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Nonimbedding Theorems of Algebras

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Introduction

Recently several theorems have been shown on nonimbedding of nilpotent Lie algebras. Chao [2] has proved that a non-commutative Lie algebra L such that (1) the center is 1-dimensional or (2) L/L^2 is 2-dimensional cannot be any C^kN, $k\geq 1$ of a nilpotent algebra N where C^kN, $k\geq 0$ is the descending central series of N. This result has been improved by Ravisankar [6] as follows: Such an algebra L cannot be any $N^{[k]\prime}$, $k\geq 1$ of a $\mathfrak{D}$-nilpotent algebra N for a collection $\mathfrak{D}$ of derivations of N containing all inner derivations. Here the $N^{[k]\prime}$ are inductively defined by $N^{[0]\prime}=N$ and $N^{[k]\prime}=N^{[k-1]\prime}\mathfrak{D}$ for $k>0$, and N is called $\mathfrak{D}$-nilpotent if $N^{[n]\prime}=(0)$ for some n, generalizing the definitions of $N^{[k]}$ and characteristic nilpotency. On the other hand, it is known [4] that a characteristically nilpotent Lie algebra L such that (3) the center is annihilated by $\mathfrak{D}(L)$ or (4) $L\mathfrak{D}(L)\subseteq L^2$ cannot be the derived algebra of any Lie algebra H. It seems natural to ask other conditions put on L and a larger class of subalgebras of N or H to which L cannot be equal. The purpose of this paper is to investigate nonimbedding of algebras on this line and to generalize and sharpen these results.

In Section 1, we shall give two equivalent definitions of $\mathfrak{D}$-nilpotency of nonassociative algebras and show some fundamental lemmas. In Section 2, we shall search for a class of subalgebras of such an N as stated above, to which any non-commutative Lie algebra L satisfying each of the conditions (1)-(4) and the condition (5) dim $L/L^3=3$ cannot belong. The results for Lie algebras satisfying the condition (1) or (2) will generalize the results, stated above, of Chao and Ravisankar. We shall make similar investigation on non-commutative associative algebras. In Section 3, we shall show that any characteristically nilpotent Lie algebra L satisfying each of the conditions (1)-(5) cannot belong to a certain type of class of subalgebras of any Lie algebra H. Especially for Lie algebras satisfying the condition (3) or (4) it will generalize the result in [4] stated above. Similar results for characteristically nilpotent non-commutative associative algebras will be also given. In Section 4, we shall show some results on nonimbedding of nonassociative algebras satisfying the condition similar to (1) or (5). Section 5 will be devoted to discussing several examples of algebras in connection with the results in Sections 2-4 and the conditions (1)-(5). Especially the examples of a characteristically nilpotent commutative and a characteristically nilpotent non-commutative associative algebras will be shown there.

§ 1.

By a nonassociative algebra we mean an algebra which is not necessarily associative, that is, a distributive algebra. For a nonassociative algebra A over a field Φ, we denote by $Z(A)$ the center of A and by $\mathfrak{D}(A)$ the set of all derivations of A. Especially when A is Lie (resp. associative), we denote by $\mathfrak{J}(A)$ the set of all inner derivations of A, that is, the set of D_x, $x \in A$, with $yD_x=[y, x]$ (resp. $=yx-xy$) for $y \in A$.

Ravisankar [6] has introduced the notion of $\mathfrak{D}$-nilpotency of an algebra for a set $\mathfrak{D}$ of its derivations which coincides with nilpotency and characteristic nilpotency in special cases. We shall here give his definition of $\mathfrak{D}$-nilpotency [6, Definition 4.1] and another equivalent definition by making use of an ascending series, with our notations which seem to be more useful for our purpose.

Let $\mathfrak{D}$ be a set of derivations of a nonassociative algebra A over a field Φ. For a set $M \subseteq A$, we denote by $M\mathfrak{D}$ the subspace of A spanned by all xD with $x \in M$, $D \in \mathfrak{D}$. A subalgebra B of A is said to be a $\mathfrak{D}$-*ideal* provided $B\mathfrak{D} \subseteq B$. Put $A\mathfrak{D}^0=A$ and define $A\mathfrak{D}^k=(A\mathfrak{D}^{k-1})\mathfrak{D}$ inductively for $k \geq 1$. Put $A\mathfrak{D}_0=(0)$ and define $A\mathfrak{D}_k$ inductively for $k \geq 1$ as the subspace of A consisting of all x such that $xD \in A\mathfrak{D}_{k-1}$ for every $D \in \mathfrak{D}$. Then it is immediate that $A\mathfrak{D}^k=(0)$ if and only if $A\mathfrak{D}_k=A$. A is called $\mathfrak{D}$-*nilpotent* provided it satisfies one of the following equivalent conditions:

(1) $A\mathfrak{D}^n=(0)$ for some n.

