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Energy Inequalities and Cauchy Problems for a System of Linear Partial Differential Equations

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In the present paper we consider a system of linear partial differential operators with variable coefficients written in matrix notation

$$Lu=D_t^m u+\sum_{j=0}^{m-1}\sum_{j+|p|\leq m}A_{j,p}(t,\ x)D_t^jD_x^pu, \qquad (m\geqq 1)$$

where the $A_{j,p}(t, x)$ are $N\times N$ matrices whose entries lie in $\mathscr{B}(H)$, H being a slab $0\leqq t\leqq T$, $x\in R_n$. By $u\in\mathscr{D}'(\mathring{H})$ we mean that each component of u lies in $\mathscr{D}'(\mathring{H})$. To simplify the notation, similar abbreviations will constantly be used for vector distributions. A Cauchy problem for L with $t=0$ as initial hyperplane has been formulated in a generalized sense in a related paper [6]: To find in $\mathscr{D}'(\mathring{H})$ a solution u satisfying

$$Lu=f \qquad \text{in } \mathring{H}$$

under the condition

$$u_0\equiv\lim_{t\downarrow 0}(u,\ D_tu,\ \cdots,\ D_t^{m-1}u)=\alpha$$

for preassigned $f\in\mathscr{D}'(\mathring{H})$ and $\alpha\in\mathscr{D}'(R_n)$. Here $\lim_{t\downarrow 0} u$ denotes the distributional boundary value of u which was defined in [6] in accord with S. Łojasiewicz [10]. If a solution u exists, f must have a canonical extension over $t=0$. If this is a case and u satisfies $Lu=f$ in $\mathring{H}$, then u_0 exists if and only if u has an extension over $t=0$, that is, u is a restriction to $\mathring{H}$ of a $U\in\mathscr{D}'((-\infty,\ T)\times R_n)$. Most spaces of distributions encountered in the usual treatments of partial differential equations have such an extension property. For example, as for $\mathscr{H}_{(\sigma,s)}(H)$, the property is involved in its definition [4].

The purpose of this paper is to investigate Cauchy problems for L from our stand-points, imposing on L or L^* some additional conditions such as energy inequalities of Friedrichs-Levy type. While we regard such inequalities as a priori estimates, they are usually deduced from the properties involved in a differential system called hyperbolic.

In Section 2 we deal with energy inequalities with the aid of the lemmas given in Section 1. The equivalence of $[E_{(0)}]$ and $[E_{(s)}]$ are shown. In Section 3 we consider a Cauchy problem in a generalized sense to make clear our approach to studying Cauchy problems from our view-points. Section 4 is devoted to some remarks to the spaces $\mathscr{H}_{(\sigma,s)}(H)$ in connection with distributional boundary value, canonical extension and so on. In Section 5 we show an approximation theorem, which, together with energy inequalities, enables us to obtain uniqueness and existence theorems for Cauchy problems. This was done in Section 6 through a Hilbert space approach. The results are so arranged as to be compared with those recently established by T. Bałaban [1] for a hyperbolic pseudo-differential operator. In Section 7 we consider more strict energy inequalities which we can deduce from assuming $[E_{(0)}]$ for both L and L^*. In the final section, some investigations are made on uniqueness and existence of a solution to a Cauchy problem in a general sense already mentioned.

1. Preliminary lemmas

We denote by $x=(x_1, \cdots, x_n)$, y, z points of Euclidean space R_n, and by $p=(p_1, \cdots, p_n)$ the multi-indices. We write Ξ_n for the dual Euclidean space with points $\xi=(\xi_1, \cdots, \xi_n)$. $<x, \xi>=\sum_{j=1}^{n} x_j\xi_j$ is the dual pairing. As usual, we write $|x|=(\sum_{j=1}^{n}|x_j|^2)^{\frac{1}{2}}$, $|p|=\sum_{j=1}^{n}p_j$, $x^p=x_1^{p_1}\cdots x_n^{p_n}$, $D_x^p=(-i)^{|p|}\frac{\partial^{|p|}}{\partial x_1^{p_1}\cdots\partial x_n^{p_n}}$, and so on.

In what follows, unless otherwise stated, we shall use the notations of L. Hörmander [4], where the Fourier transform $\hat{\phi}$, $\phi\in\mathscr{S}(R_n)$, is defined by

$$\hat{\phi}(\xi)=\int\phi(x)e^{-i<x,\xi>}dx,$$

which is extended by continuity to a temperate distribution $u\in\mathscr{S}'(R_n)$ by the formula

$$<\hat{u}, \phi>=<u, \hat{\phi}>.$$

Here $\mathscr{S}(R_n)$ denotes the space of complex valued C^∞-functions on R_n such that $\sup_{|p|\leq l}\sup_{x\in R_n}(1+|x|^2)^{\frac{l}{2}}|D_x^p\phi(x)|<\infty$ for all non-negative integers l. The topology in $\mathscr{S}(R_n)$ is defined by the semi-norms $\sup_{|p|\leq l}\sup_{x\in R_n}(1+|x|^2)^{\frac{l}{2}}|D_x^p\phi(x)|$. By $\mathscr{S}'(R_n)$ we mean the space of the temperate distributions on R_n, that is, the strong dual of $\mathscr{S}(R_n)$.

Let $\mathscr{B}(R_n)$ denote the space of complex valued C^∞-functions defined on R_n which possess bounded derivatives of all orders. The topology is defined by the family of all semi-norms

$$|||a|||_l = \sup_{|p|\leq l} \sup_{x\in R_n} |D_x^p a|, \qquad a \in \mathscr{B}(R_n).$$

Let $\chi \in C_0^\infty(R_n)$ and assume that for some integer $k \geqq 0$

(i) $$\hat{\chi}(\xi) = O(|\xi|^k), \qquad \xi \to 0,$$

but that

(ii) $\hat{\chi}(t\xi) = 0$ for all real t implies $\xi = 0$ if $\xi \in \Xi_n$.

For our later purpose we shall need the following three lemmas generalizing the corresponding results of Hörmander [4, Corollary 2.4.1, Theorem 2.4.2 and Theorem 2.4.3]. The proofs of these lemmas will be omitted since it is not difficult to carry out them by modifying the methods exhibited there.

Let us now recall that $\mathscr{H}_{(s)}$, $-\infty < s < +\infty$, is the space of temperate distributions u on R_n such that $\hat{u}$ is a function and

$$||u||_{(s)}^2 = \frac{1}{(2\pi)^n} \int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty.$$

LEMMA 1. *If* (i) *and* (ii) *are valid and* $s < k$, *it follows that there exist positive constants* C_1 *and* C_2, *independent of* u *but depending on* s, σ *and* χ, *such that*

$$C_1 ||u||_{(s+\sigma)}^2 \leqq \int_0^1 ||u * \chi_\varepsilon||_{(\sigma)}^2 \varepsilon^{-2s-1} d\varepsilon + ||u||_{(\sigma)}^2 \leqq C_2 ||u||_{(s+\sigma)}^2 \qquad \text{for } s \geqq 0,$$

$$C_1 ||u||_{(s+\sigma)}^2 \leqq \int_0^1 ||u * \chi_\varepsilon||_{(\sigma)}^2 \varepsilon^{-2s-1} d\varepsilon \leqq C_2 ||u||_{(s+\sigma)}^2 \qquad \text{for } s < 0$$

for any $u \in \mathscr{H}_{(s+\sigma)}$.

LEMMA 2. *Let* $a \in \mathscr{B}(R_n)$ *and let* χ *satisfy* (i). *If* $s < k$, *there exist a non-negative integer* l *and a constant* C, *independent of* u *but depending on* s *and* σ *such that we have*

$$\int_0^1 ||a(u * \chi_\varepsilon) - (au) * \chi_\varepsilon||_{(\sigma)}^2 \varepsilon^{-2s-1} d\varepsilon \leqq C |||a|||_l^2 ||u||_{(\sigma+s-1)}^2$$

for any $u \in \mathscr{H}_{(\sigma+s-1)}$.

LEMMA 3. *Let* $a \in \mathscr{B}(R_n)$ *and* $\chi \in C_0^\infty(R_n)$. *Then for any* $u \in \mathscr{H}_{(s-1)}$ *there exist a non-negative integer* l *and a constant* C, *independent of* ε *and* u *but depending on* s, *such that when* $0 < \varepsilon < 1$

$$||a(u * \chi_\varepsilon) - (au) * \chi_\varepsilon||_{(s)} \leqq C |||a|||_l ||u||_{(s-1)},$$

and we have

$$a(u*\chi_\varepsilon)-(au)*\chi_\varepsilon\to 0 \textit{ in } \mathcal{H}_{(s)} \textit{ when } \varepsilon\to 0.$$

2. Energy inequalities

Let H be the slab $[0, T]\times R_n$, $T>0$. We shall consider a system of linear partial differential operators:

$$L\equiv D_t^m+\sum_{j=0}^{m-1}\sum_{j+|p|\leq m}A_{j,p}(t, x)D_t^jD_x^p,$$

where each $A_{j,p}$ is an $N\times N$ matrix with entries in $\mathcal{B}(H)$, the space of C^∞-functions a on H such that a is bounded with its derivatives of every order. Any function in $\mathcal{B}(H)$ is, as easily verified by an argument due to R. T. Seeley [13], a restriction of function in $\mathcal{B}(R_{n+1})$. Thus we assume that the entries of $A_{j,p}$ belong to $\mathcal{B}(R_{n+1})$.

We shall denote by $C_0^\infty(H)=\mathcal{D}(H)$ the space of the restrictions to H of the functions in $C_0^\infty(R_{n+1})$, equipped with the quotient topology. For a vector function $u=(u_1, \dots, u_N)$ we write $u\in C_0^\infty(R_{n+1})$ if $u_j\in C_0^\infty(R_{n+1})$, $j=1, 2,\dots, N$. Then, by $\|u(t, \cdot)\|_{(s)}$ we mean the norm defined by $\|u(t, \cdot)\|_{(s)}^2=\sum_{j=1}^N\|u_j(t, \cdot)\|_{(s)}^2$.

Now we shall introduce an inequality of Friedrichs-Levy type:

$$[E_{(0)}]\qquad \sum_{j=0}^{m-1}\|D_t^ju(t, \cdot)\|_{(m-1-j)}^2\leqq C\Big(\sum_{j=0}^{m-1}\|D_t^ju(0, \cdot)\|_{(m-1-j)}^2+ +\int_0^t\|Lu(t', \cdot)\|_{(0)}^2dt'\Big),\qquad 0\leqq t\leqq T,\quad u\in C_0^\infty(R_{n+1}),$$

where C is a constant independent of u. Similarly, for any s we shall introduce

$$[E_{(s)}]\qquad \sum_{j=0}^{m-1}\|D_t^ju(t, \cdot)\|_{(s+m-1-j)}^2\leqq C\Big(\sum_{j=0}^{m-1}\|D_t^ju(0, \cdot)\|_{(s+m-1-j)}^2+ +\int_0^t\|Lu(t', \cdot)\|_{(s)}^2dt'\Big),\qquad 0\leqq t\leqq T,\quad u\in C_0^\infty(R_{n+1}),$$

where C may depend on s. In what follows, by C with or without index we shall denote a constant having a different meaning according to the cases.

In the proof of Theorem 1 below we shall need the following lemma (cf. [2, p. 72]) which is easily verified.

LEMMA 4. *Let $r(t)$ and $\rho(t)$ be two real-valued functions defined in the interval $0\leqq t\leqq T$ and suppose that r is continuous and ρ is non-decreasing. Then the inequality*

$$r(t) \leqq C(\rho(t) + \int_0^t r(t')dt') \qquad (C>0 \textit{ is a constant})$$

implies

$$r(t) \leqq Ce^{Ct}\rho(t).$$

THEOREM 1. *$[E_{(0)}]$ implies $[E_{(s)}]$ and vice versa.*

PROOF. First we show that $[E_{(0)}]$ implies $[E_{(s)}]$. Let χ be a function satisfying the conditions (i) and (ii) in Section 1. For any $u \in C_0^\infty(R_{n+1})$ we put $u_\varepsilon(t) = u *' \chi_\varepsilon$, $\varepsilon > 0$, where by $*'$ we mean the partial convolution with respect to the variable x. By our hypothesis u_ε satisfies the inequality $[E_{(0)}]$. Hence we have

$$\sum_{j=0}^{m-1} \|D_t^j u_\varepsilon(t, \cdot)\|^2_{(m-1-j)} \leqq C(\sum_{j=0}^{m-1} \|D_t^j u_\varepsilon(0, \cdot)\|^2_{(m-1-j)} +$$

$$+ 2\int_0^t \|((Lu)*'\chi_\varepsilon)(t', \cdot)\|^2_{(0)} dt' + 2\int_0^t \|(L(u*'\chi_\varepsilon) - (Lu)*'\chi_\varepsilon)(t', \cdot)\|^2_{(0)} dt').$$

Here we can write with a constant C_0

$$\|(L(u*'\chi_\varepsilon) - (Lu)*'\chi_\varepsilon)(t', \cdot)\|^2_{(0)}$$

$$\leqq C_0 \sum_{j=0}^{m-1} \sum_{j+|p| \leq m} \|(A_{j,p} D_t^j D_x^p (u*'\chi_\varepsilon) - (A_{j,p} D_t^j D_x^p u)*'\chi_\varepsilon)(t', \cdot)\|^2_{(0)}$$

$$= C_0 \sum_{j=0}^{m-1} \sum_{j+|p| \leq m} \|(A_{j,p}((D_t^j D_x^p u)*'\chi_\varepsilon) - (A_{j,p} D_t^j D_x^p u)*'\chi_\varepsilon)(t', \cdot)\|^2_{(0)}.$$

We first confine ourselves to the case $s<0$. Owing to Lemmas 1 and 2 the following estimates are valid:

$$\sum_{j=0}^{m-1} \int_0^1 \|D_t^j u_\varepsilon(t, \cdot)\|^2_{(m-1-j)} \varepsilon^{-2s-1} d\varepsilon \geqq C_1 \sum_{j=0}^{m-1} \|D_t^j u(t, \cdot)\|^2_{(s+m-1-j)},$$

$$\sum_{j=1}^{m-1} \int_0^1 \|D_t^j u_\varepsilon(0, \cdot)\|^2_{(m-1-j)} \varepsilon^{-2s-1} d\varepsilon \leqq C_2 \sum_{j=0}^{m-1} \|D_t^j u(0, \cdot)\|^2_{(s+m-1-j)},$$

$$\int_0^1 \int_0^t \|((Lu)*'\chi_\varepsilon)(t', \cdot)\|^2_{(0)} \varepsilon^{-2s-1} dt' d\varepsilon \leqq C_2 \int_0^t \|Lu(t', \cdot)\|^2_{(s)} dt',$$

and

$$\int_0^1 \int_0^t \|(L(u*'\chi_\varepsilon) - (Lu)*'\chi_\varepsilon)(t', \cdot)\|^2_{(0)} \varepsilon^{-2s-1} dt' d\varepsilon$$

$$\leqq C_3 \sum_{j=0}^{m-1} \int_0^t \|D_t^j u(t', \cdot)\|^2_{(s+m-1-j)} dt',$$

where constants C_j, $j=1, 2, 3$ are independent of u but depending on s.

