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The Reduced Symmetric Product of a Complex Projective Space and the Embedding Problem

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§0. Introduction

A given embedding f of a topological space X in the real m-space R^m induces the continuous map F of the space $X \times X - \Delta$ (Δ is the diagonal of $X \times X$) into the unit $(m-1)$-sphere S^{m-1} in R^m, which is defined as follows:

$$F(x, y) = \frac{f(x)-f(y)}{\|f(x)-f(y)\|} \text{ for any distinct points } x, y \text{ of } X.$$

Then it is clear that F is equivariant with respect to the symmetry which interchanges the factors in $X \times X - \Delta$ and the antipodal map of S^{m-1}. Also, an isotopy $f_t (t \in [0, 1])$ of two embeddings f_0, f_1 of X in R^m induces the equivariant homotopy F_t.

A. Haefliger [3] investigated the embeddings of compact differentiable manifolds in Euclidean spaces using the above equivariant maps and proved

THEOREM (Haefliger). *Let M be an n-dimensional compact differentiable manifold. Consider the correspondence which associates with an isotopy class of a differentiable embedding $f: M \longrightarrow R^m$ the equivariant homotopy class of the map F defined as above. Then this correspondence is surjective if $2m \geq 3(n+1)$ and bijective if $2m > 3(n+1)$.*

Let the reduced symmetric product space M^* be the quotient space obtained from $M \times M - \Delta$ by identifying $(x, y) \sim (y, x)$. Then the projection $M \times M - \Delta \longrightarrow M^*$ is a double covering, and there exists a sphere bundle $S^{m-1} \longrightarrow (M \times M - \Delta) \times_{Z_2} S^{m-1} \longrightarrow M^*$ associated with this covering. Since there is a one-to-one correspondence between the equivariant homotopy classes of equivariant maps $M \times M - \Delta \longrightarrow S^{m-1}$ and the homotopy classes of cross sections of the above sphere bundle $S^{m-1} \longrightarrow (M \times M - \Delta) \times_{Z_2} S^{m-1} \longrightarrow M^*$, the study of this sphere bundle and so the cohomology of M^* play an important part in studying embeddings of M in R^m. In fact, D. Handel [4] and S. Feder [2] studied the cohomology of $(RP^n)^*$ and applied it to the existence and the classification of embeddings of the real projective spaces RP^n in Euclidean spaces.

In this paper, we try to determine the cohomology of $(CP^n)^*$ and to study the double covering $CP^n \times CP^n - \Delta \longrightarrow (CP^n)^*$ and to apply it to the embedding problem of the complex projective spaces CP^n.

This paper is organized as follows: In §1, we construct the double covering $Z_{n+1,2}\longrightarrow SZ_{n+1,2}$ in (1.3-4) which is homotopy equivalent to the double covering $CP^n\times CP^n-\Delta\longrightarrow(CP^n)^*$ of above. We prepare some results concerning the cohomology of real and complex projective bundles in §2. In §3, we determine the cohomology of $Z_{n+1,2}$ in Theorem 3.1 using the results of §2. In §4, we determine the cohomology of $SZ_{n+1,2}$ and so the reduced symmetric product space $(CP^n)^*$ in Theorems 4.9, 4.10, 4.15. In §5, we consider the isotopy classification of embeddings of CP^n in R^m ($m=4n$, $4n-1$, $4n-2$) and so we have the main theorem:

THEOREM 5.5. *Let* $n\geq 4$.

(1) *There exists a unique isotopy class of embeddings of* CP^n *in* R^{4n}.

(2) *There exist just two isotopy classes of embeddings of* CP^n *in* R^{4n-1}.

(3) *There exist just two isotopy classes of embeddings of* CP^n *in* R^{4n-2} *for* $n\neq 2^r$.

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§1. Construction of the double covering $Z_{n+1,2}\longrightarrow SZ_{n+1,2}$

Let $U(2)$ be the unitary group on the complex 2-space C^2 and $T^2=S^1\times S^1$ be the maximal torus of $U(2)$ and let

$$S^1=\{e^{i\theta}\mid 0\leq\theta<2\pi\},$$

$$G=\left\{\begin{pmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{pmatrix}, \begin{pmatrix}0 & \gamma_3\\ \gamma_4 & 0\end{pmatrix}\middle|\, \gamma_i\in S^1,\quad i=1, 2, 3, 4\right\}.$$

Then we have a sequence of inclusions

$$S^1\subset T^2\subset G\subset U(2), \tag{1.1}$$

where S^1 is embedded in T^2 by the diagonal map.

It is clear that $G/T^2=Z_2$ and we have the following

LEMMA 1.2. *The quotient spaces* $U(2)/T^2$ *and* $U(2)/G$ *are diffeomorphic to* S^2 *and* RP^2 *respectively, and natural projection* $U(2)/T^2\longrightarrow U(2)/G$ *corresponds to the double covering* $S^2\longrightarrow RP^2$.

Set $W_{n,2}=U(n)/U(n-2)$. Then $W_{n,2}$ is the complex Stiefel manifold of orthonormal 2-frames in C^n, and $U(2)$ acts freely on $W_{n,2}$ as follows: If $\alpha=\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}$ is an element of $U(2)$ and $(u_1, u_2)\in W_{n,2}$, then

$$\alpha(u_1, u_2)=(\alpha_1 u_1+\alpha_2 u_2, \alpha_3 u_1+\alpha_4 u_2).$$

We consider the following quotient manifolds:

$$(1.3)\qquad \begin{aligned} &X_{n,2}=W_{n,2}/S^1, \qquad Z_{n,2}=W_{n,2}/T^2 \\ &SZ_{n,2}=W_{n,2}/G, \qquad G_{n,2}(C)=W_{n,2}/U(2). \end{aligned}$$

Here $X_{n,2}$ is called the complex projective Stiefel manifold [7] and $G_{n,2}(C)$ is the complex Grassmann manifold of complex 2-spaces in C^n.