(2) $A\mathfrak{D}_n=A$ for some n.

A $\mathfrak{D}$-nilpotent algebra A is said to be of class n provided $A\mathfrak{D}^{n-1} \neq (0)$ but $A\mathfrak{D}^n=(0)$.

For a $\mathfrak{D}$-nilpotent algebra A of class n, we have obviously

$$A\mathfrak{D}^i \subseteq A\mathfrak{D}_{n-i} \qquad \text{for } 0 \leq i \leq n.$$

In the case where $\mathfrak{D}=\mathfrak{D}(A)$, $A\mathfrak{D}(A)^k$ coincides with $A^{[k]}$ and $\mathfrak{D}(A)$-nilpotency is nothing but characteristic nilpotency. We shall denote $A\mathfrak{D}(A)_k$ by $A_{[k]}$. In the case where A is a Lie algebra and $\mathfrak{D}=\mathfrak{J}(A)$, we have

$$A\mathfrak{J}(A)^k=A^{k+1}=C^kA \quad \text{and} \quad A\mathfrak{J}(A)_k=C_kA$$

and $\mathfrak{J}(A)$-nilpotency is just nilpotency.

Every associative algebra A may be endowed with the structure of a Lie algebra by employing the commutator of two elements as their new product. The Lie algebra thus obtained from A is denoted by A_L. Then every subalgebra of A is that of A_L. Every derivation of A is that of A_L. Every inner derivation of A is that of A_L and conversely. For subsets M, N of A, the subspace $[M, N]$ of A spanned by all the commutators $xy-yx$ of $x \in M$ and $y \in N$ is obviously the product of M and N in A_L.

We shall here show the following two lemmas which are fundamental for the development of our main results.

LEMMA 1. *Let A be a Lie or associative algebra over a field Φ. Let $\mathfrak{D}$ be a set of derivations of A containing $\mathfrak{J}(A)$. Then*

(1) $[A\mathfrak{D}^i, A\mathfrak{D}^j] \subseteq A\mathfrak{D}^{i+j+1}$.

(2) $[A\mathfrak{D}^i, A\mathfrak{D}_j] \subseteq A\mathfrak{D}_{j-i-1}$.

PROOF. We shall show the assertions by induction on i.

As for the assertion (1), when $i=0$,

$$[A, A\mathfrak{D}^j] \subseteq (A\mathfrak{D}^j)\mathfrak{J}(A) \subseteq (A\mathfrak{D}^j)\mathfrak{D} = A\mathfrak{D}^{j+1}.$$

Assume that (1) holds for $i=k$. Then by taking account of the remark preceding the lemma, we infer

$$\begin{aligned}[A\mathfrak{D}^{k+1}, A\mathfrak{D}^j] &\subseteq [A\mathfrak{D}^k, A\mathfrak{D}^j]\mathfrak{D} + [A\mathfrak{D}^k, A\mathfrak{D}^{j+1}] \\ &\subseteq (A\mathfrak{D}^{k+j+1})\mathfrak{D} + A\mathfrak{D}^{k+(j+1)+1} \\ &= A\mathfrak{D}^{(k+1)+j+1},\end{aligned}$$

which shows that (1) holds for $i=k+1$.

As for the assertion (2), when $i=0$,

$$[A, A\mathfrak{D}_j] \subseteq (A\mathfrak{D}_j)\mathfrak{J}(A) \subseteq (A\mathfrak{D}_j)\mathfrak{D} \subseteq A\mathfrak{D}_{j-1}.$$

Assume that (2) holds for $i=k$. Then we infer

$$\begin{aligned}[A\mathfrak{D}^{k+1}, A\mathfrak{D}_j] &\subseteq [A\mathfrak{D}^k, A\mathfrak{D}_j]\mathfrak{D} + [A\mathfrak{D}^k, A\mathfrak{D}_{j-1}] \\ &\subseteq (A\mathfrak{D}_{j-k-1})\mathfrak{D} + A\mathfrak{D}_{(j-1)-k-1} \\ &\subseteq A\mathfrak{D}_{j-(k+1)-1}.\end{aligned}$$

Hence (2) holds for $i=k+1$, and the proof is complete.

