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# *Some Exact Sequences of Modules over the Steenrod Algebra*

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## §1. Introduction

In this paper, $p$ denotes always an odd prime number. Let $A^*$ denote the Steenrod algebra mod $p$, and $A_*$ the dual Hopf algebra of $A^*$.

In [6] and [7; Lemma 3.9 and (3.7)], H. Toda has calculated the kernels of several right translations of $A^*$ (or left $A^*$-modules in general). These results can be written in the form of the exact sequences of left $A^*$-modules. For example, the following circular exact sequence is proved in Proposition 1.5 of [6]:

$$\begin{array}{c} A^* \xrightarrow{(R_{p-2})^*} A^* \xrightarrow{(R_{p-3})^*} \cdots \xrightarrow{(R_2)^*} A^* \xrightarrow{(R_1)^*} A^* \\ {\scriptstyle R'} \nwarrow \qquad\qquad\qquad\qquad \swarrow {\scriptstyle R} \\ A^*/A^*\Delta \oplus A^*/A^*\Delta, \end{array}$$

where $R_k=(k+1)\mathscr{P}^1\Delta-k\Delta\mathscr{P}^1$, $(R_k)^*(\alpha)=\alpha R_k$, $R'(\alpha_1, \alpha_2)=\alpha_1\Delta\mathscr{P}^1-\alpha_2\Delta\mathscr{P}^1\Delta$ and $R(\alpha)=(\alpha\Delta\mathscr{P}^1, \alpha\mathscr{P}^1)$, using the Adem relations directly.

The above exact sequences are used in the calculations of the $p$-primary components of the stable homotopy groups of spheres (see [7] and [8]). For example, the above circular sequence is used in the determination of the elements $\alpha_k$ of $(2k(p-1)-1)$-stem in the $J$-image (see [8]).

In [1] and [2], J. Cohen has discussed the dual maps of the right (or left) translations by the elements of $A^*$. Calculating the kernel and the image of the dual map of $(\mathscr{P}^i)^*$, $i<p$, the exactness of

$$A^* \xrightarrow{(\mathscr{P}^i)^*} A^* \xrightarrow{(\mathscr{P}^{p-i})^*} A^*, \qquad 1\leqq i<p, \tag{1.1}$$

has been proved in [1; (7.11)]. By making use of Cohen's methods, the exactness of the above circular sequence can be reproved more simply.

In this paper, we shall prove several exact sequences (3.3–10), (4.2) and (4.8–15) in §§3–4 below by use of Cohen's methods. For example, we have the following exact sequence:

$$\begin{aligned} A^*\oplus A^*\oplus A^*\oplus A^* &\xrightarrow{\Delta^*+(\mathscr{P}^1)^*+(\mathscr{P}^{p+1})^*+(\mathscr{P}^{2p})^*} A^* \\ &\xrightarrow{(c^*(\mathscr{P}^{(p-1)p})\Delta)^*} A^*/A^*\mathscr{P}^1+A^*(2\mathscr{P}^p\mathscr{P}^1\Delta-\mathscr{P}^{p+1}\Delta), \end{aligned} \tag{4.11}$$

($c^*$ denotes the conjugation of $A^*$), which was proved partially in [7; (3.7)].

Our results will be applied to the calculations of the $p$-components of the stable homotopy groups of spheres in forthcoming paper [5].

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## §2 Cohen's methods

Let $c^*: A^* \to A^*$ (resp. $c_*: A_* \to A_*$) be the anti-automorphism (or conjugation) of $A^*$ (resp. $A_*$) of [3]. For the multiplication $\phi^*$ and the comultiplication $\psi^*$ of $A^*$, $c^*$ satisfies the following (see [4; Propositions 8.6 and 8.7])

$$c^*\phi^* = \phi^* T(c^* \otimes c^*),\ \psi^* c^* = (c^* \otimes c^*) T\psi^*, \tag{2.1}$$

*where* $T$ *denotes the switching map, i.e.,* $T(\alpha \otimes \beta) = (-1)^{\deg\alpha \deg\beta} \beta \otimes \alpha$.

For the simplicity, we shall denote

$$P^i = c^*(\mathscr{P}^i) \quad \text{and} \quad \beta = c^*(\Delta),$$

the conjugation of the reduced power operation $\mathscr{P}^i$ and the Bockstein operation $\Delta$, respectively. Then the Adem relations can be rewritten as follows, by use of (2.1):

$$P^r P^s = \sum_i (-1)^{s+i} \binom{(r-i)(p-1)-1}{s-pi} P^i P^{r+s-i} \qquad \text{for } s < rp, \tag{2.2}$$

$$P^r \beta P^s = \sum_i (-1)^{s+i} \binom{(r-i)(p-1)}{s-pi} P^i P^{r+s-i} \beta$$

$$+ \sum_i (-1)^{s+i+1} \binom{(r-i)(p-1)-1}{s-pi-1} P^i \beta P^{r+s-i} \qquad \text{for } s \leqq rp.$$

The following lemma is easy.