The sequence (1.1) induces the following commutative diagram of fibrations:

$$(1.4)\qquad \begin{array}{ccccccc} S^1 & \longrightarrow & T^2 & \longrightarrow & G & \longrightarrow & U(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ W_{n,2} & = & W_{n,2} & = & W_{n,2} & = & W_{n,2} \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow \\ X_{n,2} & \xrightarrow{\pi_1} & Z_{n,2} & \xrightarrow{\pi_2} & SZ_{n,2} & \xrightarrow{\pi_3} & G_{n,2}(C), \end{array}$$

where $\pi_2: Z_{n,2}\longrightarrow SZ_{n,2}$ is a double covering.

Let f: $Z_{n+1,2}\longrightarrow CP^n\times CP^n-\Delta$ be a map defined by

$$f(\pi(u_1, u_2))=([u_1], [u_2]),$$

where $[u_i](i=1, 2)$ is the element of CP^n determined by $u_i\in S^{2n+1}$. Then f is well-defined and is an equivariant map, which induces the map $\bar{f}$: $SZ_{n+1,2}\longrightarrow(CP^n)^*$ and so we obtain the map of double coverings

$$(1.5)\qquad \begin{array}{ccc} Z_{n+1,2} & \xrightarrow{f} & CP^n\times CP^n-\Delta \\ \downarrow{\scriptstyle \pi_2} & & \downarrow \\ SZ_{n+1,2} & \xrightarrow{\bar{f}} & (CP^n)^* \end{array}$$

PROPOSITION 1.6. *In (1.5), the map f is a homotopy equivalence and $\bar{f}$ is a weak homotopy equivalence.*

PROOF. Let (u_1, u_2) be a pair of linearly independent unit vectors in C^{n+1}. Then $\left(u_1, \frac{u_2-<u_2, u_1>u_1}{||u_2-<u_2, u_1>u_1||}\right)$ is a pair of orthonormal vectors in C^{n+1} which is obtained from (u_1, u_2) by the Gram-Schmidt process, where $<u_2, u_1>$ stands for the inner product of u_2 and u_1. We define a map g: $CP^n\times CP^n-\Delta\longrightarrow Z_{n+1,2}$ by

$$g([u_1], [u_2])=\pi\left(u_1, \frac{u_2-<u_2, u_1>u_1}{||u_2-<u_2, u_1>u_1||}\right).$$

Then g is a well-defined map such that gf is the identity map. Let f_t:

$CP^n \times CP^n - \Delta \longrightarrow CP^n \times CP^n - \Delta$ be the homotopy defined by

$$f_t([u_1], [u_2]) = \left([u_1], \left[\frac{u_2 - t < u_2, u_1 > u_1}{\|u_2 - t < u_2, u_1 > u_1\|}\right]\right).$$

Then f_t is a well-defined homotopy between the identity map and fg. Hence f is a homotopy equivalence.

By the exact sequences of homotopy groups of fibrations and the five lemma, $\bar{f}$ induces isomorphisms of all homotopy groups of $SZ_{n+1,2}$ and $(CP^n)^*$ and so $\bar{f}$ is a weak homotopy equivalence. Q. E. D.

Let $V_{n,2}$ be the real Stiefel manifold of orthonormal 2-frames in the real n-space R^n. The orthogonal group $O(2)$ acts on $V_{n,2}$ as follows: If $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$ is an element of $O(2)$ and $(v_1, v_2) \in V_{n,2}$, then

$$\alpha(v_1, v_2) = (\alpha_1 v_1 + \alpha_2 v_2, \alpha_3 v_1 + \alpha_4 v_2).$$

Let

$$G' = \left\{\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_3 \\ \varepsilon_4 & 0 \end{pmatrix} \middle| \varepsilon_i = \pm 1, i = 1, 2, 3, 4\right\},$$

$$O(1) \times O(1) = \left\{\begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \middle| \varepsilon_i = \pm 1, i = 1, 2\right\}, \quad D = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\},$$

and consider the quotient manifolds

$$X'_{n,2} = V_{n,2}/D, \quad Z'_{n,2} = V_{n,2}/O(1) \times O(1), \quad SZ'_{n,2} = V_{n,2}/G',$$

and the double coverings $X'_{n,2} \longrightarrow Z'_{n,2}$, $Z'_{n,2} \longrightarrow SZ'_{n,2}$. Considering the 2-frame in R^n as that in C^n, we have a map $h: V_{n,2} \longrightarrow W_{n,2}$. The map h induces the equivariant map $Z'_{n,2} \longrightarrow Z_{n,2}$ and so the map of double coverings. Also, let $g: X'_{n,2} \longrightarrow Z'_{n,2}$ be the equivariant map defined by

$$g(\pi'(v_1, v_2)) = \pi''\left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{v_1 - v_2}{\sqrt{2}}\right)$$

where $(v_1, v_2) \in V_{n,2}$ and $\pi': V_{n,2} \longrightarrow X'_{n,2}$, $\pi'': V_{n,2} \longrightarrow Z'_{n,2}$ are the projections. Then we obtain the following commutative diagram of double coverings:

(1.7)
$$\begin{array}{ccccc} X'_{n+1,2} & \xrightarrow{g} & Z'_{n+1,2} & \xrightarrow{h} & Z_{n+1,2} \\ \downarrow & & \downarrow & & \downarrow \pi_2 \\ Z'_{n+1,2} & \xrightarrow{\bar{g}} & SZ'_{n+1,2} & \xrightarrow{\bar{h}} & SZ_{n+1,2}. \end{array}$$

REMARK. D. Handel [4] treated the spaces $Z'_{n,2}$ and $SZ'_{n,2}$ and applied them to embedding problem for real projective spaces. Our notations are due to D. Handel.