LEMMA 2. *Let A be a Lie or associative algebra over a field Φ. Assume that A is $\mathfrak{D}$-nilpotent of class n for a set $\mathfrak{D}$ of derivations of A containing $\mathfrak{J}(A)$. Then*

(1) $A\mathfrak{D}^{n-1} \subseteq A\mathfrak{D}_1 \subseteq Z(A)$.

(2) $A\mathfrak{D}^{n-2} \subseteq Z(A\mathfrak{D}^1) \cap Z(A\mathfrak{D}_{n-1})$ *for* $n \geq 3$.

Proof. It is obvious that $A\mathfrak{D}^{n-1}\subseteq A\mathfrak{D}_1$ and

$$[A\mathfrak{D}_1, A]\subseteq (A\mathfrak{D}_1)\mathfrak{I}(A)\subseteq (A\mathfrak{D}_1)\mathfrak{D}\subseteq A\mathfrak{D}_0=(0),$$

proving the assertion (1). For $n\geq 3$, $A\mathfrak{D}^{n-2}\subseteq A\mathfrak{D}^1\subseteq A\mathfrak{D}_{n-1}$ and Lemma 1 (2) shows that

$$[A\mathfrak{D}^{n-2}, A\mathfrak{D}_{n-1}]\subseteq A\mathfrak{D}_0=(0).$$

Hence we have the assertion (2).

§2.

In this section we study nonimbedding of non-commutative nilpotent Lie or associative algebras satisfying each of the conditions (1)–(5) stated in the introduction.

We begin with

Theorem 1. *Let A be a non-commutative Lie or associative algebra over a field Φ satisfying the condition:*

(a) *The center of A is 1-dimensional.*

Then A cannot be any subalgebra B of a $\mathfrak{D}$-nilpotent algebra N of class $n\geq 3$ (resp. $=2$) for a set $\mathfrak{D}$ of derivations of N containing $\mathfrak{I}(N)$ such that

$$N\mathfrak{D}_{n-1}\supseteq B\supseteq N\mathfrak{D}^{n-2} \quad or \quad B\subseteq N\mathfrak{D}^{n-2} \quad or \quad B\subseteq N\mathfrak{D}_1 \qquad (resp.\ B\subseteq N\mathfrak{D}_1).$$

Proof. By Lemma 2, $N\mathfrak{D}^{n-2}(n\geq 3)$ and $N\mathfrak{D}_1$ are commutative. Hence $A\not\subseteq N\mathfrak{D}^{n-2}$ and $A\not\subseteq N\mathfrak{D}_1$. Now assume that $N\mathfrak{D}_{n-1}\supseteq A\supset N\mathfrak{D}^{n-2}$ for $n\geq 3$. Then by Lemma 2 (2) we have $N\mathfrak{D}^{n-2}\subseteq Z(N\mathfrak{D}_{n-1})$ and therefore $N\mathfrak{D}^{n-2}\subseteq Z(A)$. It follows that

$$\dim Z(A)\geq \dim N\mathfrak{D}^{n-2}\geq 2,$$

which contradicts the condition (a). Thus the assertion is proved.

Theorem 2. *Let A be a non-commutative Lie or associative algebra over a field Φ satisfying the condition:*

(b) *$\mathfrak{D}(A)$ annihilates the center of A.*

Then A cannot be any $\mathfrak{D}$-ideal B of a $\mathfrak{D}$-nilpotent algebra N of class $n\geq 3$ (resp. $=2$) for a set $\mathfrak{D}$ of derivations of N containing $\mathfrak{I}(N)$ such that

$$N\mathfrak{D}_{n-1}\supseteq B\supseteq N\mathfrak{D}^{n-2} \quad or \quad B\subseteq N\mathfrak{D}^{n-2} \quad or \quad B\subseteq N\mathfrak{D}_1 \qquad (resp.\ B\subseteq N\mathfrak{D}_1).$$

Proof. We have $A\not\subseteq N\mathfrak{D}^{n-2}(n\geq 3)$ and $A\not\subseteq N\mathfrak{D}_1$ since $N\mathfrak{D}^{n-2}$ and $N\mathfrak{D}_1$ are commutative by Lemma 2. Now assume that A is a $\mathfrak{D}$-ideal of N such that $N\mathfrak{D}_{n-1}\supseteq A\supset N\mathfrak{D}^{n-2}$ for $n\geq 3$. Then by Lemma 2 (2), we have $N\mathfrak{D}^{n-2}\subseteq$

$Z(N\mathfrak{D}_{n-1})$ and therefore $N\mathfrak{D}^{n-2} \subseteq Z(A)$. Since A is a $\mathfrak{D}$-ideal of N, it follows that

$$N\mathfrak{D}^{n-1} = (N\mathfrak{D}^{n-2})\mathfrak{D} \subseteq Z(A)\mathfrak{D} \subseteq Z(A)\mathfrak{D}(A) .$$

Therefore by the condition (b) $N\mathfrak{D}^{n-1} = (0)$, which contradicts the definition of n. Thus the theorem is proved.

