^{2mp+1};p) \\ \approx\uparrow\Omega & & & & \approx\uparrow\Omega \\ \pi_{i+1}(S^{2m+1};p) & \xrightarrow{H^{(2)}} & \pi_{i-2}(Q_2^{2m-1};p) & \xrightarrow{I} & \pi_{i+1}(S^{2mp+1};p). \end{array}$$

PROPOSITION 3.3 (*see* (2.1)$'$ of [8]). *There exists a map h: $Q_2^{2m-1}\longrightarrow \Omega^3 S^{2mp+1}$ such that*

(3.3) $$I=\Omega^{-3}h_*:\ \pi_i(Q_2^{2m-1};p)\longrightarrow \pi_i(\Omega^3 S^{2mp+1};p)\xleftarrow{\approx}\pi_{i+3}(S^{2mp+1};p).$$

For such a map h the following diagram is homotopy commutative:

(3.4)
$$\begin{array}{ccc} Q_2^{2m-1} & \xrightarrow{h} & \Omega^3 S^{2mp+1} \\ \uparrow G & & \uparrow i_1 \\ Y^{2mp-2} & \xrightarrow{\pi} & S^{2mp-2}, \end{array}$$

where i_1 denotes the inclusion.

PROPOSITION 3.4 ((2.6) of [8]). *The homomorphisms I and I' satisfy the following relations:*

(3.5) $$I(\alpha\circ\beta)=I\alpha\circ S^3\beta \text{ and } I'(\alpha'\circ S^2\beta)=I'\alpha'\circ\beta \quad \text{for } \beta\in\pi_j(S^i;p).$$

By Lemma 2.3 for $\Delta=\bar{\partial}_n$ and (3.5), we have

(3.6) $$I\bar{\partial}_n I'(\alpha\circ S^3\beta)=(I\bar{\partial}_n I'\alpha)\circ S^3\beta.$$

Therefore we can assume that

$$\gamma=\iota_{2(n+p)p-4} \text{ in } (0.11),$$

where $\iota_m\in\pi_m(S^m)\approx Z$ is represented by the identity map. By Proposition 3.1, we obtain the following

(3.7) $\pi_{2(n+p)p-3}(Q_2^{2n+2p-1};p)\approx Z_p$ *is generated by* $I'\iota_{2(n+p)p-1}$. *For isomorphisms* $\Omega:\pi_i(Q_2^{2n+2p-1};p)\longrightarrow\pi_{i-1}(\Omega Q_2^{2n+2p-1};p)$, $\Omega I'\iota_{2(n+p)p-1}$ *is represented by the map* Gi_0, *where i_0 denotes the inclusion* $S^{2(n+p)p-4}\subset Y^{2(n+p)p-3}$.

Let ρ_n: $\Omega S^{2n+2p-1}\longrightarrow S^{2n+1}$ be a map of (2.5) with respect to the fibering $B_n(p)\longrightarrow S^{2n+2p-1}$. Since the diagram (0.9) implies $S^2\partial_n=\partial_{n+1}S^2$, we have the following commutative diagram:

$$\begin{array}{ccc} \Omega^3 S^{2n+2p+1} & \xrightarrow{\Omega^2\rho_{n+1}} & \Omega^2 S^{2n+3} \\ \uparrow & & \uparrow \\ \Omega S^{2n+2p-1} & \xrightarrow{\rho_n} & S^{2n+1}, \end{array}$$

where vertical arrows are inclusions. Then we can define a map

$$Q_2(\rho_n):\ \Omega Q_2^{2n+2p-1}\approx\Omega(\Omega^3 S^{2n+2p+1},\ \Omega S^{2n+2p-1})\longrightarrow\Omega(\Omega^2 S^{2n+3},\ S^{2n+1})=Q_2^{2n+1},$$

which coincides with the map of (2.5) with respect to the fiberig $QB_n(p)\longrightarrow Q_2^{2n+2p-1}$. Since the homomorphism G_*: $\pi_{2(n+p)p-4}(Y^{2(n+1)p-2};p)\longrightarrow\pi_{2(n+p)p-4}(Q_2^{2n+1};p)$ is an isomorphism by (3.1) and $[Y^{2(n+p)p-3},\ Y^{2(n+1)p-2}]\longrightarrow\pi_{2(n+p)p-4}(Y^{2(n+1)p-2};p)$ is an epimorphism, we have

(3.8) *There exists a map* λ_n: $Y^{2(n+1)p+2p(p-1)-3}\longrightarrow Y^{2(n+1)p-2}$ *such that the following diagram is homotopy commutative*:

$$\begin{array}{ccc}\Omega Q_2^{2n+2p-1} & \xrightarrow{Q_2(\rho_n)} & Q_2^{2n+1}\\ \uparrow G & & \uparrow G\\ Y^{2(n+p)p-3} & \xrightarrow{\lambda_n} & Y^{2(n+1)p-2}.\end{array}$$

By (3.7), (3.8) and (3.1), $I\bar{\partial}_n I'(\iota_{2(n+p)p-1})\in\pi_{2(n+p)p-1}(S^{2(n+1)p+1};p)$ is represented by the map $S^3(\pi_0\lambda_n i_0)$ for the pinching map π_0: $Y^{2(n+1)p-2}\longrightarrow S^{2(n+1)p-2}$.

According to Toda [8], $\pi_{2m+1+2p(p-1)-2}(S^{2m+1};p)\,(m\geqq p)$ are in the stable range and isomorphic to Z_p. We put

(3.9) $\beta_1(2p+1)\in\pi_{2p+1+2p(p-1)-2}(S^{2p+1};p)\approx Z_p$ *is a generator and* $\beta_1(m)=S^{m-2p-1}\beta_1(2p+1)\in\pi_{m+2p(p-1)-2}(S^m;p)$ *for* $m\geqq 2p+1$.

PROPOSITION 3.5. *Let* f: $Y^{m+2p(p-1)-1}\longrightarrow S^m(m\geqq 2p+1)$ *be a map and let* $K=S^m\cup e^{m+2p(p-1)-1}\cup e^{m+2p(p-1)}$ *be the mapping cone of* f. *Assume that*

$$\mathscr{P}^p:\ H^m(K;Z_p)\longrightarrow H^{m+2p(p-1)}(K;Z_p)\tag{3.10}$$

is non-trivial. Then the map fi *is essential*, i.e., fi *represents* $x_m\beta_1(m)$ *for some* $x_m\not\equiv 0\,(\mathrm{mod}\ p)$, *where* i *denotes the inclusion* $S^{m+2p(p-1)-2}\subset Y^{m+2p(p-1)-1}$.

PROOF. If $fi\simeq 0$, then K has the same homotopy type as $K_1=(S^m\vee S^{m+2p(p-1)-1})\cup e^{m+2p(p-1)}$ ($\vee$denotes the one point union), and $\mathscr{P}^p\neq 0$ holds in K_1. Smashing a subcomplex $S^{m+2p(p-1)-1}$ to a point in K_1, we get a complex $K_2=S^m\cup e^{m+2p(p-1)}$ with non-trivial $\mathscr{P}^p$. This contradicts the triviality of mod p Hopf invariant. q.e.d.

REMARK. Additionaly, the converse of above Proposition 3.5 holds. And so, we can choose $\beta_1(2p+1)$ satisfying $\mathscr{P}^p(S^m)=(-1)^m e^{m+2p(p-1)}$ in (3.10) and $\beta_1(2p+1)=\{fi\}$ for $m=2p+1$. Thus, the elements $\beta_1(m)\,(m\geqq 2p+1)$ in (3.9) are determined uniquely.

The map

$$\pi_0\lambda_n i_0:\ S^{2(n+p)p-4}\longrightarrow Y^{2(n+p)p-3}\longrightarrow Y^{2(n+1)p-2}\longrightarrow S^{2(n+1)p-2}\tag{3.11}$$

represents the element $x_n\beta_1(2(n+1)p-2)$ and this coefficient $x_n\in Z_p$ coincides with one in (0.11). From Proposition 3.5, we have

(3.12) *If* $\mathscr{P}^p \neq 0$ *holds in the mapping cone of* $\pi_0\lambda_n$, *then* $x_n \not\equiv 0 \pmod p$, i.e., Theorem 0.5 *is proved.*

Now we consider the following fibering:

$$(3.13) \qquad \Omega^2 S^{2n+2p+1} \xrightarrow{\Omega\rho_{n+1}} \Omega S^{2n+3} \xrightarrow{\Omega i} \Omega B_{n+1}(p).$$

The map Ωi has an extension $\bar{i}$: $C_{\Omega\rho_{n+1}} = \Omega S^{2n+3} \cup C\Omega^2 S^{2n+2p+1} \longrightarrow \Omega B_{n+1}(p)$, where C_f denotes the mapping cone of a map f. The cohomology ring $H^*(\Omega^2 S^{2n+2p+1}; Z_p)$ is generated as Z_p-module by the elements 1, $z_0 = \sigma y_0$, z_1 and $\Delta z_1 = \sigma y_1$ for $\deg < 2(n+p)(p+1)-3$, where Δ denotes the cohomology Bockstein operation and y_i are the same as (2.7). Therefore $H^*(C_{\Omega\rho_{n+1}}; Z_p)$ is spanned by the following elements for low degrees:

$$(3.14) \qquad 1,\ \bar{x}_0,\ \dots,\ \bar{x}_0^{p-1},\ \bar{x}_1,\ \dots,\ \bar{x}_1^{p-1},\ \dots;\ \bar{z}_0 = \mathscr{P}^1\bar{x}_0,\ \bar{z}_1,\ \Delta\bar{z}_1,\ \cdots,$$

where $\bar{\gamma}$ denotes a corresponding element of γ for $\gamma \in H^*(\Omega S^{2n+3}; Z_p)$ or $\gamma \in H^*(\Omega^2 S^{2n+2p+1}; Z_p)$ and x_i are the same as (2.7). For the homomorphism $\bar{i}^*$: $H^*(\Omega B_{n+1}(p); Z_p) \longrightarrow H^*(C_{\Omega\rho_{n+1}}; Z_p)$ and the elements a_i and b_i in $H^*(\Omega B_{n+1}(p); Z_p)$, we obtain the following relations:

(3.15) $\bar{i}^*(a_0) = \bar{x}_0$, $\bar{i}^*(a_1) = \bar{x}_1$, $\bar{i}^*(b_0) = \bar{z}_0$ *and* $\bar{i}^*(b_1) = \Delta\bar{z}_1$, *up to non-zero coefficients.*

The last relation is obtained by comparing two spectral sequences associated with the fibering (3.13) and the fibering $\Omega(\Omega S^{2n+2p+1}, \Omega S^{2n+2p+1}) \longrightarrow \Omega S^{2n+2p+1}$, and others are obvious.

Applying $\bar{i}^*$ to the relation (2.13) and using (3.15), we have $\mathscr{P}^p\bar{x}_1 = \Delta\bar{z}_1$ up to non-zero coefficient. Since the map h_p^* in Proposition 3.2 for $m = n+1$ is an isomorphism of $H^{2(n+1)p}(\ ; Z_p)$, we have

(3.16) *In the mapping cone* C_g *of the map* $g = h_p\Omega\rho_{n+1}$: $\Omega^2 S^{2n+2p+1} \longrightarrow \Omega S^{2n+3} \longrightarrow \Omega S^{2(n+1)p+1}$,

$$\mathscr{P}^p: H^{2(n+1)p}(C_g; Z_p) \longrightarrow H^{2(n+p)p}(C_g; Z_p)$$

is non-trivial.

The map g is homotopic to $g'k$ for some g': $(\Omega^2 S^{2n+2p+1}, S^{2n+2p-1}) \longrightarrow (\Omega S^{2(n+1)p+1}, *)$ and inclusion k: $(\Omega^2 S^{2n+2p+1}, *) \longrightarrow (\Omega^2 S^{2n+2p+1}, S^{2n+2p-1})$. Put $g'' = \Omega^2 g'$: $\Omega Q_2^{2n+2p-1} \longrightarrow \Omega^3 S^{2(n+1)p+1}$. From definition of $H^{(2)}$ and Propositions 3.1, 3.2 and 3.3, we obtain the following

(3.17) *The map* g'' *is homotopic to* $hQ_2(\rho_n)$.

