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On Some Secondary Cohomology Operations

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§1. Introduction

Let $\mathcal{P}^i$ be the ith reduced power operation mod p, and let $\varDelta$ be the Bockstein operation associated with the exact coefficient sequence: $0\rightarrow Z_p\rightarrow Z_{p^2}\rightarrow Z_p\rightarrow 0$, where p is an odd prime and Z_m denotes the cyclic group of order m. Let $\varPhi_i$, $\varPhi'_i$ and $\varPhi''_i$ $(i>1)$ be the stable secondary cohomology operations associated with the following Adem relations:

$$(1.1)\qquad (\mathcal{P}^1\varDelta)\mathcal{P}^{i-1}-(i-1)\varDelta\mathcal{P}^i-\mathcal{P}^i\varDelta=0,$$

$$(1.2)\qquad (\mathcal{P}^2\varDelta)\mathcal{P}^{i-1}-\binom{(p-1)(i-1)}{2}\varDelta\mathcal{P}^{i+1}-i\mathcal{P}^{i+1}\varDelta=0,$$

$$(1.3)\qquad (\mathcal{P}^3\varDelta)\mathcal{P}^{i-1}+\binom{(p-1)(i-1)}{3}\varDelta\mathcal{P}^{i+2}-\varepsilon\varDelta(\mathcal{P}^{i+1}\mathcal{P}^1)$$
$$-\binom{(p-1)(i-1)-1}{2}\mathcal{P}^{i+2}\varDelta=0,$$

respectively, where $\varepsilon=1$ if $p=3$, and $\varepsilon=0$ if $p>3$. For each space X and each integer $q>0$, the operation $\varPhi_i$, for example, is a homomorphism:

$$\varPhi_i:\ K^q(\varPhi_i;X)\rightarrow H^{q+2(p-1)i}(X;Z_p)/Q^{q+2(p-1)i}(\varPhi_i;X)$$

where

$$K^q(\varPhi_i;X)=\{u\in H^q(X;Z_p)\,|\,\mathcal{P}^{i-1}u=0,\ \mathcal{P}^iu=0,\ \varDelta u=0\},$$

$$Q^{q+2(p-1)i}(\varPhi_i;X)=\mathcal{P}^1\varDelta H^{q+2(p-1)(i-1)-1}(X;Z_p)$$
$$-(i-1)\varDelta H^{q+2(p-1)i-1}(X;Z_p)-\mathcal{P}^iH^q(X;Z_p).$$

It is known [1, Chapter 3] that the secondary operation associated with the Adem relation is natural with respect to maps, that the operation is stable, i.e., it commutes with suspension, and that it satisfies the second formula of Peterson-Stein [9].

One of our purposes is to give some cup product formulas concerning these operations $\varPhi_i$, $\varPhi'_i$, $\varPhi''_i$. For example, we have the following

THEOREM 3.4. *Let k and j be given integers with $0<j<k$, and let $u\in H^l$*

$(X; Z_p)$ and $v \in H^m(X; Z_p)$ be mod p reductions of integral classes. If $\Phi_i(u)$ for $j<i\leqq k$ and $\Phi_{k-i}(v)$ for $0\leqq i<j$ are defined, then $\Phi_k(u\cup v)$ is defined, and we have

$$\Phi_k(u\cup v)=\sum_{i=j+1}^{k}(\Phi_i(u)\cup\mathcal{P}^{k-i}v)+\sum_{i=0}^{j-1}(\mathcal{P}^i u\cup\Phi_{k-i}(v))$$

in $H^{l+m+2(p-1)k}(X; Z_p)$ modulo the indeterminacy Q. (The definition of Q is given in §3.)

In §2 we prove three formulas (Theorems 2.4, 2.10 and 2.16) concerning mod p functional cohomology operations associated with the relations (1.1–3). Combining the results in §2 with the second formula of Peterson-Stein [3, Theorem 5.2], we have, in §3, Theorems 3.4, 3.11 and 3.14 which are our main theorems. In §4 we discuss the operations Φ_i, Φ'_i and Φ''_i in the infinite dimensional complex projective space CP^∞. In §5 we calculate the values of the operations on the Thom class of the tangent bundle of the real $2n$-dimensional complex projective space CP^n in case $p=3$ and $n=3^r-1$, by the method of Adem-Gitler [4]. Using the results in §5, we study the mod 3 secondary operations in CP^n in §6.

We consider the double secondary cohomology operations Θ_i associated with the relations (1.1) and (1.3). Adams [2] has applied the double secondary operations associated with the relations of squaring operations to the problem of vector fields on spheres, and Adem-Gitler [4] to the immersion problem for real projective spaces. We have the results on Θ_i for the complex projective space CP^∞ and the mod 3 lens space $L^\infty(3)$ of infinite dimension in §7.

In §8 we apply the results in §7 to the stable vector field problem for some $(2n+1)$-dimensional mod 3 lens space $L^n(3)$, and we have a non-immersion theorem for $L^n(3)$ as follows: *$L^n(3)$ cannot be immersed in $(3n-3^t-1)$-dimensional Euclidean space for $n=2\cdot 3^s+3^t-1$ $(s>t>1)$* (Theorem 8.4).

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§2. Formulas on mod p functional cohomology operations

2–1. We denote the Adem relation (1.1) by

$$(1.1)\qquad \alpha_i\beta_i=0,\quad \alpha_i=\mathcal{P}^1\Delta-(i-1)\Delta-\mathcal{P}^i,\quad \beta_i=(\mathcal{P}^{i-1}, \mathcal{P}^i, \Delta),$$

where $i>1$. (By these notations we mean that $\beta_i(w)=(\mathcal{P}^{i-1}w, \mathcal{P}^i w, \Delta w)$ and $\alpha_i(x, y, z)=\mathcal{P}^1\Delta x-(i-1)\Delta y-\mathcal{P}^i z$.) Let k and j be given integers such that $0<j<k$. Let X and Y be spaces and $f: X\to Y$ be a map. Suppose that the elements $c\in H^l(Y; Z_p)$ and $d\in H^m(Y; Z_p)$ satisfy the following conditions, where $l>0$ and $m>0$.

(2.1) $\Delta c=0, \quad \Delta d=0,$

(2.2) $f^*\mathcal{P}^i c=0 \qquad \text{for } i=j, j+1,\cdots, k,$

(2.3) $f^*\mathcal{P}^{k-i} d=0 \qquad \text{for } i=0, 1,\cdots, j.$

Then we can define the functional cohomology operations $(\alpha_i)_f\beta_i(c)$ for $j<i\leqq k$, $(\alpha_{k-i})_f\beta_{k-i}(d)$ for $0\leqq i<j$, and $(\alpha_k)_f\beta_k(c\cup d)$. Moreover, we have

THEOREM 2.4. $(\alpha_k)_f\beta_k(c\cup d)$

$$=\sum_{i=j+1}^{k}\{((\alpha_i)_f\beta_i(c))\cup\mathcal{P}^{k-i}f^*d\}+\sum_{i=0}^{j-1}\{\mathcal{P}^i f^*c\cup((\alpha_{k-i})_f\beta_{k-i}(d))\}$$

in $H^{l+m+2(p-1)k}(X;\, Z_p)/Q$, *where*

$$Q=f^*H^{l+m+2(p-1)k}(Y;\, Z_p)+Q^{l+m+2(p-1)k}(\Phi_k;\, X)$$
$$+\sum_{i=j+1}^{k}\{Q^{l+2(p-1)i}(\Phi_i;\, X)\cup\mathcal{P}^{k-i}f^*d+H^{l+2(p-1)i-1}(X;\, Z_p)\cup\Delta\mathcal{P}^{k-i}f^*d\}$$
$$+\sum_{i=0}^{j-1}\{\mathcal{P}^i f^*c\cup Q^{m+2(p-1)(k-i)}(\Phi_{k-i};\, X)+\Delta\mathcal{P}^i f^*c\cup H^{m+2(p-1)(k-i)-1}(X;\, Z_p)\}.$$

We may suppose that X is a subspace of Y and that f is the inclusion, by the mapping cylinder construction. Consider the following exact sequence:

$$\cdots\to H^{q-1}(X;\, Z_p)\xrightarrow{\delta}H^q(Y,\, X;\, Z_p)\xrightarrow{j^*}H^q(Y;\, Z_p)\xrightarrow{f^*}H^q(X;\, Z_p)\to\cdots$$

where $j\colon Y\to(Y,\, X)$ is the inclusion and δ is the coboundary homomorphism. Put $q=l+2(p-1)i$. Since $f^*\mathcal{P}^i c=0$ for $j\leqq i\leqq k$ by (2.2), it follows that there is an element $x_i\in H^{l+2(p-1)i}(Y,\, X;\, Z_p)$ such that

(2.5) $j^*x_i=\mathcal{P}^i c \qquad \text{for } i=j, j+1,\cdots, k.$

Similarly by (2.3) we have an element $y_{k-i}\in H^{m+2(p-1)(k-i)}(Y,\, X;\, Z_p)$ such that

(2.6) $j^*y_{k-i}=\mathcal{P}^{k-i}d \qquad \text{for } i=0, 1,\cdots, j.$

LEMMA 2.7. *Let z_i denote x_i or y_i. The following relations hold modulo Image δ.*

(2.7.1) $\mathcal{P}^1 z_i=(i+1)z_{i+1},$

(2.7.2) $\mathcal{P}^2 z_i=\binom{(p-1)i-1}{2}z_{i+2},$

$$(2.7.3) \qquad \mathcal{P}^3 z_i = -\binom{(p-1)i-1}{3} z_{i+3} + \varepsilon \mathcal{P}^{i+2} z_1.$$

PROOF. These formulas follow easily from the definition of z_i and the Adem relations. Q. E. D.

PROOF OF THEOREM 2.4. By the Cartan formula and (2.1–3) we have $f^*\beta_k(c \cup d)=0$, and by (1.1) $\alpha_k\beta_k(c\cup d)=0$. Thus $(\alpha_k)_f\beta_k(c\cup d)$ is defined. Similarly $(\alpha_i)_f\beta_i(c)$ for $j<i\leqq k$, and $(\alpha_{k-i})_f\beta_{k-i}(d)$ for $0\leqq i<j$ are defined by (2.1–3) and (1.1).

We consider the diagram 1, where each row is the cohomology exact sequence of the pair (Y, X) and the coefficient group Z_p is omitted.

$$\begin{array}{ccccccc}
\xrightarrow{f^*} & H^{n-1}(X) & \xrightarrow{\delta} & H^n(Y, X) & \xrightarrow{j^*} & H^n(Y) & \xrightarrow{f^*} \\
 & \downarrow\beta_i & & \downarrow\beta_i & & \downarrow\beta_i & \\
\xrightarrow{f^*} & H^{n+2(p-1)(i-1)-1}(X) & \xrightarrow{\delta} & H^{n+2(p-1)(i-1)}(Y, X) & \xrightarrow{j^*} & H^{n+2(p-1)(i-1)}(Y) & \xrightarrow{f^*} \\
 & \oplus & & \oplus & & \oplus & \\
\xrightarrow{f^*} & H^{n+2(p-1)i-1}(X) & \xrightarrow{\delta} & H^{n+2(p-1)i}(Y, X) & \xrightarrow{j^*} & H^{n+2(p-1)i}(Y) & \xrightarrow{f^*} \\
 & \oplus & & \oplus & & \oplus & \\
\xrightarrow{f^*} & H^{n}(X) & \xrightarrow{\delta} & H^{n+1}(Y, X) & \xrightarrow{j^*} & H^{n+1}(Y) & \xrightarrow{f^*} \\
 & \downarrow\alpha_i & & \downarrow\alpha_i & & \downarrow\alpha_i & \\
\xrightarrow{f^*} & H^{n+2(p-1)i}(X) & \xrightarrow{\delta} & H^{n+2(p-1)i+1}(Y, X) & \xrightarrow{j^*} & H^{n+2(p-1)i+1}(Y) & \xrightarrow{f^*}
\end{array}$$

Diagram 1

Consider the case $n=l$. The functional operation $(\alpha_i)_f\beta_i(c)$ is determined, modulo the indeterminacy, by

$$(2.8) \qquad \delta((\alpha_i)_f\beta_i(c)) = \mathcal{P}^1 \Delta x_{i-1} - (i-1)\Delta x_i,$$

where δ is the coboundary homomorphism in the bottom of the diagram 1 and x_i is the element defined by (2.5). Similarly, in case $n=m$, $(\alpha_{k-i})_f\beta_{k-i}(d)$ is determined, modulo the indeterminacy, by

$$(2.9) \qquad \delta((\alpha_{k-i})_f\beta_{k-i}(d)) = \mathcal{P}^1 \Delta y_{k-i-1} - (k-i-1)\Delta y_{k-i},$$

where y_{k-i} is the element defined by (2.6). Next, consider the case $n=l+m$ and $i=k$ in the diagram 1. We have

$$j^*\{\sum_{i=j+1}^{k}(x_{i-1}\cup \mathcal{P}^{k-i}d) + \sum_{i=0}^{j-1}(\mathcal{P}^i c \cup y_{k-i-1})\} = \mathcal{P}^{k-1}(c\cup d),$$

$$j^*\{\sum_{i=j+1}^{k}(x_{i}\cup \mathcal{P}^{k-i}d) + \sum_{i=0}^{j}(\mathcal{P}^i c \cup y_{k-i})\} = \mathcal{P}^{k}(c\cup d),$$

by (2.5–6) and the Cartan formulas. Put

$$y = \mathcal{P}^1 \Delta \{\sum_{i=j+1}^{k}(x_{i-1}\cup \mathcal{P}^{k-i}d) + \sum_{i=0}^{j-1}(\mathcal{P}^i c \cup y_{k-i-1})\}$$

$$-(k-1)\Delta\{\sum_{i=j+1}^{k}(x_i\cup\mathcal{P}^{k-i}d)+\sum_{i=0}^{j}(\mathcal{P}^i c\cup y_{k-i})\}.$$

Then $j^*y=\alpha_k\beta_k(c\cup d)=0$ by (1.1) and (2.1). Thus we have $\delta((\alpha_k)_f\beta_k(c\cup d))=y$ by the definition of the functional operation.

Let us calculate the element y by means of the Cartan formulas, the Adem relations, (2.1) and (2.7.1). We have

$$\begin{aligned}y=&\mathcal{P}^1\Delta\sum_{i=j+1}^{k}(x_{i-1}\cup\mathcal{P}^{k-i}d)-(k-1)\Delta\sum_{i=j+1}^{k}(x_i\cup\mathcal{P}^{k-i}d)\\&+\mathcal{P}^1\Delta\sum_{i=0}^{j-1}(\mathcal{P}^i c\cup y_{k-i-1})-(k-1)\Delta\sum_{i=0}^{j}(\mathcal{P}^i c\cup y_{k-i})\\=&\sum_{i=j+1}^{k}\{(\mathcal{P}^1\Delta x_{i-1}-(i-1)\Delta x_i)\cup\mathcal{P}^{k-i}d\}\\&+(k-j)\Delta x_j\cup\mathcal{P}^{k-j}d+(-1)^l(k-j-1)x_j\cup\Delta\mathcal{P}^{k-j}d\\&+(-1)^l\sum_{i=0}^{j-1}\{\mathcal{P}^i c\cup(\mathcal{P}^1\Delta y_{k-i-1}-(k-i-1)\Delta y_{k-i})\}\\&-(k-j)\Delta\mathcal{P}^j c\cup y_{k-j}-(-1)^l(k-j-1)\mathcal{P}^j c\cup\Delta y_{k-j}\end{aligned}$$

modulo $\sum_{i=j+1}^{k}(\text{Image }\delta\cup\Delta\mathcal{P}^{k-i}d)+\sum_{i=0}^{j-1}(\Delta\mathcal{P}^i c\cup\text{ Image }\delta)$.