COROLLARY 1. *Let A be a non-commutative Lie or associative algebra over a field Φ satisfying one of the conditions* (a) *and* (b). *Then A cannot be any subalgebra B of a $\mathfrak{D}$-nilpotent algebra N of class n for a set $\mathfrak{D}$ of derivations of N containing $\mathfrak{J}(N)$ such that*

$$N\mathfrak{D}^i \supseteq B \supseteq N\mathfrak{D}^{i+1} \qquad \textit{for } i \geq 1, \textit{ or}$$

$$N\mathfrak{D}_i \supseteq B \supseteq N\mathfrak{D}_{i-1} \qquad \textit{for } i \leq n-1 \textit{ and } i \neq 2 .$$

PROOF. A subalgebra B of N in the statement is obviously a $\mathfrak{D}$-ideal. Hence the statement follows from Theorems 1 and 2.

COROLLARY 2. *Let L be a non-commutative Lie algebra over a field Φ satisfying the condition* (a) (*resp.* (b)). *Then L cannot be any subalgebra* (*resp. ideal*) *B of a nilpotent algebra N of class $n \geq 3$ such that*

$$C_{n-1}N \supseteq B \supseteq C^{n-2}N \quad \textit{or} \quad B \subseteq C^{n-2}N \quad \textit{or} \quad B \subseteq C_1 N$$

and of class $n=2$ such that $B \subseteq C_1 N$, and L cannot be any subalgebra (*resp. characteristic ideal*) *B of a characteristically nilpotent algebra N of class $n \geq 3$ such that*

$$N_{[n-1]} \supseteq B \supseteq N^{[n-2]} \quad \textit{or} \quad B \subseteq N^{[n-2]} \quad \textit{or} \quad B \subseteq N_{[1]}$$

and of class $n=2$ such that $B \subseteq N_{[1]}$.

PROOF. This follows from Theorems 1 and 2 as a special case where $\mathfrak{D}$ is $\mathfrak{J}(N)$ or $\mathfrak{D}(N)$.

COROLLARY 3. *Let L be a non-commutative Lie algebra over a field Φ satisfying one of the conditions* (a) *and* (b). *Then L cannot be any subalgebra B of a nilpotent algebra N of class n such that*

$$C^i N \supseteq B \supseteq C^{i+1} N \qquad \textit{for } i \geq 1, \textit{ or}$$

$$C_i N \supseteq B \supseteq C_{i-1} N \qquad \textit{for } i \leq n-1 \textit{ and } i \neq 2,$$

and any subalgebra B of a characteristically nilpotent algebra N of class n such that

$$N^{[i]} \supseteq B \supseteq N^{[i+1]} \qquad \textit{for } i \geq 1, \textit{ or}$$

$$N_{[i]} \supseteq B \supseteq N_{[i-1]} \qquad \textit{for } i \leq n-1 \textit{ and } i \neq 2.$$

Proof. This follows from Corollary 1 if we take $\mathfrak{J}(N)$, $\mathfrak{D}(N)$ for $\mathfrak{D}$.

It may be here remarked that the part of Corollary 1 related to (a) and $B=N\mathfrak{D}^i$ is [6, Theorems 4.6 and 4.8] and the part of Corollary 3 related to (a) and $B=C^iN$ is [2, Theorem 1].

Theorem 3. *Let A be a non-commutative Lie or associative algebra over a field Φ satisfying one of the conditions*:

(c) $\dim A = \dim [[A, A], A]+3$.

(d) $\dim A = \dim [A, A]+2$.