From (3.16) we have easily

(3.18) $\mathscr{P}^p$: $H^{2(n+1)p-2}(C_{g''}; Z_p) \longrightarrow H^{2(n+p)p-2}(C_{g''}; Z_p)$ *is non-trivial.*

By the maps i_1: $S^{2(n+1)p-2} \longrightarrow \Omega^3 S^{2(n+1)p+1}$ in (3.4) and G: $Y^{2(n+p)p-3} \longrightarrow \Omega Q_2^{2n+2p-1}$

in (3.8), we can define a map i': $C_{\pi_0\lambda_n} \longrightarrow C_{i_1\pi_0\lambda_n}$ and a map G': $C_{hQ_2(\rho_n)G} \longrightarrow C_{hQ_2(\rho_n)}$ such that $i'|S^{2(n+1)p-2}=i_1$ and that $G'|\Omega^3 S^{2(n+1)p+1}=$ identity of $\Omega^3 S^{2(n+1)p+1}$. Such maps i' and G' satisfy

(3.19) *i'^* and G'^* are isomorphisms of $H^{2(n+1)p-2}(\ ;Z_p)$.*

Therefore by (3.17), (3.18) and (3.19), the assumption of (3.12) is proved. Thus Theorem 0.5 is established.

§ 4 The Homotopy Groups of Spheres.

In this section, we shall quote the main results of [7], [8] and [9].

Let G_k be the k-stem group $\varinjlim_N \pi_{k+N}(S^N)$ and ${}_pG_k$ be its p-primary component. Then $G_*=\sum_k G_k$ and ${}_pG_*=\sum_k {}_pG_k$ admit a graded ring structure with respect to the composition.

THEOREM 4.1 (*see* Theorems 4.14 and 4.15 of [7], Proposition 4.18 of [7] and Theorems 15.1 and 15.2 of [8]).

(I) *For $k<2(p^2+p)(p-1)-5$, the group ${}_pG_k$ is as follows:*

(4.1) ${}_pG_k \approx Z_{p^3}$ *for $k=2p^2(p-1)-1$ (generator α'_{p^2})*

$\approx Z_{p^2}+Z_p$ *for $k=2(p^2-p)(p-1)-1$ (generators α'_{p^2-p} and $\alpha_1\beta_1^{p-1}$)*

$\approx Z_{p^2}$ *for $k=2sp(p-1)-1$ and $1\leqq s<p-1$ (generator α'_{sp})*

$\approx Z_p+Z_p$ *for $k=2(p^2+1)(p-1)-1$ (generators α_{p^2+1} and $\alpha_1\beta_1^{p-2}\beta_2$)*

$\approx Z_p$ *for $k=2r(p-1)-1$, $r\not\equiv 0 \pmod p$ and $r\neq p^2+1$ (generator α_r)*

$\approx Z_p$ *for $k=2((r+s)p+s-1)(p-1)-2(r+1)$, $r\geqq 0$ and $1\leqq s<p$ (generator $\beta_1^r\beta_s$)*

$\approx Z_p$ *for $k=2((r+s)p+s)(p-1)-2(r+1)-1$, $r\geqq 0$ and $1\leqq s<p$ except the cases $(r,s)=(p-2,1)$, $(p-1,1)$ and $(p-2,2)$ (generator $\alpha_1\beta_1^r\beta_s$)*

$\approx Z_p$ *for $k=2(p^2+1)(p-1)-3$ (generator ε')*

$\approx Z_p$ *for $k=2(p^2+i)(p-1)-2$ and $1\leqq i<p$ (generator ε_i)*

$\approx Z_p$ *for $k=2(p^2+i+1)(p-1)-3$ and $1\leqq i<p-2$ (generator $\alpha_1\varepsilon_i$)*

$=0$ *for otherwise $k<2(p^2+p)(p-1)-5$.*

(II) *Using the secondary composition, the elements* $\alpha_r (=r\alpha'_r$ *if* $r\equiv 0 \pmod p)$) *and* ε_i *are defined inductively as follows*:

(4.2) $$\alpha_{r+1} \in \{\alpha_r, p\iota, \alpha_1\} \text{ and } \varepsilon_{i+1}=\{\varepsilon_i, p\iota, \alpha_1\}.$$

And the following relations hold:

(4.3) $\alpha_1\alpha_r=\alpha_1\alpha'_{sp}=0$ *for* $r\geqq 1$, $s\geqq 1$ *and* $\beta_1\alpha_r=\beta_1\alpha'_{sp}=0$ *for* $r>1$, $s\geqq 1$.

(4.3)′ $\alpha_1\varepsilon'=0$ *for* $p>3$ *and* $\alpha_1\varepsilon'=\beta_1^4$ *for* $p=3$.

(4.3)″ $\alpha_1\varepsilon_{p-2}=0$.

(4.3)‴ $\frac{1}{r+1}\alpha_{r+1}\in\{\alpha_1, \alpha_r, p\iota\}$ *for* $r\not\equiv -1, 0 \pmod p$, $\left(p+\frac{1}{s}\right)\alpha'_{sp}\in\{\alpha_1, \alpha_{sp-1}, p\iota\}$ *for* $s<p$, $(p^2+1)\alpha'_{p^2}\in\{\alpha_1, \alpha_{p^2-1}, p\iota\}$, $\alpha_{sp+1}\in\{\alpha_1, \alpha'_{sp}, p^2\iota\}$ *for* $s<p$, $\alpha_{p^2+1}\in\{\alpha_1, \alpha'_{p^2}, p^3\iota\}$ *and* $\varepsilon_{p-1}=\{\alpha_1, \varepsilon_{p-2}, p\iota\}$.

We mention that ε' and $\alpha_1\varepsilon_i (1\leqq i<p-2)$ correspond to ε'_1 and ε'_{i+1} of [8] respectively and that the proofs of the non-triviality of $\alpha_1\varepsilon_i (1\leqq i\leqq p-3)$ and $\{\varepsilon_i, p\iota, \alpha_1\}$ $(1\leqq i\leqq p-2)$ and the relations (4.3)‴, $\deg\geqq 2p^2(p-1)-3$, are not given in [7] and [8]. But we can prove those by the similar methods in [7] with simple calculations of exact sequences in Steenrod algebra. Details may appear elsewhere.

According to Toda [8], there are elements

(4.4) $\alpha_r(3)\in\pi_{3+2r(p-1)-1}(S^3; p)$ *of order* p, $S^\infty\alpha_r(3)=\alpha_r$ *for* $r\geqq 1$,

$\alpha'_{sp}(5)\in\pi_{5+2sp(p-1)-1}(S^5; p)$ *of order* p^2, $S^\infty\alpha'_{sp}(5)=\alpha'_{sp}$ *for* $1\leqq s<p$,

$\alpha'_{p^2}(7)\in\pi_{7+2p^2(p-1)-1}(S^7; p)$ *of order* p^3, $S^\infty\alpha'_{p^2}(7)=\alpha'_{p^2}$,

$\beta_1(2p-1)\in\pi_{2p-1+2p(p-1)-2}(S^{2p-1}; p)$ *of order* p^2, $S^2\beta_1(2p-1)=\beta_1(2p+1)$ *in* (3.9) *and* $S^\infty\beta_1(2p-1)=\beta_1$,

$\beta_s(2p+3)\in\pi_{2p+3+2(sp+s-1)(p-1)-2}(S^{2p+3}; p)$ *of order* p, $S^\infty\beta_s(2p+3)=\beta_s$ *for* $1<s<p$,

$\alpha_1\beta_s(5)\in\pi_{5+2(sp+s)(p-1)-3}(S^5; p)$ *of order* p, $S^2\alpha_1\beta_s(5)=\alpha_1(7)\circ S\beta_s(2p+3)$ *for* $1<s<p$,

$\varepsilon'(2p(p-2)+1)\in\pi_{2p(p-2)+1+2(p^2+1)(p-1)-3}(S^{2p(p-2)+1}; p)$ *of order* p, $S^\infty\varepsilon'(2p(p-2)+1)=\varepsilon'$,

$\varepsilon_i(2p(p-i)+3)\in\pi_{2p(p-i)+3+2(p^2+i)(p-1)-2}(S^{2p(p-i)+3}; p)$ *of order* p, $S^\infty\varepsilon_i(2p(p-i)+3)=\varepsilon_i$ *for* $1\leqq i\leqq p-2$ *and for* $p>3$, $i=p-1$,

$\varepsilon_2(11)\in\pi_{11+42}(S^{11}; 3)$ *of order* 3, $S^\infty\varepsilon_2(11)=\varepsilon_2$ *for* $p=3$,

$\alpha_1\varepsilon_i(2p(p-i-2)+1)\in\pi_{2p(p-i-2)+1+2(p^2+i+1)(p-1)-3}(S^{2p(p-i-2)+1};p)$ *of order* p, $S^{2p+6}\alpha_1\varepsilon_i(2p(p-i-2)+1)=\alpha_1(2p(p-i-1)+7)\circ S\varepsilon_i(2p(p-i)+3)$ *for* $1\leqq i<p-2$.

Here all above elements are not in the S^2-image.

We define the elements in $\pi_{i+m}(S^m;p)$ for suitable i as follows:

(4.5) $\alpha_r(m)=S^{m-3}\alpha_r(3)\,(m\geqq 3)$, $\alpha'_{sp}(m)=S^{m-5}\alpha'_{sp}(5)\,(m\geqq 5,\ 1\leqq s<p)$, $\alpha'_{p^2}(m)=S^{m-7}\alpha'_{p^2}(7)\,(m\geqq 7)$, $\beta_1^r(m)=\beta_1^{r-1}(m)\circ\beta_1(m+2(r-1)p(p-1)-2(r-1))$ $(m\geqq 2p-1)$, $\beta_s(m)=S^{m-2p-3}\beta_s(2p+3)\,(m\geqq 2p+3, 1<s<p)$, $\beta_1^r\beta_s(m)=\beta_1^r(m)\circ\beta_s(m+2rp(p-1)-2r)\,(m\geqq 2p-1,\ 1<s<p)$, $\alpha_1\beta_1^r(m)=\alpha_1(m)\circ\beta_1^r(m+2p-3)$ $(m\geqq 3)$, $\alpha_1\beta_1^r\beta_s(m)=\alpha_1\beta_1^r(m)\circ\beta_s(m+2(rp+1)(p-1)-2r-1)\,(m\geqq 3,\ r>1)$, $\alpha_1\beta_s(m)=S^{m-5}\alpha_1\beta_s(5)\,(m\geqq 5,\ 1<s<p)$,……*etc.*

In addition, we shall use the following notations:

(4.6) (i) *For* $\gamma\in S^\infty\pi_{i+2}(S^{2mp-1};p)=\mathrm{Im}\ S^\infty\cap{}_pG_{i-2mp+3}$, $Q^m(\gamma)\in\pi_i(Q_2^{2m-1};p)$ *denotes an element such that* $Q^m(\gamma)=I'\gamma(2mp-1)$ *and* $S^\infty\gamma(2mp-1)=\gamma$ *for some* $\gamma(2mp-1)\in\pi_{i+2}(S^{2mp-1};p)$.

(ii) *For* $\gamma\in{}_pG_{i-2mp+2}$, $\bar{Q}^m(\gamma)\in\pi_i(Q_2^{2m-1};p)$ *denotes an element (if it exists) such that* $S^\infty I(\bar{Q}^m(\gamma))=\gamma$.

THEOREM 4.2 (Theorems 11.1, 15.1 and 15.2 in [8]).

(I) *For* $m\geqq 1$ *and* $k<2(p^2+p)(p-1)-5$, *we have the following direct sum decomposition*:

$$\pi_{2m+1+k}(S^{2m+1};p)=A(m,k)+B(m,k)+E(m,k)+\sum_{t=1}^{4}U_t(m,k).$$

(4.7) $A(m,k)$ *is defined as follows*:

$A(m,2p^2(p-1)-1)\approx Z_{p^3}$ *generated by* $\alpha'_{p^2}(2m+1)$ *for* $m\geqq 3$.

$A(2,2p^2(p-1)-1)\approx Z_{p^2}$ *generated (formally) by* $p\,\alpha'_{p^2}(5)$ *(in this case, the element* $\alpha_{p^2}(5)$ *exists and is divisible by* p, *but not divisible by* p^2, *and an element* $\alpha'_{p^2}(5)$ *such that* $p^2\alpha'_{p^2}(5)=\alpha_{p^2}(5)$ *does not exist).*

$A(m,2sp(p-1)-1)\approx Z_{p^2}$ *generated by* $\alpha'_{sp}(2m+1)$ *for* $m\geqq 2$, $1\leqq s<p$.