§ 2. Projective bundles

In this section, we prepare some results concerning the cohomology of projective bundles, which will be applied in §§3–4.

For a complex (or real) n-plane bundle $\xi=(E(\xi), p(\xi), B(\xi))$, there determines the associated sphere bundle $S(\xi)=(S(\xi), p_0(\xi), B(\xi))$ with S^{2n-1} (or S^{n-1}) as the fiber. Let $P(\xi)$ be the quotient space of $S(\xi)$ where two unit vectors in the same fiber in $S(\xi)$ are identified by the standard free action of S^1 (or Z_2) on S^{2n-1} (or S^{n-1}), and let $q(\xi): P(\xi)\longrightarrow B(\xi)$ be the factorization of $p_0(\xi): S(\xi)\longrightarrow B(\xi)$ through $P(\xi)$ by the natural projection $q'(\xi): S(\xi)\longrightarrow P(\xi)$. The bundle $P(\xi)=(P(\xi), q(\xi), B(\xi))$ with CP^{n-1} (or RP^{n-1}) as the fiber is the projective bundle associated with ξ.

Let λ_ξ be the complex (or real) line bundle associated with the S^1-bundle (or double covering) $(S(\xi), q'(\xi), P(\xi))$. Then, for the inclusion $i: CP^{n-1}\longrightarrow P(\xi)$ (or $i: RP^{n-1}\longrightarrow P(\xi)$) in any fiber of $P(\xi)$, $i^*\lambda_\xi$ is the canonical line bundle of CP^{n-1} (or RP^{n-1}).

Under the above situations, we have

THEOREM 2.1. *Let ξ be a complex n-plane bundle and let $a_\xi \in H^2(P(\xi); Z)$ be the first Chern class of λ_ξ^*, the dual of λ_ξ. Then $1, a_\xi,\cdots,a_\xi^{n-1}$ form a base of $H^*(B(\xi); Z)$-module $H^*(P(\xi); Z)$. Moreover $q(\xi)^*: H^*(B(\xi); Z)\longrightarrow H^*(P(\xi); Z)$ is a monomorphism. The ring structure of $H^*(P(\xi); Z)$ is given by*

$$a_\xi^n=-\sum_{i=1}^{n} c_i(\xi)a_\xi^{n-i}$$

where $c_i(\xi)$ is the i-th Chern class of ξ. If $H^i(B(\xi); Z)=0$ for $i>2n$, then there is the following relation:

$$\text{(2.2)}\qquad a_\xi^{n+k}=-\sum_{i=1}^{n-k}\sum_{j=0}^{k}\bar{c}_j(\xi)c_{i+k-j}(\xi)a_\xi^{n-i}\qquad \text{for } k\geq 0,$$

where $\bar{c}_j(\xi)$ is the j-th dual Chern class of ξ.

Similarly, we have

THEOREM 2.3. *Let ξ be a real n-plane bundle and let $a_\xi \in H^1(P(\xi); Z_2)$ be the first Stiefel-Whitney class of λ_ξ and let $w_i(\xi)$ (resp. $\bar{w}_i(\xi)$) be the i-th Stiefel-Whitney class (resp. dual Stiefel-Whitney class) of ξ. Then $1, a_\xi,\cdots, a_\xi^{n-1}$ form a base of $H^*(B(\xi); Z_2)$-module $H^*(P(\xi); Z_2)$. Moreover $q(\xi)^*: H^*(B(\xi); Z_2)\longrightarrow H^*(P(\xi); Z_2)$ is a monomorphism. The ring structure of $H^*(P(\xi); Z_2)$ is given by*

$$a_\xi^n = \sum_{i=1}^{n} w_i(\xi) a_\xi^{n-i}.$$

If $H^i(B(\xi); Z_2)=0$ for $i>n$, then there is the following relation:

$$a_\xi^{n+k} = \sum_{i=1}^{n-k} \sum_{j=0}^{k} \bar{w}_j(\xi) w_{i+k-j}(\xi) a_\xi^{n-i} \qquad \text{for } k \geq 0. \tag{2.4}$$

PROOF OF THEOREMS 2.1, 2.3. The first half of each theorem is well-known (e.g. [5]), and the straightforward induction provides the proofs of (2.2) and (2.4) (see [4]). Q. E. D.

§3. Cohomology of $Z_{n+1,2}$

It is easily seen that $X_{n+1,2}$ of (1.3) is the total space of the tangent sphere bundle of CP^n and $Z_{n+1,2}$ of (1.3) is the total space of the complex projective bundle associated with the tangent bundle of CP^n. Also, it is well-known that the i-th Chern class $c_i(CP^n)$ and the i-th dual Chern class $\bar{c}_i(CP^n)$ of the tangent bundle of CP^n are equal to $\binom{n+1}{i} z^i$ and $(-1)^i \binom{n+i}{i} z^i$, respectively, where z is the generator of $H^2(CP^n; Z)$. Therefore the cohomology $H^*(Z_{n+1,2}; Z)$ is determined by Theorem 2.1 as follows:

THEOREM 3.1. *As $H^*(CP^n; Z)$-module, $H^*(Z_{n+1,2}; Z)$ has $\{1, a, \dots, a^{n-1}\}$ as basis, where a ($\neq 0$) $\in H^2(Z_{n+1,2}; Z)$ is the first Chern class of the dual of the complex line bundle associated with the S^1-bundle $\pi_1: X_{n+1,2} \longrightarrow Z_{n+1,2}$. The ring structure is given by*

$$a^{n+k} = -\sum_{i=1}^{n-k} \sum_{j=0}^{k} (-1)^j \binom{n+j}{j} \binom{n+1}{i+k-j} z^{i+k} a^{n-i} \qquad \text{for } k \geq 0,$$

where z is the generator of $H^2(CP^n; Z)$.