LEMMA 2.1. *Let* $r < p$. *Then we have*

$$P^r = (-1)^r \mathscr{P}^r,\ P^{p+r} = (-1)^{r+1} \mathscr{P}^p \mathscr{P}^r,\ \beta = -\Delta. \tag{2.3}$$

$$P^{rp} \equiv (-1)^r \mathscr{P}^{rp} \bmod A^* \mathscr{P}^1,\ P^{rp} \equiv (-1)^r \mathscr{P}^{rp} \bmod \mathscr{P}^1 A^*. \tag{2.4}$$

PROOF. The formula (2.3) is (1.7) and (1.8) of [6]. In particular, we have

$$P^1 = -\mathscr{P}^1,\ P^p = -\mathscr{P}^p. \tag{$*$}$$

By the Adem relations and (2.2), we have

$$r!\mathscr{P}^{rp} \equiv (\mathscr{P}^p)^r \text{ mod } A^*\mathscr{P}^1 \text{ and mod } \mathscr{P}^1 A^*,$$

$$r!P^{rp} \equiv (P^p)^r \text{ mod } A^*\mathscr{P}^1 \text{ and mod } \mathscr{P}^1 A^*.$$

By (*), $(P^p)^r = (-1)^r(\mathscr{P}^p)^r$ hence

$P^{rp} \equiv (-1)^r \mathscr{P}^{rp}$ mod $A^*\mathscr{P}^1$ and mod $\mathscr{P}^1 A^*$. q. e. d.

Let $\xi_i \in A_{2p^i-2}(\xi_0=1)$ and $\tau_i \in A_{2p^i-1}$ be the dual elements to $P^1P^p\cdots P^{p^{i-1}}$ and $\beta P^1P^p\cdots P^{p^{i-1}}$, respectively, with respect to the basis of the conjugations of the admissible monomials. These are the conjugations ($c_*$-images) of Milnor's $\xi_i$ and $\tau_i$ of [3]. According to Milnor's results [3] and (2.1), the Hopf algebra structure of $A_*$ is given by

(2.5) $$A_* = Z_p[\xi_1, \xi_2, \cdots] \otimes \Lambda(\tau_0, \tau_1, \cdots),$$

$$\phi_*(\xi_k) = \sum_{i=0}^{k} \xi_i \otimes \xi_{k-i}^{p^i}, \quad \phi_*(\tau_k) = 1 \otimes \tau_k + \sum_{i=0}^{k} \tau_i \otimes \xi_{k-i}^{p^i},$$

*where $\phi_*$ is the dual map of $\phi^*$, i. e., the comultiplication of $A_*$.*

For any element $\alpha \in A^*$, we shall denote

$$\alpha^*: A^* \to A^* \text{ and } \alpha_*: A_* \to A_*$$

the right translation by $\alpha$ and the dual map of $\alpha^*$, respectively. Then $\alpha_*$ is given by

(2.6) $$\langle \alpha_*(a), \alpha' \rangle = \langle a, \alpha'\alpha \rangle \qquad \text{for any } \alpha' \in A^*.$$

Obviously, $(\ )^*$ is contravariant and $(\ )_*$ is covariant, i.e.,

(2.7) $$(\alpha\alpha')^* = \alpha'^*\alpha^* \text{ and } (\alpha\alpha')_* = \alpha_*\alpha'_* \qquad \text{for any } \alpha, \alpha' \in A^*.$$

The following theorem is due to J. Cohen ([1; Theorem 5.18] and [2; Theorem 5.8]).

THEOREM 2.2 (Cohen). *The following formulas are established.*

(2.8) $$P_*^n(ab) = \sum_{i+j=n} P_*^i(a) P_*^j(b) \qquad \text{for } a, b \in A_*.$$

(2.9) $$P_*^n(\xi_k) = \begin{cases} \xi_k & \text{for } n=0 \\ \xi_{k-1} & \text{for } n=p^{k-1} \\ 0 & \text{for other } n. \end{cases}$$

$$(2.10)\qquad P_*^n(\tau_k)=\begin{cases}\tau_k & \text{for } n=0\\ \tau_{k-1} & \text{for } n=p^{k-1}\\ 0 & \text{for other } n.\end{cases}$$

$$(2.11)\qquad \beta_*(ab)=\beta_*(a)b+(-1)^{\deg a}a\beta_*(b),\qquad \text{for } a,\ b\in A_*.$$

$$(2.12)\qquad \beta_*(\xi_k)=\beta_*(\tau_l)=0\qquad \text{for } k\geqq 0,\ l\geqq 1,\qquad \beta_*(\tau_0)=1.$$

*In the above, we interpret* $\xi_{-1}=\tau_{-1}=0$, $\xi_0=1$.

REMARK. By (2.7) and this theorem, $\alpha_*$ is determined entirely for any $\alpha\in A^*$.

Let $n$ be a non-negative integer,

$$A_*(n)=Z_p[\xi_1^{p^n},\cdots,\ \xi_n^p,\ \xi_{n+1},\ \xi_{n+2},\cdots]\otimes\Lambda(\tau_{n+1},\ \tau_{n+2},\cdots)$$

be the subalgebra of $A_*$ and let

$$A'_*(n)=\{\tau_0^{\varepsilon_0}\cdots\tau_n^{\varepsilon_n}\xi_1^{r_1}\cdots\xi_n^{r_n};\ \varepsilon_i=0,\ 1,\ r_i<p^{n+1-i}\},$$

where $\{a_1,\cdots,a_n\}$ denotes the vector space over $Z_p$ spanned by the elements $a_1,\cdots,a_n$. Then

$$A_*=A'_*(n)\otimes A_*(n),\qquad n\geqq 0,$$

as vector space over $Z_p$. Also let

(2.13) $A^*(n)$ be the subalgebra of $A^*$ generated by $\beta$, $P^1,\cdots,P^{p^{n-1}}$.