$A(m,2r(p-1)-1)\approx Z_p$ *generated by* $\alpha_r(2m+1)$ *for* $m=1$ *and for* $r\not\equiv 0$ (mod p).

$A(m,k)=0$ *for* $k\not\equiv -1\,(\mathrm{mod}\ 2p-2)$.

(4.8) $B(m,k)$ *is defined as follows*:

$B(m,2((r+s)p+s-1)(p-1)-2(r+1))\approx Z_p$ *generated by* $\beta_1^r\beta_s(2m+1)$ *for* $m\geqq p-1$ *if* $r\geqq 1$ *and* $s\geqq 1$, *for* $m\geqq p$ *if* $r=0$ *and* $s=1$, *for* $m\geqq p+1$ *if* $r=0$

and $s\geqq 2$, *and for* $m=1$ *if* $(p, r, s)=(3, 3, 1)$.

$B(m, 2((r+s)p+s)(p-1)-2(r+1)-1)\approx Z_p$ *generated by* $\alpha_1\beta_1^r\beta_s(2m+1)$ *for* $m\geqq 1$ *if* $r\geqq 1$ *or* $s=1$, *and for* $m\geqq 2$ *if* $r=0$ *and* $s\geqq 2$, *except the case* $(r, s)=(p-1, 1)$.

$B(m, 2p^2(p-1)-3)\approx Z_p$ *generated by* $\alpha_1\beta_1^p(2m+1)$ *for* $1\leqq m<p^2-p$ (*remark that, for* $p^2-p\leqq m\leqq p^2-3$, $\alpha_1\beta_1^p(2m+1)$ *is non-vanishing and divisible by* p, *and that* $\alpha_1\beta_1^p(2p^2-3)=0$ (*see* (4.17) (iii))).

$B(m, k)=0$ *for the other cases.*

(4.9) $E(m, k)$ *is defined as follows*:

$E(m, 2(p^2+1)(p-1)-3)\approx Z_p$ *generated by* $\varepsilon'(2m+1)$ *for* $m\geqq p(p-2)$.

$E(m, 2(p^2+i)(p-1)-2)\approx Z_p$ *generated by* $\varepsilon_i(2m+1)$ *for* $m\geqq p(p-i)+1$ *and* $1\leqq i<p$, *except the case* $(p, i, m)=(3, 2, 4)$.

$E(m, 2(p^2+i+1)(p-1)-3)\approx Z_p$ *generated by* $\alpha_1\varepsilon_i(2m+1)$ *for* $m\geqq p(p-i-2)$ *and* $1\leqq i<p-2$.

$E(m, k)=0$ *for the other cases.*

(4.10) $U_1(m, k)$ *is defined as follows*:

(i) $U_1(m, 2(p^2-p+m)(p-1)-2)\approx Z_p+Z_p$ *generated by* $p_*\bar{Q}^{m+1}(\alpha_{p^2-p-1})$ *and* $p_*Q^{m+1}(\beta_1^{p-1})$ *for* $1\leqq m<2p-1$ *and* $m\neq p-1, p$.

(ii) $U_1(m, 2(p^2+m+1)(p-1)-2)\approx Z_p+Z_p$ *generated by* $p_*\bar{Q}^{m+1}(\alpha_{p^2})$ *and* $p_*Q^{m+1}(\beta_1^{p-2}\beta_2)$ *for* $1\leqq m<p-1$.

(iii) $U_1(m, 2r(p-1)-2)\approx Z_p$ *generated by* $p_*\bar{Q}^{m+1}(\alpha_{r-m-1})$ (*by* $p_*Q^{m+1}(\iota)$ *if* $m=r-1$) *for* $1\leqq m<r$, $r\not\equiv 0 \pmod p$ *and* $r-m\neq p^2-p, p^2+1$.

(iv)*) $U_1(m, 2((r+s)p+s+m)(p-1)-2(r+2))\approx Z_p$ *generated by* $p_*Q^{m+1}(\beta_1^r\beta_s)$ *for* $m\not\equiv -1 \pmod p$, $r\geqq 0$, $1\leqq s<p$, $(r, s)\neq(p-2, 1), (p-2, 2)$ *and for* $(m, r, s)=(p, p-2, 1)$.

(v)*) $U_1(m, 2((r+s)p+s+m)(p-1)-2(r+1)-1)\approx Z_p$ *generated by* $p_*\bar{Q}^{m+1}(\beta_1^r\beta_s)$ *for* $m\not\equiv 0 \pmod p$, $r\geqq 0$ *and* $1\leqq s<p$.

(vi) $U_1(3, 41)\approx Z_3$ *generated by* $p_*\bar{Q}^4(\beta_2)$ *for* $p=3$.

(vii) $U_1(m, 2(tp+t)(p-1)-4)\approx Z_p$ *for* $2\leqq m<t<p$.

(viii) $U_1(m, k)=0$ *for the other cases.*

Remark that any element γ of $U_1(m, k)$ is characterized by the relations $S^2\gamma$

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*) In the third and fourth cases of (11.9) in [8], the cases $m=1$, $r=0$, $s\geqq 2$ should not be excluded.

$=0$ and $\gamma \notin \operatorname{Im} S^2$.

(4.11) *$U_2(m, k)$ is defined as follows*:

$U_2(m, 2p^2(p-1)-2) \approx Z_{p^3}$ *generated by an element* $\gamma_p(2m+1)$ *for* $3 \leqq m < p^2-2$.

$U_2(m, 2sp(p-1)-2) \approx Z_{p^2}$ *generated by an element* $\gamma_s(2m+1)$ $(\gamma_p(2p^2-3) = S^2\gamma_p(2p^2-5))$ *for* $2 \leqq m < sp-1$ *and for* $m=p-1$, $s=1$ $(\gamma_1(2p-1)=\beta_1(2p-1))$ *except the case* $3 \leqq m < p^2-2$, $s=p$.

$U_2(1, 2sp(p-1)-2) \approx Z_p$ *generated by an element* $\gamma_s(3)$.

$U_2(sp-1, 2sp(p-1)-2) \approx Z_p$ *generated by* $S^2\gamma_s(2sp-3)$, $s \geqq 2$.

$U_2(m, k)=0$ *for the other cases.*

(4.12) *$U_3(m, k)$ is defined as follows*:

$U_3(lp+j, 2((r+s+l)p+s-1)(p-1)-2(r+1)-1) \approx Z_p$ *generated by an element* $S^{2j}u_3(l, \beta_1^r\beta_s)$ *for* $r \geqq 0$, $s \geqq 1$, $l \geqq 1$, $0 \leqq j \leqq p-2$ *except the case* $r=0$, $s \geqq 2$, *the case* $l=p-1$, $r=0$, $s=1$, $j<p-2$ *and the case* $p=3$, $l=1$, $r=2$, $s=1$.

$U_3((p-1)p+j, 2p^2(p-1)-3) \approx Z_{p^2}$ *generated by an element* $S^{2j}u_3(p-1, \beta_1)$ *for* $0 \leqq j \leqq p-3$ *(the element* $S^{2p-4}u_3(p-1, \beta_1)$ *is of order* p*).*

$U_3(lp+j+1, 2((r+s+l)p+s)(p-1)-2(r+1)) \approx Z_p$ *generated by an element* $S^{2j}\bar{u}_3(l, \beta_1^r\beta_s)$ *for* $r \geqq 1$, $s \geqq 1$, $l \geqq 0$ *and* $0 \leqq j \leqq p-2$.

$U_3(m, k)=0$ *for the other cases.*

(4.13) *$U_4(m, k)$ is defined as follows*:

$U_4(lp+j, 2((s+l)p+s-1)(p-1)-3) \approx Z_p$ *generated by an element* $S^{2j}u_4(l, \beta_s)$ *for* $l \geqq 1$, $s \geqq 2$, $s+l<p$ *and* $0 \leqq j \leqq p$.

$U_4(m, k)=0$ *for the other cases.*

(II) *For the elements in* $E(m, k)$ *and* $U_t(m, k)$ $t=1, 2, 3, 4$, *we have the following relations up to non-zero coefficients*:

(4.14) $H^{(2)}\varepsilon'(2p(p-2)+1)=Q^{p(p-2)}(\beta_2)$ *and*

$$H^{(2)}\alpha_1\varepsilon_i(2p(p-i-2)+1)=Q^{p(p-i-2)}(\beta_{i+2}).$$

(4.15)

(i) $H^{(2)}p_*\bar{Q}^{m+1}(\alpha_j)=Q^m(\alpha'_{j+1})$ *and* $H^{(2)}p_*Q^{m+1}(\iota)=Q^m(\alpha_1)$.

(ii) $H^{(2)}p_*Q^{m+1}(\beta_1^r\beta_s)=Q^m(\alpha_1\beta_1^r\beta_s)$, $H^{(2)}p_*\bar{Q}^{m+1}(\beta_1^r\beta_s)=\bar{Q}^m(\alpha_1\beta_1^r\beta_s)$ *and* $H^{(2)}p_*\bar{Q}^4(\beta_2)=Q^3(\beta_1^3)$ *for* $p=3$.

(iii) $p_*\bar{Q}^2(\alpha_{sp-1})=\alpha_1(3)\circ\alpha'_{sp}(2p)$ *except the case* $p=s=3$, $p_*\bar{Q}^2(\alpha_8)=\alpha_2(3)\circ\alpha_8(10)$ *for* $p=3$ (*in this case,* $\alpha'_9(6)$ *does not exist*) *and* $p_*\bar{Q}^2(\alpha_r)=\alpha_1(3)\circ\alpha_{r+1}(2p)$ *for* $r\not\equiv -1, -2 \pmod p$.

(iv) $p_*Q^2(\beta_1^r\beta_s)=\alpha_1(3)\circ\alpha_1\beta_1^r\beta_s(2p)$.

(4.16)

(i) $S^2\gamma_s(2m+1)=p\gamma_s(2m+3)$ *for* $1\leqq m<sp-3$, *for* $m=sp-3$, $s<p$ *and for* $m=p-2$, $s=1$.

(ii) $H^{(2)}\gamma_s(2m+1)=Q^m(\alpha'_{sp-m})$ *for* $1\leqq m<sp-2$, *for* $m=sp-2$, $s<p$ *and for* $m=p-1$, $s=1$.

(iii) $p_*\bar{Q}^{m+1}(\alpha_{sp-m-1})=p\gamma_s(2m+1)\,(=0$ *if* $m=1)$ *for* $1\leqq m<sp-1$, $s<p$,

$p_*Q^{m+1}(\iota)=S^2\gamma_s(2sp-3)$ *for* $m=sp-1$,

$p_*\bar{Q}^{m+1}(\alpha_{p^2-m-1})=p^2\gamma_p(2m+1)\,(=0$ *if* $m=1, 2)$ *for* $1\leqq m<p^2-2$,

$p_*\bar{Q}^{m+1}(\alpha_1)=S^2\gamma_p(2p^2-5)=\gamma_p(2p^2-3)$ *for* $m=p^2-2$.

(iv) $\gamma_s(3)=\alpha_1(3)\circ\alpha_{sp-1}(2p)$, $p\gamma_s(5)=\alpha_1(5)\circ\alpha_{sp-1}(2p+2)$ *and* $p^2\gamma_p(7)=\alpha_1(7)\circ\alpha_{p^2-1}(2p+4)$.

Note that, in the above two cases (4.15) and (4.16), $\alpha'_r=\alpha_r$ for $r\not\equiv 0 \pmod p$.

(4.17)

(i) $H^{(2)}u_3(l, \beta_1^r\beta_s)=Q^{lp}(\beta_1^r\beta_s)$ *and* $H^{(2)}\bar{u}_3(l, \beta_1^r\beta_s)=\bar{Q}^{lp+1}(\beta_1^r\beta_s)$.