Similarly, $Z'_{n+1,2}$ is the total space of the real projective bundle associated with the tangent bundle of RP^n. Therefore, by Theorem 2.3 we have

PROPOSITION 3.2 [4, Proposition 3.1]. *In $H^*(Z'_{n+1,2}; Z_2)$, the following relation holds:*

$$v'^{n+k} = \sum_{i=1}^{n-k} \sum_{j=0}^{k} \bar{w}_j(RP^n) w_{i+k-j}(RP^n) v'^{n-1} \qquad \text{for } k \geq 0,$$

where v' ($\neq 0$) is the first Stiefel-Whitney class of the double covering $X'_{n+1,2} \longrightarrow Z'_{n+1,2}$ and $w_j(RP^n)$ and $\bar{w}_j(RP^n)$ are the j-th Stiefel-Whitney class and the j-th dual Stiefel-Whitney class of RP^n, respectively.

COROLLARY 3.3 [4, Corollary 3.2]. *If* $k = \max\left\{i \,\middle|\, \binom{n+i}{i} \not\equiv 0 \bmod 2,\right.$

$\left.0\leq i\leq n\right\}$, *then* $v'^{n+k-1}\neq 0$, $v'^{n+k}=0$.

LEMMA 3.4 [4, Lemma 3.3]. *Let* u' *denote the first Stiefel-Whitney class of the double covering* $Z'_{n+1,2}\longrightarrow SZ'_{n+1,2}$, *and* $k=\max\left\{i\,\middle|\binom{n+i}{i}\neq 0 \bmod 2,\right.$ $\left.0\leq i\leq n\right\}$. *Then* $u'^{n+k-1}\neq 0$.

PROOF. By the diagram (1.7), it is evident. Q. E. D.

COROLLARY 3.5. *If* $n\geq 4$, *then* $u'^{4}\neq 0$.

§4. Cohomology of $(CP^n)^*$

By the mapping cylinder considerations, the diagram (1.4) gives rise to the commutative diagram of fibrations:

$$(4.1)\qquad \begin{array}{ccccccc}
W_{n+1,2} & = & W_{n+1,2} & = & W_{n+1,2} & = & W_{n+1,2}\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
X_{n+1,2} & \xrightarrow{\pi_1} & Z_{n+1,2} & \xrightarrow{\pi_2} & SZ_{n+1,2} & \xrightarrow{\pi_3} & G_{n+1,2}(C)\\
\downarrow p_1 & & \downarrow p_2 & & \downarrow p_3 & & \downarrow p_4\\
BU(1) & \xrightarrow{i_1} & BT^2 & \xrightarrow{i_2} & BG & \xrightarrow{i_3} & BU(2).
\end{array}$$

The cohomology structures of $SZ_{n+1,2}$ and BG are unknown. On the other hand, the cohomology of $Z_{n+1,2}$ has been determined in §3 and the cohomology of $X_{n+1,2}$ was determined by C.A. Ruiz [7], and the others are well-known:

(4.2) $H^*(W_{n+1,2};Z)=\wedge(w_n, w_{n+1})$ where deg $w_i=2i-1$ $(i=n, n+1)$.

(4.3) $H^*(BU(2);Z)=Z[c_1, c_2]$

where $c_i(i=1, 2)$ is the universal i-th Chern class.

(4.4) $H^*(BT^2;Z)=Z[x_1, x_2]$ where deg $x_i=2$ $(i=1, 2)$,

and there are the relations

$$(4.5)\qquad i_2^*i_3^*c_1=x_1+x_2, \qquad i_2^*i_3^*c_2=x_1x_2.$$

For $G_{n+1,2}(C)$, it is known that

$$H^*(G_{n+1,2}(C);Z)=S(y_1, y_2)\otimes S(y_3,\cdots, y_{n+1})/S^+(y_1,\cdots, y_{n+1})$$

where deg $y_i=2(i=1,\cdots, n+1)$ and $S(y_1,\cdots, y_k)$ is the ring of symmetric polynomials of k variables $y_1,\cdots, y_k$ with integral coefficients and $S^+(y_1,\cdots, y_k)$

is the ideal generated by the elements of positive degree [1, Proposition 31.1].

Let $\sigma_i\,(i=1,\cdots,n-1)$ be the i-th elementary symmetric function with respect to $n-1$ variables $y_3,\cdots,y_{n+1}$ and let $c_1=y_1+y_2$, $c_2=y_1y_2$. Then the ideal $S^+(y_1,\cdots,y_{n+1})$ is generated by the elements σ_1+c_1, $\sigma_2+\sigma_1c_1+c_2$, $\sigma_i+\sigma_{i-1}c_1+\sigma_{i-2}c_2\,(i>2)$, where $\sigma_i=0$ for $i\geq n$. By a straightforward induction, we obtain

$$\sigma_r=\sum_{i\geq 0}(-1)^{r-i}\binom{r-i}{i}c_1^{r-2i}c_2^i \qquad \text{for } r\geq 1, \tag{4.6}$$

and

$$H^*(G_{n+1,2}(C);Z)=Z[c_1,c_2]/(\sigma_n,\sigma_{n+1}). \tag{4.7}$$

From now on, we shall study the cohomology of $SZ_{n+1,2}$ and BG. Consider the following commutative diagram of fibrations:

$$\begin{array}{ccccc} T^2 & \longrightarrow & U(2) & \longrightarrow & U(2)/T^2=S^2 \\ \downarrow & & \| & & \downarrow \\ G & \longrightarrow & U(2) & \longrightarrow & U(2)/G=RP^2. \end{array}$$