Then A cannot be any subalgebra B of a $\mathfrak{D}$-nilpotent algebra N for a set $\mathfrak{D}$ of derivations of N containing $\mathfrak{J}(N)$ such that

$$N\mathfrak{D}^i \supseteq B \supseteq N\mathfrak{D}^{i+1} \qquad \textit{for } i \geq 1.$$

Proof. Let A satisfy the condition (c). Assume that $N\mathfrak{D}^i \supseteq A \supset N\mathfrak{D}^{i+1}$ for $i \geq 1$. Let N be $\mathfrak{D}$-nilpotent of class n, that is, $N\mathfrak{D}^{n-1} \supset N\mathfrak{D}^n=(0)$. Then $n \geq 2$ and $i \leq n-1$. We have $i \neq n-1$ since $N\mathfrak{D}^{n-1}$ is commutative by Lemma 2. Hence $i \leq n-2$ and therefore $n \geq 3$. Then by Lemma 2 $N\mathfrak{D}^{n-2}$ is commutative and therefore $i \neq n-2$. It follows that $i+3 \leq n$. Hence $N\mathfrak{D}^{i+2} \supset N\mathfrak{D}^{i+3} \supseteq (0)$. Thus

$$\dim A \geq \dim N\mathfrak{D}^{i+3}+3.$$

But by Lemma 1

$$[[A, A], A] \subseteq [[N\mathfrak{D}^i, N\mathfrak{D}^i], N\mathfrak{D}^i] \subseteq N\mathfrak{D}^{3i+2}.$$

Therefore by using the condition (c) we have

$$\dim N\mathfrak{D}^{3i+2} \geq \dim [[A, A], A] \geq \dim N\mathfrak{D}^{i+3}.$$

Hence $N\mathfrak{D}^{3i+2} \supseteq N\mathfrak{D}^{i+3}$. It follows that $3i+2 \leq i+3$ and therefore $2i \leq 1$, which is a contradiction. Thus A is not such a subalgebra B of N as in the statement.

We here remark that, for a non-commutative nilpotent Lie algebra L, the conditions (c) and (d) are equivalent. This is easily seen by observing the fact that the dimension of L/L^2 is ≥ 2. Hence we have the statement for a non-commutative Lie algebra satisfying the condition (d). We use this to see the statement for a non-commutative associative algebra satisfying the condition (d) by considering the Lie algebras associated to associative algebras and by taking account of the related remark in Section 1. This completes the proof.

Theorem 4. *Let A be a non-commutative Lie or associative (resp. Lie) algebra over a field Φ satisfying the condition*:

(e) $A\mathfrak{D}(A)\subseteq[A, A]$.

Then A cannot be any subalgebra B of a $\mathfrak{D}$-nilpotent algebra N for a set $\mathfrak{D}$ of derivations of N containing $\mathfrak{J}(N)$ such that

$$B=N\mathfrak{D}^1 \quad or \quad N\mathfrak{D}^i\supseteq B\supseteq N\mathfrak{D}^{i+1} \qquad for\ i\geq 2$$

$$(resp.\ \ N\mathfrak{D}^i\supseteq B\supseteq N\mathfrak{D}^{i+1} \qquad for\ i\geq 1).$$

Proof. Assume that N is $\mathfrak{D}$-nilpotent of class n. In the case where $A=N\mathfrak{D}^1$, since A is a $\mathfrak{D}$-ideal of N, by using the condition (e) and Lemma 1 we have

$$N\mathfrak{D}^2=A\mathfrak{D}\subseteq A\mathfrak{D}(A)\subseteq[A, A]\subseteq N\mathfrak{D}^3,$$

from which it follows that $N\mathfrak{D}^2=(0)$ and therefore $[A, A]=(0)$, that is, A is commutative.

Now suppose that $N\mathfrak{D}^i\supseteq A\supset N\mathfrak{D}^{i+1}$ for $i\geq 2$. Since A is a $\mathfrak{D}$-ideal of N, we have

$$N\mathfrak{D}^{i+2}=(N\mathfrak{D}^{i+1})\mathfrak{D}\subseteq A\mathfrak{D}(A).$$

By using the condition (e) and Lemma 1 we have

$$N\mathfrak{D}^{i+2}\subseteq[A, A]\subseteq[N\mathfrak{D}^i, N\mathfrak{D}^i]\subseteq N\mathfrak{D}^{2i+1},$$

from which it follows that $i+2\geq 2i+1$ and therefore $i\leq 1$, which contradicts our supposition $i\geq 2$.

