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Note on the Canonical Extensions and the Boundary Values for Distributions in the Space H^{μ}

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In our previous paper [3], the multiplicative product between distributions was investigated together with the related topics. With the viewpoints mentioned there in mind, in this paper we shall study the problems centering around the notions of the trace, the section, the boundary value and the canonical extension, for distributions especially in H^{μ}. The present paper is in a sense a continuation of our related paper [2].

The general discussions about these notions are made in Section 1 with reference especially to the canonical extension. Sections 2 and 3 are devoted to the discussions about the trace mapping and the canonical extension for distributions in the space H^{μ} and we have tried to make clear the close relationship between them. Some complements to our previous paper [2] are given with new results. In the final section, the notions of $\mathscr{S}'$-boundary value and $\mathscr{S}'$-canonical extension are introduced and discussed. We can speak of $\mathscr{D}'_{L^2}$-boundary value and $\mathscr{D}'_{L^2}$-canonical extension and so on. However, we do not proceed to the study about these matters, because the treatment involves no essential difficulty, of course, though it is necessary to introduce modifications into our considerations given in this section in order to obtain the analogues.

1. Preliminaries

We first recall some notions concerning multiplicative product (or simply product) between distributions closely connected with the discussions in the subsequent sections. Let $u, v \in \mathscr{D}'(R_N)$, where R_N is an N-dimensional Euclidean space. If the distributional limit $\lim_{j\to\infty}(u*\rho_j)v$ exists for any δ-sequence $\{\rho_j\}$, the limit is uniquely determined, which is called the product in the strict sense and denoted by $u\cdot v$. We have shown in [9, p. 225] that if the limit exists, then $\lim_{j\to\infty} u(v*\rho_j)$ exists and

$$(1) \qquad \lim_{j\to\infty}(u*\rho_j)v=\lim_{j\to\infty}u(v*\rho_j)$$

for any δ-sequence $\{\rho_j\}$. The above definition with $\{\rho_j\}$ replaced by $\{\phi_\varepsilon\}$, $\varepsilon>0$, where $\phi \in \mathcal{D}(R_N)$, $\phi \geqq 0$, $\int \phi(x)dx = 1$ and $\phi_\varepsilon(x) = \frac{1}{\varepsilon^N}\phi\left(\frac{x}{\varepsilon}\right)$, yields the product in the weak sense denoted by uv [3, p. 161]. This product determines a multiplication called normal in the sense descrived in [3]. R. Shiraishi has shown that the product in the weak sense is also obtained when δ-sequences are confined to the restricted δ-sequences in his sense [8, p. 95], and that (1) holds also for the product in the weak sense. For the one-dimensional distributions, when ρ_j are subjected to the condition $\operatorname{supp}\rho_j \subset (0, \infty)$ in either cases, strict or weak, it can be shown along the line in his proof [8, p. 95] that if $\lim_{j\to\infty}(u*\rho_j)v$ is defined for any $\{\rho_j\}$, then $\lim_{j\to\infty} u(v*\check{\rho}_j)$ exists and

$$\lim_{j\to\infty}(u*\rho_j)v = \lim_{j\to\infty} u(v*\check{\rho}_j). \tag{2}$$

This is certainly the case if we replace $\{\rho_j\}$ by $\{\phi_\varepsilon\}$ with $\operatorname{supp}\phi \subset (0, \infty)$; if $\lim_{\varepsilon\downarrow 0}(u*\phi_\varepsilon)v$ is defined, then $\lim_{\varepsilon\downarrow 0} u(v*\check{\phi}_\varepsilon)$ exists and

$$\lim_{\varepsilon\downarrow 0}(u*\phi_\varepsilon)v = \lim_{\varepsilon\downarrow 0} u(v*\check{\phi}_\varepsilon). \tag{3}$$

For the sake of simplicity, we shall discuss in the rest of this section the related notions considered only in the weak sense.