(ii) $S^{2p-4}u_3(l, \beta_1^r\beta_s)=p_*Q^{lp+p-1}(\alpha_1\beta_1^{r-1}\beta_s)\,(=p_*Q^{lp+p-1}(\alpha_1)$ *if* $r=0$, $s=1)$, $S^{2p-4}\bar{u}_3(l, \beta_1^r\beta_s)=p_*\bar{Q}^{lp+p}(\alpha_1\beta_1^{r-1}\beta_s)$ *and* $S^{2p-2}u_3(l, \beta_1^r\beta_s)=S^{2p-2}\bar{u}_3(l, \beta_1^r\beta_s)=0$.

(iii) $pS^{2j}u_3(p-1, \beta_1)=\alpha_1\beta_1^p(2(p-1)p+2j+1)$ *for* $0\leqq j\leqq p-3$ *and* $pS^{2p-6}u_3(p-1, \beta_1)=\alpha_1\beta_1^p(2p^2-5)=p_*Q^{p^2-2}(\alpha_2)$.

(4.18) $H^{(2)}u_4(l, \beta_s)=Q^{lp}(\beta_s)$, $S^{2p}u_4(l, \beta_s)=p_*\bar{Q}^{lp+p+1}(\beta_{s-1})$ *and* $S^{2p+2}u_4(l, \beta_s)=0$.

Now let π_k be the limit group $\varinjlim_N [Y^{k+N}, Y^N]$ and π_* the direct sum $\sum_k \pi_k$. Then π_* admits a ring structure with respect to the composition. Moreover, π_* admits an algebra structure over Z_p, since $\iota=\{\text{identity map}\}$ generates $\pi_0\approx Z_p$. π_* can be computed from the results on ${}_pG_*$ by the following isomorphism:

$$\pi_k\approx G_{k+1}\otimes Z_p+G_k\otimes Z_p+\mathrm{Tor}(G_k, Z_p)+\mathrm{Tor}(G_{k-1}, Z_p).$$

Let $\delta\in\pi_{-1}$ be the class represented by the map $i\pi$; $Y^{N-1}\longrightarrow S^{N-1}\longrightarrow Y^N$. Concerning the map $i^*\pi_*: \pi_k\longrightarrow {}_pG_{k-1}$, we have (*see* Yamamoto [9])

(4.19) *There are elements* α *and* $\beta_{(s)}$ *in* π_* *uniquely, satisfying the following conditions*:

$i^*\pi_*(\alpha)=\alpha_1$, $i^*\pi_*(\beta_{(s)})=\beta_s(1\leqq s<p)$, $\alpha\beta_{(s)}=\beta_{(s)}\alpha=0\,(1\leqq s\leqq p-1)$ *and* $\beta_{(s)}\in\{\beta_{(s-1)},\alpha,\beta_{(1)}\}$.

THEOREM 4.3 (Theorem II of [9]). *The ring* π_*, *in* $\dim<2p^2(p-1)-4$, *has multiplicative generators* $\delta\in\pi_{-1}$, $\iota\in\pi_0$, $\alpha\in\pi_{2p-2}$ *and* $\beta_{(s)}\in\pi_{2(sp+s-1)(p-1)-1}$ $(1\leqq s<p)$. *These elements satisfy the following fundamental relations*:

(4.20)

(i) $\delta^2=0$ *and* $2\alpha\delta\alpha=\alpha^2\delta+\delta\alpha^2$.

(ii) $\alpha\beta_{(s)}=\beta_{(s)}\alpha=0$ *and* $\alpha\delta\beta_{(s)}=\beta_{(s)}\delta\alpha$ *for* $s<p-1$ *and for* $p>3$, $s=p-1$. *For* $p=3$, $s=p-1=2$, $\alpha\beta_{(2)}=-\beta_{(2)}\alpha=\pm(\beta_{(1)}\delta)^2\beta_{(1)}$ *and* $\alpha\delta\beta_{(2)}\equiv\beta_{(2)}\delta\alpha$ *modulo the elements* $(\delta\beta_{(1)})^3$ *and* $(\beta_{(1)}\delta)^3$.

(iii) $\beta_{(s)}\beta_{(t)}=0$ *for* $p>3$, $s+t<p$ *and* $\beta_{(s)}\delta\beta_{(t)}=\dfrac{st}{s+t-1}\,\beta_{(1)}\delta\beta_{(s+t-1)}$ *for* $s+t-1<p$. *For* $p=3$, $\beta_{(1)}\beta_{(1)}\equiv0$ *modulo the element* $\delta\alpha\delta\,(\beta_{(1)}\delta)^2$.

REMARK (i) Strictly speaking, in the case $p=3$, $s=2$ of the third relation of (4.19), the equality should be understood modulo $(\beta_{(1)}\delta)^2\beta_{(1)}$.

REMARK (ii) The relation (4.19) (i) implies

$$\alpha^s\delta\alpha^t=t\alpha^{s+t-1}\delta\alpha+(1-t)\alpha^{s+t}\delta \text{ and } \alpha^s\delta\alpha^t\delta=\delta\alpha^t\delta\alpha^s=t\alpha^{s+t-1}\delta\alpha\delta.$$

REMARK (iii) By the map $i^*\pi_*:\pi_k\longrightarrow{}_pG_{k-1}$, we have

(4.21) $i^*\pi_*(\alpha^r)=\alpha_r$, $i^*\pi_*((\beta_{(1)}\delta)^r\beta_{(s)})=\beta_1^r\beta_s$ *and* $i^*\pi_*(\alpha\delta(\beta_{(1)}\delta)^r\beta_{(s)})=\alpha_1\beta_1^r\beta_s$.

REMARK (iv) Since $\pi_{2p(p-1)-1}$ has a Z_p-basis $\{\beta_{(1)},\alpha^p\delta,\alpha^{p-1}\delta\alpha\}$, the class of λ_n in (3.8) is described as follows:

(4.22) $\{\lambda_n\}=x_n\beta_{(1)}+y_n\alpha^p\delta+z_n\alpha^{p-1}\delta\alpha$ *for some* $x_n, y_n, z_n\in Z_p$ *and* $x_n\not\equiv0$ *by* Theorem 0.5.

In addition, we shall use the following two results (*see* §4, §6 of [8] and [9]):

(4.23) *Let* $\gamma\in{}_pG_{k-1}=\pi_{N+k-1}(S^N;p)$ *be of order* p. *Then there is an element* $\gamma'\in\pi_k$ *such that* $i^*\pi_*(\gamma')=\gamma$. *Furthermore the element* $\delta\gamma'\in\pi_{k-1}=[Y^{N+k},Y^{N+1}]$ *is an extension of* $i_*(\gamma)\in\pi_{N+k-1}(Y^{N+1};p)$.

(4.24) *The element* $\alpha^{k-1}\delta\alpha$ *is an extension of* $i_*(\alpha'_k)$ *up to non-zero coefficient*, i.e., $i^*(\alpha^{k-1}\delta\alpha)=xi_*(\alpha'_k)$ *for some* $x\not\equiv0\,(\mathrm{mod}\,p)$, *where* $\alpha'_k=\alpha_k$ *for* $k\not\equiv0\,(\mathrm{mod}\,p)$.

§ 5 Determination of ∂_n.

In this section, we always assume that $k<2(p^2+p)(p-1)-5$ and $m=n+p-1$.

We shall determine the boundary homomorphism

$$(5.1)_k \quad \partial_n: \pi_{2m+1+k-(2p-3)}(S^{2m+1}; p) \longrightarrow \pi_{2n+1+k}(S^{2n+1}; p)$$

in the homotopy exact sequence of the bundle $B_n(p)$, using mainly Theorems 0.3, 0.4 and 0.5 as follows:

$$(5.2) \qquad \partial_n(S^2\gamma)=\alpha_1(2n+1)\circ S\gamma \quad \textit{for any } \gamma \in \pi_{2m-1+k-(2p-3)}(S^{2m-1}; p).$$

$$(5.3) \qquad S^2\partial_n=\partial_{n+1}S^2.$$

$$(5.4) \qquad H^{(2)}\partial_{n+1}=\bar{\partial}_n H^{(2)}.$$

$$(5.5) \qquad p_*\bar{\partial}_n=\partial_n p_*.$$

$$(5.6) \qquad I\bar{\partial}_n I'(S^3\gamma)=x_n\beta_1(2(n+1)p+1)\circ S^3\gamma, \quad x_n \not\equiv 0 \ (\mathrm{mod}\ p),$$
$$\textit{for any } \gamma \in \pi_{2n+k}(S^{2(n+p)p-4}; p).$$

By Theorem 4.2, we have

$$(5.7)_k \quad \pi_{2m+1+k-2p+3}(S^{2m+1}; p)$$
$$=A(m, k-2p+3)+B(m, k-2p+3)+E(m, k-2p+3)$$
$$+\sum_{t=1}^{4} U_t(m, k-2p+3).$$

Note that the elements in (4.10) (vii) do not appear in $(5.7)_k$ since $m \geqq p$.

Using properties (5.2) and (5.3) of ∂_n and relations (4.3), (4.3)$'$, (4.3)$''$, (4.15) (iii), (4.15) (iv) and (4.16) (iv), we have easily

Proposition 5.1. *For the stable elements, ∂_n of $(5.1)_k$ satisfies the followings up to non-zero coefficients:*

(i) $\partial_n(\iota_{2m+1})=\alpha_1(2n+1)$.

(ii) $\partial_n(\alpha_r(2m+1))=p_*\bar{Q}^2(\alpha_{r-1})$ *for* $n=1$, $r \not\equiv 0, -1 \ (\mathrm{mod}\ p)$,

$=\gamma_s(3)$ *for* $n=1$, $r=sp-1$,

$=p\gamma_s(5)$ *for* $n=2$, $r=sp-1$,

$=p^2\gamma_p(7)$ *for* $n=3$, $r=p^2-1$,

$=0$ *for the other cases.*

(iii) $\partial_n(\alpha'_{sp}(2m+1))=p_*\bar{Q}^2(\alpha_{sp-1})$ *for* $n=1$ *except the case* $p=s=3$,

$$n=1,$$
$$=p_*\bar{Q}^2(\alpha_8) \text{ or } 0 \quad \text{for } p=s=3,\ n=1,$$
$$=0 \quad \text{for the other cases.}$$

(iv) $\partial_n(\beta_1^r\beta_s(2m+1))=\alpha_1\beta_1^r\beta_s(2n+1)$ *except the case* $r=p-1,\ s=1,\ n>p^2-3,$

$$=0 \quad \text{for } r=p-1,\ s=1,\ n>p^2-3.$$

(v) $\partial_n(\alpha_1\beta_1^r\beta_s(2m+1))=p_*Q^2(\beta_1^r\beta_s)$ *for* $n=1,$

$$=0 \quad \text{for } n>1.$$

(vi) $\partial_n(\varepsilon'(2m+1))=\beta_1^4(2n+1)$ *for* $p=3,\ m\geqq p(p-2)=3,$

$$=0 \quad \text{for } p>3,\ m\geqq p(p-1)-1.$$

(vii) $\partial_n(\varepsilon_i(2m+1))=\alpha_1\varepsilon_i(2n+1)$ *for* $1\leqq i\leqq p-3,$

$$=0 \quad \text{for } i=p-2.$$

(viii) $\partial_n(\alpha_1\varepsilon_i(2m+1))=0$ *for* $m\geqq p(p-i-1)-1.$

For the case $p>3$, $p(p-2)\leqq m\leqq p(p-1)-2$ of (vi) and the case $p(p-i-2)\leqq m\leqq p(p-i-1)-2$ of (viii), we shall discuss in Proposition 5.2.

PROOF. First, we consider (i), (ii), (iii), (iv) (except $r=0$, $s\geqq 2$, $m=p+1(n=2)$) and (v). Since any element γ which is mapped by ∂_n is in the S^2-image, $\partial_n(\gamma)=\alpha_1(2n+1)\circ\gamma'(S\gamma'=\gamma)$ holds by (5.2). So, the above results follow from (4.3), (4.15) (iii), (4.15) (iv) and (4.16) (iv).

Second, we consider the case $r=0$, $s\geqq 2$, $m=p+1$ of (iv). By (5.3), $S^2\partial_2(\beta_s(2p+3))=\partial_3(\beta_s(2p+5))=\alpha_1\beta_s(7)=S^2\alpha_1\beta_s(5)$. And S^2: $\pi_{5+j}(S^5;p)\longrightarrow\pi_{7+j}(S^7;p)(j=2(sp+s)(p-1)-3)$ is monomorphic by Theorem 4.2. So the above result follows. Similarly, the cases $p=3$ of (vi) and (vii) are proved.