But by the naturality of cup products and (2.5–6) we have $\Delta x_j\cup\mathcal{P}^{k-j}d=\Delta x_j\cup j^*y_{k-j}=\Delta x_j\cup y_{k-j}=j^*\Delta x_j\cup y_{k-j}=\Delta\mathcal{P}^j c\cup y_{k-j}$, and $x_j\cup\Delta\mathcal{P}^{k-j}d=\mathcal{P}^j c\cup\Delta y_{k-j}$. Therefore y is given, modulo the above indeterminacy, as follows:

$$\begin{aligned}y=&\sum_{i=j+1}^{k}\{(\mathcal{P}^1\Delta x_{i-1}-(i-1)\Delta x_i)\cup\mathcal{P}^{k-i}d\}\\&+(-1)^l\sum_{i=0}^{j-1}\{\mathcal{P}^i c\cup(\mathcal{P}^1\Delta y_{k-i-1}-(k-i-1)\Delta y_{k-i})\}.\end{aligned}$$

On the other hand, by the property of δ and (2.8–9), we see

$$\begin{aligned}&\delta[\sum_{i=j+1}^{k}\{((\alpha_i)_f\beta_i(c))\cup f^*\mathcal{P}^{k-i}d\}+\sum_{i=0}^{j-1}\{f^*\mathcal{P}^i c\cup((\alpha_{k-i})_f\beta_{k-i}(d))\}]\\&=\sum_{i=j+1}^{k}\{\delta((\alpha_i)_f\beta_i(c))\cup\mathcal{P}^{k-i}d\}+(-1)^l\sum_{i=0}^{j-1}\{\mathcal{P}^i c\cup\delta((\alpha_{k-i})_f\beta_{k-i}(d))\}=y\end{aligned}$$

modulo the indeterminacy. Hence we have the desired result. Q. E. D.

2–2. We denote the Adem relation (1.2) by

$$(1.2)\qquad \alpha'_i\beta'_i=0,\quad \alpha'_i=\mathcal{P}^2\Delta-\left(\frac{(p-1)(i-1)}{2}\right)\Delta-i\mathcal{P}^{i+1},$$

$$\beta'_i=(\mathcal{P}^{i-1},\ \mathcal{P}^{i+1},\ \Delta),\quad \text{where } i>1.$$

Let k and j be given integers such that $0<j<k$. Let X and Y be spaces and $f\colon X\to Y$ be a map. Suppose that $c\in H^l(Y;Z_p)$ and $d\in H^m(Y;Z_p)$ satisfy the following conditions, where $l>0$ and $m>0$.

$$\Delta c=0,\quad \Delta d=0, \tag{2.1}$$

$$f^*\mathcal{P}^i c=0 \qquad \text{for } i=j,\ j+1,\cdots,\ k+1, \tag{2.2$'$}$$

$$f^*\mathcal{P}^{k-i} d=0 \qquad \text{for } i=-1,\ 0,\cdots,\ j. \tag{2.3$'$}$$

Then we can define the functional cohomology operations $(\alpha_i)_f\beta_i(c)$ and $(\alpha'_i)_f\beta'_i(c)$ for $j<i\leqq k$; $(\alpha_{k-i})_f\beta_{k-i}(d)$ and $(\alpha'_{k-i})_f\beta'_{k-i}(d)$ for $0\leqq i<j$; and $(\alpha_k)_f\beta_k(c\cup d)$ and $(\alpha'_k)_f\beta'_k(c\cup d)$. Furthermore, we have

THEOREM 2.10. $(\alpha'_k)_f\beta'_k(c\cup d)$

$$=\sum_{i=j+1}^{k}\{((\alpha'_i)_f\beta'_i(c))\cup\mathcal{P}^{k-i}f^*d+(k-i+1)\,((\alpha_i)_f\beta_i(c))\cup\mathcal{P}^{k-i+1}f^*d\}$$

$$+\sum_{i=0}^{j-1}\{\mathcal{P}^i f^*c\cup((\alpha'_{k-i})_f\beta'_{k-i}(d))+(i+1)\mathcal{P}^{i+1}f^*c\cup((\alpha_{k-i})_f\beta_{k-i}(d))\}$$

in $H^{l+m+2(p-1)(k+1)}(X;Z_p)/Q'$, *where*

$$Q'=f^*H^{l+m+2(p-1)(k+1)}(Y;Z_p)+Q^{l+m+2(p-1)(k+1)}(\Phi'_k;X)$$

$$+\sum_{i=j+1}^{k}\{Q^{l+2(p-1)(i+1)}(\Phi'_i;X)\cup\mathcal{P}^{k-i}f^*d+(k-i+1)Q^{l+2(p-1)i}(\Phi_i;X)\cup\mathcal{P}^{k-i+1}f^*d\}$$

$$+\sum_{i=0}^{j-1}\{\mathcal{P}^i f^*c\cup Q^{m+2(p-1)(k-i+1)}(\Phi'_{k-i};X)+(i+1)\mathcal{P}^{i+1}f^*c\cup Q^{m+2(p-1)(k-i)}(\Phi_{k-i};X)\}$$

$$+\sum_{i=j}^{k-1}\{H^{l+2(p-1)(i+1)-1}(X;Z_p)\cup\Delta\mathcal{P}^{k-i}f^*d\}$$

$$+\sum_{i=1}^{j}\{\Delta\mathcal{P}^i f^*c\cup H^{m+2(p-1)(k-i+1)-1}(X;Z_p)\}.$$

PROOF. As in the previous case, we see from (2.2–3)$'$ that there exist elements $x_i\in H^{l+2(p-1)i}(Y,X;Z_p)$ and $y_{k-i}\in H^{m+2(p-1)(k-i)}(Y,X;Z_p)$ such that

$$j^*x_i=\mathcal{P}^i c \qquad \text{for } i=j,\ j+1,\cdots,\ k+1, \tag{2.5$'$}$$

$$j^*y_{k-i}=\mathcal{P}^{k-i} d \qquad \text{for } i=-1,\ 0,\cdots,\ j. \tag{2.6$'$}$$

Then we notice that Lemma 2.7 holds in this case.

By the Cartan formula, (2.1) and (2.2–3)$'$, we have $f^*\beta'_k(c\cup d)=0$, and by (1.2), $\alpha'_k\beta'_k(c\cup d)=0$. Hence $(\alpha'_k)_f\beta'_k(c\cup d)$ is defined. Similarly $(\alpha'_i)_f\beta'_i(c)$ for $j<i\leqq k$, and $(\alpha'_{k-i})_f\beta'_{k-i}(d)$ for $0\leqq i<j$ are defined, by (2.1), (2.2–3)$'$ and

(1.2).

We consider the diagram 2, where the coefficient group Z_p is omitted.

$$\begin{array}{ccccccc}
\xrightarrow{f^*} H^{n-1}(X) & \xrightarrow{\delta} & H^n(Y, X) & \xrightarrow{j^*} & H^n(Y) & \xrightarrow{f^*} & \\
\downarrow \beta'_i & & \downarrow \beta'_i & & \downarrow \beta'_i & & \\
\xrightarrow{f^*} H^{n+2(p-1)(i-1)-1}(X) & \xrightarrow{\delta} & H^{n+2(p-1)(i-1)}(Y, X) & \xrightarrow{j^*} & H^{n+2(p-1)(i-1)}(Y) & \xrightarrow{f^*} & \\
\oplus & & \oplus & & \oplus & & \\
\xrightarrow{f^*} H^{n+2(p-1)(i+1)-1}(X) & \xrightarrow{\delta} & H^{n+2(p-1)(i+1)}(Y,X) & \xrightarrow{j^*} & H^{n+2(p-1)(i+1)}(Y) & \xrightarrow{f^*} & \\
\oplus & & \oplus & & \oplus & & \\
\xrightarrow{f^*} H^n(X) & \xrightarrow{\delta} & H^{n+1}(Y, X) & \xrightarrow{j^*} & H^{n+1}(Y) & \xrightarrow{f^*} & \\
\downarrow \alpha'_i & & \downarrow \alpha'_i & & \downarrow \alpha'_i & & \\
\xrightarrow{f^*} H^{n+2(p-1)(i+1)}(X) & \xrightarrow{\delta} & H^{n+2(p-1)(i+1)+1}(Y, X) & \xrightarrow{j^*} & H^{n+2(p-1)(i+1)+1}(Y) & \xrightarrow{f^*} &
\end{array}$$

Diagram 2

Consider the case $n=l$. The functional operation $(\alpha'_i)_f\beta'_i(c)$ is determined, modulo the indeterminacy, by

$$(2.11) \qquad \delta((\alpha'_i)_f\beta'_i(c)) = \mathcal{P}^2\Delta x_{i-1} - \binom{(p-1)(i-1)}{2}\Delta x_{i+1},$$

where δ is the coboundary homomorphism in the bottom of the diagram 2 and x_i is the element defined by (2.5)$'$. Similarly in case $n=m$, $(\alpha'_{k-i})_f\beta'_{k-i}(d)$ is determined, modulo the indeterminacy, by

$$(2.12) \quad \delta((\alpha'_{k-i})_f\beta'_{k-i}(d)) = \mathcal{P}^2\Delta y_{k-i-1} - \binom{(p-1)(k-i-1)}{2}\Delta y_{k-i+1},$$

where y_{k-i} is the element defined by (2.6)$'$. Next, consider the case $n=l+m$ and $i=k$ in the diagram 2. We have

$$j^*\{\sum_{i=j+1}^{k}(x_{i-1}\cup\mathcal{P}^{k-i}d) + \sum_{i=0}^{j-1}(\mathcal{P}^i c\cup y_{k-i-1})\} = \mathcal{P}^{k-1}(c\cup d),$$

$$j^*\{\sum_{i=j+1}^{k}(x_{i+1}\cup\mathcal{P}^{k-i}d) + \sum_{i=0}^{j+1}(\mathcal{P}^i c\cup y_{k-i+1})\} = \mathcal{P}^{k+1}(c\cup d),$$

by (2.5–6)$'$ and the Cartan formulas. Put

$$y' = \mathcal{P}^2\Delta\{\sum_{i=j+1}^{k}(x_{i-1}\cup\mathcal{P}^{k-i}d) + \sum_{i=0}^{j-1}(\mathcal{P}^i c\cup y_{k-i-1})\}$$
$$-\binom{(p-1)(k-1)}{2}\Delta\{\sum_{i=j+1}^{k}(x_{i+1}\cup\mathcal{P}^{k-i}d) + \sum_{i=0}^{j+1}(\mathcal{P}^i c\cup y_{k-i+1})\}.$$

Then $j^*y' = \alpha'_k\beta'_k(c\cup d) = 0$ by (1.2) and (2.1). Hence we have $\delta((\alpha'_k)_f\beta'_k(c\cup d)) = y'$ by the definition of the functional operation. Put $y' = A+B$, where

$$A=\mathcal{P}^2\Delta\sum_{i=j+1}^{k}(x_{i-1}\cup\mathcal{P}^{k-i}d)-\binom{(p-1)(k-1)}{2}\Delta\sum_{i=j+1}^{k}(x_{i+1}\cup\mathcal{P}^{k-i}d),$$

$$B=\mathcal{P}^2\Delta\sum_{i=0}^{j-1}(\mathcal{P}^i c\cup y_{k-i-1})-\binom{(p-1)(k-1)}{2}\Delta\sum_{i=0}^{j+1}(\mathcal{P}^i c\cup y_{k-i+1}).$$

Let us calculate A and B by means of the Cartan formulas, the Adem relations, (2.1), Lemma 2.7 and the next lemma.

LEMMA 2.13. *The following congruence holds for any integers k and i.*

$$\binom{(p-1)(i-1)}{2}+i(k-i)+\binom{(p-1)(k-i-2)-1}{2}\equiv\binom{(p-1)(k-1)}{2}\quad(\text{mod }p).$$

The proof is easy.

By calculations we obtain

$$\begin{aligned}A=\sum_{i=j+1}^{k}\Big[&\Big\{\mathcal{P}^2\Delta x_{i-1}-\binom{(p-1)(i-1)}{2}\Delta x_{i+1}\Big\}\cup\mathcal{P}^{k-i}d\\&+(k-i+1)\{\mathcal{P}^1\Delta x_{i-1}-(i-1)\Delta x_i\}\cup\mathcal{P}^{k-i+1}d\Big]\\&+\binom{(p-1)(k-j-1)-1}{2}\Delta x_j\cup\mathcal{P}^{k-j+1}d\\&+\Big\{j(k-j)+\binom{(p-1)(k-j-2)-1}{2}\Big\}\Delta x_{j+1}\cup\mathcal{P}^{k-j}d\\&+(-1)^l\binom{(p-1)(k-j-1)}{2}x_j\cup\Delta\mathcal{P}^{k-j+1}d\\&+(-1)^l\Big\{(k-j-1)(j+1)+\binom{(p-1)(k-j-2)}{2}\Big\}x_{j+1}\cup\Delta\mathcal{P}^{k-j}d\end{aligned}$$

modulo $\sum_{i=j}^{k-1}(\text{Image }\delta\cup\Delta\mathcal{P}^{k-i}d)$,

$$\begin{aligned}B=(-1)^l\sum_{i=0}^{j-1}\Big[&\mathcal{P}^i c\cup\Big\{\mathcal{P}^2\Delta y_{k-i-1}-\binom{(p-1)(k-i-1)}{2}\Delta y_{k-i+1}\Big\}\\&+(i+1)\mathcal{P}^{i+1}c\cup\{\mathcal{P}^1\Delta y_{k-i-1}-(k-i-1)\Delta y_{k-i}\}\Big]\\+(-1)^l&\Big\{j(k-j)+\binom{(p-1)(j-2)-1}{2}-\binom{(p-1)(k-1)}{2}\Big\}\mathcal{P}^j c\cup\Delta y_{k-j-1}\\+(-1)^l&\Big\{\binom{(p-1)(j-1)-1}{2}-\binom{(p-1)(k-1)}{2}\Big\}\mathcal{P}^{j+1}c\cup\Delta y_{k-j}\\+&\Big\{(j-1)(k-j+1)+\binom{(p-1)(j-2)}{2}-\binom{(p-1)(k-1)}{2}\Big\}\Delta\mathcal{P}^j c\cup y_{k-j+1}\end{aligned}$$

$$+\left\{\binom{(p-1)(j-1)}{2}-\binom{(p-1)(k-1)}{2}\right\}\Delta\mathcal{P}^{j+1}\cup y_{k-j}$$

modulo $\sum_{i=1}^{j}(\Delta\mathcal{P}^i c\cup \text{Image }\delta)$.