In these notations, we have the following

PROPOSITION 2.3. *Let* $L$ *be a linear subspace of* $A'_*(n)$ *and* $\alpha\in A^*(n)$, $a\in A_*(n)$, $a'\in A'_*(n)$. *Then,*

$$(2.14)\qquad \alpha_*(a'a)=\alpha_*(a')a,\ \alpha_*A'_*(n)\subset A'_*(n),$$

$$(2.15)\qquad \operatorname{Ker}(\alpha_*|L\otimes A_*(n))=\operatorname{Ker}(\alpha_*|L)\otimes A_*(n),$$

$$\alpha_*(L\otimes A_*(n))=(\alpha_*L)\otimes A_*(n).$$

PROOF. Let $0<k\leqq p^{n-1}$. By (2.9) and (2.10),

$$P_*^k(a)=0\qquad \text{for } a=\xi_i^{p^{n+1-i}}(i\leqq n),\ \xi_j \text{ and } \tau_j(j\geqq n+1),$$

and by (2.8), $P_*^kA_*(n)=0$. Then $P_*^k(a'a)=P_*^k(a')a$ by (2.8). Similarly, $\beta_*(a'a)=\beta_*(a')a$ by (2.11) and (2.12). By use of (2.7), we obtain the first statement of (2.14). Similarly, we obtain the second statement of (2.14).

Next, assume that $a=\sum a'_ia_i\in\operatorname{Ker}(\alpha_*|L\otimes A_*(n))$, $a'_i\in L$, $a_i\in A_*(n)$. To prove $a \in \operatorname{Ker}(\alpha_* | L) \otimes A_*(n)$, we may assume further that $\deg a'_1 = \deg a'_2 = \cdots$, $\deg a_1 = \deg a_2 = \cdots$ and that the elements $a_i$ are the monomials, $a_i \neq a_j (i \neq j)$. Then such an expression of $a = \sum a'_i a_i$ is unique. By the hypothesis and by (2.14), $0 = \alpha_*(a) = \sum (\alpha_* a'_i) a_i$, $\alpha_* a'_i \in A'_*(n)$. The uniqueness of this expression implies $\alpha_* a'_i = 0$, $i = 1, 2, \cdots$. Thus,

$$\operatorname{Ker}(\alpha_* | L \otimes A_*(n)) \subset \operatorname{Ker}(\alpha_* | L) \otimes A_*(n).$$

The rest of (2.15) is an easy consequence of (2.14). q. e. d.

## §3 Exact sequences

In §§3–4, we shall use the following

CONVENTION 3.1. Let $\alpha, \beta_1, \cdots, \beta_n \in A^*$. We shall denote $A^*(\beta_1, \cdots, \beta_n)$ the left $A^*$-submodule of $A^*$ generated by $\beta_1, \cdots, \beta_n$. Composing the map $\alpha^*$ with the natural projection, we obtain the map of left $A^*$-modules $A^* \to A^*/M$, $M = A^*(\beta_1, \cdots, \beta_n)$. Then we shall use the same notation $\alpha^*$ for this map. Furthermore, let $\gamma_1, \cdots, \gamma_m \in A^*$ with $\gamma_i \alpha \in M$. Then $\alpha^*: A^* \to M$ induces the map $A^*/A^*(\gamma_1, \cdots, \gamma_m) \to A^*/M$, and we shall also use the same notation $\alpha^*$ for this map.

The following lemma is immediate.

LEMMA 3.2. *Let $\alpha, \beta_1, \cdots, \beta_l, \gamma_1, \cdots, \gamma_m, \delta_1, \cdots, \delta_n \in A^*$ with $\gamma_i \alpha \in A^*(\delta_1, \cdots, \delta_n)$. Then the exactness of the following sequence*

$$A^* \oplus \cdots \oplus A^* \xrightarrow{\beta_1^* + \cdots + \beta_l^*} A^*/A^*(\gamma_1, \cdots, \gamma_m) \xrightarrow{\alpha^*} A^*/A^*(\delta_1, \cdots, \delta_n)$$

*is equivalent to*

$$\text{(3.1)}\quad \operatorname{Ker}\beta_{1*} \cap \cdots \cap \operatorname{Ker}\beta_{l*} \cap \operatorname{Ker}\gamma_{1*} \cap \cdots \cap \operatorname{Ker}\gamma_{m*} = \alpha_*(\operatorname{Ker}\delta_{1*} \cap \cdots \cap \operatorname{Ker}\delta_{n*}).$$

To prove the exactness of the sequences of left $A^*$-modules, we shall show the equalities of the form of (3.1) by making use of Theorem 2.2 and Proposition 2.3.

In this section, we shall prove several exact sequences with respect to the elements in $A^*(1)$ of (2.13). For the element $\alpha \in A^*(1)$, the map $\alpha_*$ is determined by (2.7), (2.14), (2.15) and

$$\text{(3.2)}\quad P_*^k(\xi_1^r) = \binom{r}{k} \xi_1^{r-k}, \quad P_*^k(\tau_0 \xi_1^r) = \binom{r}{k} \tau_0 \xi_1^{r-k},$$

$$P_*^k(\tau_1 \xi_1^r) = \binom{r}{k} \tau_1 \xi_1^{r-k} + \binom{r}{k-1} \tau_0 \xi_1^{r-k+1}, \quad P_*^k(\tau_0 \tau_1 \xi_1^r) = \binom{r}{k} \tau_0 \tau_1 \xi_1^{r-k},$$

$$\beta_*(\xi_1^r)=\beta_*(\tau_1\xi_1^r)=0,\ \beta_*(\tau_0\xi_1^r)=\xi_1^r,\ \beta_*(\tau_0\tau_1\xi_1^r)=\tau_1\xi_1^r.$$

This is an easy consequence of Theorem 2.2.