Finally, the triviality of $\partial_n(\gamma)$ for $\gamma=\varepsilon'(2m+1)(p>3)$, $\varepsilon_{p-2}(2m+1)$ or $\alpha_1\varepsilon_i(2m+1)$ is obtained from the triviality of the homotopy groups containing $\partial_n(\gamma)$. q.e.d.

PROPOSITION 5.2. *Up to non-zero coefficients, ∂_n of $(5.1)_k$ satisfies the followings:*

(i) $\partial_n(\varepsilon'(2m+1))=S^{2j}\bar{u}_3(p-3,\ \beta_1\beta_2)$ *for* $m=p(p-2)+j,\ 0\leqq j\leqq p-2,\ p>3.$

$\partial_n(\alpha_1\varepsilon_i(2m+1))=S^{2j}\bar{u}_3(p-i-3,\ \beta_1\beta_{i+2})$ *for* $m=p(p-i-2)+j,\ 0\leqq j\leqq p-2.$

(ii) $\partial_n(S^{2j}u_3(l,\ \beta_1^r\beta_s))=S^{2j}\bar{u}_3(l-1,\ \beta_1^{r+1}\beta_s)$ *for* $m=lp+j,\ 0\leqq j\leqq p-2.$

(iii) $\partial_n(S^{2j}\bar{u}_3(l,\ \beta_1^r\beta_s))=0$ *for* $m=lp+1+j,\ 0\leqq j\leqq p-2.$

(iv) $\partial_n(S^{2j}u_4(l,\ \beta_s))=S^{2j}\bar{u}_3(l-1,\ \beta_1\beta_s)$ *for* $m=lp+j,\ 0\leqq j\leqq p-2,$

$$=0 \quad \textit{for } m=lp+j,\ j=p-1,\ p.$$

Proof. By (4.14), (5.4) and (5.6), we obtain

$$IH^{(2)}\partial_n(\varepsilon_i')=I\bar{\partial}_{n-1}H^{(2)}\varepsilon_i'=I\bar{\partial}_{n-1}Q^{p(p-i-1)}(\beta_{i+1})$$
$$=I\bar{\partial}_{n-1}I'\beta_{i+1}(2p^2(p-i-1)-1)=\beta_1\beta_{i+1}(2np+1),$$

where $\varepsilon_1'=\varepsilon'(2p(p-2)+1)$, $\varepsilon_i'=\alpha_1\varepsilon_{i-1}(2p(p-i-1)+1)(2\leqq i\leqq p-2)$ and $n=p(p-i-2)+1$. Similarly, we have

$$IH^{(2)}\bar{u}_3(p-i-2,\ \beta_1\beta_{i+1})=\beta_1\beta_{i+1}(2np+1).$$

Since $IH^{(2)}\colon \pi_{2n+1+k}(S^{2n+1};p)\longrightarrow\pi_{2n+1+k}(S^{2np+1};p)\ (n=p(p-i-2)+1,\ k=2(p^2+i+1)(p-1)-4)$ is isomorphic by Theorem 4.2, $\partial_n(\varepsilon_i')=\bar{u}_3(p-i-2,\ \beta_1\beta_{i+1})$ holds. Thus, (i) is proved.

By (5.6), we have

$$I\bar{\partial}_nQ^{lp+p-1}(\alpha_1\beta_1^r\beta_s)=I\bar{Q}^{lp}(\alpha_1\beta_1^{r+1}\beta_s),\ n=lp-1,$$

and by Theorem 4.2, we can verify the triviality of the kernel of

$$I\colon \pi_{2n+1+k}(Q_2^{2n+1};p)\longrightarrow\pi_{2n+4+k}(S^{2(n+1)p+1};p),$$
$$n=lp-1,\ k=2((r+s+l)p+s)(p-1)-2(r+2).$$

So, $\bar{\partial}_nQ^{lp+p-1}(\alpha_1\beta_1^r\beta_s)=\bar{Q}^{lp}(\alpha_1\beta_1^{r+1}\beta_s)$. Applying p_* to this and using (4.17) (ii) and (5.5), we get $\partial_n(S^{2p-4}u_3(l,\ \beta_1^r\beta_s))=S^{2p-4}\bar{u}_3(l-1,\ \beta_1^{r+1}\beta_s)$. The kernel of the $(2p-4-2j)$-fold iterated suspension into $\pi_{2lp-1+k}(S^{2lp-1};p)\ (0\leqq j\leqq p-2)$ is trivial by Theorem 4.2, provided that $(r,\ s,\ l)\neq(p-2,\ 1,\ 1)$. Thus (ii) $((r,\ s,\ l)\neq(p-2,\ 1,\ 1))$ is proved. By (3.5) and the relation

$$H^{(2)}(\alpha\circ S^3\beta)=H^{(2)}\alpha\circ\beta \quad \textit{for } \alpha\in\pi_{i+2}(S^{u+2}),\ \beta\in\pi_{j-1}(S^{i-1}),$$

we can choose the elements in $U_3(m,\ k)$ as follows:

$$u_3(l,\ \beta_1^r\beta_s)=u_3(l,\ \beta_1^{r-1}\beta_s)\circ\beta_1(2lp+1+2((r+s+l-1)p+s-1)(p-1)-2r-1) \quad \textit{for } r\geqq 1,$$

$$\bar{u}_3(l,\ \beta_1^r\beta_s)=\bar{u}_3(l,\ \beta_1^{r-1}\beta_s)\circ\beta_1(2lp+3+2((r+s+l-1)p+s)(p-1)-2r) \quad \textit{for } r\geqq 2.$$

Then the case $(r, s, l)=(p-2, 1, 1)$ of (ii) is obtained from Lemma 2.3 for $\Delta=\partial_n$.

The element $S^{2j}\bar{u}_3(l, \beta_1^r\beta_s)$ is contained in the group $\pi_{2n+1+k}(S^{2n+1}; p)\cap \mathrm{Ker}\, S^{2p-2}$, $n=(l-1)p+j+2$, $k=2((r+s+l)p+s+1)(p-1)-2(r+1)-1$, and this group vanishes by Theorem 4.2. So, (iii) is proved.

The relation (iv) is similar to (i). q.e.d.

PROPOSITION 5.3. *Up to non-zero coefficients, ∂_n of $(5.1)_k$ satisfies followings*:

(i) $\partial_n(p_*Q^{m+1}(\iota))=p_*\bar{Q}^{n+1}(\beta_1)$.

(ii) $\partial_n(p_*\bar{Q}^{m+1}(\alpha_{r-m-1}))=0$.

(iii) $\partial_n(p_*Q^{m+1}(\beta_1^r\beta_s))=p_*\bar{Q}^{n+1}(\beta_1^{r+1}\beta_s)$.

(iv) $\partial_n(p_*\bar{Q}^{m+1}(\beta_1^r\beta_s))=0$.

(v) $\partial_n(\gamma_s(2m+1))=S^{2p-2}u_4(s-2, \beta_2)$ *for* $m=sp-2, 3\leqq s<p$,

$=0$ *for the other cases.*

(vi) $\partial_n(S^2\gamma_s(2sp-3))=S^{2p}u_4(s-2, \beta_2)$ *for* $m=sp-1, 3\leqq s<p$,

$=0$ *for* $m=sp-1, s=2, p$.

PROOF. (i), (iii) and (vi) are similar to (ii) in Proposition 5.2, and (iv) is similar to (iii) in Proposition 5.2. The first half of (v) follows from (vi).

To prove (ii) and the second half of (v), we put

$$\mu_r(2m+1)=p_*\bar{Q}^{m+1}(\alpha_{r-m-1}) \quad \textit{for } r\not\equiv 0 \pmod p) \textit{ and } \mu_{sp}(2m+1)=\gamma_s(2m+1).$$

For the case $(r, m)=(p^2, p^2-2)$, we have $\partial_n(\mu_r(2m+1))\in\pi_{2n+1+2(p^2+1)(p-1)-3}(S^{2n+1}; p)\cap\mathrm{Ker}\, S^\infty=0$, $n=p^2-p-1$. And so, in the following, we assume $(r, m)\neq(p^2, p^2-2)$. By (4.15) (i) and (4.16) (ii), we have $H^{(2)}\mu_r(2m+1)=Q^m(\alpha'_{r-m})=I'\alpha'_{r-m}(2mp-1)$. The composition $\beta_1(2np+1)\circ\alpha'_{r-m}(2mp-1)$ is in the stable range and vanishes by (4.3). Therefore we have $IH^{(2)}\partial_n(\mu_r(2m+1))=0$. Thus we get

$$\partial_n(\mu_r(2m+1))\in\pi_{2n+1+2(r+1)(p-1)-3}(S^{2n+1}; p)\cap\mathrm{Ker}\, S^\infty\cap\mathrm{Ker}\, IH^{(2)}=\pi.$$

By Theorem 4.2, this group π is as follows:

$\pi\approx Z_p$ *generated by* $S^{2j}u_3(l, \beta_1)$ *for* $r=(l+1)p-1$,

$\approx Z_{p^2}$ *generated by* $S^{2j}u_3(p-1, \beta_1)$ *for* $r=p^2-1$,

$\approx Z_p$ *generated by* $\alpha_1\beta_1^p(2n+1)$ *for* $r=p^2-1$,

$\approx Z_p$ *generated by* $S^{2j}u_3(1, \beta_1^p)(p>3)$ *for* $r=(p+1)p-2$,

$\approx Z_p$ *generated by* $S^{2j}u_4(l, \beta_s)$ *for* $r=(s+l)p+s-2$,

$=0$ *for the other cases.*

According to the facts $\partial_n(\mu_r(2m+1)) \in \operatorname{Ker} S^2$ for $r \not\equiv 0 \pmod{p}$ and $\partial_n(\mu_{sp}(2m+1)) \in \operatorname{Ker} S^4$ for $s<p$, we have the following non-zero possibilities:

(A) $\partial_n(p_*\bar{Q}^{3p-2}(\alpha_{p^2-2p}))=aS^{2p-4}u_3(1, \beta_1^p)=a'p_*Q^{2p-1}(\alpha_1\beta_1^{p-1})$ *for some* a, $a' \in Z_p$, $p>3$.

(B) $\partial_n(p_*Q^{(l+2)p}(\alpha_{(s-2)(p+1)}))=bS^{2p}u_4(l, \beta_s)=b'p_*Q^{(l+1)p+1}(\beta_{s-1})$ *for some* b, $b' \in Z_p$, $2<s<p$, $l\geqq 1$, $s+l<p$.

(C) $\partial_n(\gamma_s(2sp-3))=cS^{2p-2}u_4(s-2, \beta_2)$ *for some* $c \in Z_p$, $2<s<p$.

Since $Q^{2p-1}(\alpha'_{p^2-p})$ generates the kernel of

$$p_*: \pi_{2(p^2+p+1)(p-1)-2}(Q_2^{4p-3}; p) \longrightarrow \pi_{4p-3+2(p^2+p-1)(p-1)-3}(S^{4p-3}; p),$$

the relation (A) implies

$$\partial_n(\bar{Q}^{3p-2}(\alpha_{p^2-2p}))=a'Q^{2p-1}(\alpha_1\beta_1^{p-1})+a''Q^{2p-1}(\alpha'_{p^2-p}) \quad \textit{for some } a'' \in Z_p.$$

Since the kernel of

$$G_*: \pi_{2(p^2+p+1)(p-1)-2}(Y^{4p^2-2p-2}; p) \longrightarrow \pi_{2(p^2+p+1)(p-1)-2}(Q_2^{4p-3}; p)$$

is trivial, we have

$$\lambda_{n*}i^*\alpha^{p^2-2p}=i_{1*}(xa'\alpha_1\beta_1^{p-1}+a'''\alpha'_{p^2-p}), \quad x,\ a''' \in Z_p,\ x \not\equiv 0,$$

where i: $S^{2(p^2+p+1)(p-1)-2} \subset Y^{2(p^2+p+1)(p-1)-1}$ and i_1: $S^{4p^2-2p-3} \subset Y^{4p^2-2p-2}$ are inclusions and λ_n is the same as (4.22). By (4.23) and (4.24), we have

$$i_{1*}(xa'\alpha_1\beta_1^{p-1}+a'''\alpha'_{p^2-p})=i^*(xa'\delta\alpha(\delta\beta_{(1)})^{p-1}+a'''\alpha^{p^2-p-1}\delta\alpha),$$

and the kernel of

$$i^*: \pi_{2(p^2-p)(p-1)-1} \longrightarrow \pi_{2(p^2+p+1)(p-1)-2}(Y^{4p^2-2p-2}; p)$$

is generated by $\alpha^{p^2-p}\delta$ and $\alpha\delta(\beta_{(1)}\delta)^{p-1}$. Thus we have

$$\lambda_{n*}\alpha^{p^2-2p} \equiv xa'\delta\alpha(\delta\beta_{(1)})^{p-1}+a'''\alpha^{p^2-p-1}\delta\alpha \quad \textit{modulo } \alpha^{p^2-p}\delta \textit{ and } \alpha\delta(\beta_{(1)}\delta)^{p-1}.$$

In this relation, the linearly independency of $\delta\alpha(\delta\beta_{(1)})^{p-1}$ implies $a'=0$.