But by the naturality of cup products and (2.5–6)′, we have $\Delta x_i\cup\mathcal{P}^{k-i+1}d=\Delta\mathcal{P}^i c\cup y_{k-i+1}$ and $x_i\cup\Delta\mathcal{P}^{k-i+1}d=\mathcal{P}^i c\cup\Delta y_{k-i-1}$ for $i=j$ and $j+1$. According to Lemma 2.13, each of the sums of the coefficients of the corresponding terms in A and B is zero. Therefore y' is given, modulo the indeterminacy, as follows:

$$y'=\sum_{i=j+1}^{k}\Big[\Big\{\mathcal{P}^2\Delta x_{i-1}-\binom{(p-1)(i-1)}{2}\Delta x_{i+1}\Big\}\cup\mathcal{P}^{k-i}d$$

$$+(k-i+1)\{\mathcal{P}^1\Delta x_{i-1}-(i-1)\Delta x_i\}\cup\mathcal{P}^{k-i+1}d\Big]$$

$$+(-1)^l\sum_{i=0}^{j-1}\Big[\mathcal{P}^i c\cup\Big\{\mathcal{P}^2\Delta y_{k-i-1}-\binom{(p-1)(k-i-1)}{2}\Delta y_{k-i+1}\Big\}$$

$$+(i+1)\mathcal{P}^{i+1}c\cup\{\mathcal{P}^1\Delta y_{k-i-1}-(k-i-1)\Delta y_{k-i}\}\Big].$$

On the other hand, by the property of δ we have

$$\delta\Big[\sum_{i=j+1}^{k}\{((\alpha_i')_f\beta_i'(c))\cup f^*\mathcal{P}^{k-i}d+(k-i+1)((\alpha_i)_f\beta_i(c))\cup f^*\mathcal{P}^{k-i+1}d\}$$

$$+\sum_{i=0}^{j-1}\{f^*\mathcal{P}^i c\cup((\alpha_{k-i}')_f\beta_{k-i}'(d))+(i+1)f^*\mathcal{P}^{i+1}c\cup((\alpha_{k-i})_f\beta_{k-i}(d))\}\Big]$$

$$=\sum_{i=j+1}^{k}\{\delta((\alpha_i')_f\beta_i'(c))\cup\mathcal{P}^{k-i}d+(k-i+1)\delta((\alpha_i)_f\beta_i(c))\cup\mathcal{P}^{k-i+1}d\}$$

$$+(-1)^l\sum_{i=0}^{j-1}\{\mathcal{P}^i c\cup\delta((\alpha_{k-i}')_f\beta_{k-i}'(d))+(i+1)\mathcal{P}^{i+1}c\cup\delta((\alpha_{k-i})_f\beta_{k-i}(d))\}.$$

Then this is equal to y', modulo the indeterminacy, by (2.8–9) and (2.11–12). Therefore we get the desired result. Q. E. D.

2–3. We denote the Adem relation (1.3) by

(1.3) $\alpha_i''\beta_i''=0,$

$$\alpha_i''=\mathcal{P}^3\Delta+\binom{(p-1)(i-1)}{3}\Delta-\varepsilon\Delta-\binom{(p-1)(i-1)-1}{2}\mathcal{P}^{i+2},$$

$$\beta_i''=(\mathcal{P}^{i-1},\ \mathcal{P}^{i+2},\ \varepsilon\mathcal{P}^{i+1}\mathcal{P}^1,\ \Delta),\quad \text{where } i>1.$$

Let k and j be integers such that $0<j<k$. Let X and Y be spaces and f:

$X \to Y$ be a map. Assume that $c \in H^l(Y; Z_p)$ and $d \in H^m(Y; Z_p)$ satisfy the folowing conditions ($l>0$, $m>0$).

(2.1) $\quad \Delta c=0, \quad \Delta d=0,$

(2.2)'' $\quad f^*\mathcal{P}^i c=0 \quad$ for $i=j, j+1, \dots, k+2,$

(2.3)'' $\quad f^*\mathcal{P}^{k-i} d=0 \quad$ for $i=-2, -1, \dots, j,$

(2.14) $\quad \varepsilon f^*\mathcal{P}^{i+1}\mathcal{P}^1 c=0 \quad$ for $i=j+1, j+2, \dots, k,$

(2.15) $\quad \varepsilon f^*\mathcal{P}^{k-i+1}\mathcal{P}^1 d=0 \quad$ for $i=0, 1, \dots, j-1.$

Then we can define $(\alpha_i)_f\beta_i(c)$, $(\alpha'_i)_f\beta'_i(c)$ and $(\alpha''_i)_f\beta''_i(c)$ for $j<i\leqq k$; $(\alpha_{k-i})_f\beta_{k-i}(d)$, $(\alpha'_{k-i})_f\beta'_{k-i}(d)$ and $(\alpha''_{k-i})_f\beta''_{k-i}(d)$ for $0\leqq i<j$; $(\alpha_k)_f\beta_k(c\cup d)$, $(\alpha'_k)_f\beta'_k(c\cup d)$ and $(\alpha''_k)_f\beta''_k(c\cup d)$. Moreover, we have

THEOREM 2.16. $(\alpha''_k)_f\beta''_k(c\cup d)$

$$= \sum_{i=j+1}^{k}\Big\{((\alpha''_i)_f\beta''_i(c))\cup\mathcal{P}^{k-i}f^*d+(k-i+1)((\alpha'_i)_f\beta'_i(c))\cup\mathcal{P}^{k-i+1}f^*d$$

$$+\left(\frac{(p-1)(k-i)-1}{2}\right)((\alpha_i)_f\beta_i(c))\cup\mathcal{P}^{k-i+2}f^*d\Big\}$$

$$+\sum_{i=0}^{j-1}\Big\{\mathcal{P}^i f^*c\cup((\alpha''_{k-i})_f\beta''_{k-i}(d))+(i+1)\mathcal{P}^{i+1}f^*c\cup((\alpha'_{k-i})_f\beta'_{k-i}(d))$$

$$+\left(\frac{(p-1)i-1}{2}\right)\mathcal{P}^{i+2}f^*c\cup((\alpha_{k-i})_f\beta_{k-i}(d))\Big\}$$

in $H^{l+m+2(p-1)(k+2)}(X; Z_p)/Q''$, *where*

$$Q''=f^*H^{l+m+2(p-1)(k+2)}(Y; Z_p)+Q^{l+m+2(p-1)(k+2)}(\Phi''_k; X)$$

$$+\sum_{i=j+1}^{k}\Big\{Q^{l+2(p-1)(i+2)}(\Phi''_i; X)\cup\mathcal{P}^{k-i}f^*d$$

$$+(k-i+1)Q^{l+2(p-1)(i+1)}(\Phi'_i; X)\cup\mathcal{P}^{k-i+1}f^*d$$

$$+\left(\frac{(p-1)(k-i)-1}{2}\right)Q^{l+2(p-1)i}(\Phi_i; X)\cup\mathcal{P}^{k-i+2}f^*d\Big\}$$

$$+\sum_{i=0}^{j-1}\Big\{\mathcal{P}^i f^*c\cup Q^{m+2(p-1)(k-i+2)}(\Phi''_{k-i}; X)$$

$$+(i+1)\mathcal{P}^{i+1}f^*c\cup Q^{m+2(p-1)(k-i+1)}(\Phi'_{k-i}; X)$$

$$+\left(\frac{(p-1)i-1}{2}\right)\mathcal{P}^{i+2}f^*c\cup Q^{m+2(p-1)(k-i)}(\Phi_{k-i}; X)\Big\}$$

$$+\sum_{i=j-1}^{k-1}\{H^{l+2(p-1)(i+2)-1}(X;\,Z_p)\cup\Delta\mathcal{P}^{k-i}f^*d\}$$

$$+\sum_{i=1}^{j+1}\{\Delta\mathcal{P}^if^*c\cup H^{m+2(p-1)(k-i+2)-1}(X;\,Z_p)\}.$$

PROOF. The method of the proof is the same as that of Theorem 2.10. From (2.2–3)$''$ we see that there are elements $x_i\in H^{l+2(p-1)i}(Y,X;\,Z_p)$ and $y_{k-i}\in H^{m+2(p-1)(k-i)}(Y,X;\,Z_p)$ such that

$$j^*x_i=\mathcal{P}^ic \qquad \text{for } i=j,\,j+1,\dots,\,k+2, \tag{2.5$''$}$$

$$j^*y_{k-i}=\mathcal{P}^{k-i}d \qquad \text{for } i=-2,\,-1,\dots,\,j. \tag{2.6$''$}$$

Notice that Lemma 2.7 holds in this case.

As is easily seen, all functional operations in the theorem are defined by the assumptions.

We consider the diagram 3, where the coefficient group Z_p is omitted.

$$\begin{array}{ccccccc}
\xrightarrow{f^*} & H^{n-1}(X) & \xrightarrow{\delta} & H^n(Y,X) & \xrightarrow{j^*} & H^n(Y) & \xrightarrow{f^*}\\
 & \downarrow\beta''_i & & \downarrow\beta''_i & & \downarrow\beta''_i & \\
\xrightarrow{f^*} & H^{n+2(p-1)(i-1)-1}(X) & \xrightarrow{\delta} & H^{n+2(p-1)(i-1)}(Y,X) & \xrightarrow{j^*} & H^{n+2(p-1)(i-1)}(Y) & \xrightarrow{f^*}\\
 & \oplus & & \oplus & & \oplus & \\
\xrightarrow{f^*} & H^{n+2(p-1)(i+2)-1}(X) & \xrightarrow{\delta} & H^{n+2(p-1)(i+2)}(Y,X) & \xrightarrow{j^*} & H^{n+2(p-1)(i+2)}(Y) & \xrightarrow{f^*}\\
 & \oplus & & \oplus & & \oplus & \\
\xrightarrow{f^*} & \varepsilon H^{n+2(p-1)(i+2)-1}(X) & \xrightarrow{\delta} & \varepsilon H^{n+2(p-1)(i+2)}(Y,X) & \xrightarrow{j^*} & \varepsilon H^{n+2(p-1)(i+2)}(Y) & \xrightarrow{f^*}\\
 & \oplus & & \oplus & & \oplus & \\
\xrightarrow{f^*} & H^{n}(X) & \xrightarrow{\delta} & H^{n+1}(Y,X) & \xrightarrow{j^*} & H^{n+1}(Y) & \xrightarrow{f^*}\\
 & \downarrow\alpha''_i & & \downarrow\alpha''_i & & \downarrow\alpha''_i & \\
\xrightarrow{f^*} & H^{n+2(p-1)(i+2)}(X) & \xrightarrow{\delta} & H^{n+2(p-1)(i+2)+1}(Y,X) & \xrightarrow{j^*} & H^{n+2(p-1)(i+2)+1}(Y) & \xrightarrow{f^*}
\end{array}$$

Diagram 3

Consider the case $n=l$. The functional operation $(\alpha''_i)_f\beta''_i(c)$ is determined, modulo the indeterminacy, by

$$\delta((\alpha''_i)_f\beta''_i(c))=\mathcal{P}^3\Delta x_{i-1}+\binom{(p-1)(i-1)}{3}\Delta x_{i+2}-\varepsilon\Delta\mathcal{P}^{i+1}x_1, \tag{2.17}$$

where δ is the coboundary homomorphism in the bottom of the diagram 3 and x_i is the element defined by (2.5)$''$. Similarly, in case $n=m$, $(\alpha''_{k-i})_f\beta''_{k-i}(d)$ is determined, modulo the indeterminacy, by

$$\delta((\alpha''_{k-i})_f\beta''_{k-i}(d))=\mathcal{P}^3\Delta y_{k-i-1}+\binom{(p-1)(k-i-1)}{3}\Delta y_{k-i+2}-\varepsilon\Delta\mathcal{P}^{k-i+1}y_1, \tag{2.18}$$

where y_{k-i} is the element defined by (2.6)$''$. Next, consider the case $n=l+m$ and $i=k$ in the diagram 3. We have

$$j^*\{\sum_{i=j+1}^{k}(x_{i-1}\cup\mathcal{P}^{k-i}d)+\sum_{i=0}^{j-1}(\mathcal{P}^{i}c\cup y_{k-i-1})\}=\mathcal{P}^{k-1}(c\cup d),$$

$$j^*\{\sum_{i=j+1}^{k}(x_{i+2}\cup\mathcal{P}^{k-i}d)+\sum_{i=0}^{j+2}(\mathcal{P}^{i}c\cup y_{k-i+2})\}=\mathcal{P}^{k+2}(c\cup d),$$

$$j^*\varepsilon\{\sum_{i=j+1}^{k}(\mathcal{P}^{i+1}x_1\cup\mathcal{P}^{k-i}d)+\sum_{i=0}^{j+1}(\mathcal{P}^{i}\mathcal{P}^{1}c\cup y_{k-i+1})$$

$$+\sum_{i=j}^{k}(x_{i+1}\cup\mathcal{P}^{k-i}\mathcal{P}^{1}d)+\sum_{i=0}^{j}(\mathcal{P}^{i}c\cup\mathcal{P}^{k-i+1}y_1)\}=\varepsilon\mathcal{P}^{k+1}\mathcal{P}^{1}(c\cup d),$$

by (2.5–6)$''$ and the Cartan formulas. Put

$$y''=\mathcal{P}^3\Delta\{\sum_{i=j+1}^{k}(x_{i-1}\cup\mathcal{P}^{k-i}d)+\sum_{i=0}^{j-1}(\mathcal{P}^{i}c\cup y_{k-i-1})\}$$

$$+\binom{(p-1)(k-1)}{3}\Delta\{\sum_{i=j+1}^{k}(x_{i+2}\cup\mathcal{P}^{k-i}d)+\sum_{i=0}^{j+2}(\mathcal{P}^{i}c\cup y_{k-i+2})\}$$

$$-\varepsilon\Delta\{\sum_{i=j+1}^{k}(\mathcal{P}^{i+1}x_1\cup\mathcal{P}^{k-i}d)+\sum_{i=0}^{j+1}(\mathcal{P}^{i}\mathcal{P}^{1}c\cup y_{k-i+1})$$

$$+\sum_{i=j}^{k}(x_{i+1}\cup\mathcal{P}^{k-i}\mathcal{P}^{1}d)+\sum_{i=0}^{j}(\mathcal{P}^{i}c\cup\mathcal{P}^{k-i+1}y_1)\}.$$

Then $j^*y''=\alpha''_k\beta''_k(c\cup d)=0$ by (1.3) and (2.1). Hence we have $\delta((\alpha''_k)_f\beta''_k(c\cup d))=y''$.

We calculate y'' by means of the Cartan formulas, the Adem relations, (2.1), Lemma 2.7 and the next lemma.

LEMMA 2.19. *The following congruence holds for any integers k and i.*

$$\binom{(p-1)(i-1)-1}{3}-\binom{(p-1)i-1}{2}(k-i-1)$$

$$-(i+2)\binom{(p-1)(k-i-2)}{2}+\binom{(p-1)(k-i-3)}{3}$$

$$\equiv\binom{(p-1)(k-1)}{3}\qquad(\text{mod } p).$$

The proof is not difficult.