THEOREM 3.3. *The followings are the exact sequences of left $A^*$-modules.*

$$A^*/A^*\mathscr{P}^1 \xrightarrow{(\mathscr{P}^1\Delta)^*} A^*/A^*\mathscr{P}^1 \xrightarrow{(\mathscr{P}^1\Delta)^*} A^*/A^*\mathscr{P}^1. \tag{3.3}$$

$$A^*\oplus A^* \xrightarrow{(\mathscr{P}^1)^*+(\mathscr{P}^1\Delta)^*} A^* \xrightarrow{(\mathscr{P}^i\Delta)^*} A^*/A^*\mathscr{P}^i, \qquad 2\leqq i<p. \tag{3.4}$$

$$A^*\oplus A^* \xrightarrow{(\mathscr{P}^1)^*+(\mathscr{P}^1\Delta)^*} A^* \xrightarrow{(\mathscr{P}^1\Delta\mathscr{P}^{p-1})^*} A^*. \tag{3.5}$$

PROOF. By (2.3), the exactness of these sequences is equivalent to the exactness of the sequences, which are given by the replacement of $\mathscr{P}^i$ and $\Delta$ by $P^i$ and $\beta$. Using (2.14), (2.15) and (3.2), we have

$$\mathrm{Ker}P^1_* = \{1,\ \tau_0,\ \tau_1-\tau_0\xi_1,\ \tau_0\tau_1\}\otimes A_*(1),$$

$$(P^1\beta)_*(\mathrm{Ker}P^1_*)=\{1,\ \tau_0\}\otimes A_*(1)=\mathrm{Ker}P^1_*\cap\mathrm{Ker}(P^1\beta)_*,$$

where $\{a_1,\cdots, a_n\}$ denotes the vector space over $Z_p$ spanned by the elements $a_1, \cdots, a_n$. By Lemma 3.2 we have the exact sequence

$$A^*/A^*P^1 \xrightarrow{(P^1\beta)^*} A^*/A^*P^1 \xrightarrow{(P^1\beta)^*} A^*/A^*P^1.$$

Thus (3.3) is proved.

To prove (3.4) and (3.5), we shall use the following known fact, which is equivalent to (1.1) of §1:

$$\mathrm{Ker}P^i_*=\mathrm{Im}P^{p-i}_*, \qquad 1\leqq i<p. \tag{*}$$

Then,

$$\mathrm{Im}(P^1\beta P^{p-1})_*=(P^1\beta)_*(\mathrm{Im}P^{p-1}_*)$$
$$=(P^1\beta)_*(\mathrm{Ker}P^1_*)=\mathrm{Ker}P^1_*\cap\mathrm{Ker}(P^1\beta)_*.$$

Thus (3.5) is proved.

By (2.2) and the simple calculations on binomial coefficients, we have

$$P^iP^j=\binom{i+j}{i}P^{i+j} \qquad \textit{for } j<p,$$

(**)

$$P^i\beta=P^1\beta P^{i-1}-(i-1)\beta P^i \qquad \textit{for } i\leqq p.$$

Therefore, for $i<p$

$$(P^i\beta)_*(\mathrm{Ker}P^i_*)=(P^1\beta P^{i-1})_*(\mathrm{Ker}P^i_*) \qquad \text{by } (**)$$

$$=(P^1\beta P^{i-1})_*(\mathrm{Im}P_*^{p-i}) \qquad \text{by } (*)$$

$$=\mathrm{Im}(P^1\beta P^{p-1})_*=\mathrm{Ker}P_*^1\cap\mathrm{Ker}(P^1\beta)_* \qquad \text{by (3.5).}$$

Thus, (3.4) is proved. q. e. d.

THEOREM 3.4. *The following sequences are exact.*

$$A^*/A^*R_k\xrightarrow{(\mathscr{P}^k)^*}A^*/A^*\varDelta\xrightarrow{(\mathscr{P}^{p-k})^*}A^*/A^*R_k\xrightarrow{(\mathscr{P}^k)^*}A^*/A^*\varDelta, \tag{3.6}$$

*for* $1\leqq k<p$, *where* $R_k=(k+1)\mathscr{P}^1\varDelta-k\varDelta\mathscr{P}^1$.

$$A^*\oplus A^*\xrightarrow{\varDelta^*+(\varDelta\mathscr{P}^1)^*}A^*\xrightarrow{(\varDelta\mathscr{P}^1\varDelta)^*}A^*. \tag{3.7}$$

$$A^*\oplus A^*\xrightarrow{\varDelta^*+(\mathscr{P}^1)^*}A^*\xrightarrow{(\mathscr{P}^1\varDelta\mathscr{P}^{p-1})^*}A^*/A^*\varDelta. \tag{3.8}$$

$$A^*\oplus A^*\xrightarrow{(\mathscr{P}^1)^*+(\varDelta\mathscr{P}^1\varDelta)^*}A^*\xrightarrow{(\mathscr{P}^{p-1})^*}A^*/A^*\varDelta. \tag{3.9}$$

$$A^*\oplus A^*\xrightarrow{\varDelta^*+(\mathscr{P}^1\varDelta\mathscr{P}^{p-1})^*}A^*\xrightarrow{(\varDelta\mathscr{P}^1)^*}A^*. \tag{3.10}$$