Now setting $r(t)=\sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(s+m-1-j)}^2$ and $\rho(t)=r(0)+\int_0^t\|Lu(t',\cdot)\|_{(s)}^2dt'$, we obtain from these estimates the following inequality

$$r(t)\leqq C(\rho(t)+\int_0^t r(t')dt').$$

We can therefore apply Lemma 4 to obtain the estimate $[E_{(s)}]$, as desired.

We now turn to the case $s>0$. We assume that χ is chosen to satisfy the condition $s<k$ as in Lemma 2. A slight modification of the arguments given above will lead us to the same inequality:

$$r(t)\leqq C(\rho(t)+\int_0^t r(t')dt'+\sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(m-1-j)}^2).$$

Applying Lemma 4 with $\rho(t)$ replaced by $\rho(t)+\sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(m-1-j)}^2$, we obtain the estimate $[E_{(s)}]$ as before.

Conversely, assume that $[E_{(s)}]$ holds. Put $s'=-s$. Making use of the estimate $[E_{(s)}]$, we have with a constant C'

$$\sum_{j=0}^{m-1}\int_0^1\|D_t^j u_\varepsilon(t,\cdot)\|_{(s+m-1-j)}^2\varepsilon^{-2s'-1}d\varepsilon$$

$$\leqq C'(\sum_{j=0}^{m-1}\int_0^1\|D_t^j u_\varepsilon(0,\cdot)\|_{(s+m-1-j)}^2\varepsilon^{-2s'-1}d\varepsilon+\int_0^1\int_0^t\|((Lu)*'\chi_\varepsilon))(t',\cdot)\|_{(s)}^2\varepsilon^{-2s'-1}dt'd\varepsilon+$$

$$+\sum_{j=0}^{m-1}\sum_{j+|p|\leq m}\int_0^1\int_0^t\|(A_{j,p}((D_t^jD_x^p u)*'\chi_\varepsilon)-(A_{j,p}D_t^jD_x^p u)*'\chi_\varepsilon)(t',\cdot)\|_{(s)}^2\varepsilon^{-2s'-1}dt'd\varepsilon).$$

Suppose $s'<0$. Then, applying Lemmas 1 and 2, we shall obtain with a constant C''

$$\sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(m-1-j)}^2\leqq C''(\sum_{j=0}^{m-1}\|D_t^j u(0,\cdot)\|_{(m-1-j)}^2+\int_0^t\|Lu(t',\cdot)\|_{(0)}^2dt'+$$

$$+\sum_{j=0}^{m-1}\int_o^t\|D_t^j u(t',\cdot)\|_{(m-1-j)}^2dt').$$

Consequently, by Lemma 4, we have the estimate $[E_{(0)}]$. When $s'>0$, we assume that χ is chosen to satisfy the condition $s'<k$ as in Lemma 2. If we take into account the corresponding part of the proof of the implication $[E_{(0)}]\Rightarrow[E_{(s)}]$, it is easy to see that the same arguments given above will lead to the estimate $[E_{(0)}]$. Thus the proof is complete.

We also consider the energy inequalities

$$[E_{(s)}:\downarrow]\quad \sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(s+m-1-j)}^2\leqq C(\sum_{j=0}^{m-1}\|D_t^j u(T,\cdot)\|_{(s+m-1-j)}^2+$$

$$+\int_t^T \|Lu(t', \cdot)\|_{(s)}^2 dt').$$

From the above discussions it will be evident that Theorem 1 remains valid for these energy inequalities.

COROLLARY 1. *L satisfies $[E_{(0)}]$ if and only if the principal part L_m of L satisfies $[E_{(0)}]$.*

PROOF. It is easy to verify the assertion from the arguments given in the proof of Theorem 1.

3. Some observations about the Cauchy problems

In [6] one of the authors investigated in a general framework Cauchy problem for a system of linear partial differential equations with $t=0$ as initial hyperplane. Let us first recall some notions introduced there to make clear our stand-point. Given a distribution $u \in \mathcal{D}'(\mathring{H})$ we understand the distributional boundary value $\lim_{t\downarrow 0} u=\alpha \in \mathcal{D}'(R_n)$ as follows: Let $\phi \in C_0^\infty(R_t^+)$ be arbitrarily chosen in such a way that $\phi \geqq 0$, $\int \phi dt=1$ and supp $\phi \subset [1,2]$. We put $\phi_\varepsilon(t)=\frac{1}{\varepsilon}\phi\left(\frac{t}{\varepsilon}\right)$, $\varepsilon>0$. If ε is small enough, $\phi_\varepsilon(t)u$ is a distribution $\in \mathcal{D}'(R_{n+1})$. If $\lim_{\varepsilon\to 0}\phi_\varepsilon u$ exists in $\mathcal{D}'(R_{n+1})$ and equals $\delta\otimes\alpha$ for any choice of ϕ with the properties just indicated, we define $\alpha=\lim_{t\downarrow 0} u$. Similarly we can speak of the boundary value $\lim_{t\uparrow T} u$.

Another important notion is a canonical extension of a distribution $u \in \mathcal{D}'(\mathring{H})$. Let $\rho(t)=\int_0^t \phi(t')dt'$ and $\rho_{(\varepsilon)}(t)=\rho\left(\frac{t}{\varepsilon}\right)$, where ϕ is a function chosen as above. $\rho_{(\varepsilon)}u$ will be understood as a distribution $\in \mathcal{D}'((-\infty, T)\times R_n)$ vanishing for $t<\varepsilon$. If there exists a distribution $u_\smile \in \mathcal{D}'((-\infty, T)\times R_n)$ such that $\lim_{\varepsilon\to 0}\rho_{(\varepsilon)}u=u_\smile$ in $\mathcal{D}'((-\infty, T)\times R_n)$ for any choice of ϕ, $u_\smile$ is called a canonical extension of u over $t=0$. In this connection we note that if $v=D_t u$ in $\mathring{H}$, then v has the canonical extension over $t=0$ if and only if $\lim_{t\downarrow 0} u$ exists [6, p. 14]. Similarly we can speak of a canonical extension over $t=T$, denoted by $u^\frown$, and therefore a two-sided canonical extension which we shall denote by $(u_\smile)^\frown$ or $(u^\frown)_\smile$. We shall frequently make use of the following facts which will be easily verified: If $\lim_{t\downarrow 0} u$ exists, then

$$\lim_{t\downarrow 0}(u*'\chi)=(\lim_{t\downarrow 0} u)*\chi, \qquad \chi \in C_0^\infty(R_n),$$

$$\lim_{t\downarrow 0}(\phi u)=\phi(0,\cdot)\lim_{t\downarrow 0} u,\qquad \phi\in C^{\infty}(R_{n+1}),$$

and

$$\lim_{t\downarrow 0} D_x^p u=D_x^p(\lim_{t\downarrow 0} u).$$

Analogues hold also for canonical extensions.

The Cauchy problem for the differential operator L is formulated in a generalized sense. It is to find a solution $u=(u_1, \cdots, u_N)$, $u_j\in\mathcal{D}'(\mathring{H})$ to the equation

$$Lu\equiv D_t^m u+\sum_{j=0}^{m-1}\sum_{j+|p|\leq m} A_{j,p}(t, x)D_t^j D_x^p u=f\qquad \text{in } \mathring{H},$$

with the initial condition

$$u_0\equiv\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)=\alpha,$$

where $f=(f_1, \cdots, f_N)$, $f_j\in\mathcal{D}'(\mathring{H})$ and $\alpha=(\alpha_0, \cdots, \alpha_{m-1})$, $\alpha_{j,k}\in\mathcal{D}'(R_n)$ are given. For the sake of simplicity, we shall write $u\in\mathcal{D}'(\mathring{H})$ if each component $u_j\in\mathcal{D}'(\mathring{H})$. A similar abbreviation for a vector distribution will be used when there occurs no fear of confusions.

Suppose there exists a solution $u\in\mathcal{D}'(\mathring{H})$. As shown in [6, p. 18], u and f must have the canonical extensions $u_\sim$, $f_\sim$. We shall see that $u_\sim$ satisfies the equation

$$L(u_\sim)=f_\sim+\sum_{k=0}^{m-1} D_t^k\delta\otimes\gamma_k(0)\qquad \text{in } (-\infty, T)\times R_n.$$

Here we put

$$\gamma_k(t)=-i\sum_{j=k+1}^{m}\sum_{l=1}^{j-k}(-1)^{j-l-k}\binom{j-l}{k}D_t^{j-l-k}A_j(t, x, D_x)\alpha_{l-1},$$

where $A_j(t, x, D_x)$ abbreviates $\sum_{|p|\leq m-j} A_{j,p}(t, x)D_x^p$ for $j<m$ and A_m is the unit matrix. In fact, from the identity

$$\rho_{(\varepsilon)}D_t^j u=D_t^j(\rho_{(\varepsilon)}u)+D_t^{j-1}(i\phi_\varepsilon u)+D_t^{j-2}(i\phi_\varepsilon D_t u)+\cdots+(i\phi_\varepsilon D_t^{j-1}u)\qquad (1)$$

it follows that

$$(D_t^j u)_\sim=D_t^j(u_\sim)+iD_t^{j-1}(\delta\otimes\alpha_0)+iD_t^{j-2}(\delta\otimes\alpha_1)+\cdots+i\delta\otimes\alpha_{j-1}.$$

Consequently

$$L(u_\sim)=f_\sim-i\sum_{j=1}^{m}\sum_{l=1}^{j} A_j(t, x, D_x)(D_t^{j-l}\delta\otimes\alpha_{l-1})$$

$$=f_\sim - i\sum_{j=1}^{m}\sum_{l=1}^{j}\sum_{k=0}^{j-l} D_t^k\delta\otimes\{(-1)^{j-l-k}\binom{j-l}{k}D_t^{j-l-k}A_j(0,\ x,\ D_x)\alpha_{l-1}\}$$

$$=f_\sim + \sum_{k=0}^{m-1} D_t^k\delta\otimes\{-i\sum_{j=k+1}^{m}\sum_{l=1}^{j-k}(-1)^{j-l-k}\binom{j-l}{k}D_t^{j-l-k}A_j(0,\ x,\ D_x)\alpha_{l-1}\}$$

$$=f_\sim + \sum_{k=0}^{m-1} D_t^k\delta\otimes\gamma_k(0).$$

Before proving the converse, we observe that γ_{m-k-1} is rewritten in the form:

$$\gamma_{m-k-1}(t) = -i\alpha_k + \sum_{j=0}^{k-1}\sum_{|q|+j\le k} B_{k,q,j}(t,\ x)D_x^q\alpha_j, \qquad k=0,\ \cdots,\ m-1, \tag{2}$$

where $B_{k,q,j}$ is a linear combination of derivatives of the coefficient matrices $A_{j,p}$ of order up to at most $k-1$. This implies that in a special case $m=1$ we have $\gamma_0(t) = -i\alpha_0$. In what follows, we shall use the notation

$$\Gamma_t(\alpha) = (\gamma_0(t),\ \cdots,\ \gamma_{m-1}(t)).$$

Now it is easy to see that the mapping $\mathcal{D}'(R_n) \ni \alpha \to \gamma(0) = (\gamma_0(0),\ \cdots,\ \gamma_{m-1}(0)) \in \mathcal{D}'(R_n)$ is a linear isomorphism.

Suppose that a vector distribution $v \in \mathcal{D}'((-\infty,\ T)\times R_n)$ with support in $[0,\ T)\times R_n$ is a solution to the equation

$$Lv = f_\sim + \sum_{j=0}^{m-1} D_t^k\delta\otimes\gamma_k(0). \tag{3}$$

Then, substituting

$$v_1 = v,\ v_2 = D_tv_1 + i\delta\otimes\alpha_0,\ \cdots,\ v_m = D_tv_{m-1} + i\delta\otimes\alpha_{m-2},$$

the equation (3) can be written in the equivalent form

$$D_tv_1 = v_2 - i\delta\otimes\alpha_0,$$
$$\vdots$$
$$D_tv_{m-1} = v_m - i\delta\otimes\alpha_{m-2},$$
$$D_tv_m = -\sum_{j=0}^{m-1}\sum_{j+|p|\le m} A_{j,p}D_x^pv_{j+1} - i\delta\otimes\alpha_{m-1} + f_\sim.$$

If we apply Theorem 1 in [6, p. 18], we can conclude that $u = v\,|\,\mathring{H}$ is a solution to the Cauchy problem in question with the initial data α, and that v is the canonical extension of u.

In view of Proposition 7 in [6, p. 21] we can also conclude the following

PROPOSITION 1. *Let u, f be vector distributions $\in \mathcal{D}'(\mathring{H})$ and assume that f has the canonical extension $f_\sim$ over $t=0$ and that u satisfies the equation Lu*

$=f$ *in* $\mathring{H}$. *Then* $\lim_{t\downarrow 0} u$ *exists if and only if* u *can be extended to a vector distribution* $\in \mathcal{D}'((-\infty, T)\times R_n)$.

Let L^* be the formal adjoint of L, that is,

$$L^*=\sum_{j=0}^{m-1} D_t^j A_j^*,$$

where

$$A_j^*=\sum_{|p|\leq m-j} D_x^p A_{j,p}^*(t, x) \qquad \text{for } j<m, \qquad A_m^*=I$$

and by $A_{j,p}^*$ we mean the adjoint of $A_{j,p}$.