This diagram induces the following two commutative diagrams such that each row is a fibration and each column is a double covering:

$$\begin{array}{ccccc} S^2 & \longrightarrow & Z_{n+1,2} & \xrightarrow{\pi_3\pi_2} & G_{n+1,2}(C) \\ \downarrow & & \downarrow{\scriptstyle \pi_2} & & \| \\ RP^2 & \longrightarrow & SZ_{n+1,2} & \xrightarrow{\pi_3} & G_{n+1,2}(C), \end{array} \qquad \begin{array}{ccccc} S^2 & \longrightarrow & BT^2 & \xrightarrow{i_3i_2} & BU(2) \\ \downarrow & & \downarrow{\scriptstyle i_2} & & \| \\ RP^2 & \longrightarrow & BG & \xrightarrow{i_3} & BU(2). \end{array} \tag{4.8}$$

Therefore $SZ_{n+1,2}$ and BG are the total spaces of the real projective bundles over $G_{n+1,2}(C)$ and $BU(2)$, respectively.

Since $H^*(G_{n+1,2}(C);Z)$ and $H^*(BU(2);Z)$ have no torsion, we adopt the same symbol for each element of $H^*(G_{n+1,2}(C);Z)$ and $H^*(BU(2);Z)$ and its image in $H^*(G_{n+1,2}(C);Z_2)$ and $H^*(BU(2);Z_2)$ by the mod 2 reduction, in the rest of this paper.

THEOREM 4.9. *Let $n\geq 4$ and let $v\in H^1(SZ_{n+1,2};Z_2)$ be the first Stiefel-Whitney class of the double covering $Z_{n+1,2}\xrightarrow{\pi_2}SZ_{n+1,2}$. Then, as $H^*(G_{n+1,2}(C);Z_2)$-module, $H^*(SZ_{n+1,2};Z_2)$ has $\{1, v, v^2\}$ as basis and $\pi_3^*: H^*(G_{n+1,2}(C);Z_2)\longrightarrow H^*(SZ_{n+1,2};Z_2)$ is a monomorphism. Moreover the ring structure of $H^*(SZ_{n+1,2};Z_2)$ is given by*

$$v^3=c_1v$$

where $c_1\in H^*(G_{n+1,2}(C);Z_2)$ is the mod 2 *reduction of the element of* (4.7).

PROOF. The first half follows from Theorem 2.3. Hence it is sufficient to show that $v^3=c_1v$. By (1.7), we have $\bar{h}^*v=u'$, the first Stiefel-Whitney class of the double covering $Z'_{n+1,2}\longrightarrow SZ'_{n+1,2}$. Since $u'^3\neq 0$ for $n\geq 4$ by Corollary 3.5, we have $v^3\neq 0$. On the other hand, $H^3(SZ_{n+1,2};Z_2)=Z_2$ and its generator is c_1v by the first half of this theorem. Therefore we have $v^3=c_1v$. Q. E. D.

Let $\delta_2\colon H^*(\ ;Z_2)\longrightarrow H^{*+1}(\ ;Z)$ be the Bockstein homomorphism associated with the exact sequence $0\longrightarrow Z\xrightarrow{\times 2}Z\xrightarrow{\rho_2}Z_2\longrightarrow 0$.

Since $\rho_2\delta_2=Sq^1$ and $Sq^1v=v^2\neq 0$ in $H^*(SZ_{n+1,2};Z_2)$, we have $\delta_2v\neq 0$. Put $\delta_2v=u\in H^2(SZ_{n+1,2};Z)$. Then we have

THEOREM 4.10. *Let $n\geq 4$. Then $H^*(G_{n+1,2}(C);Z)$-module $H^*(SZ_{n+1,2};Z)$ has $\{1, u\}$ as generators and $\pi_3^*\colon H^*(G_{n+1,2}(C);Z)\longrightarrow H^*(SZ_{n+1,2};Z)$ is a monomorphism. Moreover there are the following relations:*

$$2u=0,\qquad \rho_2u=v^2,\qquad u^2=c_1u.$$

PROOF. The first two relations follow from the fact that $\delta_2v=u$.

In the integral cohomology spectral sequence of the fibration $RP^2\longrightarrow SZ_{n+1,2}\xrightarrow{\pi_3}G_{n+1,2}(C)$, E_2-term is given as follows:

$$E_2^{s,t}=H^s(G_{n+1,2}(C);\ H^t(RP^2;\ Z))=\begin{cases}H^s(G_{n+1,2}(C);Z) & \text{for } t=0\\ H^s(G_{n+1,2}(C);Z_2) & \text{for } t=2\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, each differential is trivial and so we have $E_2=E_\infty$. Hence we obtain the following exact sequence:

$$0\longrightarrow E_\infty^{s,0}\longrightarrow H^s(SZ_{n+1,2};Z)\longrightarrow E_\infty^{s-2,2}\longrightarrow 0.$$

This gives rise to the exact sequence

$$(4.11)\qquad 0\longrightarrow H^s(G_{n+1,2}(C);Z)\longrightarrow H^s(SZ_{n+1,2};Z)\longrightarrow H^{s-2}(G_{n+1,2}(C);Z_2)\longrightarrow 0.$$

(4.11) induces that $H^{2s-1}(SZ_{n+1,2};Z)=0$ for all s and $H^{2s}(SZ_{n+1,2};Z)$ has no p-torsion for odd prime p. Since $H^{2s-1}(SZ_{n+1,2};Z)=0$, the Bockstein cohomology exact sequence associated with the exact sequence of coefficients $0\longrightarrow Z\xrightarrow{\times 2}Z\xrightarrow{\rho_2}Z_2\longrightarrow 0$ induces the exact sequence

$$0\longrightarrow H^{2s-1}(SZ_{n+1,2};Z_2)\xrightarrow{\delta_2}H^{2s}(SZ_{n+1,2};Z)\xrightarrow{\times 2}$$
$$H^{2s}(SZ_{n+1,2};Z)\xrightarrow{\rho_2}H^{2s}(SZ_{n+1,2};Z_2)\longrightarrow 0.$$