By tedious calculations we obtain

$$y''=\sum_{i=j+1}^{k}\Big[\Big\{\mathcal{P}^3\Delta x_{i-1}+\binom{(p-1)(i-1)}{3}\Delta x_{i+2}-\varepsilon\Delta\mathcal{P}^{i+1}x_1\Big\}\cup\mathcal{P}^{k-i}d$$

$$+(k-i+1)\Big\{\mathcal{P}^2\Delta x_{i-1}-\binom{(p-1)(i-1)}{2}\Delta x_{i+1}\Big\}\cup\mathcal{P}^{k-i+1}d$$

$$+\binom{(p-1)(k-i)-1}{2}\{\mathcal{P}^1\varDelta x_{i-1}-(i-1)\varDelta x_i\}\cup\mathcal{P}^{k-i+2}d\Big]$$

$$+(-1)^l\sum_{i=0}^{j-1}\Big[\mathcal{P}^i c\cup\Big\{\mathcal{P}^3\varDelta y_{k-i-1}+\binom{(p-1)(k-i-1)}{3}\varDelta y_{k-i+2}-\varepsilon\varDelta\mathcal{P}^{k-i+1}y_1\Big\}$$

$$+(i+1)\mathcal{P}^{i+1}c\cup\Big\{\mathcal{P}^2\varDelta y_{k-i-1}-\binom{(p-1)(k-i-1)}{2}\varDelta y_{k-i+1}\Big\}$$

$$+\binom{(p-1)i-1}{2}\mathcal{P}^{i+2}c\cup\{\mathcal{P}^1\varDelta y_{k-i-1}-(k-i-1)\varDelta y_{k-i}\}\Big]$$

modulo $\sum_{i=j-1}^{k-1}(\text{Image }\delta\cup\varDelta\mathcal{P}^{k-i}d)+\sum_{i=1}^{j+1}(\varDelta\mathcal{P}^i c\cup\text{ Image }\delta)$.

By the property of δ, (2.8–9), (2.11–12) and (2.17–18), we have the desired result, as in the proof of Theorem 2.10. Q. E. D.

§3. Formulas on mod p secondary cohomology operations

3–1. We denote by $\varPhi_i$ the stable secondary cohomology operation associated with the Adem relation (1.1) of degree $2(p-1)i+1$. Let k and j be given integers with $0<j<k$. Let X be a space. Assume that the elements $u\in H^l(X; Z_p)$ and $v\in H^m(X; Z_p)$ have the following properties ($l>0$, $m>0$):

(3.1) u and v are mod p reductions of integral classes.

(3.2) $\mathcal{P}^i u=0$ for $i=j, j+1,\cdots, k$,

(3.3) $\mathcal{P}^{k-i}v=0$ for $i=0, 1,\cdots,j$.

Then we can define the secondary cohomology operations $\varPhi_i(u)$ for $j<i\leqq k$, $\varPhi_{k-i}(v)$ for $0\leqq i<j$, and $\varPhi_k(u\cup v)$. Moreover, we have

THEOREM 3.4. $\varPhi_k(u\cup v)=\sum_{i=j+1}^{k}(\varPhi_i(u)\cup\mathcal{P}^{k-i}v)+\sum_{i=0}^{j-1}(\mathcal{P}^i u\cup\varPhi_{k-i}(v))$

in $H^{l+m+2(p-1)k}(X; Z_p)$ modulo the indeterminacy Q.

If $\varPhi_i(u)$ for $j<i\leqq k$ and $\varPhi_{k-i}(v)$ for $0\leqq i<j$ are defined, clearly the conditions (3.2–3) are satisfied. Therefore Theorem 3.4 is equivalent to the theorem in §1.

The indeterminacy Q is given as follows. Let $g\colon X\to K(Z, l)$ and $h\colon X\to K(Z, m)$ be maps such that $g^*\gamma=u$ and $h^*\kappa=v$, where γ and κ are the mod p reductions of the fundamental classes of $H^l(K(Z, l); Z)$ and $H^m(K(Z, m); Z)$ respectively. Such maps g and h exist because of (3.1). Define a map $f\colon X\to K(Z, l)\times K(Z, m)$ by $f(x)=(g(x), h(x))$ for each $x\in X$. Then

$$Q=f^*H^{l+m+2(p-1)k}(K(Z,\ l)\times K(Z,\ m);\ Z_p)+Q^{l+m+2(p-1)k}(\Phi_k;\ X)$$

$$+\sum_{i=j+1}^{k}\{Q^{l+2(p-1)i}(\Phi_i;\ X)\cup\mathcal{P}^{k-i}v+H^{l+2(p-1)i-1}(X;\ Z_p)\cup\varDelta\mathcal{P}^{k-i}v\}$$

$$+\sum_{i=0}^{j-1}\{\mathcal{P}^i u\cup Q^{m+2(p-1)(k-i)}(\Phi_{k-i};\ X)+\varDelta\mathcal{P}^i u\cup H^{m+2(p-1)(k-i)-1}(X;\ Z_p)\}.$$

PROOF OF THEOREM 3.4. Let g, h and f be defined as above. Then $f^*(\gamma\times 1)=g^*\gamma=u$, $f^*(1\times\kappa)=h^*\kappa=v$ and $f^*(\gamma\times\kappa)=f^*((\gamma\times 1)\cup(1\times\kappa))=u\cup v$. Properties (2.1–3) hold for $Y=K(Z,\ l)\times K(Z,\ m)$, $c=\gamma\times 1$, $d=1\times\kappa$ by (3.1–3). Hence we can define $(\alpha_i)_f\beta_i(\gamma\times 1)$ for $j<i\leqq k$, $(\alpha_{k-i})_f\beta_{k-i}(1\times\kappa)$ for $0\leqq i<j$, and $(\alpha_k)_f\beta_k(\gamma\times\kappa)$, and we have

$$(3.5)\qquad (\alpha_k)_f\beta_k(\gamma\times\kappa)=\sum_{i=j+1}^{k}\{((\alpha_i)_f\beta_i(\gamma\times 1))\cup\mathcal{P}^{k-i}f^*(1\times\kappa)\}$$
$$+\sum_{i=0}^{j-1}\{\mathcal{P}^i f^*(\gamma\times 1)\cup((\alpha_{k-i})_f\beta_{k-i}(1\times\kappa))\}$$

in $H^{l+m+2(p-1)k}(X;\ Z_p)/Q$ by Theorem 2.4.

On the other hand, the second formula of Peterson-Stein [3, Theorem 5.2] implies that

$$(3.6)\qquad \Phi_k(u\cup v)=-(\alpha_k)_f\beta_k(\gamma\times\kappa)$$

modulo $f^*H^{l+m+2(p-1)k}(K(Z,\ l)\times K(Z,\ m);\ Z_p)+Q^{l+m+2(p-1)k}(\Phi_k;\ X)$,

$$(3.7)\qquad \Phi_i(u)=-(\alpha_i)_f\beta_i(\gamma\times 1)$$

modulo $f^*H^{l+2(p-1)i}(K(Z,\ l)\times K(Z,\ m);\ Z_p)+Q^{l+2(p-1)i}(\Phi_i;\ X)$,

$$(3.8)\qquad \Phi_{k-i}(v)=-(\alpha_{k-i})_f\beta_{k-i}(1\times\kappa)$$

modulo $f^*H^{m+2(p-1)(k-i)}(K(Z,\ l)\times K(Z,\ m);\ Z_p)+Q^{m+2(p-1)(k-i)}(\Phi_{k-i};\ X)$.

Thus we have the desired formula with the indeterminacy Q from (3.5–8). Q. E. D.

COROLLARY 3.9. *Assume that the elements* $u\in H^l(X;\ Z_p)$ *and* $v\in H^m(X;\ Z_p)$ $(l>0,\ m>0)$ *satisfy* (3.1), *and*

$$(3.9.1)\qquad \mathcal{P}^i u=0\qquad \textit{for } i=1,\ 2,\cdots,\ k,$$

$$(3.9.2)\qquad m<2(k-1).$$

Then $\Phi_i(u)$ *for* $1<i\leqq k$ *and* $\Phi_k(u\cup v)$ *are defined, and we have*

$$\Phi_k(u\cup v)=\sum_{i=2}^{k}(\Phi_i(u)\cup\mathcal{P}^{k-i}v)$$

in $H^{l+m+2(p-1)k}(X;\, Z_p)/Q_1$, *where*

$$Q_1=f^*H^{l+m+2(p-1)k}(K(Z,\,l)\times K(Z,\,m);\,Z_p)+Q^{l+m+2(p-1)k}(\Phi_k;\,X)$$
$$+\sum_{i=2}^{k}\{Q^{l+2(p-1)i}(\Phi_i;\,X)\cup\mathcal{P}^{k-i}v+H^{l+2(p-1)i-1}(X;\,Z_p)\cup\Delta\mathcal{P}^{k-i}v\}.$$

PROOF. If we put $j=1$, the conditions (3.2–3) are satisfied by (3.9.1–2). Clearly, we have $\Phi_k(v)=0$ by (3.9.2). Since $u\cup Q^{m+2(p-1)k}(\Phi_k;X)$ is contained in $Q^{l+m+2(p-1)k}(\Phi_k;\,X)$ by (3.9.1), the result follows immediately from Theorem 3.4. Q. E. D.

COROLLARY 3.10. *Assume that the elements* $u\in H^l(X;\,Z_p)$ *and* $v\in H^m(X;\,Z_p)$ $(l>0,\ m>0)$ *satisfy* (3.1), *and*

(3.10.1) $\qquad \mathcal{P}^i u=0 \qquad$ *for* $i=1,\,2,\cdots,\,k,$

(3.10.2) $\qquad \mathcal{P}^i v=0 \qquad$ *for* $i=1,\,2,\cdots,\,k.$

Then $\Phi_k(u)$, $\Phi_k(v)$ *and* $\Phi_k(u\cup v)$ *are defined, and we have*

$$\Phi_k(u\cup v)=\Phi_k(u)\cup v+u\cup\Phi_k(v)$$

in $H^{l+m+2(p-1)k}(X;\,Z_p)/Q_2$, *where*

$$Q_2=f^*H^{l+m+2(p-1)k}(K(Z,\,l)\times K(Z,\,m);\,Z_p)+Q^{l+m+2(p-1)k}(\Phi_k;\,X).$$

This result follows also from the formula of Adem [3, Theorem 8.4].

PROOF. The conditions (3.2–3) are satisfied by (3.10.1–2). Since $Q^{l+2(p-1)k}(\Phi_k;\,X)\cup v$ and $u\cup Q^{m+2(p-1)k}(\Phi_k;\,X)$ are contained in $Q^{l+m+2(p-1)k}(\Phi_k;\,X)$ by (3.10.1–2), the result follows from Theorem 3.4. Q. E. D.

3–2. Let us denote by Φ'_i the secondary operation associated with the Adem relation (1.2) of degree $2(p-1)(i+1)+1$. Let k and j be integers with $0<j<k$. Let X be a space. Suppose that $u\in H^l(X;\,Z_p)$ and $v\in H^m(X;\,Z_p)$ $(l>0,\ m>0)$ are mod p reductions of integral classes, and that they have the following properties:

(3.2)′ $\qquad \mathcal{P}^i u=0 \qquad$ for $i=j,\,j+1,\cdots,\,k+1,$

(3.3)′ $\qquad \mathcal{P}^{k-i}v=0 \qquad$ for $i=-1,\,0,\cdots,\,j.$

Then we can define the secondary cohomology operations $\Phi_i(u)$ and $\Phi'_i(u)$ for $j<i\leqq k$; $\Phi_{k-i}(v)$ and $\Phi'_{k-i}(v)$ for $0\leqq i<j$; and $\Phi_k(u\cup v)$ and $\Phi'_k(u\cup v)$. Furthermore, we have

THEOREM 3.11. $\Phi'_k(u\cup v)=\sum_{i=j+1}^{k}\{\Phi'_i(u)\cup\mathcal{P}^{k-i}v+(k-i+1)\Phi_i(u)\cup\mathcal{P}^{k-i+1}v\}$

$$+\sum_{i=0}^{j-1}\{\mathcal{P}^i u\cup\Phi'_{k-i}(v)+(i+1)\mathcal{P}^{i+1}u\cup\Phi_{k-i}(v)\}$$

in $H^{l+m+2(p-1)(k+1)}(X;\ Z_p)/Q'$, *where*

$$Q'=f^*H^{l+m+2(p-1)(k+1)}(K(Z,\ l)\times K(Z,\ m);\ Z_p)+Q^{l+m+2(p-1)(k+1)}(\Phi'_k;\ X)$$

$$+\sum_{i=j+1}^{k}\{Q^{l+2(p-1)(i+1)}(\Phi'_i;\ X)\cup\mathcal{P}^{k-i}v+(k-i+1)Q^{l+2(p-1)i}(\Phi_i;\ X)\cup\mathcal{P}^{k-i+1}v\}$$

$$+\sum_{i=0}^{j-1}\{\mathcal{P}^i u\cup Q^{m+2(p-1)(k-i+1)}(\Phi'_{k-i};\ X)+(i+1)\mathcal{P}^{i+1}u\cup Q^{m+2(p-1)(k-i)}(\Phi_{k-i};\ X)\}$$

$$+\sum_{i=j}^{k-1}\{H^{l+2(p-1)(i+1)-1}(X;\ Z_p)\cup\Delta\mathcal{P}^{k-i}v\}+\sum_{i=1}^{j}\{\Delta\mathcal{P}^i u\cup H^{m+2(p-1)(k-i+1)-1}(X;\ Z_p)\}.$$

PROOF. The result follows from Theorem 2.10 and the second formula of Peterson-Stein, as in the proof of Theorem 3.4, so we omit the details.

Q. E. D.

3-3. We denote by Φ''_i the secondary operation associated with the Adem relation (1.3) of degree $2(p-1)(i+2)+1$. Let k and j be integers with $0<j<k$. Suppose that $u\in H^l(X;\ Z_p)$ and $v\in H^m(X;\ Z_p)$ ($l>0$, $m>0$) are mod p reductions of integral classes, and that they have the following properties:

(3.2)'' $\qquad \mathcal{P}^i u=0 \quad$ for $i=j,\ j+1,\cdots,\ k+2$,

(3.3)'' $\qquad \mathcal{P}^{k-i}v=0 \quad$ for $i=-2,\ -1,\cdots,j$,

(3.12) $\qquad \varepsilon\mathcal{P}^{i+1}\mathcal{P}^1 u=0 \quad$ for $i=j+1,\ j+2,\cdots,\ k$,

(3.13) $\qquad \varepsilon\mathcal{P}^{k-i+1}\mathcal{P}^1 v=0 \quad$ for $i=0,\ 1,\cdots,\ j-1$.