From these considerations we obtain the following

PROPOSITION 2. *Let* $f\in\mathcal{D}'(\mathring{H})$, $\alpha\in\mathcal{D}'(R_n)$ *and assume that* f *has a canonical extension* $f_\sim$. *To find a solution* $u\in\mathcal{D}'(\mathring{H})$ *to the equation*

$$Lu=f$$

with the condition

$$u_0\equiv\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)=\alpha$$

is reduced to the problem of finding $v\in\mathcal{D}'((-\infty, T)\times R_n)$ *with* supp $v\subset [0,T)\times R_n$ *such that*

$$(v, L^*w)=(f_\sim, w)+(\Gamma_0(\alpha), w_0), \qquad w\in C_0^\infty((-\infty, T)\times R_n) \qquad (4)$$

where $w_0=\lim_{t\downarrow 0}(w, D_t w, \cdots, D_t^{m-1}w)$. u *and* v *are related by* $u=v\,|\,\mathring{H}$.

REMARK. We note that (4) leads us to Green's formula

$$((Lu)_\sim, w)-(u_\sim, L^*w)=-(\Gamma_0(u_0), w_0).$$

In a similar way we also obtain

PROPOSITION 3. *Let* $f\in\mathcal{D}'(\mathring{H})$ *and* $\alpha, \beta\in\mathcal{D}'(R_n)$ *and assume that* f *has a two-sided canonical extension* $(f_\sim)^\sim$. *To find a solution* $u\in\mathcal{D}'(\mathring{H})$ *to the equation*

$$Lu=f$$

with the conditions

$$u_0\equiv\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)=\alpha,$$

$$u_T\equiv\lim_{t\uparrow T}(u, D_t u, \cdots, D_t^{m-1}u)=\beta$$

is reduced to the problem of finding $v \in \mathring{\mathscr{D}}'(H)$ *such that*

$$(v,\ L^*w)=((f_\smile)^\frown,\ w)+(\Gamma_0(\alpha),\ w_0)-(\Gamma_T(\beta),\ w_T),\ w \in C_0^\infty(R_{n+1}), \quad (5)$$

where

$$w_0=\lim_{t\downarrow 0}(w,\ D_t w,\ \cdots,\ D_t^{m-1}w) \text{ and } w_T=\lim_{t\uparrow T}(w,\ D_t w,\ \cdots,\ D_t^{m-1}w).$$

Remark. We also note that (5) leads to Green's formula

$$(((Lu)_\smile)^\frown,\ w)-((u_\smile)^\frown,\ L^*w)=(\Gamma_T(u_T),\ w_T)-(\Gamma_0(u_0),\ w_0).$$

We shall denote by $\mathscr{S}'(H)$ the set of distributions $\in \mathscr{D}'(\mathring{H})$, which can be extensible to temperate distributions $\in \mathscr{S}'(R_{n+1})$. The quotient topology is introduced there. Similarly for $\mathscr{S}'((-\infty,\ T]\times R_n)$ or $\mathscr{S}'([0,\ \infty)\times R_n)$. By $\mathscr{S}(H)$ we also mean the space of the restrictions to H of the functions $\in \mathscr{S}(R_{n+1})$, equipped with the quotient topology. Now let $u \in \mathscr{S}'(H)$. If u has the canonical extension $u_\smile$ (resp. $u^\frown$) and $u_\smile \in \mathscr{S}'((-\infty,\ T]\times R_n)$ (resp. $u^\frown \in \mathscr{S}'([0,\ \infty)\times R_n)$), we shall say that u has the canonical $\mathscr{S}'$-extension over $t=0$ (resp. $t=T$). A two-sided canonical $\mathscr{S}'$-extension will be similarly defined.

Corollary 2. *Let* $u \in \mathscr{S}'(H)$ *and* $u_0,\ u_T \in \mathscr{S}'(R_n)$ *and assume that* u, Lu *have two-sided canonical* $\mathscr{S}'$*-extensions. Then for any* $w \in \mathscr{S}(R_{n+1})$ *we have*

$$(((Lu)_\smile)^\frown,\ w)-((u_\smile)^\frown,\ L^*w)=(\Gamma_T(u_T),\ w_T)-(\Gamma_0(u_0),\ w_0).$$

Proof. Take a sequence $\{\psi_j\}$, $\psi_j \in C_0^\infty(R_{n+1})$, such that it converges in $\mathscr{S}(R_{n+1})$ to w. Then from the preceding remark

$$((Lu)_\smile)^\frown,\ \psi_j)-((u_\smile)^\frown,\ L^*\psi_j)=(\Gamma_T(u_T),\ (\psi_j)_T)-(\Gamma_0(u_0),\ (\psi_j)_0).$$

Passing to the limit as $j\to\infty$, we obtain the equation which was to be proved.

4. Some remarks on the space $\mathscr{H}_{(\sigma,s)}(H)$

In the sequel the space $\mathscr{H}_{(\sigma,s)}(H)$, $-\infty<\sigma,\ s<+\infty$, will play a central rôle. $\mathscr{H}_{(\sigma,s)}(H)$ is the space of all distributions $u \in \mathscr{D}'(\mathring{H})$ such that there exists a distribution $U \in \mathscr{H}_{(\sigma,s)}(R_{n+1})$ with $U=u$ in $\mathring{H}$. The norm of u is defined by $\|u\|_{(\sigma,s)}=\inf \|U\|_{(\sigma,s)}$, the infimum being taken over all such U. We also consider the space $\mathring{\mathscr{H}}_{(\sigma,s)}(H)$, the space of all $u \in \mathscr{H}_{(\sigma,s)}(R_{n+1})$ with $\operatorname{supp} u \subset H$. Then $\mathscr{H}_{(\sigma,s)}(H)$ and $\mathring{\mathscr{H}}_{(-\sigma,-s)}(H)$ are anti-dual Hilbert spaces with respect to an extension of the sesquilinear form

$$\int_{R_n}\int_0^T u\bar{v}\,dt\,dx, \qquad u \in C_0^\infty(H), \qquad v \in C_0^\infty(\mathring{H}).$$

The scalar product between them will be denoted by $(\cdot, \cdot)$. These spaces are stable under the multiplication by the elements of $\mathcal{B}(H)$. In fact, this follows immediately from repeated use of the interpolation theorem for the Hilbert scales [8, p. 150]. A similar reasoning will be employed in the proof of Corollary 4 to Theorem 3 in Section 6. When $\sigma > \frac{1}{2}$ it is known that any $u \in \mathscr{H}_{(\sigma,s)}(H)$ has a trace $u(0, \cdot) \in \mathscr{H}_{(\sigma+s-\frac{1}{2})}(R_n)$ [4, p. 55; 5, p. 14]. From the definition of the trace it can be easily verified that it coincides with the boundary value. In the case where $\sigma = k$ is a non-negative integer, we may assume that $\mathscr{H}_{(k,s)}(H)$ is equipped with the equivalent norm:

$$\Big(\sum_{j=0}^{k}\int_0^T \|D_t^j u(t', \cdot)\|_{(k+s-j)}\,dt'\Big)^{\frac{1}{2}},$$

which will also be denoted by the same symbol $\|u\|_{(k,s)}$. As shown in [6, p. 16] we can see that $u \in \mathscr{H}_{(k,s)}(H)$, $k>0$, has the canonical extension $u_\sim \in \mathscr{H}_{(k,s)}((-\infty, T] \times R_n)$ if and only if $\lim_{t\downarrow 0} u = \lim_{t\downarrow 0} D_t u = \cdots = \lim_{t\downarrow 0} D_t^{k-1}u = 0$, and that every $u \in \mathscr{H}_{(0,s)}(H)$ has the canonical extension $u_\sim \in \mathscr{H}_{(0,s)}((-\infty, T] \times R_n)$. Similar statements about the other canonical extensions remain valid. We note that $\mathscr{H}_{(0,s)}(H)$ may be identified with $\mathring{\mathscr{H}}_{(0,s)}(H)$.

From now on, to simplify notations, we shall denote by u, f vector distributions. The rest of this section is devoted to show the following

PROPOSITION 4. *If* $u \in \mathscr{H}_{(0,k+s+m-1)}(H)$ *satisfies* $Lu = f \in \mathscr{H}_{(k,s-1)}(H)$, *then* $u \in \mathscr{H}_{(k+m,s-1)}(H)$.

For the proof we need Lemma 6 below. This will in turn be shown by making use of the following

LEMMA 5. *If* $u \in \mathscr{H}_{(0,k+s+m)}(H)$ *and* $Lu = f \in \mathscr{H}_{(k,s)}(H)$, *then* $\psi u \in \mathscr{H}_{(k+m,s)}(R_{n+1})$ *for any* $\psi \in C_0^\infty(\mathring{H})$.

PROOF. This follows by the same argument as used in [4, Theorem 4.3.1].

In what follows, we shall use the notations

$$\mathbf{H}_{(s)}(R_n) = \mathscr{H}_{(s+m-1)}(R_n) \times \mathscr{H}_{(s+m-2)}(R_n) \times \cdots \times \mathscr{H}_{(s)}(R_n),$$

$$\mathbf{H}^{\#}_{(s)}(R_n) = \mathscr{H}_{(s)}(R_n) \times \mathscr{H}_{(s+1)}(R_n) \times \cdots \times \mathscr{H}_{(s+m-1)}(R_n).$$

Owing to the relations (2) we can easily verify that Γ_t is an isomorphism of $\mathbf{H}_{(s)}$ onto $\mathbf{H}^{\#}_{(s)}(R_n)$ and that the sets $\{\Gamma_t\}_{0\le t\le T}$ and $\{\Gamma_t^{-1}\}_{0\le t\le T}$ are equicontinuous.

LEMMA 6. *Let* $u \in \mathscr{H}_{(0,k+s+m)}(H)$ *and assume that* $Lu=f \in \mathscr{H}_{(k,s)}(H)$ *and* $\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)=\alpha \in \mathbf{H}_{(k+s+\frac{1}{2})}(R_n)$ (*resp.* $\lim_{t\uparrow T}(u, D_t u, \cdots, D_t^{m-1}u)=\beta \in \mathbf{H}_{(k+s+\frac{1}{2})}(R_n)$) *exists. Then* $\phi u \in \mathscr{H}_{(k+m,s)}(H)$ *for every* $\phi \in C_0^\infty(R_t)$ *with* $\operatorname{supp}\phi \subset (-\infty, T)$ (*resp.* $\operatorname{supp}\phi \subset (0, \infty)$).

PROOF. It will be sufficient to prove the case where $Lu=f \in \mathscr{H}_{(k,s)}(H)$ and $\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)=\alpha \in \mathbf{H}_{(k+s+\frac{1}{2})}(R_n)$. In virtue of the trace theorem [4, p. 55] we can find a $v \in \mathscr{H}_{(k+m,s)}(\bar{R}_{n+1}^-)$ such that

$$\lim_{t\uparrow 0} D_t^j v=\alpha_j, \qquad 0\leqq j\leqq m-1$$

and

$$\lim_{t\uparrow 0} Lv=\lim_{t\downarrow 0} f,\ \lim_{t\uparrow 0} D_t(Lv)=\lim_{t\downarrow 0} D_t f,\ \cdots,\ \lim_{t\uparrow 0} D_t^{k-1}(Lv)=\lim_{t\downarrow 0} D_t^{k-1}f.$$

Let $v^\sim$ (resp. $(Lv)^\sim$) be the canonical extension of v (resp. Lv) over $t=0$. If we put $w=u_\sim+v^\sim$, then $w \in \mathscr{H}_{(0,k+s+m)}((-\infty, T]\times R_n)$ and

$$\begin{aligned} Lw&=L(u_\sim)+L(v^\sim)\\ &=(Lu)_\sim+\sum_{j=0}^{m-1} D_t^j\delta\otimes\gamma_j(0)+(Lv)^\sim-\sum_{j=0}^{m-1} D_t^j\delta\otimes\gamma_j(0)\\ &=(Lu)_\sim+(Lv)^\sim\\ &=f_\sim+(Lv)^\sim \in \mathscr{H}_{(0,k+s)}((-\infty, T]\times R_n). \end{aligned}$$

It follows from Lemma 5 that we have only to show that

$$\begin{aligned} D_t Lw&=D_t(f_\sim)+D_t((Lv)^\sim)\\ &=(D_t f)_\sim-i\delta\otimes(\lim_{t\downarrow 0} f)+(D_t Lv)^\sim+i\delta\otimes(\lim_{t\downarrow 0} f)\\ &=(D_t f)_\sim+(D_t Lv)^\sim \in \mathscr{H}_{(0,k+s-1)}((-\infty, T]\times R_n). \end{aligned}$$

Repeating this procedure, we obtain for $j\leqq k$

$$D_t^j Lw=(D_t^j f)_\sim+(D_t^j Lv)^\sim \in \mathscr{H}_{(0,k+s-j)}((-\infty, T]\times R_n).$$

which implies $Lw \in \mathscr{H}_{(k,s)}((-\infty, T]\times R_n)$, as desired.

We now turn to the proof of Proposition 4. In view of Proposition 1 we see that $\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1} u)=\alpha$ exists. Let $w \in C_0^\infty((-\infty, T)\times R_n)$. Applying Green's formula, we obtain

$$(f_\sim, w)-(u_\sim, L^*w)=-(\Gamma_0(\alpha), w_0),$$

where $w_0=(w(0, \cdot), D_t w(0, \cdot), \cdots, D_t^{m-1}w(0, \cdot))$.

Let us consider the space $X=\mathring{\mathscr{H}}_{(m,-k-s-m+1)}((-\infty, T]\times R_n)$, where

$C_0^\infty((-\infty, T)\times R_n)$ is dense. The anti-linear form

$$l:\quad C^\infty((-\infty, T)\times R_n)\ni w\rightarrow(f_\sim, w)-(u_\sim, L^*w)$$

is continuous in the topology of X, so it can be continuously extended to the whole space X.

Let X_0 be the set of $v\in X$ such that $\lim_{t\downarrow 0}(v, D_tv, \cdots, D_t^{m-1}v)$, the trace of $(v, D_tv, \cdots, D_t^{m-1}v)$ on $t=0$, vanishes. Then $v\in X_0$ can be approximated in X by a sequence $\{w_j\}$, $w_j\in C_0^\infty((-\infty, T)\times R_n)$ such that w_j vanishes near $t=0$. Indeed, v can be written as $v=v_1+v_2$, $v_1\in\mathring{\mathscr{H}}_{(m,-k-s-m+1)}(H)$, $v_2\in\mathring{\mathscr{H}}_{(m,-k-s-m+1)}(\bar{R}_{n+1}^-)$ and we know that $C_0^\infty(\mathring{H})$ (resp. $C_0^\infty(R_{n+1}^-)$) is dense in $\mathring{\mathscr{H}}_{(m,-k-s-m+1)}(H)$ (resp. $\mathring{\mathscr{H}}_{(m,-k-s-m+1)}(\bar{R}_{n+1}^-)$). Then we must have $l(v)=0$. Since by the trace theorem $X/X_0\simeq\mathbf{H}_{(-k-s-m+\frac{3}{2})}(R_n)$, we can conclude that $w_0\rightarrow(\Gamma_0(\alpha), w_0)$ can be continuously extended to an anti-linear form on $\mathbf{H}_{(-k-s-m+\frac{3}{2})}(R_n)$. It then follows that $\Gamma_0(\alpha)\in\mathbf{H}^\#_{(k+s-\frac{1}{2})}(R_n)=\mathscr{H}_{(k+s-\frac{1}{2})}\times\cdots\times\mathscr{H}_{(k+s+m-\frac{3}{2})}$ and then from the relations (2) we can see that $\alpha\in\mathbf{H}_{(k+s-\frac{1}{2})}(R_n)$.