This exact sequence implies that the torsion part of $H^{2s}(SZ_{n+1,2};Z)$ is isomorphic to $H^{2s-1}(SZ_{n+1,2};Z_2)$ by δ_2. Since $H^{2s-2}(G_{n+1,2}(C);Z_2)$ is isomorphic to $H^{2s-1}(SZ_{n+1,2};Z_2)$ by the cup product with v, $H^{2s-2}(G_{n+1,2}(C);Z_2)$ is isomorphic to the torsion part of $H^{2s}(SZ_{n+1,2};Z)$, which is given by

$uH^{2s-2}(G_{n+1,2}(C); Z)$. Therefore the exact sequence (4.11) is split. Thus $H^*(G_{n+1,2}(C); Z)$-module $H^*(SZ_{n+1,2}; Z)$ has $\{1, u\}$ as generators and π_3^*: $H^*(G_{n+1,2}(C); Z) \longrightarrow H^*(SZ_{n+1,2}; Z)$ is a monomorphism.

Since $\rho_2 u^2 = v^4$ in $H^*(SZ_{n+1,2}; Z_2)$ and $\bar{h}^* v^4 = u'^4 \neq 0$ by (1.7) and Corollary 3.5, we have $u^2 \neq 0$ in $H^4(SZ_{n+1,2}; Z)$. On the other hand, the torsion part of $H^4(SZ_{n+1,2}; Z)$ is Z_2 and its generator is $c_1 u$. Therefore we have the last relation $u^2 = c_1 u$. Q. E. D.

The integral and the mod 2 cohomology of BG are given by the same way as Theorems 4.9–10 and we omit the details.

THEOREM 4.12. *Let $n \geq 4$ and let $v \in H^1(BG; Z_2)$ be the first Stiefel-Whitney class of the double covering $BT^2 \xrightarrow{i_2} BG$ and let $u = \delta_2 v$. Then $H^*(BU(2); Z_2)$-module $H^*(BG; Z_2)$ has $\{1, v, v^2\}$ as basis and $H^*(BU(2); Z)$-module $H^*(BG; Z)$ has $\{1, u\}$ as generators, and $i_3^*: H(BU(2); Z_2) \longrightarrow H^*(BG; Z_2)$ and $i_3^*: H^*(BU(2); Z) \longrightarrow H^*(BG; Z)$ are both monomorphic. Moreover the following relations hold*:

$$v^3 = c_1 v, \qquad u^2 = c_1 u, \qquad p_3^* v = v, \qquad p_3^* u = u.$$

REMARK. If we notice that the transgression of the fibration $W_{n+1,2} \longrightarrow G_{n+1,2}(C) \longrightarrow BU(2)$ is given by $\tau w_i = \bar{c}_i (i = n, n+1)$, the universal i-th dual Chern class of the complex 2-plane bundle, and that i_3^* is a monomorphism because $i_2^* i_3^*$ is so, we see easily

$$H^*(SZ_{n+1,2}; Z) = H^*(BG; Z)/(i_3^* \bar{c}_n, i_3^* \bar{c}_{n+1}) \qquad \text{for } n \geq 1,$$

$$H^*(SZ_{n+1,2}; Z_2) = H^*(BG; Z_2)/(i_3^* \bar{c}_n, i_3^* \bar{c}_{n+1}) \qquad \text{for } n \geq 1.$$

LEMMA 4.13. *Let $n \geq 4$. Then the homomorphism $\pi_2^*: H^*(SZ_{n+1,2}; Z_2) \longrightarrow H^*(Z_{n+1,2}; Z_2)$ is given by*

$$\pi_2^* c_1 = a, \qquad \pi_2^* c_2 = az + z^2, \qquad \pi_2^* v = 0,$$

where a, z in $H^*(Z_{n+1,2}; Z_2)$ are the images of a, z in $H^*(Z_{n+1,2}; Z)$ respectively, by the mod 2 *reduction.*

PROOF. It is easily seen that $\pi_2^* v = 0$. Since $W_{n+1,2}$ is 6-connected for $n \geq 4$, $p_i^* (i = 1, 2, 3, 4)$ is isomorphic in degree smaller than 7. Therefore there exists a unique element a' in $H^2(BT^2; Z_2)$ such that $p_2^* a' = a$. Since $0 = \pi_1^* a = p_1^* i_1^* a'$ and p_1^* is isomorphic in degree 2, we have $i_1^* a' = 0$. On the other hand, the generator of $H^2(BU(1); Z_2)$ is $i_1^* x_1 = i_1^* x_2$. Therefore the kernel of i_1^* of degree 2 is generated by $x_1 + x_2$. Hence we have $a' = x_1 + x_2 = i_2^* c_1$ by (4.5) and so we have $\pi_2^* c_1 = a$.