Then we can define $\Phi_i(u)$, $\Phi'_i(u)$ and $\Phi''_i(u)$ for $j<i\leqq k$; $\Phi_{k-i}(v)$, $\Phi'_{k-i}(v)$ and $\Phi''_{k-i}(v)$ for $0\leqq i<j$; $\Phi_k(u\cup v)$, $\Phi'_k(u\cup v)$ and $\Phi''_k(u\cup v)$. Moreover, we have

THEOREM 3.14. $\Phi''_k(u\cup v)$

$$=\sum_{i=j+1}^{k}\Big\{\Phi''_i(u)\cup\mathcal{P}^{k-i}v+(k-i+1)\Phi'_i(u)\cup\mathcal{P}^{k-i+1}v$$

$$+\binom{(p-1)(k-i)-1}{2}\Phi_i(u)\cup\mathcal{P}^{k-i+2}v\Big\}$$

$$+\sum_{i=0}^{j-1}\Big\{\mathcal{P}^i u\cup\Phi''_{k-i}(v)+(i+1)\mathcal{P}^{i+1}u\cup\Phi'_{k-i}(v)$$

$$+\binom{(p-1)i-1}{2}\mathcal{P}^{i+2}u\cup\Phi_{k-i}(v)\Big\}$$

in $H^{l+m+2(p-1)(k+2)}(X;\ Z_p)/Q''$, *where*

$$Q^{\prime\prime}=f^{*}H^{l+m+2(p-1)(k+2)}(K(Z,l)\times K(Z,m);Z_p)+Q^{l+m+2(p-1)(k+2)}(\Phi''_k;X)$$

$$+\sum_{i=j+1}^{k}\Big\{Q^{l+2(p-1)(i+2)}(\Phi''_i;X)\cup\mathcal{P}^{k-i}v+(k-i+1)Q^{l+2(p-1)(i+1)}(\Phi'_i;X)\cup\mathcal{P}^{k-i+1}v$$

$$+\binom{\frac{(p-1)(k-i)-1}{2}}{}Q^{l+2(p-1)i}(\Phi_i;X)\cup\mathcal{P}^{k-i+2}v\Big\}$$

$$+\sum_{i=0}^{j-1}\Big\{\mathcal{P}^{i}u\cup Q^{m+2(p-1)(k-i+2)}(\Phi''_{k-i};X)+(i+1)\mathcal{P}^{i+1}u\cup Q^{m+2(p-1)(k-i+1)}(\Phi'_{k-i};X)$$

$$+\binom{\frac{(p-1)i-1}{2}}{}\mathcal{P}^{i+2}u\cup Q^{m+2(p-1)(k-i)}(\Phi_{k-i};X)\Big\}$$

$$+\sum_{i=j-1}^{k-1}\{H^{l+2(p-1)(i+2)-1}(X;Z_p)\cup\Delta\mathcal{P}^{k-i}v\}$$

$$+\sum_{i=1}^{j+1}\{\Delta\mathcal{P}^{i}u\cup H^{m+2(p-1)(k-i+2)-1}(X;Z_p)\}.$$

Proof. This result follows from Theorem 2.16 and the second formula of Peterson-Stein. We omit the detailed proof. Q. E. D.

We have corollaries of Theorems 3.11 and 3.14 similar to Corollaries 3.9–10.

§4. Mod p secondary cohomology operations in complex projective space

Let CP^∞ denote the infinite dimensional complex projective space. The cohomology algebra $H^*(CP^\infty;Z_p)$ is a polynomial algebra over Z_p generated by $z\in H^2(CP^\infty;Z_p)\cong Z_p$, where z is the mod p reduction of a generator z_0 of $H^2(CP^\infty;Z)\cong Z$. We are going to calculate the secondary cohomology operations Φ_i, Φ'_i, and Φ''_i $(i>1)$ in CP^∞ associated with the Adem relations (1.1–3).

Lemma 4.1. *If* $\binom{n}{i}\equiv 0 \pmod p$, *then* $Q^{2n+2(p-1)i}(\Phi_i;CP^\infty)=0$.

Proof. Since $\mathcal{P}^i z^n=\binom{n}{i}z^{n+(p-1)i}=0$ for a generator $z^n\in H^{2n}(CP^\infty;Z_p)$, and $H^q(CP^\infty;Z_p)=0$ for odd q, we get the desired result. Q. E. D.

Lemma 4.2. *If* $i\binom{n}{i+1}\equiv 0 \pmod p$, *then* $Q^{2n+2(p-1)(i+1)}(\Phi'_i;CP^\infty)=0$.

Proof. Since $i\mathcal{P}^{i+1}z^n=i\binom{n}{i+1}z^{n+(p-1)(i+1)}=0$ for a generator z^n, and $H^q(CP^\infty;Z_p)=0$ for odd q, we have the above result. Q. E. D.

LEMMA 4.3. *If* $\binom{(p-1)(i-1)-1}{2}\binom{n}{i+2}\equiv 0 \pmod p$, *then* $Q^{2n+2(p-1)(i+2)}(\Phi_i''; CP^\infty)=0$.

PROOF. Since $\binom{(p-1)(i-1)-1}{2}\mathcal{P}^{i+2}z^n=0$, the result follows similarly.

Q. E. D.

Let l and m be positive integers, and let g: $CP^\infty \to K(Z, 2l)$ and h: $CP^\infty \to K(Z, 2m)$ be maps such that $g^*\gamma=z^l$ and $h^*\kappa=z^m$, where γ and κ are the mod p reductions of the fundamental classes of $H^{2l}(K(Z, 2l); Z)$ and $H^{2m}(K(Z, 2m); Z)$ respectively. Define a map f: $CP^\infty \to K(Z, 2l)\times K(Z, 2m)$ by $f(x)=(g(x), h(x))$ for each $x \in CP^\infty$.

THEOREM 4.4. *Let k and j be integers such that $0<j<k$, and l and m be positive integers satisfying the following conditions*:

(4.4.1) $$\binom{l}{i}\equiv 0 \pmod p \qquad \textit{for } i=j, j+1, \cdots, k,$$

(4.4.2) $$\binom{m}{k-i}\equiv 0 \pmod p \qquad \textit{for } i=0, 1, \cdots, j.$$

Then we can define $\Phi_i(z^l)$ for $j<i\leqq k$, $\Phi_{k-i}(z^m)$ for $0\leqq i<j$, and $\Phi_k(z^{l+m})$. Moreover, we have

$$\Phi_k(z^{l+m})=\sum_{i=j+1}^{k}(\Phi_i(z^l)\cup\mathcal{P}^{k-i}z^m)+\sum_{i=0}^{j-1}(\mathcal{P}^i z^l\cup\Phi_{k-i}(z^m))$$

in $H^{2l+2m+2(p-1)k}(CP^\infty; Z_p)$ modulo

$$f^*H^{2l+2m+2(p-1)k}(K(Z, 2l)\times K(Z, 2m); Z_p)+Q^{2l+2m+2(p-1)k}(\Phi_k; CP^\infty).$$

PROOF. We apply Theorem 3.4, setting $X=CP^\infty$, $u=z^l$ and $v=z^m$. The conditions (3.1–3) are satisfied by (4.4.1–2) and the definition of z. We have $Q^{2l+2(p-1)i}(\Phi_i; CP^\infty)=0$ for $j<i\leqq k$ by (4.4.1) and Lemma 4.1, $Q^{2m+2(p-1)(k-i)}(\Phi_{k-i}; CP^\infty)=0$ for $0\leqq i<j$ by (4.4.2) and Lemma 4.1. Therefore we get the desired indeterminacy. Q. E. D.

In addition, if we assume that

(4.5) $$\binom{l+m}{k}\equiv 0 \pmod p,$$

then we see that the second term of the indeterminacy of Theorem 4.4, $Q^{2l+2m+2(p-1)k}(\Phi_k; CP^\infty)$, is zero by Lemma 4.1.

Now we investigate the first term of the indeterminacy.

LEMMA 4.6. *Let s and t be integers with $0<t<s$, N be a positive integer, and $a=0$, 1 or 2. Set $l=Np^{s+1}$, $m=p^s-p^t$ and $k=p^s$. Then we have*

$$f^*H^{2l+2m+2(p-1)(k+a)}(K(Z, 2l)\times K(Z, 2m); Z_p)=0.$$

PROOF. Since $f=(g, h)$, we have, by the Künneth formula,

$$f^*H^{2l+2m+2(p-1)(k+a)}(K(Z, 2l)\times K(Z, 2m); Z_p)$$

$$=\sum_j g^*H^j(K(Z, 2l); Z_p)\otimes h^*H^{2l+2m+2(p-1)(k+a)-j}(K(Z, 2m); Z_p).$$

In case $0<j<2l$, clearly $H^j(K(Z, 2l); Z_p)=0$.

In case $j>2l$, for the fundamental class $\gamma\in H^{2l}(K(Z, 2l); Z_p)$, we have $\Delta\gamma=0$, and $g^*\mathcal{P}^i\gamma=\mathcal{P}^i z^l=0$ for any i with $0<i<p^{s+1}$. Then $g^*H^{2l+2(p-1)i}(K(Z, 2l); Z_p)=0$. Since $2(p-1)(p^{s+1}-1)>2m+2(p-1)(k+a)$, we have the lemma in this case.

In case $j=2l$, we must show $h^*H^{2m+2(p-1)(k+a)}(K(Z, 2m); Z_p)=0$. For the fundamental class $\kappa\in H^{2m}(K(Z, 2m); Z_p)$, clearly $\mathcal{P}^k\kappa=0$, and $h^*\mathcal{P}^i\kappa=\mathcal{P}^i z^m=0$ for each i with $0<i<p^t$. Put $u=p^q$, where q is any integer such that $t\leqq q<s$. Then we see easily $h^*\mathcal{P}^i\mathcal{P}^u\kappa=0$ for all i with $p^{q+1}\leqq i<p^s$. As $2(p-1)(p^s-1+p^q)\geqq 2(p-1)(k+a)$, we get the desired result.

Finally, we consider the case $j=0$. Let I be any sequence $\{i_1, i_2,\cdots, i_n\}$ of positive integers, whose degree is $2l+2(p-1)(k+a)$ and put $\mathcal{P}^I=\mathcal{P}^{i_1}\mathcal{P}^{i_2}\cdots\mathcal{P}^{i_n}$. We are going to show $h^*\mathcal{P}^I\kappa=0$. If $N\not\equiv 0 \pmod{p-1}$, there is no sequence I such that degree $\mathcal{P}^I\kappa=2m+2l+2(p-1)(k+a)=2m+2(p-1)(i_1+\cdots+i_n)$, and so we have $H^{2l+2m+2(p-1)(k+a)}(K(Z, 2m); Z_p)=0$. If $N\equiv 0 \pmod{p-1}$, then $i_1+\cdots+i_n\equiv a \pmod{p^s}$ by the assumptions. Let

$$i_j=\sum_q a_q^j p^q \qquad (j=1, 2,\cdots, n)$$

be the p-adic expansion of i_j, where $0\leqq a_q^j<p$. Let r be the least integer such that $a_q^j=0$ for any q with $q<r$ and any j, and such that at least one of the coefficients $a_r^1,\cdots, a_r^n$ is non-zero. By M we denote the maximum of the integers j with $a_r^j\neq 0$.

In the rest of the proof, we use the symbol $z[n]$ instead of z^n. If $r+1\leqq t$, $h^*\mathcal{P}^I\kappa=\mathcal{P}^{i_1}\cdots\mathcal{P}^{i_M}(Kz[\cdots+Ap^{r+1}])$ for some integers K and A. Since $0<a_r^M<p$, we have $\mathcal{P}^{i_M}z[\cdots+Ap^{r+1}]=0$, and hence $h^*\mathcal{P}^I\kappa=0$. If $r\geqq s$, clearly $h^*\mathcal{P}^I\kappa=0$. If $t\leqq r<s$, there exists a positive integer R such that $a_r^1+\cdots+a_r^n=Rp$, because $i_1+\cdots+i_n\equiv a \pmod{p^s}$. Suppose $a=0$. Take a positive integer Q such that $a_r^{Q+1}+a_r^{Q+2}+\cdots+a_r^n<p$ and $a_r^Q+a_r^{Q+1}+\cdots+a_r^n\geqq p$. Now we have

$$h^*\mathcal{P}^I\kappa=\mathcal{P}^{i_1}\cdots\mathcal{P}^{i_M}(Hz[\cdots+(p-1)p^r+\cdots])$$

$$=H\mathcal{P}^{i_1}\cdots\mathcal{P}^{i_Q}\cdots\mathcal{P}^{i_{M-1}}\binom{\cdots+(p-1)p^r+\cdots}{\cdots+a_r^M p^r}z[\cdots+(p-1-a_r^M)p^r+\cdots]$$

$$=G\mathcal{P}^{i_1}\cdots\mathcal{P}^{i_{Q-1}}\begin{pmatrix}\cdots+(p-1-a_r^M-\cdots-a_r^{Q+1})p^r+\cdots \\ \cdots+a_r^Q p^r\end{pmatrix}z[\cdots]$$

for some integers H and G. In the above equalities the last binomial coefficient is congruent to zero modulo p, since $p-1-a_r^M-\cdots-a_r^{Q+1}<a_r^Q$. Therefore we get $h^*\mathcal{P}^l\kappa=0$. The proof in case $a=1$ or 2 is similar. Q. E. D.

COROLLARY 4.7. *Let s and t be integers with $0<t<s$, and N be a positive integer. Set $l=Np^{s+1}$, $m=p^s-p^t$ and $k=p^s$. Then $\Phi_i(z^l)$ for $1<i\leqq k$ and $\Phi_k(z^{l+m})$ are defined, and with zero indeterminacy we have*

$$\Phi_k(z^{l+m})=\sum_{i=2}^{k}(\Phi_i(z^l)\cup\mathcal{P}^{k-i}z^m)$$

in $H^{2l+2m+2(p-1)k}(CP^\infty;Z_p)$.

PROOF. The conditions (4.4.1–2) for $j=1$ and (4.5) are satisfied by the assumptions. Since $m<k-1$, we have $\Phi_k(z^m)=0$. By (4.5) and Lemma 4.6 the indeterminacy is zero, and hence we have the desired result from Theorem 4.4. (We may use also Corollary 3.9.) Q. E. D.

COROLLARY 4.8. *Let s and N be positive integers. Put $l=Np^{s+1}$ and $k=p^s$. Then $\Phi_i(z^{pk})$ and $\Phi_i(z^l)$ are defined for $1<i\leqq k$, and with zero indeterminacy we have*

$$\Phi_i(z^l)=N\Phi_i(z^{pk})\cup z^{l-pk}.$$

PROOF. If $N=1$, the result is trivial. Suppose $N>1$. We apply Corollary 3.10, setting $X=CP^\infty$, $u=z^{pk}$ and $v=z^{l-pk}$. The conditions (3.10.1–2) are satisfied by the assumptions. Thus by Corollary 3.10, for any i with $1<i\leqq k$, $\Phi_i(z^{pk})$, $\Phi_i(z^{l-pk})$ and $\Phi_i(z^l)$ are defined, and we have

$$\Phi_i(z^l)=\Phi_i(z^{pk})\cup z^{l-pk}+z^{pk}\cup\Phi_i(z^{l-pk}).$$

It can be shown that the indeterminacy is zero. The result follows by induction. Q. E. D.

THEOREM 4.9. *Let k and j be integers such that $0<j<k$, and l and m be positive integers satisfying the following conditions*:

(4.9.1) $\binom{l}{i}\equiv 0 \pmod p$ *for* $i=j, j+1,\dots, k+1$,

(4.9.2) $\binom{m}{k-i}\equiv 0 \pmod p$ *for* $i=-1, 0,\dots, j$.