In virtue of Lemma 6, if $\phi\in C_0^\infty((-\infty, T))$ such that $\phi=1$ near $t=0$, we can conclude that $\phi u\in\mathscr{H}_{(k+m,s-1)}(H)$. Similarly we can show that $(1-\phi)u\in\mathscr{H}_{(k+m,s-1)}(H)$. It then follows that $u=\phi u+(1-\phi)u\in\mathscr{H}_{(k+m,s-1)}(H)$. The proof is complete.

5. Approximation theorem

It will be shown in the next section that the energy inequalities combined with the following approximation theorem will play an essential rôle in our approach to the Cauchy problem for differential operators. We shall apply this theorem to obtain some generalizations of the energy inequalities considered in Section 2.

Let k be a non-negative integer, m the order of differential operator L and s a real number.

THEOREM 2 (*Approximation Theorem*). *Let $u\in\mathscr{H}_{(0,k+s+m-1)}(H)$ and assume that*

$$Lu=f\in\mathscr{H}_{(k,s)}(H),$$

$$u_0\equiv\lim_{t\downarrow 0}(u, D_tu, \cdots, D_t^{m-1}u)=\alpha\in\mathbf{H}_{(k+s)}(R_n).$$

Then there exists a sequence $\{u_j\}$, $u_j\in C_0^\infty(R_{n+1})$ such that

(i) $\quad u_j\rightarrow u\quad$ *in* $\mathscr{H}_{(k+m,s-1)}(H)$,

(ii) $\quad(u_j(0, x), D_tu_j(0, x),\cdots, D_t^{m-1}u_j(0, x))\rightarrow\alpha\quad$ *in* $\mathbf{H}_{(k+s)}(R_n)$,

(iii) $\qquad L(u_j)\to f \qquad in\ \mathcal{H}_{(k,s)}(H)$

as $j\to\infty$.

Furthermore if $\alpha=0$, *we can take* u_j *in* $C_0^\infty(R_{n+1}^+)$.

PROOF. In view of Proposition 4, we see that $u\in\mathcal{H}_{(k+m,s-1)}(H)$. Let $\chi\in C_0^\infty(R_n)$ be so chosen that $\chi\geqq 0$ and $\int\chi dt=1$. If we put $u_\varepsilon=u*'\chi_\varepsilon$, $\varepsilon>0$, then $u_\varepsilon\in\mathcal{H}_{(k+m,\infty)}(H)$ and u_ε tends to u in $\mathcal{H}_{(k+m,s-1)}(H)$ as $\varepsilon\to 0$. Furthermore the distributional limit of $D_t^j u_\varepsilon$, $j=0,\cdots,m-1$, exists and will be written as

$$\lim_{t\downarrow 0}(D_t^j u_\varepsilon)=(\lim_{t\downarrow 0} D_t^j u)*\chi_\varepsilon=\alpha_j*\chi_\varepsilon\in\mathcal{H}_{(\infty)}(R_n).$$

Now we can write

$$L(u_\varepsilon)-f=(L(u_\varepsilon)-(Lu)*'\chi_\varepsilon)+(f*'\chi_\varepsilon-f)$$
$$=\sum_{j=1}^{m-1}\sum_{j+|p|\le m}\{A_{j,p}((D_t^j D_x^p u)*'\chi_\varepsilon)-(A_{j,p}D_t^j D_x^p u)*'\chi_\varepsilon\}+(f*'\chi_\varepsilon-f).$$

$f*'\chi_\varepsilon-f$ tends to 0 in $\mathcal{H}_{(k,s)}(H)$ as $\varepsilon\to 0$. Since $D_t^j D_x^p u\in\mathcal{H}_{(k,s-1)}(H)$, it follows from Lemma 3 that

$$\|A_{j,p}((D_t^j D_x^p u)*'\chi_\varepsilon)-(A_{j,p}D_t^j D_x^p u)*'\chi_\varepsilon\|_{(k,s)}\to 0$$

as $\varepsilon\to 0$.

From these considerations we have only to show that, for any fixed $\varepsilon>0$, there exists a sequence $\{v_l\}$, $v_l\in C_0^\infty(R_{n+1})$, such that $u_\varepsilon=\lim_{l\to\infty} v_l$ in $\mathcal{H}_{(k+m,\infty)}(H)$. Indeed, in view of the trace theorem [4, p. 55], $(D_t^j v_l)(0,x)$ tends to $\alpha_j*'\chi_\varepsilon$ in $\mathcal{H}_{(\infty)}(R_n)$ as $l\to\infty$. The existence of $\{v_l\}$ is evident since $C_0^\infty(H)$ is dense in $\mathcal{H}_{(k+m,\infty)}(H)$.

Let us denote by $\mathcal{E}_t^0(\mathcal{H}_{(s)})$ the space of $\mathcal{H}_{(s)}(R_n)$-valued continuous function of t defined on the interval $[0, T]$. Applying the approximation theorem to the case $k=0$, we obtain the following

PROPOSITION 5. *Suppose that the energy inequality* $[E_{(0)}]$ *holds for* L *and that* $u\in\mathcal{H}_{(0,s+m-1)}(H)$ *satisfies* $Lu=f\in\mathcal{H}_{(0,s)}(H)$ *and* $u_0\equiv\lim_{t\downarrow 0}(u, D_t u,\cdots, D_t^{m-1})$ $\in\mathbf{H}_{(s)}(R_n)$. *Then* $u\in\mathcal{H}_{(m,s-1)}(H)$. *Furthermore* u *possesses the properties*:

(i) $\qquad (u, D_t u,\cdots, D_t^{m-1}u)\in\mathcal{E}_t^0(\mathcal{H}_{(s+m-1)})\times\cdots\times\mathcal{E}_t^0(\mathcal{H}_{(s)}) \qquad (6)$

(ii)
$$\sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(s+m-1-j)}^2\leqq C(\sum_{j=0}^{m-1}\|D_t^j u(0,\cdot)\|_{(s+m-1-j)}^2+\int_0^t\|f(t',\cdot)\|_{(s)}^2 dt'). \qquad (7)$$

Therefore if $f=0$, *and* $u_0=0$, *then* u *must be* 0.

PROOF. Applying Theorem 2 for the case $k=0$, we can find a sequence $\{u_l\}$, $u_l \in C_0^\infty(R_{n+1})$, with the properties mentioned there. It then follows from Theorem 1 that

$$\sum_{j=0}^{m-1}\|D_t^j u_l(t,\cdot)\|_{(s+m-1-j)}^2 \leqq C(\sum_{j=0}^{m-1}\|D_t^j u_l(0,\cdot)\|_{(s+m-1-j)}^2 + \int_0^t \|(L(u_l))(t',\cdot)\|_{(s)}^2 dt') \tag{8}$$

$$\sum_{j=0}^{m-1}\|D_t^j u_l(t,\cdot)-D_t^j u_{l'}(t,\cdot)\|_{(s+m-1-j)}^2$$
$$\leqq C(\sum_{j=0}^{m-1}\|D_t^j u_l(0,\cdot)-D_t^j u_{l'}(0,\cdot)\|_{(s+m-1-j)}^2 + \int_0^t \|(L(u_l))(t',\cdot)-(L(u_{l'}))(t',\cdot)\|_{(s)}^2 dt'). \tag{9}$$

(9) means that $\{D_t^j u_l(t,\cdot)\}$ is a Cauchy sequence in $\mathcal{E}_t^0(\mathcal{H}_{(s+m-1-j)})$. Let v be the limit of the sequence $\{u_l\}$. Clearly $u\equiv v$. It follows from (8) that the inequality (7) must hold true.

COROLLARY 3. *Suppose that the energy inequality* $[E_{(0)}]$ *holds for* L *and that* $u \in \mathcal{H}_{(0,k+s+m-1)}(H)$ *satisfies* $Lu=f\in\mathcal{H}_{(k,s)}(H)$ *and* $u_0\equiv\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)\in \mathbf{H}_{(k+s)}(R_n)$. *Then* $u\in\mathcal{H}_{(k+m,s-1)}(H)$. *Furthermore* u *possesses the properties*:

(i) $$(u, D_t u, \cdots, D_t^{k+m-1}u)\in\mathcal{E}_t^0(\mathcal{H}_{(k+s+m-1)})\times\cdots\times\mathcal{E}_t^0(\mathcal{H}_{(s)}), \tag{10}$$

(ii) $$\sum_{j=0}^{k+m-1}\|D_t^j u(t,\cdot)\|_{(k+s+m-1-j)}^2 \leqq C(\sum_{j=0}^{m-1}\|D_t^j u(0,\cdot)\|_{(k+s+m-1-j)}^2 + \sum_{j=0}^{k-1}\|D_t^j f(0,\cdot)\|_{(k+s-1-j)}^2 + \sum_{j=0}^{k}\int_0^t\|D_t^j f(t',\cdot)\|_{(k+s-j)}^2 dt'). \tag{11}$$

Therefore if $f=0$ *and* $u_0=0$, *then* u *must be* 0.

PROOF. Let $u\in\mathcal{H}_{(0,s+m)}(H)$ and assume that $Lu=f\in\mathcal{H}_{(1,s)}(H)$ and $u_0\in\mathbf{H}_{(s+1)}(R_n)$. Then it follows from the preceding proposition that $D_t^j u\in\mathcal{E}_t^0(\mathcal{H}_{(s+m-j)})$, $j\leqq m-1$, and that

$$\sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(s+m-j)}^2 \leqq C_1(\sum_{j=0}^{m-1}\|D_t^j u(0,\cdot)\|_{(s+m-j)}^2 + \int_0^t\|f(t',\cdot)\|_{(s+1)}^2 dt'). \tag{12}$$

Put $v=D_t u$. Then $v\in\mathcal{H}_{(m,s-1)}(H)\subset\mathcal{H}_{(0,s+m-1)}(H)$ and we have

$$Lv = D_t f - \sum_{j=0}^{m-1} \sum_{j+|p|\leq m} D_t(A_{j,p}(t, x)) D_t^j D_x^p u \in \mathcal{H}_{(0,s)}(H),$$

and therefore

$$(v, D_t v, \cdots, D_t^{m-1} v) \in \mathcal{E}_t^0(\mathcal{H}_{(s+m-1)}) \times \cdots \times \mathcal{E}_t^0(\mathcal{H}_{(s)}),$$

$$\sum_{j=0}^{m-1} \|D_t^j v(t, \cdot)\|_{(s+m-1-j)}^2 \leqq C_2 \Big(\sum_{j=0}^{m-1} \|D_t^j v(0, \cdot)\|_{(s+m-1-j)}^2 + \\ + \int_0^t \|D_t f(t', \cdot)\|_{(s)}^2 dt' + \sum_{j=0}^{m-1} \int_0^t \|D_t^j u(t', \cdot)\|_{(s+m-j)}^2 dt' \Big),$$

where C_2 is a constant.

Applying Lemma 4, we obtain with a constant C_3

$$\sum_{j=0}^{m-1} \|D_t^j v(t, \cdot)\|_{(s+m-1-j)}^2 \leqq C_3 \Big(\sum_{j=0}^{m-1} \|D_t^j v(0, \cdot)\|_{(s+m-1-j)}^2 + \\ + \int_0^t \|D_t f(t', \cdot)\|_{(s)}^2 dt' \Big). \tag{13}$$

From (12) and (13) we have with a constant C

$$\sum_{j=0}^{m} \|D_t^j u(t, \cdot)\|_{(s+m-j)}^2 \leqq C \Big(\sum_{j=0}^{m-1} \|D_t^j u(0, \cdot)\|_{(s+m-j)}^2 + \|f(0, \cdot)\|_{(s)}^2 + \\ + \sum_{j=0}^{1} \int_0^t \|D_t^j f(t', \cdot)\|_{(s+1-j)}^2 dt' \Big).$$

Repeating this procedure, we obtain the inequality (11).

6. First main theorems on the Cauchy problems

For our later purpose we first show the following

Lemma 7. *Let* $u \in \mathcal{H}_{(-\infty)}(H) = \bigcup_s \mathcal{H}_{(s)}(H)$ *and assume that* $Lu = f \in \mathcal{H}_{(0,s)}(H)$. *Then* $u \in \mathcal{H}_{(m,s')}(H)$ *for some real* s'.

Proof. Let l be a positive integer such that $u \in \mathcal{H}_{(-2l)}(H)$. Now we consider a $U \in \mathcal{H}_{(-2l)}(R_{n+1})$ with $u = U|\mathring{H}$, and define $V \in \mathcal{H}_{(0)}(H)$ by the equation $U = (1-\Delta)^l V$, where Δ is the Laplacian in R_{n+1}. Then $v = V|\mathring{H}$ satisfies the equation $L(1-\Delta)^l v = f$. If we let $L_1 = L(1-\Delta)^l$, it follows from Proposition 4 that $v \in \mathcal{H}_{(m+2l,s')}(H)$, $s' = \min(s, -m-2l)$. This implies that $u \in \mathcal{H}_{(m,s')}(H)$.

Throughout this section we assume that $[E_{(0)}]$ holds for L.

Proposition 6. *Let* s *be a fixed real number. The Cauchy problem*

$$Lu=f \qquad \text{in } \mathring{H},$$

$$u_0 \equiv \lim_{t \downarrow 0} (u, D_t u, \cdots, D_t^{m-1} u) = \alpha$$

has a solution $u \in \mathscr{H}_{(0,s+m-1)}(H)$ *for any given* $f \in \mathscr{H}_{(0,s)}(H)$ *and* $\alpha \in \mathbf{H}_{(s)}(R_n)$ *if and only if*

$$L^* w = 0 \qquad \text{in } \mathring{H}$$

$$w_T \equiv \lim_{t \uparrow T} (w, D_t w, \cdots, D_t^{m-1} w) = 0$$

has a unique solution 0 *in* $\mathscr{H}_{(0,-s)}(H)$.