By the similar arguments, we obtain $b=0$ in (B). q.e.d.

§6 The Homotopy Groups of $B_n(p)$.

We start from the discussion of the stable homotopy groups

$$\pi_k^S(B;p)=\pi_{2N+1+k}(B_N(p);p),\quad N>\frac{k+2}{2(p-1)}-1.$$

The sequence (0.7) implies the following exact sequence:

(6.1) $$\cdots\xrightarrow{j_*}{}_pG_{k-(2p-3)}\xrightarrow{\partial}{}_pG_k\xrightarrow{i_*}\pi_k^S(B;p)\xrightarrow{j_*}{}_pG_{k-(2p-2)}\xrightarrow{\partial}\cdots,$$

and ∂ is composition with α_1. The group $\pi_k^S(B;p)$ is isomorphic to the stable homotopy group of the mapping cone of α_1, i.e.,

(6.2) $$\pi_{2N+1+k}(B_N(p);p)\approx\pi_{2N+1+k}(K_N;p),\ K_N=S^{2N+1}\cup_{\alpha_1(2N+1)}e^{2N+2p-1}$$

for large N.

We shall use the following notation:

(6.3) *For* $\gamma\in{}_pG_{k-(2p-2)}\cap\operatorname{Ker}\partial$, $[\gamma]\in\pi_k^S(B;p)$ *denotes an element such that* $j_*([\gamma])=\gamma$.

PROPOSITION 6.1. *For* $k<2(p^2+p)(p-1)-5$*), $\pi_k^S(B;p)$ *is generated by the following elements*:

$[\alpha_r]$ *of order* p^2 *and degree* $2(r+1)(p-1)-1$, *for* $r\not\equiv 0,\ -1 \pmod p$,

$[\alpha_{sp-1}]$ *of order* p^3 *and degree* $2sp(p-1)-1$, *for* $s<p$,

$[\alpha_{p^2-1}]$ *of order* p^4 *and degree* $2p^2(p-1)-1$,

$[\alpha'_{sp}]$ *of order* p^3 *and degree* $2(sp+1)(p-1)-1$, *for* $s<p$,

$[\alpha'_{p^2}]$ *of order* p^4 *and degree* $2(p^2+1)(p-1)-1$,

$i_*\beta_1^r\beta_s$ *of order* p *and degree* $2((r+s)p+s-1)(p-1)-2(r+1)$, *for* $r\geqq 0$, $1\leqq s<p$ *except the case* $p=3$, $r=3$, $s=1$,

$[\beta_1^p]$ *of order* p *and degree* $2p^2(p-1)-2$,

$[\alpha_1\beta_1^r\beta_s]$ *of order* p *and degree* $2((r+s)p+s+1)(p-1)-2(r+1)-1$, *for* $r\geqq 0$, $1\leqq s<p$ *except the case* $r=p-1$, $s=1$,

$i_*\varepsilon'$ *of order* p *and degree* $2(p^2+1)(p-1)-3$,

$i_*\varepsilon_i$ *of order* p *and degree* $2(p^2+i)(p-1)-2$, *for* $1\leqq i\leqq p-2$,

$[\varepsilon']$ *of order* p *and degree* $2(p^2+2)(p-1)-3$, *for* $p>3$,

$[\alpha_1\varepsilon_i]$ *of order* p *and degree* $2(p^2+i+2)(p-1)-3$, *for* $1\leqq i\leqq p-3$,

$[\varepsilon_{p-2}]$ *of order* p^2 *and degree* $2(p^2+p-1)(p-1)-2$.

---

*) For the smaller k, cf. Proposition 4.21 in [7].

This proposition follows easily from Proposition 5.1, the relation (4.3)''' and the following Lemma 6.2.

To investigate the group extensions, we shall use the following two lemmas.

LEMMA 6.2. *Let* $\gamma \in {}_pG_{k-(2p-2)}$ *be an element of order* $p^t (t \geqq 1)$, *satisfying* $\alpha_1\gamma = 0$. *Then the set of all* $-p^t[\gamma]$ *coincides with the set* $i_*\{\alpha_1, \gamma, p^t\iota\}$, *where we identify* $\pi_k^S(B; p)$ *with* $\pi_{2N+1+k}(K_N; p)$.

LEMMA 6.3. *Let* $h: Y^{k+n} \longrightarrow Y^n$ *be a map and let* $\alpha \in \pi_i(Y^{k+n})$ *be an element of order* p *such that* $h_*\alpha = 0$. *Let* $\tilde{\alpha} \in \pi_{i+1}(C_h)$ *be a coextension of* α *and* $\bar{\alpha} \in [Y^{i+1}, Y^{k+n}]$ *be an extension of* α. *Then there exists an element* $\gamma \in \pi_{i+1}(Y^n)$ *such that*

$$p\tilde{\alpha} = j_{1*}\gamma \text{ and } \pi_1^*\gamma = -h_*\bar{\alpha},$$

where $j_1: Y^n \longrightarrow C_h = Y^n \cup_h CY^{k+n}$ *is the inclusion and* $\pi_1: Y^{i+1} \longrightarrow S^{i+1}$ *is the projection.*

These lemmas are the special cases of Proposition 4.2 in [7] and Lemma 4.7 in [8], and proofs are omitted.

Now we consider the homotopy groups $\pi_{2n+1+k}(B_n(p); p)$. Results of the computations are settled as follows:

THEOREM 0.6. *For* $n \geqq 1$ *and* $k < 2(p^2+p)(p-1)-5$, *we have the following direct sum decomposition*:

$$\begin{aligned}&\pi_{2n+1+k}(B_n(p); p)\\ &\quad = \bar{A}(n, k) + \bar{B}(n, k) + \bar{E}(n, k) + U_a(n, k) + U_b(n, k) + U_u(n, k).\end{aligned}$$

To define the direct factors, the symbol [ ] is used as (6.3).

(6.4) $\bar{A}(n, k)$ *is defined as follows*:

$\bar{A}(1, 2r(p-1)-1) \approx Z_p$ *generated by* $i_*\alpha_r(3)$ *for* $r \not\equiv 1 \pmod p$,

$\bar{A}(1, 2(sp+1)(p-1)-1) \approx Z_{p^2}$ *generated by* $[\alpha_{sp}(2p+1)]$ *for* $s<p$,

$\bar{A}(1, 2(p^2+1)(p-1)-1) \approx Z_{p^3}$ *generated by* $[p\alpha'_{p^2}(2p+1)]$ *for* $p>3$,

$\approx Z_{p^3}$ *or* Z_{p^4} *generated by* $[3\alpha'_9(7)]$ *or* $[\alpha'_9(7)]$ *respectively, for* $p=3$,

$\bar{A}(2, 2sp(p-1)-1) \approx Z_{p^2}$ *generated by* $i_*\alpha'_{sp}(5)$ $(i_*(p\alpha'_{p^2}(5))$ *if* $s=p)$,

$\bar{A}(3, 2p^2(p-1)-1) \approx Z_{p^3}$ *generated by* $i_*\alpha'_{p^2}(7)$,

$\bar{A}(n, 2r(p-1)-1) \approx Z_{p^2}$ *generated by* $[\alpha_{r-1}(2n+2p-1)]$ *for* $n>1$, $r \not\equiv 0, 1 \pmod p$, $r>1$,

$\bar{A}(n, 2sp(p-1)-1) \approx Z_{p^3}$ *generated by* $[\alpha_{sp-1}(2n+2p-1)]$ *for* $n>2$, $s<p$,

$\bar{A}(n, 2p^2(p-1)-1) \approx Z_{p^4}$ *generated by* $[\alpha_{p^2-1}(2n+2p-1)]$ *for* $n>3$,

$\bar{A}(n, 2(sp+1)(p-1)-1) \approx Z_{p^3}$ *generated by* $[\alpha'_{sp}(2n+2p-1)]$ *for* $n>1$, $s<p$,

$\bar{A}(n, 2(p^2+1)(p-1)-1) \approx Z_{p^4}$ *generated by* $[\alpha'_{p^2}(2n+2p-1)]$ *for* $n>1$,

$\bar{A}(n, k)=0$ *for* $k \not\equiv -1 \pmod{2p-2}$ *and for* $k=2p-3$.

(6.5) $\bar{B}(n, k)$ *is defined as follows*:

$\bar{B}(n, 2((r+s)p+s-1)(p-1)-2(r+1)) \approx Z_p$ *generated by* $i_*\beta_1^r\beta_s(2n+1)$ *for* $n \geqq p-1$ *if* $r \geqq 1$, $s \geqq 1$, *for* $n \geqq p$ *if* $r=0$, $s=1$, *and for* $n \geqq p+1$ *if* $r=0$, $s \geqq 2$, *except the case* $(p, r, s)=(3, 3, 1)$,

$\bar{B}(n, 2((r+s)p+s+1)(p-1)-2(r+1)-1) \approx Z_p$ *generated by* $[\alpha_1\beta_1^r\beta_s(2n+2p-1)]$ *for* $n>1$, *except the case* $r=p-1$, $s=1$, $n \geqq p^2-p-1$,

$\bar{B}(n, 2p^2(p-1)-2) \approx Z_p$ *generated by* $[\beta_1^p(2n+2p-1)]$ *for* $n>p^2-3$,

$\bar{B}(n, k)=0$ *for the other cases.*

(6.6) $\bar{E}(n, k)$ *is defined as follows*:

$\bar{E}(n, 2(p^2+1)(p-1)-3) \approx Z_p$ *generated by* $i_*\varepsilon'(2n+1)$ *for* $n \geqq p^2-2p$,

$\bar{E}(n, 2(p^2+i)(p-1)-2) \approx Z_p$ *generated by* $i_*\varepsilon_i(2n+1)$ *for* $1 \leqq i \leqq p-2$, $n \geqq p(p-i)+1$,

$\bar{E}(n, 2(p^2+2)(p-1)-3) \approx Z_p$ *generated by* $[\varepsilon'(2n+2p-1)]$ *for* $p>3$, $n \geqq p(p-2)$,

$\bar{E}(n, 2(p^2+i+2)(p-1)-3) \approx Z_p$ *generated by* $[\alpha_1\varepsilon_i(2n+2p-1)]$ *for* $1 \leqq i \leqq p-3$, $n \geqq p(p-i-2)$,

$\bar{E}(p+1, 2(p^2+p-1)(p-1)-2) \approx Z_p$ *generated by* $i_*\varepsilon_{p-1}(2p+3)$ *for* $p>3$,

$\bar{E}(n, 2(p^2+p-1)(p-1)-2) \approx Z_{p^2}$ *generated by* $[\varepsilon_{p-2}(2n+2p-1)]$ *for* $n>p+1$,

$\bar{E}(n, k)=0$ *for the other cases.*

To define $U_a(n, k)$, we shall use the following notations and conventions:

(6.7) *For* $i=1, 2, 3$, G_i *denotes the group isomorphic to* $Z_{p^{i+1}}$ *or* $Z_{p^i}+Z_p$.

In a few word, we say that G_i *is generated by* γ_1 *and* γ_2, *when* G_i *is generated by* γ_1 *and* $G_i \approx Z_{p^{i+1}}$ *or by* γ_1 *and* γ_2 *and* $G_i \approx Z_{p^i}+Z_p$.