It remains only to show that we cannot have $N\mathfrak{D}^1\supset A\supset N\mathfrak{D}^2$ in the case where A is a Lie algebra. So suppose the contrary. Then A is a $\mathfrak{D}$-ideal of N. Hence by the condition (e) we infer

$$N\mathfrak{D}^3\subseteq A\mathfrak{D}\subseteq A\mathfrak{D}(A)\subseteq A^2.$$

It follows from Lemma 1 that

$$N\mathfrak{D}^4\subseteq A^2\mathfrak{D}\subseteq[A\mathfrak{D}, A]\subseteq A^3\subseteq(N\mathfrak{D}^1)^3\subseteq N\mathfrak{D}^5.$$

Hence $N\mathfrak{D}^4=(0)$ and therefore $A^3=(0)$. Take a complementary subspace U of A^2 in A. Then $A=U+U^2$ and $U\cap U^2=(0)$. Therefore the identity endomorphism of U is extended to be a semisimple derivation of A. It follows that A is not characteristically nilpotent. But, since a $\mathfrak{D}$-nilpotent algebra N for $\mathfrak{D}\supseteq\mathfrak{J}(N)$ is nilpotent, A is nilpotent and therefore by the condition (e) A is characteristically nilpotent, which is a contradiction. Thus the proof is complete.

Corollary 1. *Let L be a non-commutative Lie algebra over a field Φ satisfying one of the conditions* (c), (d) *and* (e). *Then L cannot be any subalgebra B of a nilpotent Lie algebra N such that*

$$C^iN \supseteq B \supseteq C^{i+1}N \qquad \text{for } i \geq 1,$$

and any subalgebra B of a characteristically nilpotent Lie algebra N such that

$$N^{[i]} \supseteq B \supseteq N^{[i+1]} \qquad \text{for } i \geq 1.$$

Proof. This follows from Theorems 3 and 4 as a special case where $\mathfrak{D}$ is $\mathfrak{J}(N)$ or $\mathfrak{D}(N)$.

Corollary 2. *Let L be a non-commutative Lie algebra over a field Φ satisfying one of the conditions* (a), (b), (c), (d) *and* (e). *Then L cannot be the Frattini subalgebra of any nilpotent Lie algebra.*

Proof. The Frattini subalgebra $\phi(N)$ of a Lie algebra N is the intersection of all maximal subalgebras of N. If N is nilpotent, $\phi(N)=N^2$ as can be easily seen ([2, Theorem 3]). Hence we have the assertion as an immediate consequence of Corollary 3 to Theorems 1, 2 and the preceding Corollary 1.

We here remark that the part of Theorem 3 related to a Lie algebra A, (d) and $B=N\mathfrak{D}^i$ is [6, Proposition 4.10], the part of Corollary 1 related to (d) and $B=C^iN$ is [2, Theorem 2], and the part of Corollary 2 related to (a) and (d) is [2, Corollaries 1 and 2 to Theorem 3].

§3.

In this section we shall strengthen the results in the preceding section for a special class of non-commutative algebras, characteristically nilpotent non-commutative algebras.

We first show

Lemma 3. *Let H be a nonassociative algebra over a field Φ and let $\mathfrak{D}$ be a set of derivations of H. If a characteristically nilpotent algebra A is contained in H as a $\mathfrak{D}$-ideal and contains $H\mathfrak{D}^k$ for some $k \geq 0$, then H is $\mathfrak{D}$-nilpotent.*

Proof. Since A is a $\mathfrak{D}$-ideal of H, we have

$$H\mathfrak{D}^{k+1} = (H\mathfrak{D}^k)\mathfrak{D} \subseteq A\mathfrak{D}(A) = A^{[1]}.$$

By induction on m we can establish

$$H\mathfrak{D}^{k+m} \subseteq A^{[m]} \qquad \text{for } m \geq 1.$$

Since A is characteristically nilpotent, $A^{[m]}=(0)$ for some m. Hence $H\mathfrak{D}^{k+m}=(0)$ for such an m. Therefore H is $\mathfrak{D}$-nilpotent, completing the proof.

In virtue of Lemma 3, we first establish the following

Theorem 5. *Let L be a characteristically nilpotent Lie algebra over a field Φ satisfying one of the conditions* (a), (b), (c), (d) *and* (e) *in the preceding theorems. Then L cannot be any subalgebra B of a Lie algebra H such that*

$$H\mathfrak{D}^i \supseteq B \supseteq H\mathfrak{D}^{i+1} \qquad \textit{for } i \geq 1$$

where $\mathfrak{D}$ *is a set of derivations of* H *containing* $\mathfrak{J}(H)$.