PROOF. By (2.3), we can replace $\mathscr{P}^i$ and $\varDelta$ by $P^i$ and $\beta$ respectively. By the direct calculations, we have

$$\mathrm{Ker}\beta_*\cap\mathrm{Ker}P_*^k=\{\xi_1^r(r<k),\ \tau_1\xi_1^r(r<k-1)\}\otimes A_*(1)$$

$$=(P^{p-k})_*(\mathrm{Ker}R_{k*}),$$

$$\mathrm{Ker}R_{k*}\cap\mathrm{Ker}(P^{p-k})_* \tag{3.11}$$

$$=\{\xi_1^r(r<p-k),\ k\tau_0\xi_1^r+r\tau_1\xi_1^{r-1}(r\leqq p-k)\}\otimes A_*(1)=P_*^k(\mathrm{Ker}\beta_*),$$

where $1\leqq k<p$. Thus (3.6) is proved.

Similarly we obtain (3.7–10). q. e. d.

## §4 Exact sequences (continued)

This section consists of the exact sequences related with the elements in $A^*(2)$ of (2.13). In this section, let $r$ and $s$ denote the non-negative integers smaller than $p$, and set $a_{r,s}=\xi_1^{rp}\xi_2^s$.

By Theorem 2.2, we have

$$P_*^{kp}(\tau_0^\varepsilon\tau_1^{\varepsilon'}a_{r,s})=\tau_0^\varepsilon\tau_1^{\varepsilon'}\sum_{i+j=k}\binom{r}{i}\binom{s}{j}\xi_1^ja_{r-i,s-j}, \tag{4.1}$$

$$P_*^{kp}(\tau_0^\varepsilon\tau_2a_{r,s})=\tau_0^\varepsilon\tau_2\sum_{i+j=k}\binom{r}{i}\binom{s}{j}\xi_1^ja_{r-i,s-j}$$

$$+\tau_0^{\varepsilon}\tau_1 \sum_{i+j=k-1}\binom{r}{i}\binom{s}{j}\xi_1^j a_{r-i,s-j},$$

$$P_*^{kp}(\tau_0^{\varepsilon}\tau_1\tau_2 a_{r,s})=\tau_0^{\varepsilon}\tau_1\tau_2 \sum_{i+j=k}\binom{r}{i}\binom{s}{j}\xi_1^j a_{r-i,s-j},$$

*where* $\varepsilon$, $\varepsilon'=0$ *or* 1.

THEOREM 4.1. *The following sequence is exact.*

$$(4.2)\quad A^*\oplus A^*\oplus A^*\oplus A^* \xrightarrow{\Delta^*+(\mathscr{P}^2)^*+(2\mathscr{P}^p\mathscr{P}^1-\mathscr{P}^{p+1})^*+(P^{(p-1)p})^*} A^* \xrightarrow{(\Delta\mathscr{P}^p\mathscr{P}^1\Delta)^*} A^*/A^*\Delta\mathscr{P}^1.$$

PROOF. It is proved in [6; Proposition 1.7] that the following sequence is exact:

$$A^*\oplus A^*\oplus A^*\oplus A^* \xrightarrow{\Delta^*+(\mathscr{P}^2)^*+(2\mathscr{P}^p\mathscr{P}^1-\mathscr{P}^{p+1})^*+(P^{(p-1)p})^*} A^* \xrightarrow{(\mathscr{P}^p)^*} A^*/A^*(\Delta, \mathscr{P}^1).$$

So we shall prove

$$(*)\qquad (\Delta\mathscr{P}^p\mathscr{P}^1\Delta)_*(\mathrm{Ker}(\Delta\mathscr{P}^1)_*)=\mathscr{P}_*^p(\mathrm{Ker}\Delta_*\cap\mathrm{Ker}\mathscr{P}_*^1).$$

By the direct calculations,

$$(\beta P^1\beta)_*(\mathrm{Ker}(\beta P^1)_*)=\{1\}\otimes A_*(1)=\mathrm{Ker}\beta_*\cap\mathrm{Ker}P_*^1.$$

Thus,

$$\begin{aligned}(\beta P^{p+1}\beta)_*(\mathrm{Ker}(\beta P^1)_*)&=P_*^p(\beta P^1\beta)_*(\mathrm{Ker}(\beta P^1)_*)\\&=P_*^p(\mathrm{Ker}\beta_*\cap\mathrm{Ker}P_*^1).\end{aligned}$$

Thus, by (2.3) the equality is proved. q. e. d.

REMARK. One can prove the exactness of the sequence in the above proof, by use of our method directly.

Now, we consider the elements

$$W_k=(k+1)\mathscr{P}^p\mathscr{P}^1\Delta-k\mathscr{P}^{p+1}\Delta+(k-1)\Delta\mathscr{P}^{p+1}.$$

In [7; Lemma 3.9], H. Toda has proved the exactness of the following sequences:

$$(4.3)\quad A^*/A^*\mathscr{P}^1 \xrightarrow{(W_{p-2})^*} A^*/A^*\mathscr{P}^1 \xrightarrow{(W_{p-3})^*} \cdots \xrightarrow{(W_2)^*} A^*/A^*\mathscr{P}^1.$$

(4.4) $A^*\oplus A^*\xrightarrow{(W_2)^*+(\mathscr{P}^{p-1}P^{2(p-1)p})^*}A^*/A^*\mathscr{P}^1\xrightarrow{(W_1)^*}A^*/A^*\mathscr{P}^1,$

*for* $p>3$.