(6.8) $U_a(n, k)$ *is defined as follows*:

(i) *For* $k \not\equiv -2 \pmod{2p-2}$ *and for* $k=2p-4$, $U_a(n, k)=0$.

So, in the following, we put $k=2r(p-1)-2\,(r>1)$ *and* $U=U_a(n, k)$, *and divide into eight cases by the values of* r.

(ii) $1<r<p+4$, $r\not\equiv 0 \pmod{p}$:

$U\approx Z_p$ *generated by* $[p_*\bar{Q}^{p+1}(\alpha_1)]$ *for* $n=1$, $r=p+3$,

$\approx Z_p$ *generated by* $i_*p_*\bar{Q}^{n+1}(\alpha_{r-n-1})$ *for* $2\leqq n<r-1$,

$\approx Z_p$ *generated by* $i_*p_*Q^{n+1}(\iota)$ *for* $n=r-1$,

$=0$ *for the other cases.*

(iii) $r\geqq p+4$, $r\not\equiv 0, 1 \pmod{p}$:

$U\approx Z_p$ *generated by* $[p_*\bar{Q}^{p+1}(\alpha_{r-p-2})]$ *for* $n=1$,

$\approx G_1$ *generated by* $[p_*\bar{Q}^{n+p}(\alpha_{r-p-n-1})]$ *and* $i_*p_*\bar{Q}^{n+1}(\alpha_{r-n-1})$ *for* $1<n<r-p-1$,

$\approx Z_p$ *generated by* $i_*p_*\bar{Q}^{n+1}(\alpha_{r-n-1})$ *for* $r-p-1\leqq n<r-1$,

$\approx Z_p$ *generated by* $i_*p_*Q^{n+1}(\iota)$ *for* $n=r-1$,

$=0$ *for* $n\geqq r$.

(iv) $r=p$:

$U\approx Z_p$ *generated by* $i_*\gamma_1(5)$ *for* $n=2$,

$\approx Z_{p^2}$ *generated by* $i_*\gamma_1(2n+1)$ *for* $3\leqq n\leqq p-1$,

$=0$ *for* $n=1$ *and for* $n\geqq p$.

(v) $r=sp$, $2\leqq s<p$:

$U\approx Z_p$ *generated by* $[p_*\bar{Q}^{p+1}(\alpha_{sp-p-2})]$ *for* $n=1$,

$\approx Z_p$ *generated by* $i_*\gamma_2(5)$ *for* $p=3$, $s=2$, $n=2$,

$\approx G_1$ *generated by* $[p_*\bar{Q}^{p+2}(\alpha_{sp-p-3})]$ *and* $i_*\gamma_s(5)$ *for* $n=2$ *except the case* $p=3$, $s=2$,

$\approx G_2$ *generated by* $[p_*\bar{Q}^{n+p}(\alpha_{sp-p-n-1})]$ *and* $i_*\gamma_s(2n+1)$ *for* $2<n<sp-p-1$,

$\approx Z_{p^2}$ *generated by* $i_*\gamma_s(2n+1)$ *for* $sp-p-1\leqq n<sp-1$,

$\approx Z_p$ *generated by* $i_*S^2\gamma_s(2sp-3)$ *for* $n=sp-1$,

$=0$ *for* $n\geqq sp$.

(vi) $r=p^2$:

$U\approx Z_p$ *generated by* $[p_*\bar{Q}^{p+1}(\alpha_{p^2-p-2})]$ *for* $n=1$,

$\approx G_1$ *generated by* $[p_*\bar{Q}^{p+2}(\alpha_{p^2-p-3})]$ *and* $i_*\gamma_p(5)$ *for* $n=2$,

$\approx G_2$ *generated by* $[p_*\bar{Q}^{p+3}(\alpha_{p^2-p-4})]$ *and* $i_*\gamma_p(7)$ *for* $n=3$,

$\approx G_3$ *generated by* $[p_*\bar{Q}^{n+p}(\alpha_{p^2-p-n-1})]$ *and* $i_*\gamma_p(2n+1)$
for $3<n<p^2-p-1$,

$\approx Z_{p^3}$ *generated by* $i_*\gamma_p(2n+1)$ *for* $p^2-p-1\leqq n<p^2-2$,

$\approx Z_{p^2}$ *generated by* $i_*\gamma_p(2p^2-3)$ *for* $n=p^2-2$,

$\approx Z_p$ *generated by* $i_*S^2\gamma_p(2p^2-3)$ *for* $n=p^2-1$,

$=0$ *for* $n\geqq p^2$.

(vii) $r=2p+1$:

$U\approx Z_{p^2}$ *generated by* $[\gamma_2(2p+1)]$ *for* $n=1$,

$\approx G_2$ *generated by* $[\gamma_2(2n+2p-1)]$ *and* $i_*p_*\bar{Q}^{n+1}(\alpha_{2p-n})$ *for* $1<n<p$,

$\approx G_1$ *generated by* $[S^2\gamma_2(4p-3)]$ *and* $i_*p_*\bar{Q}^{p+1}(\alpha_p)$ *for* $n=p$,

$\approx Z_p$ *generated by* $i_*p_*\bar{Q}^{n+1}(\alpha_{2p-n})$ *for* $p<n<2p$,

$\approx Z_p$ *generated by* $i_*p_*Q^{n+1}(\iota)$ *for* $n=2p$,

$=0$ *for* $n>2p$.

(viii) $r=sp+1$, $2<s<p$:

$U\approx Z_{p^2}$ *generated by* $[\gamma_s(2p+1)]$ *for* $n=1$,

$\approx G_2$ *generated by* $[\gamma_s(2n+2p-1)]$ *and* $i_*p_*\bar{Q}^{n+1}(\alpha_{sp-n})$
for $1<n<sp-p-1$,

$\approx G_1$ *generated by* $[p\gamma_s(2sp-3)]$ *and* $i_*p_*\bar{Q}^{n+1}(\alpha_{p+1})$ *for* $n=sp-p-1$,

$\approx Z_p$ *generated by* $i_*p_*\bar{Q}^{n+1}(\alpha_{sp-n})$ *for* $sp-p\leqq n<sp$,

$\approx Z_p$ *generated by* $i_*p_*Q^{n+1}(\iota)$ *for* $n=sp$,

$=0$ *for* $n\geqq sp+1$.

(ix) $r=p^2+1$:

$U\approx Z_{p^3}$ *generated by* $[\gamma_p(2p+1)]$ *for* $p>3$, $n=1$,

$\approx G_3$ *generated by* $[\gamma_p(2n+2p-1)]$ *and* $i_*p_*\bar{Q}^{n+1}(\alpha_{p^2-n})$

for $2 \leqq n < p^2-p-1$,

$\approx G_2$ *generated by* $[\gamma_p(2p^2-3)]$ *and* $i_* p_* \bar{Q}^{n+1}(\alpha_{p+1})$ *for* $n=p^2-p-1$,

$\approx G_1$ *generated by* $[S^2\gamma_p(2p^2-3)]$ *and* $i_* p_* \bar{Q}^{n+1}(\alpha_p)$ *for* $n=p^2-p$,

$\approx Z_p$ *generated by* $i_* p_* \bar{Q}^{n+1}(\alpha_{p^2-n})$ *for* $p^2-p<n<p^2$,

$\approx Z_p$ *generated by* $i_* p_* Q^{n+1}(\iota)$ *for* $n=p^2$,

$=0$ *for* $n \geqq p^2+1$.

For the case $p=3$, $n=1$, *we have either* $U \approx Z_{p^3}$ *generated by* $[\gamma_3(7)]$, *or* $U \approx G_3$ *generated by* $[\gamma_3(7)]$ *and* $i_* p_* \bar{Q}^2(\alpha_8)$.

(6.9) $U_b(n, k)$ *is defined as follows*:

(i) $U_b(n, 2((r+s+1)p+s+n)(p-1)-2(r+1)-1)$

$\approx Z_p+Z_p$ *generated by* $i_* u_3(l, \beta_1^r \beta_{s+1})$ *and* $[p_* \bar{Q}^{n+p}(\beta_1^r \beta_s)]$
for $n=lp$, $r>0$, $1 \leqq s \leqq p-2$.

$\approx Z_p+Z_p$ *generated by* $i_* u_4(l, \beta_{s+1})$ *and* $[p_* \bar{Q}^{n+p}(\beta_s)]$
for $n=lp$, $r=0$, $1 \leqq s \leqq p-2$.

$\approx Z_p$ *generated by* $[p_* \bar{Q}^{n+p}(\beta_1^r \beta_s)]$ *for* $n \equiv 0 \pmod p$, $r=0$, $s=p-1$
and for $n \not\equiv 0, 1 \pmod p$, $r \geqq 0$, $s \geqq 1$.

(ii) $U_b(n, 2((s+l)p+s-1)(p-1)-3)$

$\approx Z_p+Z_p$ *generated by* $i_* S^{2p} u_4(l, \beta_s)$ *and* $[S^{2p-4} u_4(l+1, \beta_{s-1})]$
for $l \geqq 1$, $s \geqq 3$, $s+l<p$, $n=(l+1)p$.

$\approx Z_p$ *generated by* $[S^{2p-2} u_4(1, \beta_{s-1})]$ *for* $l=0$, $s \geqq 3$, $n=(l+1)p$.

$\approx Z_p$ *generated by* $[S^{2p} u_4(l+1, \beta_{s-1})]$
for $l \geqq 0$, $s \geqq 3$, $s+l<p$, $n=(l+1)p+1$.

$\approx Z_p$ *generated by* $i_* S^{2j} u_4(l, \beta_s)$ *for* $l \geqq 1$, $s \geqq 2$, $s+l<p$,
$n=lp+j$, $0<j<p$ *except the case* $s=2$, $j=p-1$.

(iii) $U_b(n, 2((r+s+l)p+s-1)(p-1)-2(r+1)-1)$

$\approx Z_p$ *generated by* $i_* u_3(l, \beta_1^{r+1})$ *for* $n=lp$, $r \geqq 0$, $s=1$.

$\approx Z_p$ *generated by* $i_* S^{2j} u_3(l, \beta_1^r \beta_s)$
for $n=lp+j$, $1 \leqq j \leqq p-2$, $l \geqq 1$, $r \geqq 0$, $s \geqq 1$ *except* $r=0$, $s>1$.

(iv) $U_b(n, 2((r+s)p+s+n)(p-1)-2(r+2))$

$\approx Z_p$ *generated by* $i_* p_* Q^{n+1}(\beta_1^r \beta_s)$ *for* $n>1$, $n \not\equiv -1 \pmod p$,

$r \geqq 0,\ s \geqq 1$ *except the case* $n \equiv 0 \pmod p$, $s \geqq 2$.

(v) $U_b((l-1)p+2+j,\ 2((r+s+l)p+s+1)(p-1)-2(r+1))$

$\approx Z_p$ *generated by* $[S^{2j}\bar{u}_3(l,\ \beta_1^r\beta_s)]$ *for* $r \geqq 1,\ s \geqq 1,\ l \geqq 1,\ 0 \leqq j \leqq p-2$ *except* $s \leqq p-2,\ j=p-2$.

(vi) $U_b(lp,\ 2((r+s+l)p+s)(p-1)-2(r+2))$

$\approx Z_p + Z_p$ *generated by* $i_* p_* Q^{lp+1}(\beta_1^r\beta_s)$ *and* $[S^{2p-4}\bar{u}_3(l,\ \beta_1^{r+1}\beta_{s-1})]$ *for* $r \geqq 0,\ s \geqq 2,\ l \geqq 1$.

(vii) $U_b(n,\ 2(sp+s+n)(p-1)-3)$

$\approx Z_p$ *generated by* $i_* p_* \bar{Q}^{n+1}(\beta_s)$ *for* $n \not\equiv 0 \pmod p$, $s \geqq 2$ *and for* $p=3,\ n=3,\ s=2$.

(viii) *For the other cases, we put* $U_b(n,\ k)=0$.

(6.10) $U_u(n,\ k)$ *is defined as follows*:

$U_u(n,\ 2(tp+t)(p-1)-4) \approx Z_p$ *for* $2 \leqq n < t < p$.