PROOF. Necessity. Consider a $w \in \mathscr{H}_{(0,-s)}(H)$ such that $L^*w=0$ and $\lim_{t \uparrow T} (w, D_t w, \cdots, D_t^{m-1} w) = 0$. Since $w \in \mathscr{H}_{(m,-s-m)}(H)$ by Proposition 4, there exists $\lim_{t \downarrow 0} (w, D_t w, \cdots, D_t^{m-1} w) = \beta$. First we show that $\beta = 0$. Let $u \in \mathscr{H}_{(0,s+m-1)}(H)$ be a solution to the equation $Lu=0$ under the condition $\lim_{t \downarrow 0} (u, D_t u, \cdots, D_t^{m-1} u) = \alpha$, $\alpha \in C_0^\infty(R_n)$. Then, for any $\psi \in C_0^\infty(H)$ vanishing near $t=T$, Green's formula gives

$$-(u, L^*\psi) = -(\Gamma_0(\alpha), \psi_0),$$

where $\psi_0 = (\psi(0, \cdot), D_t\psi(0, \cdot), \cdots, D_t^{m-1}\psi(0, \cdot))$. In view of the approximation theorem for L^*, we have

$$(\Gamma_0(\alpha), \beta) = 0.$$

Since $\Gamma_0(\alpha)$ may be taken an arbitrary vector function in $C_0^\infty(R_n)$, it follows that $\beta = 0$. Now for any vector function $\phi \in C_0^\infty(H)$, we have from Green's formula

$$(L\phi, w) = 0.$$

Owing to our hypothesis and the approximation theorem for L, we have $(f, w) = 0$ for any vector $f \in \mathscr{H}_{(0,s)}(H)$. This implies that $w = 0$, which is the desired result.

Sufficiency. From the fact that $[E_{(s)}]$ holds for L, it is sufficient to show that the set

$$G = \{(Lu, \Gamma_0(u_0)) : \quad u \in C_0^\infty(H)\}$$

is dense in $\mathscr{H}_{(0,s)}(H) \times \mathbf{H}_{(s)}(R_n)$. Let $w \in \mathscr{H}_{(0,-s)}(H)$, $\beta \in \mathbf{H}_{(-s-m+1)}(R_n)$ such that for any $u \in C_0^\infty(H)$

$$(Lu, w) + (\Gamma_0(u_0), \beta) = 0.$$

We have only to show that $w=0$, $\beta=0$. If $\phi \in C_0^\infty(\overset{\circ}{H})$, the relation is reduced to

$$(L\phi, w)=0,$$

which implies $(\phi, L^*w)=0$, that is, $L^*w=0$ in $\overset{\circ}{H}$.

If $\phi \in C_0^\infty(H)$ and $\phi=0$ near $t=0$, then by Green's formula

$$0=(L\phi, w)=(\Gamma_T(\phi_T), w_T),$$

where $\phi_T=(\phi(T, \cdot), D_t\phi(T, \cdot), \cdots, D_t^{m-1}\phi(T, \cdot))$. Since $\Gamma_T(\phi_T)$ may be taken arbitrary in $C_0^\infty(R_n)$, it follows that $w_T=0$.

Now for any $u \in C_0^\infty(H)$, we have

$$(\Gamma_0(u_0), \beta)=0.$$

This implies $\beta=0$ by the same reason as above, which completes the proof.

We shall say that $(\mathrm{CP})_{(s)}$ holds for L if the Cauchy problem considered in Proposition 6 is always solvable in the sense given there.

Proposition 7. *If* $(\mathrm{CP})_{(s)}$ *holds for some* s, *then it does also for any* s.

Proof. Let s, s' be any two real numbers. Suppose $(\mathrm{CP})_{(s)}$ holds for L. If $s'<s$, it follows from Proposition 6 that $(\mathrm{CP})_{(s')}$ holds. Therefore we have only to show that $(\mathrm{CP})_{(s+2)}$ holds for L.

Given $f \in \mathcal{H}_{(0,s+2)}(H)$ and $\alpha \in \mathbf{H}_{(s+2)}(R_n)$, we put $h=(1-\Delta_x)f \in \mathcal{H}_{(0,s)}(H)$ and $\gamma=(1-\Delta_x)\alpha \in \mathbf{H}_{(s)}(R_n)$, where Δ_x denotes the Laplacian in R_n. Let us consider the Cauchy problem of finding $v \in \mathcal{H}_{(0,s+m-1)}(H)$ such that

$$Lv+Mv=h \qquad \text{in } \overset{\circ}{H}$$

under the condition

$$v_0 \equiv \lim_{t \downarrow 0} (v, D_tv, \cdots, D_t^{m-1}v)=\gamma,$$

where

$$M=\sum_{j=0}^{m-1} \sum_{j+|p|\leq m} \{(1-\Delta_x)A_{j,p}-A_{j,p}(1-\Delta_x)\}(1-\Delta_x)^{-1}D_t^jD_x^p.$$

First we observe that there exists a constant C such that for any $(v, D_tv, \cdots, D_t^{m-1}v) \in \mathcal{E}_t^0(\mathcal{H}_{(s+m-1)}) \times \cdots \times \mathcal{E}_t^0(\mathcal{H}_{(s)})$

$$||Mv(t, \cdot)||_{(s)} \leq C \sum_{j=0}^{m-1} ||D_t^jv(t, \cdot)||_{(s+m-1-j)}.$$

In fact, this follows from the following estimates with constants C, C_1

$$\begin{aligned}||Mv(t,\cdot)||_{(s)}&\leqq C_1\sum_{j=0}^{m-1}\sum_{j+|p|\leq m}||(1-\Delta_x)^{-1}D_x^pD_t^jv(t,\cdot)||_{(s+1)}\\&\leqq C_1\sum_{j=0}^{m-1}\sum_{j+|p|\leq m}||D_x^pD_t^jv(t,\cdot)||_{(s-1)}\\&\leqq C\sum_{j=0}^{m-1}||D_t^jv(t,\cdot)||_{(s+m-1-j)}.\end{aligned}$$

Let $v^0\in\mathcal{H}_{(0,s+m-1)}(H)$ be chosen in such a way that

$$\begin{cases}Lv^0=h\\ v_0^0\equiv\lim\limits_{t\downarrow 0}(v^0,D_tv^0,\cdots,D_t^{m-1}v^0)=\gamma.\end{cases}$$

This is possible because of our hypothesis. If there exists a $w\in\mathcal{H}_{(0,s+m-1)}(H)$ such that

$$\begin{cases}Lw=-Mw-Mv^0\\ w_0\equiv\lim\limits_{t\downarrow 0}(w,D_tw,\cdots,D_t^{m-1}w)=0,\end{cases}$$

then $v=v^0+w$ will be the solution to be found. The method of successive approximations will be successful to this end. Put $w^0=0$ and determine $w^l\in\mathcal{H}_{(0,s+m-1)}(H)$, $l=1,2,\cdots$, successively by

$$\begin{cases}L(w^{l+1})=-Mw^l-Mv^0\\ (w^{l+1})_0\equiv\lim\limits_{t\downarrow 0}(w^{l+1},D_tw^{l+1},\cdots,D_t^{m-1}w^{l+1})=0.\end{cases}$$

Consequently

$$L(w^{l+1}-w^l)=-M(w^l-w^{l-1}).$$

In view of Proposition 5, we have with a constant C_2

$$\begin{aligned}&\sum_{j=0}^{m-1}||D_t^j(w^{l+l'}-w^l)||^2_{(s+m-1-j)}\\&\quad\leqq C_2\int_0^t\sum_{j=0}^{m-1}||D_t^j(w^{l+l'-1}-w^{l-1})(t',\cdot)||^2_{(s+m-1-j)}dt'\\&\quad\leqq C_2^{l'}\int_0^t\frac{(t-t')^{l'-1}}{(l'-1)!}\sum_{j=0}^{m-1}||D_t^jw^{l'}(t',\cdot)||^2_{(s+m-1-j)}dt'\\&\quad\leqq\frac{(C_2T)^{l'}}{l'!}\sup_{0\leq t'\leq T}\sum_{j=0}^{m-1}||D_t^jw^{l'}(t',\cdot)||^2_{(s+m-1-j)}.\end{aligned}$$

From this we see that $\{D_t^jw^l\}$ is a Cauchy sequence in $\mathcal{E}_t^0(\mathcal{H}_{(s+m-1-j)})$. If we let $w=\lim\limits_{l\to\infty}w^l$, then w will be the solution as desired. If we put $u=(1-\Delta_x)^{-1}v$

$\in \mathcal{H}_{(0,s+m+1)}(H)$, it is easy to verify that u satisfies $Lu=f$ and $\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)=\alpha$. Thus the proof is complete.

REMARK. Using the method of successive approximations, as stated in the proof of Proposition 7, it is readily shown that the property "$(\mathrm{CP})_{(s)}$ holds for L" depends only on the principal part of L. This is because of the fact that the same is true of $[E_{(0)}]$.

Now we can state a uniqueness and existence theorem for the Cauchy problem for L (cf. [12, p. 221]).

THEOREM 3. *Assume that, for some s, any solution $w \in \mathcal{H}_{(0,s)}(H)$ to the equation $L^*w=0$ under the condition $\lim_{t\uparrow T}(w, D_t w, \cdots, D_t^{m-1}w)=0$ must be 0. Then for any non-negative integer k and for any real s, the solution u to the Cauchy problem*

$$\begin{cases} Lu=f & \text{in } \mathring{H}, \\ u_0=\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)=\alpha, \end{cases}$$

where $f\in \mathcal{H}_{(k,s)}(H)$, $\alpha \in \mathbf{H}_{(k+s)}(R_n)$ are arbitrarily given, uniquely exists in $\mathcal{H}_{(-\infty)}(H)$ and has the properties:

(i) $\quad D_t^j u \in \mathcal{E}_t^0(\mathcal{H}_{(k+s+m-1-j)}), \qquad j=0, 1, \cdots, k+m-1,$

(ii) $$\sum_{j=0}^{k+m-1}\|D_t^j u(t, \cdot)\|^2_{(k+s+m-1-j)} \leqq C\Big(\sum_{j=0}^{m-1}\|\alpha_j\|^2_{(k+s+m-1-j)}+ \\ +\sum_{j=0}^{k-1}\|D_t^j f(0, \cdot)\|^2_{(k+s-1-j)}+\sum_{j=0}^{k}\int_0^t \|D_t^j f(t', \cdot)\|^2_{(k+s-j)}dt'\Big).$$

PROOF. It follows from Propositions 6 and 7 that the uniqueness hypothesis for L^* implies that, for any real s, given $f\in\mathcal{H}_{(0,s)}(H)$ and $\alpha\in \mathbf{H}_{(s)}(R_n)$, there exists a solution $u\in \mathcal{H}_{(0,s+m-1)}(H)$ to the problem $Lu=f$ with $u_0=\alpha$. Thus, if we take into account the fact that $\mathcal{H}_{(k,s)}(H)\subset \mathcal{H}_{(0,k+s)}(H)$, then the existence of a solution $u\in\mathcal{H}_{(0,k+s+m-1)}(H)$ to our Cauchy problem is trivial, and the properties (i) and (ii) follow from Corollary 3. It remains to show the uniqueness in $\mathcal{H}_{(-\infty)}(H)$. Let $u\in\mathcal{H}_{(-\infty)}(H)$ be such that $Lu=0$ and $\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)=0$. From Lemma 7 we see that $u\in\mathcal{H}_{(m,s')}(H)$ for some s'. Since $\lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)=0$, it follows from Proposition 5 that $u=0$, completing the proof.

REMARK. Let $f\in\mathcal{S}(H)$ and $\alpha\in\mathcal{S}(R_n)$. Then there is a unique solution $u\in\mathcal{S}(H)$ to the Cauchy problem associated with f and α. In fact, from the preceding theorem, there exists a unique solution $u\in\mathcal{H}_{(\infty)}(H)=\bigcap_s \mathcal{H}_{(s)}(H)$.

Let k be a positive integer and put $v=(1+|x|^2)^k u$. Then $D_t^m v=(1+|x|^2)^k\cdot D_t^m u$ and

$$(1+|x|^2)^k D_x^p u = D_x^p v + \sum_{q<p}\binom{p}{q}(1+|x|^2)^k D_x^{p-q}\frac{1}{(1+|x|^2)^k} D_x^q v,$$

where $(1+|x|^2)^k D_x^{p-q}\frac{1}{(1+|x|^2)^k}\in\mathscr{B}(R_n)$. If we define $\tilde{L}$ by

$$\tilde{L}v=(1+|x|^2)^k L(1+|x|^2)^{-k}v,$$

then $\tilde{L}$ can be written in the form

$$\tilde{L}=L_m+Q,$$

where L_m is the principal part of L and Q is a lower order differential operator. The coefficients of $\tilde{L}$ are matrices whose entries lie in $\mathscr{B}(R_{n+1})$, Now, let us consider the Cauchy problem: $\tilde{L}w=(1+|x|^2)^k f$ in $\mathring{H}$ with $w_0=(1+|x|^2)^k\alpha$. Then there exists a solution $w\in\mathscr{H}_{(\infty)}(H)$. If we put $u'=\frac{w}{(1+|x|^2)^k}$, then $u'\in\mathscr{H}_{(\infty)}(H)$ and u' satisfies $Lu'=f$ in $\mathring{H}$ with $u'_0=\alpha$. From the uniqueness of a solution it is easy to verify that $u=u'$. Since k is arbitrarily chosen, u must be in $\mathscr{S}(H)$. The spaces $\mathscr{S}(H)$ and $\mathscr{S}(R_n)$ are of type (**F**). Owing to the closed graph theorem it follows that the mapping $\mathscr{S}(H)\times\mathscr{S}(R_n)\ni(f,\alpha)\rightarrow u\in\mathscr{S}(H)$ is continuous.

EXAMPLE 1. Here we consider only L with constant coefficient matrices. Recall that a Kowalevsky system L is hyperbolic if and only if the Cauchy problem $Lu=0$, $u_0=\alpha\in C_0^\infty(R_n)$ is always solvable in $\mathscr{D}'(\mathring{H})$, and note that the uniqueness theorem is valid for any Kowalevsky system. Assume that $[E_{(0)}]$ holds for L. In view of Theorem 3, L will be hyperbolic. That $[E_{(0)}]$ holds for L remains valid under any perturbation of lower order terms. Consequently the principal part L_m is strongly hyperbolic. The converse is also true. In fact, this is a consequence of the results due to K. Kasahara and M. Yamaguti [7], H. O. Kreiss [9] and G. Strang [14].