By Theorem 3.1, $\pi_2^* c_2$ has the form $\pi_2^* c_2 = \varepsilon_1 a^2 + \varepsilon_2 az + \varepsilon_3 z^2$, where $\varepsilon_i = 0$ or $1 (i = 1, 2, 3)$. Then we have

$$\pi_2^* Sq^2 c_2 = \pi_2^*(c_1 c_2) = \varepsilon_1 a^3 + \varepsilon_2 a^2 z + \varepsilon_3 a z^2.$$

However we have

$$Sq^2 \pi_2^* c_2 = Sq^2(\varepsilon_1 a^2 + \varepsilon_2 a z + \varepsilon_3 z^2) = \varepsilon_2 a^2 z + \varepsilon_2 a z^2.$$

Comparing the coefficients of the corresponding terms of $\pi_2^* Sq^2 c_2$ and $Sq^2 \pi_2^* c_2$, we obtain $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon_3$, since $n \geq 4$. Assume that $\varepsilon_2 = \varepsilon_3 = 0$. Then $0 = \pi_2^* c_2 = p_2^* i_2^* c_2 = p_2^*(x_1 x_2)$. This contradicts the fact that p_2^* is isomorphic in degree 4. Therefore we have $\pi_2^* c_2 = az + z^2$. Q. E. D.

PROPOSITION 4.14. *Let $n \geq 4$ and set $n = 2^r + s$, $0 \leq s \leq 2^r - 1$. The following relations hold in $H^*(SZ_{n+1,2}; Z_2)$:*

$$c_1^{2^{r+1}-1} = 0, \qquad c_1^{2^{r+1}-2} c_2^s v^2 \neq 0.$$

PROOF. By Lemma 4.6, we have $\sigma_r = \sum_{i \geq 0} \binom{r-i}{i} c_1^{r-2i} c_2^i$ for $r \geq 1$ in $H^*(G_{n+1,2}(C); Z_2)$. If $r \geq n$, then $\sigma_r = 0$. Therefore we obtain $c_1^{2^{r+1}-1} = 0$. To prove the second relation, it is sufficient to show that $c_1^{2^{r+1}-2} c_2^s \neq 0$ and so $\pi_2^*(c_1^{2^{r+1}-2} c_2^s) \neq 0$ in $H^*(Z_{n+1,2}; Z_2)$. By Theorem 3.1, we have

$$\pi_2^*(c_1^{2^{r+1}-2} c_2^s) = \sum_{i=0}^{n-1} b_i a^i, \qquad b_i \in H^*(CP^n; Z_2).$$

On the other hand, by Theorem 3.1 and Lemma 4.13, we have

$$\pi_2^*(c_1^{2^{r+1}-2} c_2^s) = a^{2^{r+1}-2}(a+z)^s z^s = \sum_{t=0}^{s} \binom{s}{t} a^{2^{r+1}+s-t-2} z^{s+t}$$

$$= \sum_{t=0}^{s} \binom{s}{t} \sum_{i=1}^{n-(2^r-t-2)} \sum_{j=0}^{2^r-t-2} \bar{c}_j(CP^n) c_{i+2^r-t-2-j}(CP^n) z^{s+t} a^{n-i},$$

where $c_j(CP^n)$, $\bar{c}_j(CP^n)$ are the j-th Chern and dual Chern classes of CP^n. Comparing the coefficients of a^{n-1}, we have

$$b_{n-1} = \sum_{t=0}^{s} \binom{s}{t} \sum_{j=0}^{2^r-t-2} \bar{c}_j(CP^n) c_{2^r-t-1-j}(CP^n) z^{s+t}$$

$$= \sum_{t=0}^{s} \binom{s}{t} \bar{c}_{2^r-t-1}(CP^n) z^{s+t}$$

$$= \sum_{t=0}^{s} \binom{s}{t} \binom{2^{r+1}+s-t-1}{2^r-t-1} z^{2^r+s-1}$$

By a simple calculation, we have $\binom{2^{r+1}+s-t-1}{2^r-t-1} = 0$ or $\neq 0$ according as $t \leq s-1$ or $t = s$, and so we obtain $b_{n-1} = z^{n-1} \neq 0$ in $H^*(CP^n; Z_2)$. Q. E. D.

Using the above proposition, we have

THEOREM 4.15. *Let $n\geq 4$. Then $SZ_{n+1,2}$ is an unorientable (4n-2)-dimensional manifold which is weakly homotopy equivalent to the reduced symmetric product of CP^n, and $H^{4n-2}(SZ_{n+1,2}; Z)=Z_2$ with the generator $c_1^{2^{r+1}-2}c_2^s u$ for* $n=2^r+s$, $0\leq s\leq 2^r-1$.

§5. Classification of embeddings of CP^n in Euclidean spaces

A. Haefliger investigated the embeddings in the stable range [3] and proved the following theorem.

THEOREM 5.1 (Haefliger). *Let M be an n-dimensional compact differentiable manifold. The correspondence which associates with a given differentiable embedding f: $M\longrightarrow R^m$ the equivariant map F: $M\times M-\Delta\longrightarrow S^{m-1}$ defined by $F(x, y)=\dfrac{f(x)-f(y)}{\|f(x)-f(y)\|}$ induces the correspondence which associates with a given isotopy class of f the equivariant homotopy class of F. This correspondence is surjective if $2m\geq 3(n+1)$ and bijective if $2m>3(n+1)$.*

We now know that there exists a one-to-one correspondence between the equivariant homotopy classes of equivariant maps $M\times M-\Delta\longrightarrow S^{m-1}$ and the homotopy classes of cross sections of the sphere bundle $S^{m-1}\longrightarrow(M\times M-\Delta)\times_{Z_2}S^{m-1}\longrightarrow M^*$ associated with the double covering $M\times M-\Delta\longrightarrow M^*$.

Let λ be the real line bundle over $(CP^n)^*$ associated with the double covering $CP^n\times CP^n-\Delta\longrightarrow(CP^n)^*$. Then the sphere bundle

$$S^{m-1}\longrightarrow(CP^n\times CP^n-\Delta)\times_{Z_2}S^{m-1}\longrightarrow(CP^n)^*$$

is the sphere bundle associated with $m\lambda$, the Whitney sum of m copies of λ.