Let $R_N = R_{n+1}$ and denote its points by (t, x), $x = (x_1, \cdots, x_n)$. Let Y and δ be respectively the Heaviside function and the Dirac measure in R. The partial product between distributions $w \in \mathcal{D}'(R)$ and $u \in \mathcal{D}'(R_{n+1})$ was considered in [3, p. 170] and wu means $(w\otimes 1)u$ when and only when the latter is defined. For instance, δu is defined as the unique limit $\lim_{j\to\infty}(\delta*\rho_j)u$ or equivalently $\lim_{j\to\infty}\delta(u*_t\rho_j)$ if it exists, where $\{\rho_j\}$ is an arbitrary restricted δ-sequence in R and the notation $*_t$ means the partial convolution with respect to the variable t. Here, of course, $\{\rho_j\}$ may be replaced by $\{\phi_\varepsilon(t)\}$ with the same meaning as given before. In accordance with S. Łojasiewicz [6, p. 15] $u \in \mathcal{D}'(R_{n+1})$ has a section $\in \mathcal{D}'(R_n)$ for $t=0$ if $\lim_{\varepsilon\downarrow 0} u(\varepsilon t, x)$ exists and is not depend on t. It will be equivalent to saying that the product δu exists. In fact, from its very definition, u has a section for $t=0$ if and only if $\lim_{\varepsilon\downarrow 0}\langle u, \phi_\varepsilon\rangle$ exists in $\mathcal{D}'(R_n)$ for any $\phi \in \mathcal{D}(R)$ with $\phi(t) \geqq 0$, $\int \phi(t)dt = 1$ and, furthermore, is not depend on ϕ. Since $\delta\otimes\langle u, \phi_\varepsilon\rangle = \delta(u*\check{\phi}_\varepsilon)$, the condition is, in turn, equivalent to the existence of the product δu. Then the section α is $\lim_{\varepsilon\downarrow 0}\langle u, \phi_\varepsilon\rangle$ and therefore defined also by the equation $\lim_{\varepsilon\downarrow 0} u(\varepsilon t, x) = 1_t\otimes\alpha$ or $\delta u = \delta\otimes\alpha$.

Let $R_{n+1}^+ = \{(t, x) \in R_{n+1} : t > 0\}$ and denote by $\mathcal{D}'(R_{n+1}^+)$ the space of distributions on R_{n+1}^+ and by $\mathcal{D}'(\bar{R}_{n+1}^+)$ the space of distributions $\mathcal{D}'(R_{n+1}^+)$ which can be extended to distributions $\in \mathcal{D}'(R_{n+1})$. The space $\mathcal{D}'(\bar{R}_{n+1}^+)$ is provided with the quotient topology as usual and also defined as the strong dual of $\mathring{\mathcal{D}}(\bar{R}^+_{n+1})$. In our previous paper [5], we have considered the canonical extensions and the distributional boundary values for distributions u in $\mathcal{D}'(R^+_{n+1})$. Let $\chi \in \mathcal{D}(R^+)$ be arbitrarily chosen so that $\chi(t) \geqq 0$ and $\int \chi(t)\, dt = 1$. Put $\rho = Y*\chi$ and $\rho_{(\varepsilon)}(t) = \rho\left(\frac{t}{\varepsilon}\right)$. If $\lim_{\varepsilon \downarrow 0} \rho_{(\varepsilon)} u$ exists in $\mathcal{D}'(R_{n+1})$, the limit, denoted by $u_\sim$, is called canonical extension of u over $t=0$. And if the limit $\lim_{\varepsilon \downarrow 0} < u, \chi_\varepsilon >$ exists in $\mathcal{D}'(R_n)$, the limit, denoted by $\lim_{t \downarrow 0} u$, is called the boundary value of u. If $\lim_{t \downarrow 0} u$ exists, u has the canonical extension $u_\sim$ [5, p. 12]. Let $u = \frac{\partial v}{\partial t}$ with $v \in \mathcal{D}'(R^+_{n+1})$. Then u has the canonical extension if and only if $\lim_{t \downarrow 0} v$ exists [5, p. 14]. For an n-tuple $(\alpha_1, \cdots, \alpha_n)$ of non-negative integers, the sum $\sum_{j=1}^{n} \alpha_j$ will be denoted by $|\alpha|$. With $D = (D_t, D_x) = (D_t, D_1, \cdots, D_n)$, $D_t = \frac{1}{i} \frac{\partial}{\partial t}$, $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$, we put $D^\alpha_x = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ and $\mathbf{D}^m = D_1^m \cdots D_n^m$ for an integer $m \geqq 0$. In a local representation, $\lim_{t \downarrow 0} u = \alpha$ means that for any bounded open non-empty subset $(-a, a) \times G \subset R_{n+1}$, there exists a continuous function F on R_{n+1} with support in $\bar{R}^+_{n+1}$ such that for some positive integer k we can write