If $[E_{(0)}:\downarrow]$ holds for L^*, the requirement for L^* in Theorem 3 is satisfied by Proposition 5 with $[E_{(0)}]$ replaced by $[E_{(0)}:\downarrow]$.

With the aid of the interpolation theorem for the Hilbert scales [8, p. 150], we can show

COROLLARY 4. *With the same assumption as in Theorem* 3, *there exists for any* $f\in\mathscr{H}_{(\sigma,s)}(H)$ *and* $\alpha\in\mathbf{H}_{(\sigma+s)}(R_n)$, σ *being a positive number, a unique solution* $u\in\mathscr{H}_{(-\infty)}(H)$ *to the Cauchy problem for* L *associated with* f *and* α. u *belongs to* $\mathscr{H}_{(\sigma+m,s-1)}(H)$ *and the mapping* $(f,\alpha)\rightarrow u$ *is continuous from* $\mathscr{H}_{(\sigma,s)}(H)\times\mathbf{H}_{(\sigma+s)}(R_n)$ *into* $\mathscr{H}_{(\sigma+m,s-1)}(H)$.

PROOF. Since $\mathcal{H}_{(\sigma,s)}(H)\subset\mathcal{H}_{(0,\sigma+s)}(H)$, Theorem 3 implies the uniqueness in $\mathcal{H}_{(-\infty)}(H)$ and the existence in $\mathcal{H}_{(m,\sigma+s-1)}(H)$ of the solution u. Here the mapping $\mathcal{H}_{(\sigma,s)}(H)\times\mathbf{H}_{(\sigma+s)}(R_n)\ni(f,\alpha)\rightarrow u\in\mathcal{H}_{(m,\sigma+s-1)}(H)$ is continuous.

Let k be a positive integer such that $k>\sigma$. Put $\mathfrak{H}_0=\mathcal{H}_{(0,s)}(H)$ and $\mathfrak{H}_1=\mathcal{H}_{(k,s)}(H)$. If we denote by $\|\cdot\|_0$ and $\|\cdot\|_1$ the norms of $\mathfrak{H}_0$ and $\mathfrak{H}_1$ respectively, $\mathfrak{H}_1$ is dense in $\mathfrak{H}_0$ and $\|u\|_0\leqq\|u\|_1$ for any $u\in\mathfrak{H}_1$. Then there exists an unbounded self-adjoint operator J in $\mathfrak{H}_0$ (called a generating operator) with domain $\mathfrak{H}_1$, which generates a Hilbert scale $\{\mathfrak{H}_\lambda\}_{-\infty<\lambda<+\infty}$, where we denote by $\|\cdot\|_\lambda$ the norm of $\mathfrak{H}_\lambda$. On the other hand, if μ runs through $(-\infty,\infty)$, $\mathcal{H}_{(\mu,s)}(R_{n+1})$ (resp. $\mathbf{H}_{(\mu+s)}(R_n)$) forms a Hilbert scale. Let S be the restriction mapping of $U\in\mathcal{H}_{(\mu,s)}(R_{n+1})$ to the slab $\mathring{H}$. With an obvious modification of Seeley's method [13, p. 625], we can construct a continuous linear extension T of $C_0^\infty(H)$ into $C_0^\infty(R_{n+1})$ such that for any $u\in C_0^\infty(H)$

$$\|u\|_0\leqq\|Tu\|_{(0,s)}\leqq C_0\|u\|_0,$$

$$\|u\|_1\leqq\|Tu\|_{(k,s)}\leqq C_1\|u\|_1.$$

Consequently it follows that T can be continuously extended to a monomorphism of $\mathfrak{H}_0$ into $\mathcal{H}_{(0,s)}(R_{n+1})$, which we denote by the same symbol T and that T is also a monomorphism of $\mathfrak{H}_1$ into $\mathcal{H}_{(k,s)}(R_{n+1})$. S is a continuous linear mapping of $\mathcal{H}_{(0,s)}(R_{n+1})$ into $\mathfrak{H}_0$ and of $\mathcal{H}_{(k,s)}(R_{n+1})$ into $\mathfrak{H}_1$ as well. In view of the interpolation theorem applied to the mappings S and T and to the families $\{\mathcal{H}_{(\mu,s)}(R_{n+1})\}_{0\leq\mu\leq k}$ and $\{\mathfrak{H}_\lambda\}_{0\leq\lambda\leq1}$, we can conclude that $\mathfrak{H}_\lambda=\mathcal{H}_{(\lambda k,s)}(H)$ within the equivalent norms. In fact, $T(\mathfrak{H}_\lambda)\subset\mathcal{H}_{(\lambda k,s)}(R_{n+1})$, $S(\mathcal{H}_{(k,s)}(R_{n+1}))\subset\mathfrak{H}_\lambda$ and ST is the identity on $\mathfrak{H}_\lambda$. This implies that T is a monomorphism and S an epimorphism. From Theorem 3 we know that the mapping $(f,\alpha)\rightarrow u$, which assigns a solution u to the data (f,α), is continuous for

$$\mathcal{H}_{(0,s)}(H)\times\mathbf{H}_{(s)}(R_n)\ni(f,\alpha)\rightarrow u\in\mathcal{H}_{(m,s-1)}(H)$$

and for

$$\mathcal{H}_{(k,s)}(H)\times\mathbf{H}_{(k+s)}(R_n)\ni(f,\alpha)\rightarrow u\in\mathcal{H}_{(k+m,s-1)}(H).$$

Applying again the interpolation theorem we shall reach the conclusion of Corollary 4.

COROLLARY 5. *If $u\in\mathcal{D}'(\mathring{H})$ has a bounded support, $Lu=0$ in $\mathring{H}$, and $u_0\equiv\lim_{t\downarrow0}(u,D_tu,\cdots,D_t^{m-1}u)=0$, then $u=0$ in $\mathring{H}$.*

PROOF. Let $u_\sim$ be the canonical extension of u over $t=0$. Let $\phi(t)\in C^\infty(R)$ be such that supp $\phi\subset(-\infty,T)$ and ϕ is equal to 1 for $t\leqq t_0<T$. Clearly $\phi(t)(u_\sim)\in\mathcal{E}'(R_{n+1})\subset\mathcal{H}_{(-\infty)}(R_{n+1})$ and

$$L(\psi(t)(u_\sim))=L(u_\sim)=0 \qquad \text{for } t<t_0,$$

$$(\psi(t)(u_\sim))_0=0.$$

Applying Theorem 3 to u in a slab $0<t<t_0$, we have $u=0$ for $t<t_0$. Since t_0 is arbitrarily chosen, $u=0$ in $\mathring{H}$.

We can now prove an existence theorem for the Cauchy problem for L^*.

THEOREM 4. *Given* $g\in\mathscr{H}_{(0,-s-m+1)}(H)$ *and* $\beta=(\beta_0,\cdots,\beta_{m-1})\in\mathbf{H}_{(-s-m+1)}(R_n)$, *there exists a solution* $v\in\mathscr{H}_{(0,-s)}(H)$ *to the Cauchy problem*:

$$\begin{cases} L^*v=g & \text{in } \mathring{H}, \\ v_T\equiv\lim\limits_{t\uparrow T}(v, D_tv, \cdots, D_t^{m-1}v)=\beta. \end{cases}$$

PROOF. Consider the subspace $A\subset\mathscr{H}_{(0,s)}(H)\times\mathbf{H}_{(s)}(R_n)$ consisting of (Lu,α) such that $u\in\mathscr{H}_{(0,s+m-1)}(H)$, $Lu\in\mathscr{H}_{(0,s)}(H)$, and $u_0\equiv\lim\limits_{t\downarrow 0}(u, D_tu, \cdots, D_t^{m-1}u)=\alpha\in\mathbf{H}_{(s)}(R_n)$. Consider the anti-linear form

$$l:\ A\ni(Lu,\alpha)\rightarrow(u,\ g)+(\Gamma_T(u_T),\ \beta).$$

$(Lu,\alpha)\rightarrow u$ (resp. u_T) is, by Proposition 5, continuous from A into $\mathscr{H}_{(0,s+m-1)}(H)$ (resp. $\mathbf{H}_{(s)}(R_n)$). It follows that l is continuous. Owing to the Hahn-Banach theorem, we see that there exist $v\in\mathscr{H}_{(0,-s)}(H)$ and $\gamma\in\mathbf{H}_{(-s-m+1)}(R_n)$ such that

$$(u,\ g)+(\Gamma_T(u_T),\ v_T)=(Lu,\ v)+(\Gamma_0(\alpha),\ \gamma), \tag{14}$$

and that

$$\|v\|^2_{(0,-s)}+\|\gamma\|^2_{\mathbf{H}_{(-s-m+1)}}\leqq C\|l\|^2$$

with a constant C. Thus for any $u\in C_0^\infty((0,\infty)\times R_n)$ we have

$$(Lu,\ v)=(u,\ g)+(\Gamma_T(u_T),\ \beta).$$

In virtue of Proposition 2 with L and $t=0$ replaced by L^* and $t=T$ respectively, we see that v is a solution to $L^*v=g$ with $v_T=\beta$, and this proves the theorem.

REMARK. For our later purpose we shall give an estimate for $\|l\|$. First recall that Γ_t is an isomorphism of $\mathbf{H}_{(s)}(R_n)$ onto $\mathbf{H}^\sharp_{(s)}(R_n)$ and that the sets $\{\Gamma_t\}_{0\leq t\leq T}$ and $\{\Gamma_t^{-1}\}_{0\leq t\leq T}$ are equicontinuous. Then, from the estimates

$$|(u,\ g)+(\Gamma_T(u_T),\ \beta)|$$

$$\leqq C_1(\|u\|^2_{(0,s+m-1)}+\|u_T\|^2_{\mathbf{H}_{(s)}})^{\frac{1}{2}}(\|g\|^2_{(0,-s-m+1)}+\|\beta\|^2_{\mathbf{H}_{(-s-m+1)}})^{\frac{1}{2}}$$

$$\leqq C_1C_2(\|\alpha\|_{\mathbf{H}_{(s)}}+\|Lu\|^2_{(0,s)})^{\frac{1}{2}}(\|g\|^2_{(0,-s-m+1)}+\|\beta\|^2_{\mathbf{H}_{(-s-m+1)}})^{\frac{1}{2}},$$

where C_1, C_2 are constants, we obtain with a constant C

$$\|l\|\leqq C(\|g\|_{(0,-s-m+1)}+\|\beta\|_{\mathbf{H}_{(-s-m+1)}}).$$

Before stating the next result let us introduce notations. By $\mathcal{S}_-(H)$ and $\mathcal{S}'_+(H)$ we mean the subspaces of $\mathcal{S}(H)$ and $\mathcal{S}'((-\infty, T]\times R_n)$ respectively defined in accordance with the conditions: for the former $u\in\mathcal{S}(H)$, $D_t^ju(T,\cdot)=0$, $j=0, 1, \cdots$, and for the latter $u\in\mathcal{S}'((-\infty, T]\times R_n)$, supp $u\subset H$. Then $\mathcal{S}'_+(H)$ and $\mathcal{S}_-(H)$ are regarded as anti-dual spaces with respect to an extension of the sesquilinear form

$$\int_{R_n}\int_0^T\phi\bar{\psi}\,dt\,dx, \qquad \phi\in\mathcal{D}_+(H),\ \psi\in\mathcal{D}_-(H),$$

where $\mathcal{D}_+(H)$ is the subspace of $C_0^\infty(H)$ defined according to the conditions: $u\in C_0^\infty(H)$, $D_t^ju(0,\cdot)=0$ for $j=0, 1, 2, \cdots$. Similarly for $\mathcal{D}_-(H)$.

THEOREM 5. *Assume that* $[E_{(0)}]$ *and* $[E_{(0)}:\downarrow]$ *hold for* L *and* L^* *respectively. Let* $f\in\mathcal{S}'(H)$ *and* $\alpha\in\mathcal{S}'(R_n)$ *and assume that* f *has the canonical* $\mathcal{S}'$-*extension* $f_\sim\in\mathcal{S}'((-\infty, T]\times R_n)$. *Then there exists a unique solution* $u\in\mathcal{S}'(H)$ *with the canonical* $\mathcal{S}'$-*extension* $u_\sim\in\mathcal{S}'((-\infty, T]\times R_n)$ *to the Cauchy problem* $Lu=f$ *with* $u_0=\lim_{t\downarrow 0}(u, D_tu, \cdots, D_t^{m-1}u)=\alpha$. *Here the mapping* $(f_\sim, \alpha)\to u_\sim$ *is continuous under the topology of* $\mathcal{S}'_+(H)\times\mathcal{S}'(R_n)$ *and the topology* $\mathcal{S}'_+(H)$.

PROOF. Because of the remark after Theorem 3, given $g\in\mathcal{S}_-(H)$, there can be uniquely determined a $w\in\mathcal{S}_-(H)$ such that $L^*w=g$ in $\mathring{H}$. It then follows that the mapping $g\to w$ is a topological automorphism of $\mathcal{S}_-(H)$. Now, for any given $f'\in\mathcal{S}'_+(H)$ we consider the anti-linear form $g\to(f', w)$. The continuity is easily verified. So we can find a unique $v\in\mathcal{S}'_+(H)$ satisfying $(v, g)=(f', w)$. Since $(v, g)=(v, L^*w)=(Lv, w)$, it follows that $(Lv, w)=(f', w)$, which implies that $Lv=f'$ and such a v is unique. In virtue of the closed graph theorem, the mapping $f'\to v$ of $\mathcal{S}'_+(H)$ into itself is a topological automorphism.

We now turn to the proof of the statements of the theorem. If we put $f'=f_\sim+\sum_{j=0}^{m-1}D_t^j\delta\otimes(\Gamma_0(\alpha))_j$ and consider a $v\in\mathcal{S}'_+(H)$ associated with it in the above arguments, then, as observed in Section 3, $v|\mathring{H}$ is a desired solution and v is the canonical extension of u over $t=0$. The converse is also true. Thus we see that u is unique. Since the mapping $(f_\sim, \alpha)\to u_\sim$ is decomposed into the product of the mappings $(f_\sim, \alpha)\to f'=f_\sim+\sum_{j=0}^{m-1}D_t^j\delta\otimes(\Gamma_0(\alpha))_j$, $f'\to v$ and $v\to u_\sim=(v|\mathring{H})_\sim$, we can conclude that the last statement of the theorem is true. Thus the proof is complete.