Therefore we have

PROPOSITION 5.2. (1) *Let $2m\geq 3(2n+1)$. If $m\lambda$ has a non-zero cross section, then there exists an embedding of CP^n in R^m.*

(2) *Let $2m>3(2n+1)$. Then there exists a one-to-one correspondence between the isotopy classes of embeddings of CP^n in R^m and the homotopy classes of cross sections of the sphere bundle associated with $m\lambda$ over $(CP^n)^*$.*

By Propositions 1.6 and 5.2, the obstructions for $m\lambda$ to have a non-zero cross section are the elements of $H^{i+1}(SZ_{n+1,2}; \pi_i(S^{m-1}))$ and its primary obstruction for even m is the Euler class $\chi(m\lambda)$ of $m\lambda$, and the obstructions for two given cross sections to be homotopic are the elements of $H^i(SZ_{n+1,2}; \pi_i(S^{m-1}))$.

LEMMA 5.3. *Let η be a real line bundle. Then the Euler class $\chi(2\eta)$ is given by*

$$\chi(2\eta)=\delta_2 w_1(\eta),$$

where $w_1(\eta)$ *is the first Stiefel-Whitney class of* η.

PROOF. Let ξ be the canonical line bundle over RP^∞. By the universality of ξ, it is sufficient to show that $\chi(2\xi)=\delta_2 w_1(\xi)$. Consider the Bockstein cohomology exact sequence of RP^∞

$$0 \longrightarrow H^1(RP^\infty; Z_2) \xrightarrow{\delta_2} H^2(RP^\infty; Z) \xrightarrow{\times 2} H^2(RP^\infty; Z) \xrightarrow{\rho_2} H^2(RP^\infty; Z_2) \longrightarrow 0,$$

where $H^1(RP^\infty; Z_2)=Z_2$ with the generator $w_1(\xi)$ and $H^2(RP^\infty; Z)=Z_2$ with the generator $\delta_2 w_1(\xi)$. Since $\rho_2\chi(2\xi)=w_2(2\xi)=w_1(\xi)^2\neq 0$, it follows that $\chi(2\xi)\neq 0$ in $H^2(RP^\infty; Z)$ and so we have $\chi(2\xi)=\delta_2 w_1(\xi)$. Q. E. D.

REMARK. The above lemma is generalized as follows: Let η^1 and ζ^n be a real line bundle and a real n-plane bundle over the same space with $w_1(\eta^1)=w_1(\zeta^n)$. Then we have

$$\chi(\eta^1 \oplus \zeta^n)=\delta_2 w_n(\zeta^n).$$

By the above considerations, we have the following theorem, which is already known ([6], [8], [9]):

THEOREM 5.4. (1) *CP^n is embeddable in R^{4n-2} for $n\geq 4$, $n\neq 2^r$.*
(2) *CP^{2^r} is embeddable in $R^{2^{r+2}-1}$ but not embeddable in $R^{2^{r+2}-2}$ for $r\geq 2$.*

PROOF. The obstructions for the existence of a non-zero cross section of $(4n-1)\lambda$ are in $H^{i+1}(SZ_{n+1,2}; \pi_i(S^{4n-2}))$ which is 0, since $SZ_{n+1,2}$ is a $(4n-2)$-dimensional manifold. Hence CP^n is embeddable in R^{4n-1} by Proposition 5.2, (1). The obstructions for the existence of a non-zero cross section of $(4n-2)\lambda$ are in $H^{i+1}(SZ_{n+1,2}; \pi_i(S^{4n-3}))$ and non-trivial obstruction is the Euler class $\chi((4n-2)\lambda)$ in $H^{4n-2}(SZ_{n+1,2}; Z)$. By Lemma 5.3, we have $\chi(2\lambda)=u=\delta_2 v$ and using Proposition 4.14, we have

$$\chi((4n-2)\lambda)=\chi(2\lambda)^{2n-1}=u^{2n-1}=uc_1^{2n-2}\begin{cases} =0 & \text{for } n\neq 2^r \\ \neq 0 & \text{for } n=2^r. \end{cases}$$

Therefore by Proposition 5.2 (1), it follows that CP^n is embeddable or not embeddable in R^{4n-2} according as $n\neq 2^r$ or $n=2^r$. Q. E. D.

Our main theorem is the following

THEOREM 5.5. *Let $n\geq 4$.*
(1) *There exists a unique isotopy class of embeddings of CP^n in R^{4n}.*
(2) *There exist just two isotopy classes of embeddings of CP^n in R^{4n-1}.*
(3) *There exist just two isotopy classes of embeddings of CP^n in R^{4n-2} for $n\neq 2^r$.*

PROOF. The obstructions for two non-zero cross sections of $4n\lambda$ being homotopic are the elements of $H^i(SZ_{n+1,2}; \pi_i(S^{4n-1}))$ which is 0 for all i.

This implies (1). The obstructions for two non-zero cross sections of $(4n-1)\lambda$ being homotopic are in $H^i(SZ_{n+1,2};\pi_i(S^{4n-2}))$ and

$$H^i(SZ_{n+1,2};\pi_i(S^{4n-2}))=\begin{cases}0 & \text{for } i\neq 4n-2\\ Z_2 & \text{for } i=4n-2,\end{cases}$$

by Theorem 4.15. Therefore we have (2). By Theorems 4.9–10, 4.15,

$$H^i(SZ_{n+1,2};\pi_i(S^{4n-3}))=\begin{cases}0 & \text{for } i\neq 4n-2\\ Z_2 & \text{for } i=4n-2,\end{cases}$$

and so we have (3). Q. E. D.

REMARK 1. W.-T. Wu [10] proved that any two embeddings of an n-dimensional differentiable manifold in R^{2n+1} are isotopic.

REMARK 2. T. Watabe [9] proved that any two immersions of CP^n in R^{4n-1} are regularly homotopic for even n.

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