(4.5) $A^*\oplus A^*\xrightarrow{(W_{p-1})^*+(\varDelta\mathscr{P}^{p+1}\varDelta\mathscr{P}^1\varDelta)^*}A^*/A^*\mathscr{P}^1\xrightarrow{(W_{p-2})^*}A^*/A^*\mathscr{P}^1,$

*for* $p>3$.

(4.6) $A^*\oplus A^*\oplus A^*\xrightarrow{(W_2)^*+(\varDelta\mathscr{P}^4\varDelta\mathscr{P}^1\varDelta)^*+(\mathscr{P}^2P^{12})^*}$

$A^*/A^*\mathscr{P}^1\xrightarrow{(W_1)^*}A^*/A^*\mathscr{P}^1,$ *for* $p=3$.

Let $\tau_i'-\tau_0\xi_i(i=1, 2)$, in the followings. By (3.2) and (4.1), we have

(4.7) $W_{k*}(a_{r,s})=0,\quad W_{k*}(\tau_0 a_{r,s})=sa_{r,s-1},$

$W_{k*}(\tau_1' a_{r,s})=ra_{r-1,s},\quad W_{k*}(\tau_2 a_{r,s})=-(k-1)a_{r,s},$

$W_{k*}(\tau_0\tau_1 a_{r,s})=s\tau_1' a_{r,s-1}-r\tau_0 a_{r-1,s},$

$W_{k*}(\tau_0\tau_2 a_{r,s})=k\tau_0 a_{r,s}+s\tau_2 a_{r,s-1},$

$W_{k*}(\tau_1'\tau_2 a_{r,s})=r\tau_2 a_{r-1,s}+k\tau_1' a_{r,s},$

$W_{k*}(\tau_0\tau_1\tau_2 a_{r,s})=-(k+1)\tau_0\tau_1 a_{r,s}+s\tau_1'\tau_2 a_{r,s-1}-r\tau_0\tau_2 a_{r-1,s}.$

THEOREM 4.2. *With respect to* $W_k$, *the following exact sequences also hold.*

(4.8) $A^*\oplus A^*\oplus A^*\oplus A^*\oplus A^*\xrightarrow{(\mathscr{P}^1)^*+(\varDelta\mathscr{P}^1\varDelta)^*+(W_p)^*+\gamma_1^*+\gamma_2^*}$

$A^*\xrightarrow{(W_{p-1})^*}A^*/A^*\mathscr{P}^1,$

*where* $\gamma_1=\varDelta\mathscr{P}^{p+1}\varDelta P^{(p-1)p}$, $\gamma_2=\mathscr{P}^1\varDelta\mathscr{P}^p\mathscr{P}^1\varDelta\mathscr{P}^{p-1}P^{(p-1)p}$.

(4.9) $A^*\oplus A^*\oplus A^*\xrightarrow{\varDelta^*+(\mathscr{P}^1)^*+\gamma_3^*}A^*\xrightarrow{(W_p\mathscr{P}^1)^*}A^*/A^*(R_{p-2}, \mathscr{P}^2),$

*where* $\gamma_3=\varDelta\mathscr{P}^{p+1}P^{(p-1)p}\mathscr{P}^{p-1}P^{(p-1)p}$.

(4.10) $A^*/A^*(\mathscr{P}^1, \mathscr{P}^{p+1})\xrightarrow{(W_{p-2})^*}A^*/A^*(\mathscr{P}^1, \mathscr{P}^{p+1})\xrightarrow{(W_{p-3})^*}$

$\cdots\xrightarrow{(W_1)^*}A^*/A^*(\mathscr{P}^1, \mathscr{P}^{p+1}).$

PROOF. We consider the vector spaces

$L_k=\{a_{r,s}((r, s)\neq(p-1, p-1)),\quad a_{p-1,p-1}(\text{if } k\not\equiv 1 \bmod p),$

$s\tau_1' a_{r,s-1}-r\tau_0 a_{r-1,s},\quad k\tau_0 a_{r,s}+s\tau_2 a_{r,s-1},\quad r\tau_2 a_{r-1,s}+k\tau_1' a_{r,s},$

$-(k+1)\tau_0\tau_1 a_{r,s}+s\tau_1'\tau_2 a_{r,s-1}-r\tau_0\tau_2 a_{r-1,s}\},$

$$M_k=\begin{cases}\{\tau_0\tau_1\tau_2\} & \textit{for } k\equiv -1 \bmod p,\ p>3\\ \{\tau_0\tau_1,\ \tau_0\tau_2a_{p-1,0},\ \tau_1'\tau_2a_{0,p-1}\} & \textit{for } k\equiv 0 \bmod p\\ \{\tau_0a_{p-1,0},\ \tau_1'a_{0,p-1},\ \tau_2a_{p-1,p-1}\} & \textit{for } k\equiv 1 \bmod p\\ \{a_{p-1,p-1}\} & \textit{for } k\equiv 2 \bmod p,\ p>3\\ \{a_{p-1,p-1},\ \tau_0\tau_1\tau_2\} & \textit{for } k\equiv 2 \bmod p,\ p=3\\ 0 & \textit{for } k\not\equiv -1,\ 0,\ 1,\ 2 \bmod p,\ p>3.\end{cases}$$