7. Some remarks on energy inequalities

Owing to Proposition 5, we know that generalized energy inequalities remain valid for L if we assume $[E_{(0)}]$ for L. Now we show that if, in addition, $[E_{(0)}]$ is assumed for L^*, we shall obtain more precise inequalities.

PROPOSITION 8. *Suppose that $[E_{(0)}]$ holds for L and L^*. Then for any fixed real s, there exists a constant C such that if $Lu \in \mathscr{H}_{(0,s)}(H)$ and $u_0 \equiv \lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u) \in \mathbf{H}_{(s)}(R_n)$ for $u \in \mathscr{H}_{(0,s+m-1)}(H)$, then*

$$[E'_{(s)}] \qquad \sum_{j=0}^{m-1}\|D_t^j u(t_1, \cdot)\|_{(s+m-1-j)} \leqq C\Big(\sum_{j=1}^{m-1}\|D_t^j u(t_0, \cdot)\|_{(s+m-1-j)} + \int_{t_0}^{t_1}\|Lu(t', \cdot)\|_{(s)}dt'\Big)$$

for any t_0, t_1, $0 \leqq t_0 \leqq t_1 \leqq T$, where constant C does not depend on u.

PROOF. Consider a slab $H_1 = [0, t_1] \times R_n$. In view of Theorem 4, we can find, for any given $\beta \in \mathbf{H}_{(-s-m+1)}(R_n)$, a $v \in \mathscr{H}_{(0,-s)}(H_1)$ such that $L^*v = 0$ in $\mathring{H}_1$, $v_{t_1} = \lim_{t\uparrow t_1}(v, D_t v, \cdots, D_t^{m-1}v) = \beta$. From the arguments given in the remark after Theorem 4, we may assume that there exists a constant C_1 independent of t_1 such that

$$\|v_0\|_{\mathbf{H}_{(-s-m+1)}} \leqq C_1\|\beta\|_{\mathbf{H}_{(-s-m+1)}}.$$

In the rest of the proof, $C_2, \cdots, C_5$ will denote constants independent of t_0 and t_1. That $[E_{(0)}]$ holds for L^* yields

$$\|v_t\|_{\mathbf{H}_{(-s-m+1)}} \leqq C\|v_0\|_{\mathbf{H}_{(-s-m+1)}} \leqq C_2\|\beta\|_{\mathbf{H}_{(-s-m+1)}}.$$

Applying Green's formula, we obtain

$$(\Gamma_{t_1}(u_{t_1}), \beta) = (\Gamma_{t_0}(u_{t_0}), v_{t_0}) + \int_{t_0}^{t_1}(Lu(t', \cdot), v(t', \cdot))dt',$$

and consequently

$$(\Gamma_{t_1}(u_{t_1}), \beta) \leqq C_3\{\|u_{t_0}\|_{\mathbf{H}_{(s)}}\|v_{t_0}\|_{\mathbf{H}_{(-s-m+1)}} + \int_{t_0}^{t_1}\|Lu(t', \cdot)\|_{(s)}\|v(t', \cdot)\|_{(-s-m+1)}dt'\}$$

$$\leqq C_4\|\beta\|_{\mathbf{H}_{(-s-m+1)}}(\|u_{t_0}\|_{\mathbf{H}_{(s)}} + \int_{t_0}^{t_1}\|Lu(t', \cdot)\|_{(s)}dt').$$

This implies that we have

$$\|u_{t_1}\|_{\mathbf{H}_{(s)}} \leqq C_5(\|u_{t_0}\|_{\mathbf{H}_{(s)}} + \int_{t_0}^{t_1} \|Lu(t', \cdot)\|_{(s)} dt'),$$

proving the result.

Using these inequalities we can deal with Cauchy problems for L in the case where f is assumed to be $\mathscr{H}_{(s)}(R_n)$-valued and integrable in the sense of Bochner.

In the preceding section we have dealt with the conditions under which Cauchy problems for L with initial hyperplane $t=0$ are solvable in the sense given there. In this connection we can show the following

PROPOSITION 9. *Suppose that* $[E_{(0)}]$ *holds for* L *and* L^*. *The assumption made in Theorem* 3 *for* L^* *is equivalent to the following*:

$$\sum_{j=0}^{m-1} \|D_t^j u(t_0, \cdot)\|_{(s+m-1-j)} \leqq C(\sum_{j=0}^{m-1} \|D_t^j u(t_1, \cdot)\|_{(s+m-1-j)} + \\ + \int_{t_0}^{t_1} \|L^* u(t', \cdot)\|_{(s)} dt'), \qquad u \in C_0^\infty(R_{n+1}),$$

for any t_0, t_1, $0 \leqq t_0 \leqq t_1 \leqq T$, *where constant* C *does not depend on* u.

PROOF. Since we can proceed along a similar line as in the proof of Proposition 8, the proof will be omitted.

REMARK. Suppose L is a linear differential operator, that is, $N=1$. L is regularly hyperbolic if and only if $[E_{(0)}]$ holds for L. Indeed the "only if" part is well known. So we shall consider the "if" part. The complex conjugation of coefficients transforms the principal part of L into that of L^* so that $[E_{(0)}]$ holds for L and L^*. It follows from Proposition 8 that we have for any $u \in C_0^\infty(R_{n+1})$

$$\sum_{j=0}^{m-1} \|D_t^j u(t_1, \cdot)\|_{(m-1-j)} \leqq C(\sum_{j=0}^{m-1} \|D_t^j u(t_0, \cdot)\|_{(m-1-j)} + \\ + \int_{t_0}^{t_1} \|Lu(t', \cdot)\|_{(0)} dt') \tag{15}$$

for any t_0, t_1, $0 \leqq t_0 \leqq t_1 \leqq T$, where constant C does not depend on u.

Let M be the principal part of L. When the variables are fixed, we obtain differential operators with constant coefficients. Denote by $M_{(t_0, x_0)}$ such an operator associated with M when the point $(t_0, x_0) \in H$ is fixed. We shall show that for $0 \leqq t_0 < T$ the following inequalities must hold with a constant C' independent of (t_0, x_0):

$$\sum_{j=0}^{m-1} \|D_t^j u(t, \cdot)\|_{(m-1-j)} \leqq C'(\sum_{j=0}^{m-1} \|D_t^j u(0, \cdot)\|_{(m-1-j)} +$$

$$+\int_0^t \|M_{(t_0,x_0)}u(t',\cdot)\|_{(0)}dt'), \qquad u\in C_0^\infty(R_{n+1}). \tag{16}$$

We shall confine ourselves to the case $(t_0, x_0)=(0, 0)$ and write M_0 instead of $M_{(0,0)}$. The other cases will be treated with an obvious modification.

Put $\phi_\lambda(t, x)=u(\lambda t, \lambda x)$, $\lambda>1$. For any fixed t with $0<t\leqq T$, we define t_1 by $\lambda t_1=t$. Consider (15) with u replaced by ϕ_λ and divide both sides by $\lambda^{m-1-\frac{n}{2}}$. In view of the relations

$$\|D_t^j\phi_\lambda(t_1,\cdot)\|_{(m-1-j)}^2=\sum_{|p|\leq m-1-j}\lambda^{2(j+|p|-\frac{n}{2})}\binom{m-1-j}{p}\|D_t^jD_x^pu(t,\cdot)\|_{(0)}^2,$$

$$\int_0^{t_1}\|L\phi_\lambda(t',\cdot)\|_{(0)}dt'=\int_0^t\lambda^{-1-\frac{n}{2}}\{\|\lambda^mD_t^mu(t',\cdot)+$$

$$+\sum_{j=0}^{m-1}\sum_{j+|p|\leq m}\lambda^{j+|p|}A_{j,p}\Big(\frac{t'}{\lambda},\frac{x}{\lambda}\Big)D_t^jD_x^pu(t',\cdot)\|_{(0)}dt',$$

and, letting $\lambda\to\infty$, we obtain the estimate

$$\sum_{j=0}^{m-1}\Big(\sum_{|p|=m-1-j}\binom{m-1-j}{p}\|D_t^jD_x^pu(t,\cdot)\|_{(0)}^2\Big)^{\frac{1}{2}}$$

$$\leqq C\Big\{\sum_{j=0}^{m-1}\Big(\sum_{|p|=m-1-j}\binom{m-1-j}{p}\|D_t^jD_x^pu(0,\cdot)\|_{(0)}^2\Big)^{\frac{1}{2}}+\int_0^t\|M_0u(t',\cdot)\|_{(0)}dt'\Big\},$$

or, in terms of Λ,

$$\sum_{j=0}^{m-1}\|\Lambda^{m-1-j}D_t^ju(t,\cdot)\|_{(0)}\leqq C\{\sum_{j=0}^{m-1}\|\Lambda^{m-1-j}D_t^ju(0,\cdot)\|_{(0)}+$$

$$+\int_0^t\|M_0u(t',\cdot)\|_{(0)}dt'\}, \tag{17}$$

where Λ is defined by $(\Lambda u)^\wedge=|\xi|\hat{u}(t,\xi)$. If we take $u(t, x)e^{i\langle x,\xi_0\rangle}$, $\xi_0=(1, 0, \cdots, 0)$, instead of $u(t, x)$, its partial Fourier transform can be written $\hat{u}(t, \xi-\xi_0)$, and therefore we shall obtain

$$\sum_{j=0}^{m-1}\|(\Lambda(D_x+\xi_0))^{m-1-j}D_t^ju(t,\cdot)\|_{(0)}\leqq C(\sum_{j=0}^{m-1}\|(\Lambda(D_x+\xi_0))^{m-1-j}D_t^ju(0,\cdot)\|_{(0)}+$$

$$+\int_0^t\|M_0(D_t, D_x+\xi_0)u(t',\cdot)\|_{(0)}dt'), \tag{18}$$

where $\Lambda(D_x+\xi_0)$ is defined by $(\Lambda(D_x+\xi_0)u)^\wedge=|\xi+\xi_0|\hat{u}$.

From (17) and (18), together with the estimates with constants C_1, C_2, where C_1 depends on $\max_{j+|p|=m,j<m}|A_{j,p}(0, 0)|$,

$$\|M_0(D_t, D_x+\xi_0)u(t',\cdot)\|_{(0)}\leqq\|M_0u(t',\cdot)\|_{(0)}+C_1\sum_{j=0}^{m-1}\|D_t^ju(t',\cdot)\|_{(m-1-j)},$$

$$\frac{1}{C_2^2}(1+|\xi|^2)^{m-1-j} \leqq |\xi|^{2(m-1-j)} + |\xi+\xi_0|^{2(m-1-j)}$$
$$\leqq C_2^2(1+|\xi|^2)^{m-1-j},$$

we obtain

$$\frac{1}{C_2}\sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(m-1-j)} \leqq C(C_2\sum_{j=0}^{m-1}\|D_t^j u(0,\cdot)\|_{(m-1-j)} +$$
$$+2\int_0^t\|M_0 u(t',\cdot)\|_{(0)}dt' + C_1\sum_{j=0}^{m-1}\int_0^t\|D_t^j u(t',\cdot)\|_{(m-1-j)}dt').$$

Consequently, owing to Lemma 4, we obtain the estimate (16).

Now, by letting $t_0 \uparrow T$, we see that (16) holds for any $(t_0, x_0) \in H$ with the same constant C'.

$M_{(t_0,x_0)}$ is strongly hyperbolic as shown in Example 1. Hence the coefficients of M are real. Let K be a constant such that $\sup_{j+|p|=m, j<m}\ \sup_{(t,x)\in H} |A_{j,p}(t,x)| < K$ and let $\mathfrak{M}$ represent the set of all differential operators with constant real coefficients

$$P(D) = D_t^m + \sum_{j=0}^{m-1}\sum_{j+|p|=m} a_{j,p}D_t^j D_x^p$$

such that

$$|a_{j,p}| \leqq K$$

and

$$\sum_{j=0}^{m-1}\|D_t^j u(t,\cdot)\|_{(m-1-j)} \leqq C'(\sum_{j=0}^{m-1}\|D_t^j u(0,\cdot)\|_{(m-1-j)} +$$
$$+\int_0^t\|P(D)(t',\cdot)\|_{(0)}dt'), \qquad 0\leqq t\leqq T,\ u \in C_0^\infty(R_{n+1}).$$

Every $P(D) \in \mathfrak{M}$ is strongly hyperbolic. Let l be the number of the indices (j, p) such that $j+|p|=m$, $j<m$. It is easy to see that the points $\{a_{j,p}\}$ form a compact subset of R_l. For each ξ, $|\xi|=1$, $P(\tau, \xi)$ is a polynomial in τ with simple real zeros only. Let $\Delta_P(\xi)$ be its discriminant. Since it is a continuous function of $\{a_{j,p}\}$ and ξ, it follows that $\Delta_P(\xi) \geqq d > 0$ for a constant d. Thus we see that L is regularly hyperbolic.

8. Further existence and uniqueness theorems for Cauchy problems

By $\mathcal{H}_{(0,s)}(H: loc)$ we mean the set of all $u \in \mathcal{D}'(\mathring{H})$ with the property that $\phi u \in \mathcal{H}_{(0,s)}(H)$ for any $\phi \in C_0^\infty(R_n)$. Here the topology is given as a local space. Then $\mathcal{H}_{(0,s)}(H\colon loc)$ is an (**F**)-space. Let $\mathcal{H}_{(0,-s-1)}(H\colon comp)$ be its antidual. We also consider $\mathbf{H}_{(s)}(R_n\colon loc)$ and $\mathbf{H}_{(s)}(R_n\colon comp)$ which are defined in a similar fashion.

We can prove the analogue of Proposition 4 for these local spaces:

PROPOSITION 4′. *If* $u \in \mathcal{H}_{(0,k+s+m)}(H\colon loc)$ *satisfies* $Lu=f \in \mathcal{H}_{(k,s)}(H\colon loc)$, *then* $u \in \mathcal{H}_{(k+m,s)}(H\colon loc)$.

The proof can be obtained with slight modifications of the arguments made in the proof of Proposition 4, and so will be omitted.

Throughout this section we assume that $[E_{(0)}]$ and $[E_{(0)}\colon\downarrow]$ hold for L and L^* respectively.