$U_u(n,\ k)=0$ *for other cases.*

Remark that, under the projection S^∞: $\pi_{2n+1+k}(B_n(p);p) \longrightarrow \pi_k^S(B;p)$ the subgroups $\bar{A}(n,\ k)$, $\bar{B}(n,\ k)$ $(k \neq 2(p^2+1)(p-1)-3)$ and $\bar{E}(n,\ k)$ are mapped isomorphically into the stable group $\pi_k^S(B;\ p)$, and the subgroup $U_a(n,\ k) + U_b(n,\ k) + U_u(n,\ k)$ $(+\bar{B}(n,\ k)$ if $k=2(p^2+1)(p-1)-3)$ coincides with the kernel of S^∞.

The following proposition is obtained easily from Proposition 6.1 and the above definitions (6.4), (6.5) and (6.6).

PROPOSITION 6.4. *The subgroups* $\bar{A}(n,\ k)+\bar{B}(n,\ k)+\bar{E}(n,\ k)$ *are direct factors of the groups* $\pi_{2n+1+k}(B_n(p);p)$.

To investigate $U_a(n,k)$ and $U_b(n,\ k)$, we shall discuss the exact sequence (0.8). As a consequence, we obtain

PROPOSITION 6.5. *There exists a map* $\tilde{G}$: $C_{\lambda_n} = Y^{2(n+1)p-2} \cup CY^{2(n+p)p-3} \longrightarrow QB_n(p)$, $n \geqq 1$, *such that* $\tilde{G}^*$ *are isomorphisms of* $H^{2(n+1)p-3}(\ ;Z_p)$ *and* $H^{2(n+p)p-3}(\ ;Z_p)$, *and that the following diagram is commutative*:

$$
(6.11)\quad
\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial_n} & \pi_j(Q_2^{2n+1};p) & \xrightarrow{i_*} & \pi_j(QB_n(p);p) & \xrightarrow{j_*} & \pi_j(Q_2^{2n+2p-1};p) & \xrightarrow{\partial_n} \cdots \\
& \uparrow{\scriptstyle G_*} & & \uparrow{\scriptstyle \tilde{G}_*} & & \uparrow{\scriptstyle G_*} & \\
\cdots \xrightarrow{\lambda_{n*}} & \pi_j(Y^{2(n+1)p-2};p) & \xrightarrow{j_{1*}} & \pi_j(C_{\lambda_n};p) & \xrightarrow{j_{2*}} & \pi_j(Y^{2(n+p)p-2};p) & \longrightarrow \cdots,
\end{array}
$$

where j_1 *denotes the inclusion and* j_2 *denotes the projection.*

PROOF. By Lemma 2.3 in [8], we obtain the following

(6.12) $H^*(Q_2^{2n+1}; Z_p) = \Lambda(a_0) \otimes Z_p[\Delta a_0]$ (deg $a_0 = 2(n+1)p-3$) *for* deg$<p(2(n+1)p-2)-2$ *and* $H^*(Q_2^{2n+2p-1}; Z_p) = \Lambda(b_0) \otimes Z_p[\Delta b_0]$ (deg $b_0 = 2(n+p)p-3$) *for* deg $<p(2(n+p)p-2)$.

Then the spectral sequence associated with the fibering

$$Q_2^{2n+1} \xrightarrow{i} QB_n(p) \xrightarrow{j} Q_2^{2n+2p-1}$$

is trivial for total degree$<p(2(n+1)p-2)-2$ and so, we have

(6.13) $H^*(QB_n(p); Z_p) = \Lambda(x_0, y_0) \otimes Z_p[\Delta x_0, \Delta y_0]$, $i^*(x_0) = a_0$, $y_0 = j^*(b_0)$ *for* deg$<p(2(n+1)p-2)-3$.

The map $iG\lambda_n$: $Y^{2(n+p)p-3} \longrightarrow Y^{2(n+1)p-2} \longrightarrow Q_2^{2n+1} \longrightarrow QB_n(p)$ is null-homotopic. Hence there is a map $\tilde{G}$: $C_{\lambda_n} \longrightarrow QB_n(p)$ which is an extension of iG, that is, $iG = \tilde{G}j_1$ holds. Similarly, we have a map G': $Y^{2(n+p)p-2} \longrightarrow Q_2^{2n+2p-1}$ satisfying $j\tilde{G} \simeq G'j_2$, since C_{j_1} is homotopy equivalent to $Y^{2(n+p)p-2}$. The map G' is homotopic to G: $Y^{2(n+p)p-2} \longrightarrow Q_2^{2n+2p-1}$ by the uniqueness of G in Proposition 3.1, and the required conditions of $\tilde{G}$ follow from (6.12) and (6.13). q.e.d.

Now we consider the subgroups $U_a(n, k)$ and $U_b(n, k)$.

PROPOSITION 6.6. *The subgroups* $U_a(n, k)$ *are direct factors of the groups* $\pi_{2n+1+k}(B_n(p); p)$.

PROOF. By the dimensional reason, $U_a(n, k)$ and $U_b(n, k)$ overlap in the following two cases:

(A) $k = 2(p^2-p+n)(p-1)-2$, $1<n<2p-1$, $n \neq p-1$, p. *In this case,* $\pi_{2n+1+k}(S^{2n+1}; p) / \mathrm{Im}\ \partial_n \approx Z_p + Z_p$ *is generated by* $p_* Q^{n+1}(\beta_1^{p-1})$ *and* $p_* \bar{Q}^{n+1}(\alpha_{p^2-p-1})$, *and* $\pi_{2n+1+k}(S^{2n+2p-1}; p) \cap \mathrm{Ker}\ \partial_n \approx Z_p$ *is generated by* $p_* \bar{Q}^{n+p}(\alpha_{p^2-2p-1})$.

(B) $k = 2(p^2+n+1)(p-1)-2$, $1<n<p-1$. *In this case,* $\pi_{2n+1+k}(S^{2n+1}; p) / \mathrm{Im}\ \partial_n \approx Z_p + Z_p$ *is generated by* $p_* Q^{n+1}(\beta_1^{p-2}\beta_2)$ *and* $p_* \bar{Q}^{n+1}(\alpha_{p^2})$, *and* $\pi_{2n+1+k}(S^{2n+2p-1}; p) \cap \mathrm{Ker}\ \partial_n \approx Z_p$ *is generated by* $p_* \bar{Q}^{n+p}(\alpha_{p^2-p})$.

Now we consider the case (A). By (3.1), (4.15) (i) and (4.24), we obtain the following

$$H^{(2)} p_* \bar{Q}^{n+p}(\alpha_{p^2-2p-1}) = x G_* i_1{}^* \alpha^{p^2-2p-1} \delta\alpha \quad \textit{for some } x \not\equiv 0 \pmod p,$$

where i_1 denotes the inclusion and G: $Y^{2(n+p-1)p-3} \longrightarrow Q_2^{2n+2p-3}$ is the map in (6.11) (replacing n by $n-1$). The element $i_1^* \alpha^{p^2-2p-1}\delta\alpha \in \pi_i(Y^{2(n+p-1)p-3}; p)$, $i = 2n+1+k-4$, is of order p and contained in Ker λ_{n-1}. Let $\gamma_2 \in \pi_{i+1}(C_{\lambda_{n-1}})$ be a coextension of $\gamma_1 = x i_1^* \alpha^{p^2-2p-1}\delta\alpha$. Replacing h and α by λ_{n-1} and γ_1 in Lemma 6.3, there exists $\gamma_3 \in \pi_{i+1}(Y^{2np-2}; p)$ such that $j_{1*}\gamma_3 = p\gamma_2$ and $\pi_1^* \gamma_3 = -\lambda_{n-1*} \alpha^{p^2-2p-1}\delta\alpha$ hold. Since $\lambda_{n-1*}\alpha^{p^2-2p-1}\delta\alpha = c\alpha^{p^2-p-1}\delta\alpha\delta = c\pi_1^* i_2^* \alpha^{p^2-p-1}\delta\alpha (c \in Z_p)$ and π_1^*: $\pi_{i+1}(Y^{2np-2}; p) \longrightarrow [Y^{i+1}, Y^{2np-2}] = \pi_{2(p^2-p)(p-1)-2}$

is monomorphic, we obtain $\gamma_3 = c' i_2^* \alpha^{p^2-p-1} \delta\alpha$ for the inclusion i_2: $S^{i+1} \subset Y^{i+2}$ and for some $c' \in Z_p$. Therefore $G_* \gamma_3 = c'' H^{(2)} p_* \bar{Q}^{n+1}(\alpha_{p^2-p-1})$ holds for G: $Y^{2np-2} \longrightarrow Q_2^{2n-1}$ and for some $c'' \in Z_p$. From the diagram (6.11) (replacing n by $n-1$), we can determine the group extension at $\pi_{2n+1+k}(B_n(p); p)$ by the investigation of the extension of the following groups:

$$0 \longrightarrow \pi_{i+1}(Y^{2np-2}; p) / \mathrm{Im}\ \lambda_{n-1*} \longrightarrow \pi_{i+1}(C_{\lambda_{n-1}}; p) \longrightarrow$$
$$\pi_{i+1}(Y^{2(n+p-1)p-2}; p) \cap \mathrm{Ker}\ \lambda_{n-1*} \longrightarrow 0,$$

i.e., $\pi_{2n+1+k}(B_n(p); p) \approx Z_{p^2} + Z_p$ if $c'' \not\equiv 0$, and $\approx Z_p + Z_p + Z_p$ if $c'' = 0$. Thus, we see that $i_* p_* Q^{n+1}(\beta_1^{p-1})$ generates a direct factor of $\pi_{2n+1+k}(B_n(p); p)$.

The case (B) is similar to the case (A). q.e.d.

On the groups $U_b(n, k)$, we need to investigate the group extensions in the following cases: *the first and the second cases of* (6.9) (i), *the first case of* (6.9) (ii) *and the case* (6.9) (iv).

In the case (6.9) (i), we have $H^{(2)} p_* \bar{Q}^{n+p}(\beta_1^r \beta_s) = G_* i_1^* x (\beta_{(1)} \delta)^r \beta_{(s)}$, $x \not\equiv 0$ (mod p) and $\lambda_{n-1*}(\beta_{(1)}\delta)^r \beta_{(s)} = 0$, and so the splitness of the case (6.9) (i) is established by Lemma 6.3.

By the similar arguments, we obtain the following

PROPOSITION 6.7. *The subgroups $U_b(n, k)$ defined in* (6.9) *are direct factors of the groups $\pi_{2n+1+k}(B_n(p); p)$.*

By the dimensional reason, the subgroups $U_u(n, k)$ are direct factors.

Thus, Theorem 0.6 is proved entirely.

As a corollary of Theorem 0.6, we get the following uniqueness on the homotopy type of $B_n(p)$:

PROPOSITION 6.8. *Let $n < p^2 - 2p$ and let $B = S^{2n+1} \cup e^{2n+2p-1} \cup e^{4n+2p}$ be a cell complex having the cohomology ring*

$$H^*(B; Z_p) = \Lambda(v, \mathscr{P}^1 v),\ \deg v = 2n+1.$$

Then, there is a map f: $B \longrightarrow B_n(p)$, such that f_* are isomorphisms of $\pi_i(\ ;p)$ for all i.

PROOF. Since the attaching map of the $(2n+2p-1)$-cell of B represents an element $x\alpha_1(2n+1) + \beta$, $r\beta = 0$, for some x, $r \not\equiv 0$ (mod p), there is a map f_0: $K \longrightarrow B_n(p)$ such that f_0^* are epimorphisms of $H^*(\ ; Z_p)$, where K denotes the $(2n+2p-1)$-skeleton of B. Let g: $S^{4n+2p-1} \longrightarrow K$ be the attaching map of the $(4n+2p)$-cell of B. The group $\pi_{4n+2p-1}(B_n(p))$ is finite and its order s is prime to p, since $\pi_{4n+2p-1}(B_n(p); p) = 0$ $(n < p^2 - 2p)$ by Theorem 0.6. Then we can construct a complex B' and maps f_1: $B \longrightarrow B'$ and f_2: $B' \longrightarrow B_n(p)$ such that f_1^* and f_2^* are isomorphisms of $H^*(\ ; Z_p)$ and that f_2 is an extension of f_0, where we may take B' as the mapping cone of the map gh for the map

$h: S^{4n+2p-1} \longrightarrow S^{4n+2p-1}$ of degree s. Then, the map $f=f_2 f_1$ satisfies the required conditions. q.e.d.

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