Then we can define $\Phi_i(z^l)$ and $\Phi'_i(z^l)$ for $j<i\leqq k$, $\Phi_{k-i}(z^m)$ and $\Phi'_{k-i}(z^m)$ for $0\leqq i<j$, and $\Phi_k(z^{l+m})$ and $\Phi'_k(z^{l+m})$. Moreover, we have

$$\Phi'_k(z^{l+m})=\sum_{i=j+1}^{k}\{\Phi'_i(z^l)\cup\mathcal{P}^{k-i}z^m+(k-i+1)\Phi_i(z^l)\cup\mathcal{P}^{k-i+1}z^m\}$$
$$+\sum_{i=0}^{j-1}\{\mathcal{P}^iz^l\cup\Phi'_{k-i}(z^m)+(i+1)\mathcal{P}^{i+1}z^l\cup\Phi_{k-i}(z^m)\}$$

in $H^{2l+2m+2(p-1)(k+1)}(CP^\infty;Z_p)$ *modulo*

$$f^*H^{2l+2m+2(p-1)(k+1)}(K(Z,2l)\times K(Z,2m);Z_p)+Q^{2l+2m+2(p-1)(k+1)}(\Phi'_k;CP^\infty).$$

Proof. Putting $X=CP^\infty$, $u=z^l$ and $v=z^m$, we apply Theorem 3.11. The indeterminacy is as above because of (4.9.1–2) and Lemmas 4.1–2. Q. E. D.

In addition, if we assume that

$$k\binom{l+m}{k+1}\equiv 0 \pmod p, \tag{4.10}$$

then we see that the second term of the indeterminacy of Theorem 4.9, $Q^{2l+2m+2(p-1)(k+1)}(\Phi'_k;CP^\infty)$, is zero by Lemma 4.2. Q. E. D.

Theorem 4.11. *Let k and j be integers such that $0<j<k$, and l and m be positive integers satisfying the following conditions*:

(4.11.1) $\binom{l}{i}\equiv 0 \pmod p$ *for* $i=j, j+1,\dots, k+2$,

(4.11.2) $\binom{m}{k-i}\equiv 0 \pmod p$ *for* $i=-2,-1,\dots, j$,

(4.11.3) $\varepsilon l\binom{l+p-1}{i+1}\equiv 0 \pmod p$ *for* $i=j+1, j+2,\dots, k$,

(4.11.4) $\varepsilon m\binom{m+p-1}{k-i+1}\equiv 0 \pmod p$ *for* $i=0, 1,\dots, j-1$.

Then we can define $\Phi_i(z^l)$, $\Phi'_i(z^l)$ *and* $\Phi''_i(z^l)$ *for* $j<i\leqq k$; $\Phi_{k-i}(z^m)$, $\Phi'_{k-i}(z^m)$ *and* $\Phi''_{k-i}(z^m)$ *for* $0\leqq i<j$; $\Phi_k(z^{l+m})$, $\Phi'_k(z^{l+m})$ *and* $\Phi''_k(z^{l+m})$. *Moreover, we have*

$$\Phi''_k(z^{l+m})=\sum_{i=j+1}^{k}\Big\{\Phi''_i(z^l)\cup\mathcal{P}^{k-i}z^m+(k-i+1)\Phi'_i(z^l)\cup\mathcal{P}^{k-i+1}z^m$$
$$+\binom{(p-1)(k-i)-1}{2}\Phi_i(z^l)\cup\mathcal{P}^{k-i+2}z^m\Big\}$$
$$+\sum_{i=0}^{j-1}\Big\{\mathcal{P}^iz^l\cup\Phi''_{k-i}(z^m)+(i+1)\mathcal{P}^{i+1}z^l\cup\Phi'_{k-i}(z^m)$$
$$+\binom{(p-1)i-1}{2}\mathcal{P}^{i+2}z^l\cup\Phi_{k-i}(z^m)\Big\}$$

in $H^{2l+2m+2(p-1)(k+2)}(CP^\infty; Z_p)$ *modulo*

$$f^*H^{2l+2m+2(p-1)(k+2)}(K(Z, 2l)\times K(Z, 2m); Z_p)+ Q^{2l+2m+2(p-1)(k+2)}(\Phi''_k; CP^\infty).$$

PROOF. Putting $X=CP^\infty$, $u=z^l$ and $v=z^m$, we apply Theorem 3.14. By (4.11.1–2) and Lemmas 4.1–3, we have the desired indeterminacy. Q. E. D.

Furthermore, if we assume that

$$\binom{\frac{(p-1)(k-1)-1}{2}}{}\binom{l+m}{k+2}\equiv 0 \pmod p, \tag{4.12}$$

then we see that the second term of the indeterminacy of Theorem 4.11, $Q^{2l+2m+2(p-1)(k+2)}(\Phi''_k; CP^\infty)$, is zero by Lemma 4.3.

We obtain corollaries of Theorems 4.9 and 4.11 similar to Corollaries 4.7–8.

§5. Mod 3 secondary cohomology operations on the Thom class of $\tau(CP^n)$

The next proposition is proved, using the second definition of the functional cohomology operation [9, p. 292].

PROPOSITION 5.1. *Let $f: X\to Y$ be a map, and q be a positive integer. Suppose that the element $c\in H^q(Y; Z_p)$ is the mod p reduction of an integral class $c_0\in H^q(Y; Z)$ and satisfies that $f^*c=0$. Then the functional cohomology operation $\Delta_f c$ is defined. Furthermore, there is an element $d_0\in H^q(X; Z)$ such that $f^*c_0=pd_0$, and we have*

$$\Delta_f c=d(=\text{ the mod } p \text{ reduction of } d_0).$$

Let CP^n denote the complex projective space of real dimension $2n$, and let x_0 be a generator of $H^2(CP^n; Z)\cong Z$. The cohomology algebra $H^*(CP^n; Z)$ is a polynomial algebra over Z with relation $x_0^{n+1}=0$. Set $\mu_0=x_0\times 1-1\times x_0\in H^2(CP^n\times CP^n; Z)$. There is a map $f: CP^n\times CP^n\to K(Z, 2)=CP^\infty$ such that

$$f^*z_0=\mu_0=x_0\times 1-1\times x_0, \tag{5.2}$$

where z_0 is a generator of $H^2(CP^\infty; Z)\cong Z$. Let $x\in H^2(CP^n; Z_p)$, $z\in H^2(CP^\infty; Z_p)$, $\mu\in H^2(CP^n\times CP^n; Z_p)$ be the mod p reductions of $x_0\in H^2(CP^n; Z)$, $z_0\in H^2(CP^\infty; Z)$, $\mu_0\in H^2(CP^n\times CP^n; Z)$ respectively. Then we have

PROPOSITION 5.3. *Let $n=p^{s+1}-1$ and $k=p^s$ $(s>0)$. Then, for $z^{n+i}\in H^{2n+2i}(CP^\infty; Z_p)$ $(i>0)$, the functional cohomology operation $\Delta_f z^{n+i}$ is defined. In addition, if $p=3$, the following holds with zero indeterminacy:*

$$\Delta_f z^{n+i}=-\mu^{k+i-1}\cup(x^k\times x^k)\ (\in H^{2n+2i}(CP^n\times CP^n; Z_3)).$$

PROOF. Clearly, $\Delta z^{n+i}=0$. By (5.2), we have

(5.2)′ $$f^*z=\mu=x\times 1-1\times x,$$

so that $f^*z^{n+i}=(x\times 1-1\times x)^{n+i}=(x^{n+1}\times 1-1\times x^{n+1})(x\times 1-1\times x)^{i-1}=0$, since $x^{n+1}=0$. Therefore $\Delta_f z^{n+i}$ is defined.

If $p=3$, for $n+1=3^{s+1}=3k$, we have $\mu_0^{n+1}=\{(x_0\times 1-1\times x_0)^k\}^3\equiv -3(x_0\times 1-1\times x_0)^k\cup(x_0^k\times x_0^k)$ (mod 9). Hence $f^*z_0^{n+i}=\mu_0^{i-1}\times\mu_0^{n+1}=-3\mu_0^{k+i-1}\cup(x_0^k\times x_0^k)$ (mod 9). Therefore, by Proposition 5.1, we obtain $\Delta_f z^{n+i}=-\mu^{k+i-1}\cup(x^k\times x^k)$. Since $f^*z^{n+i}=0$ and $H^q(CP^n\times CP^n;Z_3)=0$ for odd q, it follows that the indeterminacy is zero. Q. E. D.

Let $\tau=\tau(CP^n)$ denote the tangent bundle of CP^n, E the total space of τ, and $\pi: E\to CP^n$ the projection of τ. Let $\delta>0$ be a sufficiently small number. Let $E(\delta)$ (resp. $E_0(\delta)$) be the set of the pairs $(x, \vec{v})\in E$, where $x\in CP^n$ and $\|\vec{v}\|\leqq\delta$ (resp. $\|\vec{v}\|=\delta$). Let D be the diagonal in $CP^n\times CP^n$. Define a map

$$e: (E(\delta), E_0(\delta))\to(CP^n\times CP^n, CP^n\times CP^n-D)$$

by $e(x, \vec{v})=(x, y)$ for $(x, \vec{v})\in E(\delta)$, where y is the terminal point of the geodesic in CP^n which has the initial point x, the direction of the vector $\vec{v}$, and the length $\|\vec{v}\|$. Since $e(x, 0)=(x, x)$, $e(E_0(\delta))\subset CP^n\times CP^n-D$. The map e defines the isomorphism [8, pp. 46–47]

$$\psi: H^{2n}((CP^n)^\tau; Z_p)\to H^{2n}(CP^n\times CP^n, CP^n\times CP^n-D; Z_p),$$

where $(CP^n)^\tau$ denotes the Thom complex of the tangent bundle τ. Let $U\in H^{2n}((CP^n)^\tau; Z_p)$ be the Thom class of τ. For the injection $j: CP^n\times CP^n\to(CP^n\times CP^n, CP^n\times CP^n-D)$, set $\bar{U}=j^*\psi U$. From the definition of the map e, we have, for $x^i\in H^{2i}(CP^n; Z_p)$,

(5.4) $$j^*\psi(U\cup\pi^*x^i)=\bar{U}\cup(x^i\times 1).$$

According to [8, Theorem 15], $\bar{U}=\sum_{i=0}^{n}(x^{n-i}\times x^i)$.

Suppose $p=3$. If $n=3^{s+1}-1$, we have

(5.5) $$f^*z^n=\mu^n=(x\times 1-1\times x)^n=\bar{U},$$

for $z\in H^2(CP^\infty; Z_3)$, by (5.2)′.

Let Φ_i, Φ'_i and Φ''_i $(i>1)$ be the mod 3 secondary cohomology operations associated with the following Adem relations (mod 3):

$(1.1)_3$ $(\mathcal{P}^1\Delta)\mathcal{P}^{i-1}-(i-1)\Delta\mathcal{P}^i-\mathcal{P}^i\Delta=0,$

$(1.2)_3$ $(\mathcal{P}^2\Delta)\mathcal{P}^{i-1}-2i(i-1)\Delta\mathcal{P}^{i+1}-i\mathcal{P}^{i+1}\Delta=0,$

$$(1.3)_3 \quad (\mathcal{P}^3\Delta)\mathcal{P}^{i-1}+\binom{2i-2}{3}\Delta\mathcal{P}^{i+2}-\Delta(\mathcal{P}^{i+1}\mathcal{P}^1)-\binom{2i-3}{2}\mathcal{P}^{i+2}\Delta=0.$$

Let us calculate $\Phi_i(U)$, $\Phi'_i(U)$ and $\Phi''_i(U)$ by the method of Adem-Gitler [4, §7].

THEOREM 5.6. *Let $n=3^{s+1}-1$ and $k=3^s$ $(s>0)$. Let $\bar{U}=j^*\psi U$ $(\in H^{2n}(CP^n\times CP^n; Z_3))$. Then $\Phi_i(\bar{U})$ is defined for $i>1$, and with zero indeterminacy we have*

$$\Phi_k(\bar{U})=-\bar{U}\cup(x^{2k}\times 1)\neq 0, \textit{ and } \Phi_i(\bar{U})=0, \textit{ if } i\neq k.$$

PROOF. Clearly, $\Delta\bar{U}=0$. Since the mod 3 Pontrjagin class of CP^n is given by

$$(5.7) \qquad p_i=\binom{n+1}{i}x^{2i}\in H^{4i}(CP^n; Z_3),$$

it follows that $p_i=0$ for $i>0$. Thus we have

$$(5.8) \qquad \mathcal{P}^i\bar{U}=0 \qquad \text{for } i>0,$$

because $\mathcal{P}^i\bar{U}=j^*\psi\mathcal{P}^iU=j^*\psi\phi p_i$, where ϕ: $H^{4i}(CP^n; Z_3)\to H^{4i+2n}((CP^n)^\tau; Z_3)$ is the Thom isomorphism (cf. [8, p. 120]). Therefore $\Phi_i(\bar{U})$ is defined for $i>1$. By (5.5) and (5.8), $f^*\beta_i(z^n)=0$ for $i>1$, so that $(\alpha_i)_f\beta_i(z^n)$ is defined for any $i>1$. According to the second formula of Peterson-Stein [3, Theorem 5.2], we have

$$(5.9) \qquad \Phi_i(\bar{U})=-(\alpha_i)_f\beta_i(z^n)$$

in $H^{2n+4i}(CP^n\times CP^n; Z_3)$ modulo $f^*H^{2n+4i}(CP^\infty; Z_3)+Q^{2n+4i}(\Phi_i; CP^n\times CP^n)$. Since $f^*z^{n+2i}=0$ for $i>0$, it follows that $f^*H^{2i+4i}(CP^\infty; Z_3)=0$. Using the Cartan formulas we can prove that $\mathcal{P}^iH^{2n}(CP^n\times CP^n; Z_3)=0$ for any $i>0$. Thus $Q^{2n+4i}(\Phi_i; CP^n\times CP^n)=0$. Now we have

$$\begin{aligned}(\alpha_i)_f\beta_i(z^n)&=(\mathcal{P}^1\Delta-(i-1)\Delta-\mathcal{P}^i)_f(\mathcal{P}^{i-1}, \mathcal{P}^i, \Delta)(z^n)\\&=(\mathcal{P}^1\Delta)_f\mathcal{P}^{i-1}z^n-(i-1)\Delta_f\mathcal{P}^iz^n-\mathcal{P}^i_f(\Delta z^n)\\&=\mathcal{P}^1\Delta_f\mathcal{P}^{i-1}z^n-(i-1)\Delta_f\mathcal{P}^iz^n,\end{aligned}$$

by considering the definitions of functional operations. (Notice that each term is well defined.) Therefore

$$(5.10) \qquad (\alpha_i)_f\beta_i(z^n)=\binom{n}{i-1}\mathcal{P}^1\Delta_fz^{n+2i-2}-(i-1)\binom{n}{i}\Delta_fz^{n+2i}.$$

In case $i=k$, by (5.10), (5.9), (5.5) and Proposition 5.3, we obtain, with zero indeterminacy,

$$\begin{aligned}\Phi_k(\bar{U}) &= -(\alpha_k)_f\beta_k(z^n) = -\mathcal{P}^1\Delta_f z^{n+2k-2} + \Delta_f z^{n+2k} \\ &= \mathcal{P}^1(\mu^{3k-3}\cup(x^k\times x^k)) - \mu^{3k-1}\cup(x^k\times x^k) = -\mu^n\cup(x^k\times x^k) \\ &= -\bar{U}\cup(x^{2k}\times 1)\neq 0.\end{aligned}$$

Now, by (5.9) and Proposition 5.3, we have

$$\begin{aligned}\Phi_k(\bar{U}) &= -\binom{n}{i-1}\mathcal{P}^1(-\mu^{k+2i-3}\cup(x^k\times x^k)) + (i-1)\binom{n}{i}(-\mu^{k+2i-1}\cup(x^k\times x^k)) \\ &= \left\{(k+2i-3)\binom{n}{i-1} - (i-1)\binom{n}{i}\right\}\mu^{k+2i-1}\cup(x^k\times x^k).\end{aligned}$$

On the other hand, the naturality of Φ_i and (5.4) show that

$$\begin{aligned}\Phi_i(\bar{U}) &= \Phi_i(j^*\psi U) = j^*\psi\Phi_i(U) = j^*\psi(\lambda(U\cup\pi^*x^{2i})) \\ &= \lambda(\bar{U}\cup(x^{2i}\times 1)) = \lambda\mu^n\cup(x^{2i}\times 1),\end{aligned}$$

for some $\lambda\in Z_3$. Comparing the coefficients of $x^n\times x^{2i}$ on the right-hand sides of the above two expressions of $\Phi_i(\bar{U})$, we have $\lambda = a\binom{k+2i-1}{n-k}$ for some $a\in Z_3$. If $i<k$, obviously $\lambda=0$. If $i>k$ and $\binom{k+2i-1}{n-k}\not\equiv 0 \pmod 3$, we have $i=mk$ for some $m>1$, and hence $x^{2i}=0$. Thus in case $i\neq k$ we obtain $\Phi_i(\bar{U})=0$ with zero indeterminacy. Q. E. D.