PROPOSITION 10. *Let s be a fixed real number. Then the following conditions are equivalent*:

(1) *Given* $f \in \mathcal{H}_{(0,s)}(H\colon loc)$ *and* $\alpha \in \mathbf{H}_{(s)}(R_n\colon loc)$, *there exists a unique solution* $u \in \mathcal{H}_{(0,s+m-1)}(H\colon loc)$ *to the problem*

$$Lu=f \text{ in } \overset{\circ}{H}, \text{ and } u_0 \equiv \lim_{t\downarrow 0}\,(u,\, D_t u,\, \cdots,\, D_t^{m-1}u)=\alpha.$$

(2) *The condition* (1) *with* α *replaced by* 0.

(3) *Given* $g \in \mathcal{H}_{(0,-s-m+1)}(H\colon comp)$ *and* $\beta \in \mathbf{H}_{(-s-m+1)}(R_n\colon comp)$, *there exists a solution* $v \in \mathcal{H}_{(0,-s)}(H\colon comp)$ *to the problem*

$$L^*v=g \text{ in } \overset{\circ}{H}, \text{ and } v_T \equiv \lim_{t\uparrow T}\,(v,\, D_t v,\, \cdots,\, D_t^{m-1})=\beta.$$

(4) *The condition* (3) *with* β *replaced by* 0.

PROOF. It suffices to prove the implications (2)⇒(3) and (4)⇒(1).

Ad (2)⇒(3). Let u be the solution indicated in the condition (2) for given f. Let $\phi \in C_0^\infty(R_n)$. Actual calculation will show that $L(\phi u)-\phi Lu \in \mathcal{H}_{(0,s)}(H)$. Then,

$$L(\phi u)=\phi f+(L(\phi u)-\phi Lu) \in \mathcal{H}_{(0,s)}(H)$$

and

$$\lim_{t\downarrow 0}\,(\phi u,\, D_t(\phi u),\, \cdots,\, D_t^{m-1}(\phi u))=\phi\alpha \in \mathbf{H}_{(s)}(R_n).$$

Consequently, on account of Theorem 3, we obtain

$$D_t^j(\phi u) \in \mathcal{E}_t^0(\mathcal{H}_{(s+m-1-j)}), \qquad j=0,\, 1,\, \cdots,\, m-1.$$

It then follows that

$$(u,\, D_t u,\, \cdots,\, D_t^{m-1}u) \in \mathcal{E}_t^0(\mathcal{H}_{(s+m-1)}(R_n\colon loc)) \times \cdots \times \mathcal{E}_t^0(\mathcal{H}_{(s)}(R_n\colon loc)).$$

We can apply the closed graph theorem to conclude that the mapping

$$\mathcal{H}_{(0,s)}(H\colon loc) \ni f \to (u,\ u_T) \in \mathcal{H}_{(0,s+m-1)}(H\colon loc) \times \mathbf{H}_{(s)}(R_n\colon loc)$$

is continuous, because this is possible since these spaces are of type (**F**).

Let $g \in \mathcal{H}_{(0,-s-m+1)}(H\colon comp)$ and $\beta \in \mathbf{H}_{(-s-m+1)}(R_n\colon comp)$ be given, and consider a linear form

$$\mathcal{H}_{(0,s)}(H\colon loc) \ni f \to (u,\ g) + (\Gamma_T(u_T),\ \beta).$$

In virtue of the above arguments, the linear form will be continuous. Therefore we can find a $v \in \mathcal{H}_{(0,-s)}(H\colon comp)$ such that

$$(f,\ v) = (u,\ g) + (\Gamma_T(u_T),\ \beta).$$

It follows from Proposition 2 that $L^*v = g$, and $v_T = \beta$.

Ad (4)⟹(1). Let v be the solution in the condition (4) for any given g, and consider the mapping

$$l\colon \mathcal{H}_{(0,-s-m+1)}(H\colon comp) \ni g \to v \in \mathcal{H}_{(0,-s)}(H\colon comp).$$

In view of Theorem 3, the mapping is continuous from $\mathcal{H}_{(0,-s-m+1)}(H\colon comp)$ to $\mathcal{H}_{(0,-s)}(H)$. Since $\mathcal{H}_{(0,-s-m+1)}(H\colon comp)$ is of type (**LF**), we can apply Theorem B in A. Grothendieck [3, p. 17] to infer that l is continuous. Let $f \in \mathcal{H}_{(0,s)}(H\colon loc)$ and $\alpha \in \mathbf{H}_{(s)}(R_n\colon loc)$ be given. Then the anti-linear form

$$\mathcal{H}_{(0,-s-m+1)}(H\colon comp) \ni g \to (f,\ v) + (\Gamma_0(\alpha),\ v_0)$$

will be continuous, and therefore we can find a $u \in \mathcal{H}_{(0,s+m-1)}(H\colon loc)$ such that

$$(u,\ g) = (f,\ v) + (\Gamma_0(\alpha),\ v_0),$$

which implies by Proposition 2 that $Lu = f$ and $u_0 = \alpha$, completing the proof.

Another equivalent result is the following

PROPOSITION 11. *The conditions in Proposition* 10 *are equivalent to each of the following ones*:

(5) *Given* $f \ni C^\infty(H)$ *and* $\alpha \in C^\infty(R_n)$, *there exists a unique solution* $u \in C^\infty(H)$ *to the problem*

$$Lu = f \text{ in } \overset{\circ}{H}, \text{ and } u_0 = \alpha.$$

(6) *The condition* (5) *with* α *replaced by* 0.

(7) *Given* $g \in C_0^\infty(H)$ *and* $\beta \in C_0^\infty(R_n)$, *there exists a solution* $v \in C_0^\infty(H)$ *to the problem*

$$L^*v = g \text{ in } \overset{\circ}{H}, \text{ and } v_T = \beta.$$

(8) *Given* $g \in \mathcal{D}_-(H)$, *there exists a solution* $v \in \mathcal{D}_-(H)$ *to the problem*

$L^*v=g$ *in* $\mathring{H}$.

PROOF. If the condition (8) holds, then so does the condition (4) for any s. To see this, let $v\in\mathcal{D}_-(H)$ be the solution associated with $g\in\mathcal{D}_-(H)$ as indicated in the condition (8). Then, owing to Theorem A in Grothendieck [3, p. 16], there exists for any given compact subset $K\subset R_n$ a compact subset $K_1\subset R_n$ such that $\operatorname{supp} g\subset[0, T]\times K$ implies $\operatorname{supp} v\subset[0, T]\times K_1$ and the mapping $g\to v$ will be continuous from $\mathcal{D}_-(H)$ into itself. Now, let $g'\in\mathcal{H}_{(0,-s-m+1)}(H)$ and assume that $\operatorname{supp} g'$ is bounded. We can find a sequence $\{g_j'\}$, $g_j'\in\mathcal{D}_-(H)$ such that $g_j'\to g'$ in $\mathcal{H}_{(0,-s-m+1)}(H)$ as $j\to\infty$. Here we may assume that $\operatorname{supp} g_j'\subset[0, T]\times K$ for some compact subset K. Let $v_j'\in\mathcal{D}_-(H)$ be the solution corresponding to g_j' as indicated above. It follows from Theorem 3 that v_j' converges in $\mathcal{H}_{(0,-s)}(H)$ to a v'. Clearly $Lv'=g'$ and $v_T'=0$. This shows that (4) holds for any real s. As the implication (4)$\Rightarrow$(8) is trivial, we can conclude that the conditions in Proposition 10 hold for any real s whenever one of those does hold for some s.

Assuming that (1) holds for any real s, we shall prove (5). Let $f\in C^\infty(H)$ and $\alpha\in C^\infty(R_n)$. Then $f\in\mathcal{H}_{(0,s)}(H\colon loc)$ and $\alpha\in\mathbf{H}_{(s)}(R_n\colon loc)$ for any real s, and therefore (1) implies that there exists a unique solution $u\in\mathcal{H}_{(0,\infty)}(H\colon loc)$ to the problem $Lu=f$ with $u_0=\alpha$. Proposition 4′ shows that $u\in\mathcal{H}_{(\infty)}(H\colon loc)$.

The implications (5)$\Rightarrow$(6) and (7)$\Rightarrow$(8) are trivial and (6)$\Rightarrow$(7) will be proved by a similar reasoning as in the case (2)$\Rightarrow$(3). Thus the proof is complete.

Now we introduce some notations with the aid of which Theorem 6 will be stated.

By $\mathcal{D}'(H)$ we mean the set of distributions $\in\mathcal{D}'(\mathring{H})$ which can be extensible to distributions $\in\mathcal{D}'(R_{n+1})$. The quotient topology is introduced in $\mathcal{D}'(H)$. Similarly for $\mathcal{D}'((-\infty, T]\times R_n)$. It is to be noted that $\mathcal{D}'((-\infty, T)\times R_n)$ has a different meaning.

By $\mathcal{D}_+'(H)$ we mean the set of all distributions $\in\mathcal{D}'((-\infty, T]\times R_n)$ with support contained in $[0, T)\times R_n$. $\mathcal{D}_+'(H)$ is equipped with the induced topology. The space is the strong dual of $\mathcal{D}_-(H)$.

By $\mathcal{D}_+'(\mathring{H})$ we mean the set of all distributions $\in\mathcal{D}'((-\infty, T)\times R_n)$ with support contained in $[0, T)\times R_n$. $\mathcal{D}_+'(\mathring{H})$ is equipped with the induced topology. The space is the strong dual of $\mathcal{D}((-\infty, T)\times R_n)|H$, the space of the restrictions to H of the functions in $\mathcal{D}((-\infty, T)\times R_n)$.

All these spaces are ultrabornological (or (β)-) Souslin spaces. We shall make use of the Borel graph theorem of L. Schwartz [11, p. 49].

THEOREM 6. *Suppose that equivalent conditions in Proposition* 11 *hold. Then there exists a unique solution* $u\in\mathcal{D}'(\mathring{H})$ *to the Cauchy problem*

$$Lu=f \qquad in\ \mathring{H}$$

under the condition

$$u_0 \equiv \lim_{t\downarrow 0}(u, D_t u, \cdots, D_t^{m-1}u)=\alpha$$

for any preassigned $f \in \mathcal{D}'(\mathring{H})$ *and* $\alpha \in \mathcal{D}'(R_n)$, *where we assume that* f *has the canonical extension* $f_\smile$ *over* $t=0$. *The mapping* $(f_\smile, \alpha)\rightarrow u_\smile$ *is continuous under the topologies of* $\mathcal{D}'_+(\mathring{H})\times\mathcal{D}'(R_n)$ *and* $\mathcal{D}'_+(\mathring{H})$.

In addition,

(i) *if* $f \in \mathcal{D}'(H)$, *that is,* f *is extensible over* $t=T$, *then* $u \in \mathcal{D}'(H)$ *as well, and the mapping* $(f_\smile, \alpha)\rightarrow u_\smile$ *is continuous under the topologies of* $\mathcal{D}'_+(H)\times \mathcal{D}'(R_n)$ *and* $\mathcal{D}'_+(H)$.

(ii) *if* f *has the canonical extension* $f^\frown$, *then* u_T *exists.*

Proof. First we show that if $f' \in \mathcal{D}'_+(\mathring{H})$, then there exists a unique $u' \in \mathcal{D}'_+(\mathring{H})$ such that $Lu'=f'$. Take an arbitrary T_1, $0<T_1<T$, and consider a slab $H_{T_1}=[0, T_1]\times R_n$. For any given $\psi \in \mathcal{D}_-(H_{T_1})$ we can find a unique $\phi \in \mathcal{D}_-(H_{T_1})$ such that $L^*\phi=\psi$. Since $[E_{(0)}:\downarrow]$ holds for L^*, $\psi\rightarrow\phi$ defines a topological automorphism of $\mathcal{D}_-(H_{T_1})$ and therefore of $\mathcal{D}((-\infty, T)\times R_n)|H$ as well. Thus the anti-linear form $\psi\rightarrow(f', \phi)$ on $\mathcal{D}((-\infty, T)\times R_n)|H$ is continuous, and we can find a unique $u' \in \mathcal{D}'_+(\mathring{H})$ such that $(u', \psi)=(f', \phi)$, that is, $(u', L^*\phi)=(f', \phi)$. This implies $Lu'=f'$. It is easy to see that u' is a unique solution of $Lu'=f'$. Consequently the linear mapping L is a bijective and continuous endomorphism of $\mathcal{D}'_+(\mathring{H})$. Owing to the Borel graph theorem of Schwartz, we can infer that $f'\rightarrow u'$ is a topological automorphism of $\mathcal{D}'_+(\mathring{H})$.

To prove our statement, we put $f'=f_\smile+\sum_{j=0}^{m-1}D_t^j\delta\otimes(\Gamma_0(\alpha))_j$. Then the solution $v' \in \mathcal{D}'_+(\mathring{H})$ considered above gives rise to the unique solution to the Cauchy problem, that is, $u=v'|\mathring{H}$ satisfies $Lu=f$ and $u_0=\alpha$. Here we note that $v'=u_\smile$. The continuity of the mapping $(f_\smile, \alpha)\rightarrow u_\smile$ is evident from the above considerations.

Now we shall show the statement (i). If we consider $f' \in \mathcal{D}'_+(H)$ and take $T_1=T$, then the linear mapping $\psi\rightarrow\phi$ is a topological automorphism of $\mathcal{D}_-(H)$. Thus the anti-linear form $\psi\rightarrow(f', \phi)$ is continuous, and we can find a unique $v' \in \mathcal{D}'_+(H)$ such that $Lv'=f'$. Then a similar reasoning leads to the conclution of (i).

(ii) follows from Proposition 1.

Thus the proof is complete.

Remark. Making use of Propositions 10 and 11 and applying Theorem

6 we can show the following: Suppose the equivalence conditions of Proposition 10 holds. Then for any given $f \in \mathscr{H}_{(k,s)}(H: loc)$ and $\alpha \in \mathbf{H}_{(k+s)}(H: loc)$, k being a non-negative integer, the solution u to the Cauchy problem

$$\begin{cases} Lu=f & \text{in } \mathring{H}, \\ u_0 \equiv \lim_{t \downarrow 0} (u, D_t u, \cdots, D_t^{m-1}u)=\alpha \end{cases}$$

uniquely exists in $\mathscr{D}'(\mathring{H})$ and $u \in \mathscr{H}_{(k+m,s-1)}(H: loc)$. In addition, u may be chosen so that $D_t^j u \in \mathscr{E}_t^0(\mathscr{H}_{(k+s+m-1-j)}(R_n: loc))$, $j=0, 1, \cdots, k+m-1$.

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