By use of (4.7), we have

$$(*)\qquad \begin{aligned} &W_{k*}(\mathrm{Ker}\mathscr{P}^1_*)=L_k\otimes A_*(2),\\ &\mathrm{Ker}\mathscr{P}^1_*\cap\mathrm{Ker}\,W_{k*}=(L_{k-1}+M_k)\otimes A_*(2).\end{aligned}$$

Therefore to prove (4.8), it is sufficient to show the following relations:

$$\gamma_{0*}(\tau_0\tau_1)\neq 0,\ \gamma_{0*}(\tau_0\tau_2a_{p-1,0})=\gamma_{0*}(\tau_1'\tau_2a_{0,p-1})=\gamma_{0*}L_{p-1}=0,$$

$$\gamma_{1*}(\tau_0\tau_2a_{p-1,0})\neq 0,\ \gamma_{1*}(\tau_0\tau_1)=\gamma_{1*}(\tau_1'\tau_2a_{0,p-1})=\gamma_{1*}L_{p-1}=0,$$

$$\gamma_{2*}(\tau_1'\tau_2a_{0,p-1})\neq 0,\ \gamma_{2*}(\tau_0\tau_1)=\gamma_{2*}(\tau_0\tau_2a_{p-1,0})=\gamma_{2*}L_{p-1}=0,$$

where $\gamma_0=\Delta\mathscr{P}^1\Delta=-\beta P^1\beta$. By (2.3),

$$\gamma_1=\beta P^1P^p\beta P^{(p-1)p} \text{ and } \gamma_2=-P^1\beta P^pP^{1-1}P^{(p-1)p},$$

and so the above relations are verified by the routine calculations.

To prove (4.10), we consider the vector space

$$N_k=\{a_r(r<p-1),\ a_{p-1}\ (\text{if } k\not\equiv 0 \bmod p),\quad \tau_0a_r(r<p-1),\ \tau_0a_{p-1}\ (\text{if } k\not\equiv 0 \bmod p),\quad r\tau_2'a_{r-1}+(k+1)\tau_1'a_r,\ r\tau_0\tau_2a_{r-1}+(k+1)\tau_0\tau_1a_r\},$$

where $a_r=a_{r,0}=\xi_1^{rp}$. Then by (3.2), (4.1) and (4.7), we obtain

$$W_{k*}(\mathrm{Ker}\mathscr{P}^1_*\cap\mathrm{Ker}\mathscr{P}^{p+1}_*)=N_k\otimes A_*(2)\qquad \textit{for all } k,$$

$$\mathrm{Ker}\,W_{k*}\cap\mathrm{Ker}\mathscr{P}^1_*\cap\mathrm{Ker}\mathscr{P}^{p+1}_*=N_{k-1}\otimes A_*(2)\qquad \textit{for } k\not\equiv\pm 1,\ 0 \bmod p.$$

Thus (4.10) is proved.

By (3.2) and (3.11), we have

$$\mathscr{P}^1_*(\mathrm{Ker}R_{p-2*}\cap\mathrm{Ker}\mathscr{P}^2_*)=\{1,\ \tau_0,\ \tau_1'\}\otimes A_*(1)$$
$$=\{a_{r,s},\ \tau_0a_{r,s},\ \tau_1'a_{r,s},\ \tau_2a_{r,s},\ \tau_0\tau_2a_{r,s},\ \tau_1'\tau_2a_{r,s}\}\otimes A_*(2).$$

Then by (4.7), we have

$$(W_p\mathscr{P}^1)_*(\mathrm{Ker}R_{p-2*}\cap\mathrm{Ker}\mathscr{P}^2_*)=\{a_{r,s},\ \tau_2a_{r,s}\ \text{except}\ \tau_2a_{p-1,p-1}\}\otimes A_*(2).$$

On the other hand,

$$\mathrm{Ker}\Delta_*\cap\mathrm{Ker}\mathscr{P}^1_*=\{1\}\otimes A_*(1)=\{a_{r,s},\ \tau_2a_{r,s}\}\otimes A_*(2).$$

Therefore to prove (4.9), it suffices to show the following relations, which are verified by the direct calculations:

$$\gamma_{3*}(a_{r,s})=0\quad \textit{for all } (r,s),\quad \gamma_{3*}(\tau_2a_{r,s})=0\quad \textit{for } (r,s)\neq(p-1,p-1),$$
$$\gamma_{3*}(\tau_2a_{p-1,p-1})\neq 0. \qquad \text{q. e. d.}$$

REMARK. From (*) of above, the exactness of (4.3–6) can be verified immediately.

THEOREM 4.3. *The followings are exact.*

(4.11)
$$A^*\oplus A^*\oplus A^*\oplus A^*\xrightarrow{\Delta^*+(\mathscr{P}^1)^*+(\mathscr{P}^{p+1})^*+(\mathscr{P}^{2p})^*}A^*\xrightarrow{(P^{(p-1)p}\Delta)^*}A^*/A^*(\mathscr{P}^1,\ W_1).$$

(4.12)
$$A^*\oplus A^*\oplus A^*\xrightarrow{(\mathscr{P}^1)^*+(\mathscr{P}^1\Delta)^*+(\mathscr{P}^{p^2-1})^*}A^*\xrightarrow{(\mathscr{P}^{p+1}\mathscr{P}^1)^*}A^*/A^*(\mathscr{P}^2,\ \mathscr{P}^2\Delta).$$