THEOREM 5.11. *Let $n=3^{s+1}-1$ and $k=3^s$ $(s>0)$. Let $U\in H^{2n}((CP^n)^\tau; Z_3)$ denote the Thom class of the tangent bundle $\tau=\tau(CP^n)$ of CP^n. Then $\Phi_i(U)$ is defined for $i>1$, and with zero indeterminacy we have*

$$\Phi_k(U) = -U\cup(\pi^*x^{2k}\times 1)\neq 0, \textit{ and } \Phi_i(U)=0, \textit{ if } i\neq k,$$

where π is the projection of the tangent bundle $\tau(CP^n)$.

PROOF. $\Delta U=0$ is clear. $\mathcal{P}^iU=\phi p_i=0$ for $i>0$, by (5.7), where ϕ is the Thom isomorphism. Thus $\Phi_i(U)$ is defined for $i>1$. By (5.4), Theorem 5.6 and the naturality of Φ_k, we see

$$j^*\psi\Phi_k(U) = \Phi_k(\bar{U}) = -\bar{U}\cup(x^{2k}\times 1) = -j^*\psi(U\cup(\pi^*x^{2k}\times 1)).$$

Since $j^*\psi$ is a monomorphism, we have the first formula. It is easy to prove that the indeterminacy is zero.

The second part is obtained similarly. Q. E. D.

THEOREM 5.12. *Under the assumptions of Theorem 5.6, $\Phi'_i(\bar{U})$ and $\Phi''_i(\bar{U})$ are defined for $i>1$, and with zero indeterminacy we have*

$$\Phi'_i(\bar{U})=0 \text{ and } \Phi''_i(\bar{U})=0 \qquad \text{for any } i>1.$$

PROOF. Since $\Delta\bar{U}=0$ and $\mathcal{P}^i\bar{U}=0$ for $i>0$ by (5.8), it follows that $\Phi'_i(\bar{U})$ is defined for $i>1$. By (5.5) and (5.8), $f^*\beta'_i(z^n)=0$, and so $(\alpha'_i)_f\beta'_i(z^n)$ is defined for $i>1$. According to the second formula of Peterson-Stein [3, Theorem 5.2], we have

$$\Phi'_i(\bar{U})=-(\alpha'_i)_f\beta'_i(z^n) \tag{5.13}$$

in $H^{2n+4i+4}(CP^n\times CP^n; Z_3)$ modulo $f^*H^{2n+4i+4}(CP^\infty; Z_3)+Q^{2n+4i+4}(\Phi'_i; CP^n\times CP^n)$. Here we can see that the indeterminacy is zero, by the calculation using the Cartan formulas. Now we have

$$\begin{aligned}(\alpha'_i)_f\beta'_i(z^n)&=(\mathcal{P}^2\Delta-2i(i-1)\Delta-i\mathcal{P}^{i+1})_f(\mathcal{P}^{i-1}, \mathcal{P}^{i+1}, \Delta)(z^n)\\ &=\mathcal{P}^2\Delta_f\mathcal{P}^{i-1}z^n-2i(i-1)\Delta_f\mathcal{P}^{i+1}z^n\\ &=\binom{n}{i-1}\mathcal{P}^2\Delta_f z^{n+2i-2}-2i(i-1)\binom{n}{i+1}\Delta_f z^{n+2i+2}.\end{aligned}$$

Therefore, by Proposition 5.3 and (5.13), we obtain

$$\Phi'_i(\bar{U})=\left\{\binom{n}{i-1}\binom{k+2i-3}{2}-2i(i-1)\binom{n}{i+1}\right\}\mu^{k+2i+1}\cup(x^k\times x^k).$$

On the other hand, the naturality of Φ'_i and (5.4) show that

$$\Phi'_i(\bar{U})=\lambda\mu^n\cup(x^{2i+2}\times 1),$$

for some $\lambda\in Z_3$. Comparing the coefficients of $x^n\times x^{2i+2}$ on the right-hand sides of the above two equalities, we get $\lambda=0$.

The proof of the second part is quite similar. Q. E. D.

THEOREM 5.14. *Under the assumptions of Theorem* 5.11, $\Phi'_i(U)$ *and* $\Phi''_i(U)$ *are defined for* $i>1$, *and with zero indeterminacy we have*

$$\Phi'_i(U)=0 \text{ and } \Phi''_i(U)=0 \qquad \text{for any } i>1.$$

PROOF. The theorem follows from Theorem 5.12 in the same way as Theorem 5.11 follows from Theorem 5.6. Q. E. D.

§6. Mod 3 secondary cohomology operations in complex projective space

THEOREM 6.1. *Let* $k=3^s$ $(s>0)$. *Then, for* $z^{3k}\in H^{6k}(CP^\infty; Z_3)$, $\Phi_i(z^{3k})$ $(1<i\leqq k)$ *is defined, and with zero indeterminacy we have*

$$\Phi_k(z^{3k})=\pm z^{5k}, \text{ and } \Phi_i(z^{3k})=0 \qquad \text{for } 1<i<k.$$

PROOF. Put $n=3^{s+1}-1=3k-1$. Let ξ be the real restriction of the canonical complex line bundle over CP^n. We use the same notation for a vector bundle and its isomorphism class. Let X^ζ denote the Thom complex of a vector bundle ζ over a complex X. According to [5, Proposition 4.3], there exists a natural homeomorphism: $CP^{2n+1}/CP^n \approx (CP^n)^{(n+1)\xi}$, where $m\xi=\xi\oplus\cdots\oplus\xi$ (the m-fold Whitney sum of ξ). Furthermore, according to [5, Lemma 2.4], there is a natural homeomorphism: $(CP^n)^{\tau\oplus 2}\approx S^2(CP^n)^\tau$, where S^rY denotes the r-fold suspension of Y. As is well-known, $(n+1)\xi=\tau\oplus 2$. Let

$$\varphi: CP^{2n+1}/CP^n \to S^2(CP^n)^\tau$$

denote the composite of the above homeomorphisms. Consider the diagram 4, where $j: CP^{2n+1}\to CP^\infty$ is the inclusion, $q: CP^{2n+1}\to CP^{2n+1}/CP^n$ is the projection, and σ^2 is the 2-fold suspension.

$$\begin{array}{ccc}
H^{2n+2}(CP^\infty; Z_3) & \xrightarrow{\Phi_k} & H^{2n+2+4k}(CP^\infty; Z_3) \\
\downarrow j^* & & \downarrow j^* \\
H^{2n+2}(CP^{2n+1}; Z_3) & \xrightarrow{\Phi_k} & H^{2n+2+4k}(CP^{2n+1}; Z_3) \\
\uparrow q^* & & \uparrow q^* \\
H^{2n+2}(CP^{2n+1}/CP^n; Z_3) & \xrightarrow{\Phi_k} & H^{2n+2+4k}(CP^{2n+1}/CP^n; Z_3) \\
\uparrow \varphi^* & & \uparrow \varphi^* \\
H^{2n+2}(S^2(CP^n)^\tau; Z_3) & \xrightarrow{\Phi_k} & H^{2n+2+4k}(S^2(CP^n)^\tau; Z_3) \\
\uparrow \sigma^2 & & \uparrow \sigma^2 \\
H^{2n}((CP^n)^\tau; Z_3) & \xrightarrow{\Phi_k} & H^{2n+4k}((CP^n)^\tau; Z_3)
\end{array}$$

Diagram 4

It is clear that j^*, q^*, φ^* and σ^2 are isomorphisms, and that each indeterminacy of Φ_k is zero. The commutativity of the diagram 4 and Theorem 5.11 imply that $(\sigma^2)^{-1}(\varphi^*)^{-1}j^*\Phi_k(z^{n+1})=\Phi_k(\sigma^2)^{-1}(\varphi^*)^{-1}(q^*)^{-1}j^*z^{n+1}=\pm\Phi_k(U)=\mp U\cup(\pi^*x^{2k}\times 1)\neq 0$. Therefore we have $\Phi_k(z^{n+1})=\pm z^{5k}$.

The proof of the second part is similar. Q. E. D.

THEOREM 6.2. *Let $k=3^s$ $(s>0)$. Then, for $z^{3k}\in H^{6k}(CP^\infty; Z_3)$, $\Phi'_i(z^{3k})$ and $\Phi''_i(z^{3k})$ $(1<i\leqq k)$ are defined, and with zero indeterminacy we have*

$$\Phi'_i(z^{3k})=0 \text{ and } \Phi''_i(z^{3k})=0 \qquad \text{for } 1<i\leqq k.$$

PROOF. Using Theorem 5.14, we obtain the results, similarly as in the proof of Theorem 6.1. Q. E. D.

THEOREM 6.3. *Let s and t be integers with $s>t>0$, and N be a positive integer. Set $l=N3^{s+1}$, $m=3^s-3^t$ and $k=3^s$. Then, for $z^{l+m}\in H^{2l+2m}(CP^\infty; Z_3)$, $\Phi_k(z^{l+m})$ is defined, and with zero indeterminacy we have*

$$\Phi_k(z^{l+m}) = \pm N z^{l+m+2k}.$$

PROOF. By Corollary 4.7, $\Phi_i(z^l)$ for $1<i\leqq k$ and $\Phi_k(z^{l+m})$ are defined, and with zero indeterminacy we have $\Phi_k(z^{l+m})=\sum_{i=2}^{k}(\Phi_i(z^l)\cup\mathcal{P}^{k-i}z^m)$. We can also define $\Phi_i(z^{3k})$ for $1<i\leqq k$, and with zero indeterminacy we have $\Phi_i(z^l)=N\Phi_i(z^{3k})\cup z^{l-3k}$, by Corollary 4.8. These results, combined with Theorem 6.1, yield the desired formula. Q. E. D.

THEOREM 6.4. *Under the assumptions of Theorem* 6.3, $\Phi'_k(z^{l+m})$ *and* $\Phi''_k(z^{l+m})$ *are defined, and with zero indeterminacy we have*

$$\Phi'_k(z^{l+m})=0 \text{ and } \Phi''_k(z^{l+m})=0.$$

PROOF. The results follow from Lemma 4.6, Theorems 4.9, 4.11, 6.1 and 6.2. Q. E. D.

§7. Double secondary cohomology operations in complex projective space and mod 3 lens space

Let $\mathcal{P}^i$ be the ith reduced power operation mod 3, and Δ be the Bockstein operation associated with the exact coefficient sequence: $0\to Z_3\to Z_9\to Z_3\to 0$. Let $i\equiv 0 \pmod 3$. Consider the double secondary cohomology operation Θ_i associated with the Adem relations:

$$\text{(7.1)} \qquad \alpha_i\beta_i=(\mathcal{P}^1\Delta)\mathcal{P}^{i-1}+\Delta\mathcal{P}^i-\mathcal{P}^i\Delta=0,$$

$$\text{(7.2)} \qquad \bar{\alpha}_i\beta_i=(\mathcal{P}^3\Delta)\mathcal{P}^{i-1}+(2i/3-1)(\Delta\mathcal{P}^2)\mathcal{P}^i-(\Delta\mathcal{P}^{i+1})\mathcal{P}^1=0,$$

where $\alpha_i=\mathcal{P}^1\Delta+\Delta+0-\mathcal{P}^i$, $\bar{\alpha}_i=\mathcal{P}^3\Delta+(2i/3-1)\Delta\mathcal{P}^2-\Delta\mathcal{P}^{i+1}+0$, and $\beta_i=(\mathcal{P}^{i-1}, \mathcal{P}^i, \mathcal{P}^1, \Delta)$. Let Φ_i and $\bar{\Phi}_i$ be the secondary cohomology operations associated with the relations (7.1) and (7.2) respectively. Φ_i is the same one as in §§1–6 for $p=3$. As for $\bar{\Phi}_i$, if $\bar{\Phi}_i(w)$ is defined for some $w\in H^q(X; Z_3)$, the operation $\Phi''_i(w)$ in §§1–6, for $p=3$, is defined. Moreover, we have

$$\bar{\Phi}_i(w)=\Phi''_i(w) \text{ modulo } Q^{q+4i+8}(\Phi''_i; X).$$

The double secondary cohomology operation Θ_i is a stable operation constructed as follows (cf. [4, §10]). Let $\pi: E\to K(Z_3, q)$ be a fibre space determined by β_i. It is sufficient to construct Θ_i in the stable range. Let $q>4i+8$, and choose elements $a\in H^{q+4i}(E; Z_3)$ and $b\in H^{q+4i+8}(E; Z_3)$ associated with (7.1) and (7.2) respectively. Let X be a space, and $f: X\to K(Z_3, q)$ be a characteristic map for a given element $u\in H^q(X; Z_3)$. If $\beta_i(u)=0$, there is a map $g: X\to E$ such that $\pi g=f$. Define $\Theta_i(u)=(g^*(a), g^*(b))\in H^{q+4i}(X; Z_3)\oplus H^{q+4i+8}(X; Z_3)$, where $\oplus$ denotes the direct sum. The indeterminacy $Q(\Theta_i; X)$ is given by

$$(\mathcal{P}^1\Delta\oplus\mathcal{P}^3\Delta)dH^{q+4i-5}(X;\ Z_3)+(\Delta\oplus(2i/3-1)\Delta\mathcal{P}^2)dH^{q+4i-1}(X;\ Z_3)$$
$$+(0\oplus(-\Delta\mathcal{P}^{i+1}))dH^{q+3}(X;\ Z_3)+(-\mathcal{P}^i\oplus 0)dH^q(X;\ Z_3),$$

where dH is the diagonal subgroup of $H\oplus H$.