PROOF. If L is equal to a subalgebra B of H as in the statement, then H obviously satisfies the condition in Lemma 3. Hence H is $\mathfrak{D}$-nilpotent. Since a characteristically nilpotent Lie algebra is not commutative, this contradicts Corollary 1 to Theorems 1, 2 and Theorems 3, 4.

COROLLARY. *Let L be a characteristically nilpotent Lie algebra over a field Φ satisfying one of the conditions* (a), (b), (c), (d) *and* (e). *Then L cannot be any subalgebra B of a Lie algebra H such that*

$$C^iH \supseteq B \supseteq C^{i+1}H \quad \textit{or}$$

$$H^{[i]} \supseteq B \supseteq H^{[i+1]} \qquad \textit{for } i \geq 1.$$

PROOF. This follows from Theorem 5 as a special case where $\mathfrak{D}$ is $\mathfrak{J}(H)$ or $\mathfrak{D}(H)$.

It may be here remarked that the part of Corollary related to (b), (e) and $B = H^2$ is [4, Corollary to Lemma 3 and Corollary to Theorem 7].

We next establish the following

THEOREM 6. *Let A be a characteristically nilpotent non-commutative associative algebra over a field Φ satisfying one of the conditions* (a), (b), (c) *and* (d) (*resp. the condition* (e)) *in Theorems* 1–4. *Then A cannot be any subalgebra B of an associative algebra H such that*

$$H\mathfrak{D}^i \supseteq B \supseteq H\mathfrak{D}^{i+1} \qquad \textit{for } i \geq 1$$

$$(\textit{resp.} \quad B = H\mathfrak{D}^1 \textit{ or } H\mathfrak{D}^i \supseteq B \supseteq H\mathfrak{D}^{i+1} \qquad \textit{for } i \geq 2)$$

where $\mathfrak{D}$ *is a set of derivations of* H *containing* $\mathfrak{J}(H)$.

PROOF. This follows from Lemma 3 and from Corollary 1 to Theorems 1, 2 and Theorems 3, 4, as in the proof of Theorem 5.

§4.

In this section we generally deal with nonassociative algebras satisfying one of the conditions similar to (a) and (c). We note that a nilpotent nonassociative algebra A is of class n provided $A^n \neq (0)$ but $A^{n+1} = (0)$.

PROPOSITION 1. *Let A be a non-zero nonassociative algebra over a field Φ satisfying the condition*:

(a′) *The annihilator ideal I of A is* 1-*dimensional.*

Then A cannot be any subalgebra B of a nilpotent algebra N of class $n \geq 3$

(resp. $=2$*) such that*

$$N^2 \supseteq B \supseteq N^{n-1} \quad \textit{or} \quad B \subseteq N^{n-1} \qquad (\textit{resp. } B \subseteq N^2).$$

Proof. In the case $n=2$, we cannot have $A \subseteq N^2$ since $N^2N^2 \subseteq N^4 = (0)$ and A is non-zero, that is, $AA \neq (0)$. So assume that $n \geq 3$. Then $A \not\subseteq N^{n-1}$ since $N^{n-1}N^{n-1} \subseteq N^{n+1} = (0)$. Suppose that $N^2 \supseteq A \supseteq N^{n-1}$. Then it is immediate that $N^{n-1}A = AN^{n-1} = (0)$ and therefore $N^{n-1} \subseteq I$. It follows that

$$\dim I \geq \dim N^{n-1} \geq 2,$$

which contradicts the condition (a′), completing the proof.

Proposition 2. *Let A be a non-zero nonassociative algebra over a field Φ satisfying the condition:*

(c′) $\dim A = \dim A^3 + 3$.

Then A cannot be any subalgebra B of a nilpotent algebra N such that

$$N^i \supseteq B \supseteq N^{i+1} \qquad \textit{for } i \geq 2.$$

Proof. Assume that $N^i \supseteq A \supset N^{i+1}$ for $i \geq 2$. Let N be nilpotent of class n. Then $n \geq 2$. We have $i \neq n$, for if not $AA \subseteq N^{2n} = (0)$. Hence $2 \leq i \leq n-1$ and therefore $n \geq 3$. We have $i \neq n-1$, for if not $AA \subseteq N^{2n-2} \subseteq N^{n+1} = (0)$. Hence $2 \leq i \leq n-2$ and therefore $i+3 \leq n+1$. It follows that

$$\dim A \geq \dim N^{i+3} + 3.$$

But $A^3 \subseteq (N^i)^3 \subseteq N^{3i}$ and by the condition (c′) we have

$$\dim N^{3i} \geq \dim A^3 = \dim A - 3.$$

Consequently $\dim N^{3i} \geq \dim N^{i+3}$. It follows that $N^{3i} \supseteq N^{i+3}$. Hence $3i \leq i+3$ and therefore $2i \leq 3$, which is a contradiction. This completes the proof.