(4.13) *For* $p>3$,

$$A^*\oplus A^*\oplus A^*\oplus A^*\xrightarrow{(\mathscr{P}^1)^*+(\mathscr{P}^{p+1})^*+(W_2)^*+(\mathscr{P}^{(p-2)p})^*}A^*\xrightarrow{(\mathscr{P}^{2p}\mathscr{P}^1)^*}A^*/A^*(\mathscr{P}^2,\ \mathscr{P}^2\Delta,\ \mathscr{P}^{p+1}\mathscr{P}^1).$$

*For* $p=3$,

$$A^*\oplus A^*\oplus A^*\oplus A^*\xrightarrow{(\mathscr{P}^1)^*+(\mathscr{P}^3)^*+(\mathscr{P}^4\Delta)^*+(\mathscr{P}^4\Delta\mathscr{P}^1\Delta)^*}A^*\xrightarrow{(\mathscr{P}^6\mathscr{P}^1)^*}A^*/A^*(\mathscr{P}^2,\ \mathscr{P}^2\Delta,\ \mathscr{P}^4\mathscr{P}^1).$$

*Here* $W_k=(k+1)\mathscr{P}^p\mathscr{P}^1\Delta-k\mathscr{P}^{p+1}\Delta+(k-1)\Delta\mathscr{P}^{p+1}$.

PROOF. In the above, by (2.3) and (2.4), we can replace $\Delta$, $\mathscr{P}^i(i\leqq p)$, $\mathscr{P}^{p+1}$, $\mathscr{P}^{2p}$, $\mathscr{P}^{p^2-1}$, $\mathscr{P}^{(p-2)p}$, $\mathscr{P}^{2p}\mathscr{P}^1$ and $\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta$ by $\beta$, $P^i$, $P^1P^p$, $P^{2p}$, $P^{p-1}P^{(p-1)p}$, $P^{(p-2)p}$, $P^{2p+1}$ and $P^1\beta P^{p+1}\beta$ respectively. By the routine calculations, we can verify the followings:

$$(P^{(p-1)p}\beta)_*(\mathrm{Ker}P^1_*\cap\mathrm{Ker}\,W_{1*})=\{1,\ a_1\}\otimes A_*(2)$$
$$=\mathrm{Ker}\beta_*\cap\mathrm{Ker}P^1_*\cap\mathrm{Ker}(P^1P^p)_*\cap\mathrm{Ker}P^{2p}_*,$$

$$(P^1P^{p+1})_*(\mathrm{Ker}P^2_* \cap \mathrm{Ker}(P^2\beta)_*) = \{a_{r,s}(s<p-1),\ \tau_0 a_{r,s}(s<p-1),$$
$$\tau_2 a_{r,s}(s<p-2), \tau_2' a_{r,p-2},\ \tau_0\tau_2 a_{r,s}(s<p-1)\} \otimes A_*(2)$$
$$= \mathrm{Ker}P^1_* \cap \mathrm{Ker}(P^1\beta)_* \cap \mathrm{Ker}(P^{p-1}P^{(p-1)p})_*,$$

$$(P^{2p+1})_*(\mathrm{Ker}P^2_* \cap \mathrm{Ker}(P^2\beta)_* \cap \mathrm{Ker}(P^1P^{p+1})_*)$$
$$= \{a_r(r<p-2),\ \tau_0 a_r(r<p-2),\ 2\tau_1' a_r + r\tau_2' a_{r-1}(r<p-1),$$
$$2\tau_0\tau_1 a_r + r\tau_0\tau_2 a_{r-1}(r<p-1)\} \otimes A_*(2)$$
$$= \begin{cases} \mathrm{Ker}P^1_* \cap \mathrm{Ker}(P^1P^p)_* \cap \mathrm{Ker}\, W_{2*} \cap \mathrm{Ker}P^{(p-2)p}{}_* & \text{for } p>3 \\ \mathrm{Ker}P^1_* \cap \mathrm{Ker}P^3_* \cap \mathrm{Ker}(P^1P^3\beta)_* \cap \mathrm{Ker}(P^1\beta P^4\beta)_* & \text{for } p=3, \end{cases}$$

where $a_{r,0} = \xi_1^{rp}$, $\tau_i' = \tau_i - \tau_0\xi_i (i=1, 2)$.

Thus the exactness of (4.11–13) is verified. q. e. d.

THEOREM 4.4. *The followings are exact.*

$$(4.14) \qquad A^* \oplus A^* \oplus A^* \xrightarrow{\Delta^* + (\mathscr{P}^1)^* + (\mathscr{P}^p)^*} A^* \xrightarrow{(\Delta\mathscr{P}^{p+1}\Delta\mathscr{P}^1\Delta)^*} A^*/A^*(\mathscr{P}^1,\ W_{p-1}),$$

*where* $W_{p-1} = \mathscr{P}^{p+1}\Delta - 2\Delta\mathscr{P}^{p+1}$.

$$(4.15) \qquad A^* \oplus A^* \xrightarrow{(\mathscr{P}^1)^* + (\mathscr{P}^{p^2-1})^*} A^* \xrightarrow{(\mathscr{P}^{2p-1}\mathscr{P}^1)^*} A^*.$$

PROOF. The exactness of (4.15) is equivalent to the exactness of

$$(4.15)' \qquad A^* \xrightarrow{(\mathscr{P}^{p^2-1})^*} A^*/A^*\mathscr{P}^1 \xrightarrow{(\mathscr{P}^{p+1}\mathscr{P}^1)^*} A^*/A^*\mathscr{P}^1.$$

The proofs of (4.14) and (4.15)′ are similar to the proofs of theorems already stated. q. e. d.

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