THEOREM 7.3. *Let s and t be integers with $s>t>0$, and N be a positive integer such that $N\not\equiv 0$ (mod 3). Put $l=N3^{s+1}$, $m=3^s-3^t$ and $k=3^s$. Then, for $z^{l+m}\in H^{2l+2m}(CP^\infty;\ Z_3)$, $\Theta_k(z^{l+m})$ is defined, and with zero indeterminacy we have*

$$\Theta_k(z^{l+m})=\pm(z^{l+m+2k},\ 0).$$

PROOF. Under the assumptions, Φ_k, $\bar{\Phi}_k$ and Φ_k'' are defined for the element z^{l+m} with zero indeterminacy. Therefore $\bar{\Phi}_k$ coincides with Φ_k'', and Θ_k is identical with the pair $(\Phi_k,\ \bar{\Phi}_k)$ on z^{l+m}. Thus the result follows from Theorems 6.3–4. Q. E. D.

Let $L^n(p)$ be the $(2n+1)$-dimensional standard lens space mod p, and $L^\infty(p)$ be $\bigcup_n L^n(p)$. The cohomology algebra $H^*(L^\infty(p);\ Z_p)$ is given by $\Lambda[y]\otimes Z_p[w]$, where y and w are generators of $H^1(L^\infty(p);\ Z_p)\cong Z_p$ and $H^2(L^\infty(p);\ Z_p)\cong Z_p$ respectively, with relation: $\Delta y=w$.

THEOREM 7.4. *Let s and t be integers with $s>t>1$, and N be a positive integer such that $N\not\equiv 0$ (mod 3). Put $l=N3^{s+1}$, $m=3^s-3^t$ and $k=3^s$. Then, for $w^{l+m}\in H^{2l+2m}(L^\infty(3);\ Z_3)$, $\Theta_k(w^{l+m})$ is defined, and we have*

$$\Theta_k(w^{l+m})\neq 0$$

in $H^{2l+2m+4k}(L^\infty(3);\ Z_3)\oplus H^{2l+2m+4k+8}(L^\infty(3);\ Z_3)$ modulo $Q(\Theta_k;\ L^\infty(3))$.

PROOF. Let p: $L^\infty(3)\to CP^\infty$ be the natural projection. Consider the commutative diagram, where the coefficient group Z_3 is omitted, and $n=2l+2m$.

$$\begin{array}{ccc}
H^n(CP^\infty) & \xrightarrow{\Theta_k} & H^{n+4k}(CP^\infty)\oplus H^{n+4k+8}(CP^\infty) \\
\downarrow p^* & & \downarrow \bar{p}^* \\
H^n(L^\infty(3)) & \xrightarrow{\Theta_k} & H^{n+4k}(L^\infty(3))\oplus H^{n+4k+8}(L^\infty(3))/Q(\Theta_k;\ L^\infty(3))
\end{array}$$

As is well-known, p^* is an isomorphism in even degree when the coefficient group is Z_p, and $p^*z=w$, for $z\in H^2(CP^\infty;\ Z_3)$. Hence $p^*z^{l+m}=w^{l+m}$. By the commutativity of the diagram and Theorem 7.3, we have

$$\Theta_k(w^{l+m})=\Theta_k(p^*z^{l+m})=\bar{p}^*\Theta_k(z^{l+m})=\pm(w^{l+m+2k},\ 0)$$

modulo $Q(\Theta_k;\ L^\infty(3))$. However, by the simple calculation we can see that any element of $Q(\Theta_k; L^\infty(3))$ has a form $\lambda(w^{l+m+2k}, -w^{l+m+2k+4})$, where $\lambda \in Z_3$ (Here we need the assumption $t>1$). Thus $Q(\Theta_k; L^\infty(3))$ does not contain $\pm(w^{l+m+2k}, 0)$. Q. E. D.

Let ν be an m-dimensional vector bundle over $L^n(3)$, $V \in H^m((L^n(3))^\nu; Z_3)$ be the mod 3 Thom class of ν. There exists a map $f: (L^n(3))^\nu \to K(Z, m)$ such that $f^*\kappa = V$, where $\kappa \in H^m(K(Z, m); Z_3)$ is the mod 3 reduction of the fundamental class of $K(Z, m)$. Then we have

THEOREM 7.5. *Let $k=3^s$ $(s>0)$. Suppose that $m<2k-2$ and that the first Pontrjagin class mod 3, $p_1(\nu)$, is zero. Then $\Theta_k(V)$ is defined. If, in addition, $f^*H^{m+4k}(K(Z, m); Z_3)=0$ and $f^*H^{m+4k+8}(K(Z, m); Z_3)=0$, then we have*

$$\Theta_k(V)=0$$

modulo the indeterminacy $Q(\Theta_k; (L^n(3))^\nu)$.

PROOF. Clearly $\Delta V=0$. Since $m<2(k-1)$, we have $\mathcal{P}^{k-1}V=0$ and $\mathcal{P}^k V=0$. From the fact that $p_1(\nu)=0$, we have $\mathcal{P}^1 V=\phi p_1(\nu)=0$, where $\phi: H^4(L^n(3); Z_3)\to H^{m+4}((L^n(3))^\nu; Z_3)$ is the Thom isomorphism. Therefore $f^*\beta_k\kappa=\beta_k f^*\kappa =(\mathcal{P}^{k-1}, \mathcal{P}^k, \mathcal{P}^1, \Delta)(V)=0$, and hence the double secondary operation $\Theta_k(f^*\kappa)$ and the double functional operation $(\alpha_k\oplus\bar{\alpha}_k)_f(\beta_k\kappa, \beta_k\kappa)$ are defined, where $\alpha_k=\mathcal{P}^1\Delta+\Delta+0-\mathcal{P}^k$ and $\bar{\alpha}_k=\mathcal{P}^3\Delta+(2k/3-1)\Delta\mathcal{P}^2-\Delta\mathcal{P}^{k+1}+0$. According to the second formula of Peterson-Stein for double operations (cf. [4, Theorem 10.8]), we have

$$\Theta_k(f^*\kappa)=-(\alpha_k\oplus\bar{\alpha}_k)_f(\beta_k\kappa, \beta_k\kappa)$$

modulo the total indeterminacy $f^*H^{m+4k}(K(Z, m); Z_3)\oplus f^*H^{m+4k+8}(K(Z, m); Z_3)+Q(\Theta_k; (L^n(3))^\nu)$. By the assumptions, the indeterminacy is reduced to $Q(\Theta_k; (L^n(3))^\nu)$.

Since $\beta_k\kappa=(\mathcal{P}^{k-1}\kappa, \mathcal{P}^k\kappa, \mathcal{P}^1\kappa, \Delta\kappa)=(0, 0, \mathcal{P}^1\kappa, 0)$ and $(0\oplus(-\Delta\mathcal{P}^{k+1}))(y, y) =(0, -\Delta\mathcal{P}^{k+1}y)=(0, 0)$ for any element y of degree $m+4$, we may choose zero for $(\alpha_k\oplus\bar{\alpha}_k)_f(\beta_k\kappa, \beta_k\kappa)$. Therefore we have $\Theta_k(V)=0$ modulo $Q(\Theta_k; (L^n(3))^\nu)$. Q. E. D.

LEMMA 7.6. *Let s and t be integers with $0<t<s$, m be an even integer>0 and N be an integer>0. Set $M=N3^{s+1}+3^s-3^t$ and $k=3^s$. Let ν be an m-dimensional vector bundle over $L^n(3)$ such that the ith Pontrjagin class mod 3 is given by*

$$p_i(\nu)=\binom{M}{i}v^{2i},$$

where v is a generator of $H^2(L^n(3); Z_3)$. Then we have $f^*H^{m+4k}(K(Z, m); Z_3)=0$ and $f^*H^{m+4k+8}(K(Z, m); Z_3)=0$.

PROOF. For the first part, any element of $f^*H^{m+4k}(K(Z, m); Z_3)$ ($\subset H^{m+4k}((L^n(3))^\nu; Z_3)$) is of the form $f^*\mathcal{P}^I\kappa=\mathcal{P}^I f^*\kappa=\mathcal{P}^I V=\mathcal{P}^{i_1}\cdots\mathcal{P}^{i_q}(V)$, where the degree of $I=\{i_1,\cdots, i_q\}$ is $4k$. It is sufficient to prove $\mathcal{P}^I V=0$ for admissible I, that is, for the sequence $\{i_1,\cdots, i_q\}$ such that $i_j\geqq 3i_{j+1}$ for $j=1, 2,\cdots, q-1$. In case $I=\{k\}$, $\mathcal{P}^k V=\phi p_k(\nu)=0$, since $p_k(\nu)=0$. In case I is decomposable, it is clear that $\mathcal{P}^I V=0$ for $0<i_q<3^t$. If $s-1=t$, obviously $\mathcal{P}^I V=0$. So we may assume $s-2\geqq t$. Let c be any integer with $t\leqq c\leqq s-2$, and set $j=a_{s-1}3^{s-1}+\cdots+a_1 3+a_0$, where $0\leqq a_r\leqq 2$ for $r=0, 1,\cdots, s-1$, and where at least one of the integers $a_{c+1},\cdots, a_{s-1}$ is non-zero. Then $j\geqq 3^{c+1}$. Let π be the projection of the tangent bundle of $L^n(3)$. Put $3^c=l$. Then we have

$$\begin{aligned}\mathcal{P}^j\mathcal{P}^l V&=\mathcal{P}^j(\phi p_l(\nu))=\mathcal{P}^j(V\cup(-\pi^*v^{2l}))\\&=-\mathcal{P}^j V\cup\pi^*v^{2l}-\mathcal{P}^{j-l}V\cup\mathcal{P}^l\pi^*v^{2l}-\mathcal{P}^{j-2l}V\cup\mathcal{P}^{2l}\pi^*v^{2l}\\&=-\left\{\binom{M}{j}-\binom{M}{j-l}+\binom{M}{j-2l}\right\}V\cup\pi^*v^{2j+2l}.\end{aligned}$$

While, by calculation it can be proved that the coefficient of the above value is congruent to zero modulo 3. Therefore we get $\mathcal{P}^j\mathcal{P}^l V=0$ for $j\geqq 3^l$ $=3^{c+1}$, and hence $\mathcal{P}^I V=0$ for admissible I.

The second part is proved similarly. Q. E. D.

§8. Applications

Let η be the real restriction of the canonical complex line bundle over $L^n(3)$. On the number of linearly independent cross-sections of $m\eta=\eta\oplus\cdots\oplus\eta$ (m-fold), we have the next result.

THEOREM 8.1. *Let r, s and t be integers such that $r-1>s>t>1$. Put $n=2\cdot 3^s+3^t-1$. Then $(3^r-n-1)\eta$ does not have $2\cdot 3^r+3^t-3n$ independent cross-sections.*

PROOF. Suppose that $(3^r-n-1)\eta$ has b independent cross-sections, where $b=2\cdot 3^r+3^t-3n$. Then there is a $(2\cdot 3^s-3)$-dimensional vector bundle ν such that $(3^r-n-1)\eta=\nu\oplus b$.

According to [6, Theorem 1], there is a natural homeomorphism: $L^{3^r-1}(3)/L^{3^r-n-2}(3)\approx(L^n(3))^{(3^r-n-1)\eta}$. According to [5, Lemma 2.4], there is a natural homeomorphism: $(L^n(3))^{\nu\oplus b}\approx S^b(L^n(3))^\nu$. Thus we have a composite homeomorphism:

$$\varphi: L^{3^r-1}(3)/L^{3^r-n-2}(3)\approx S^b(L^n(3))^\nu.$$

We set $l=3^r-3^{s+1}=N3^{s+1}$, $m=3^s-3^t$ and $k=3^s$. Since $N\not\equiv 0$ (mod 3), by Theorem 7.4, $\Theta_k(w^{l+m})$ is defined for $w^{l+m}\in H^{2l+2m}(L^\infty(3); Z_3)$, and we have

(8.2) $$\Theta_k(w^{l+m})\neq 0 \text{ modulo } Q(\Theta_k;\, L^\infty(3)).$$

Let $j\colon L^{3^r-1}(3)\to L^\infty(3)$ be the inclusion, $g\colon L^{3^r-1}(3)\to L^{3^r-1}(3)/L^{3^r-n-2}(3)$ be the projection, and σ^b be the b-fold suspension. Now consider the diagram 5, in which we denote $L^n(3)$, $L^\infty(3)$ and $L^{3^r-1}(3)/L^{3^r-n-2}(3)$ by L^n, L^∞ and $\bar{L}$ respectively, the coefficient group Z_3 is omitted, and $q=2l+2m$.

$$\begin{array}{lcl}
H^q(L^\infty) & \xrightarrow{\Theta_k} & H^{q+4k}(L^\infty)\oplus H^{q+4k+8}(L^\infty)/Q(\Theta_k;\, L^\infty) \\
\downarrow j^* & & \downarrow \bar{j}^* \\
H^q(L^{3^r-1}) & \xrightarrow{\Theta_k} & H^{q+4k}(L^{3^r-1})\oplus H^{q+4k+8}(L^{3^r-1})/Q(\Theta_k;\, L^{3^r-1}) \\
\uparrow g^* & & \uparrow \bar{g}^* \\
H^q(\bar{L}) & \xrightarrow{\Theta_k} & H^{q+4k}(\bar{L})\oplus H^{q+4k+8}(\bar{L})/Q(\Theta_k;\, \bar{L}) \\
\uparrow \varphi^* & & \uparrow \bar{\varphi}^* \\
H^q(S^b(L^n)^\nu) & \xrightarrow{\Theta_k} & H^{q+4k}(S^b(L^n)^\nu)\oplus H^{q+4k+8}(S^b(L^n)^\nu)/Q(\Theta_k;\, S^b(L^n)^\nu) \\
\uparrow \sigma^b & & \uparrow \bar{\sigma}^b \\
H^{2k-3}((L^n)^\nu) & \xrightarrow{\Theta_k} & H^{6k-3}((L^n)^\nu)\oplus H^{6k+5}((L^n)^\nu)/Q(\Theta_k;\, (L^n)^\nu)
\end{array}$$

Diagram 5

It is clear that the assumptions of Lemma 7.6 are satisfied, and hence by Theorem 7.5, for $V\in H^{2k-3}((L^n(3))^\nu;\, Z_3)$, $\Theta_k(V)$ is defined, and we have

(8.3) $$\Theta_k(V)=0 \text{ modulo } Q(\Theta_k;\, (L^n(3))^\nu).$$

But, as is easily seen, each of the vertical homomorphisms is an isomorphism. Thus (8.2–3) give rise to a contradiction. Q. E. D.

THEOREM 8.4. *Let s and t be integers with $s>t>1$. If $n=2\cdot 3^s+3^t-1$, then $L^n(3)$ cannot be immersed in Euclidean $(3n-3^t-1)$-space.*

PROOF. Suppose that $L^n(3)$ be immersed in Euclidean $(3n-3^t-1)$-space, then $(3^{n/2}-n-1)\eta$ has $2\cdot 3^{n/2}-3n+3^t$ independent cross-sections by [7, Theorem 1]. This contradicts Theorem 8.1. Q. E. D.

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