Corollary. *Let A be a non-zero nonassociative algebra over a field Φ satisfying one of the conditions* (a′) *and* (c′). *Then A cannot be any N^i, $i \geq 2$ of a nilpotent algebra N.*

We note that the part of Corollary related to (a′) is [6, Proposition 4.9].

§5.

This final section is devoted to discussing the results obtained in the preceding sections and to observing several examples of algebras in regard to the conditions considered there.

Examples of non-commutative nilpotent Lie algebras satisfying the conditions (a), (c) and (d) have been shown in [2]. A simple example of non-commutative nilpotent associative algebras satisfying the conditions (a), (c) and (d) is the algebra of upper nil triangular matrices of degree 3. The condition $N\mathfrak{D}_{n-1}\supseteq B\supseteq N\mathfrak{D}^{n-2}$ on B in Theorem 1 and the condition $N\mathfrak{D}^{i}\supseteq B\supseteq N\mathfrak{D}^{i+1}$ for $i\geq 1$ on B in Theorem 3 cannot be weakened as can be shown by example both for Lie and associative algebras. We shall not write the examples. We only write an example showing that the case $N\mathfrak{D}_2\supset B\supset N\mathfrak{D}_1$ cannot be included in the statement of Corollary 1 to Theorem 1. Let N be a Lie algebra over a field Φ described in terms of a basis $x_1, x_2,\cdots, x_6$ by the following table [5]:

$$[x_1, x_2]=x_5,\quad [x_1, x_5]=x_6,\quad [x_3, x_4]=x_6.$$

In addition, $[x_i, x_j]=-[x_j, x_i]$ and for $i<j$ $[x_i, x_j]=0$ if it is not in the table. Take $\mathfrak{D}=\mathfrak{J}(N)$. Then N is $\mathfrak{D}$-nilpotent of class 3 and $B=(x_3, x_4, x_6)$ is a non-commutative $\mathfrak{D}$-ideal of N satisfying the condition (a) and such that $N\mathfrak{D}_2\supset B\supset N\mathfrak{D}_1$. It is open whether Theorem 4 holds with the condition $N\mathfrak{D}^1\supset B\supset N\mathfrak{D}^2$ for associative algebras as for Lie algebras.

The characteristically nilpotent Lie algebra in [1, p. 123] satisfies all the conditions (a), (b), (c), (d) and (e). The characteristically nilpotent Lie algebra in [3] satisfies the conditions (b) and (e), but not the conditions (a), (c) and (d). In connection with Theorem 6, it should be noted that a characteristically nilpotent associative algebra is not necessarily non-commutative, contrary to Lie case, and that there actually exist characteristically nilpotent non-commutative associative algebras. An example for the first fact is the 1-dimensional algebra spanned by an idempotent element, which is obviously commutative, associative and characteristically nilpotent of class 1. As an example for the second fact, we observe the following 4-dimensional nilpotent associative algebra. Let A be an associative algebra over a field Φ described in terms of a basis x_1, x_2, x_3, x_4 by the following table:

$$x_1x_1=x_4,\quad x_1x_2=x_4,\quad x_2x_2=x_3,$$

$$x_2x_3=x_4,\quad x_3x_2=x_4.$$

In addition, all other products x_ix_j are 0. Then A is not commutative. Let D be a derivation of A and put $x_iD=\sum_j \lambda_{ij}x_j$, $\lambda_{ij}\in\Phi$. Then the matrix corresponding to D is

$$\begin{pmatrix} 0 & 0 & \lambda_{13} & \lambda_{14} \\ -\lambda_{13} & 0 & \lambda_{23} & \lambda_{24} \\ 0 & 0 & 0 & -\lambda_{13}+2\lambda_{23} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence A is characteristically nilpotent of class 4.

Let A be a nonassociative algebra over a field Φ with a basis $x_1, x_2, \cdots, x_n$ ($n \geq 3$) such that $x_1 x_i = x_{i+1}$ for $i = 2, 3, \cdots, n-1$, and all other products are zero. Then A satisfies the conditions (a$'$) and (c$'$). When we consider the two characteristically nilpotent Lie algebras quoted above as nonassociative algebras, the conditions (a$'$) and (c$'$) are satisfied by the former, but not by the latter.

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