$$(4) \qquad u_\sim = Y \otimes \alpha + D_t^k \mathbf{D}_x^k F \qquad \text{on } (-\infty, a) \times G,$$

where $F = o(|t|^k)$ uniformly as $t \to 0$ [4, p. 405]. Let $u \in \mathcal{D}'(R_{n+1})$. If the restriction $u \,|\, R^+_{n+1}$ has the canonical extension (resp. the boundary value), we shall also call it in this paper the canonical extension (resp. the boundary value) of u over $t=0$ and denote it by u_+ (resp. $\lim_{t \downarrow 0} u$). If $u_+ = u$, u is called to be canonical. The same discussions are applied to the lower half space R^-_{n+1}. The notations, such as u_- and $\lim_{t \uparrow 0} u$, will then have an obvious meaning.

According to S. Łojasiewicz [6, p. 23] we shall say that $u \in \mathcal{D}'(R_{n+1})$ has no mass on the hyperplane $t=0$ if $\lim_{\varepsilon \downarrow 0} \varepsilon u(\varepsilon t, x) = 0$. One can immediately verify [6, p. 23] that, if u has the section for $t=0$, u and $D_t u$ have no mass on $t=0$. From the local representation (4) we see that u_+ has no mass on $t=0$.

Proposition 1. *Let $u \in \mathcal{D}'(R_{n+1})$ have the same boundary value α from both sides of $t=0$, that is, $\lim_{t \downarrow 0} u = \lim_{t \uparrow 0} u = \alpha$. If u has no mass on $t=0$, then u has the section α for $t=0$.*

Proof. Let $(-a, a) \times G \subset R_{n+1}$ be any bounded open subset. We can choose k sufficiently large so that we can write

$$u_+ = Y\otimes\alpha + D_t^k\mathbf{D}_x^k F_+,\ u_- = (1-Y)\otimes\alpha + D_t^k\mathbf{D}_x^k F_- \qquad \text{on } (-a, a)\times G,$$

where $F = o(|t|^k)$ uniformly as $t\to 0$ and therefore

$$u_+ + u_- = 1_t\otimes a + D_t^k\mathbf{D}_x^k F,$$

which means that $u_+ + u_-$ has the section α for $t=0$. Consider the distribution $u - u_+ - u_-$. It has no mass on $t=0$ since by hypothesis u does so, and, furthermore, its support lies on $t=0$. Consequently we see that $u = u_+ + u_-$.

PROPOSITION 2. *Suppose the canonical extension u_+ exits. Then the product Yu exists if and only if u has no mass on $t=0$.*

PROOF. Suppose the product Yu exists. Let $v\in\mathcal{D}'(R_{n+1})$ be chosen such that $u=\dfrac{\partial v}{\partial t}$. Since $\dfrac{\partial Y}{\partial x_j}=0$ for $j=1, 2, \cdots, n$, the product δv exists [3, p. 168], and therefore v has the section for $t=0$, and, in turn, u has no mass on $t=0$.

Conversely, suppose u has no mass on $t=0$. For any ϕ, $\psi\in\mathcal{D}(R)$ such that $\phi(t)\geqq 0$, $\psi(t)\geqq 0$, $\int\phi(t)dt=\int\psi(t)dt=1$, and $\operatorname{supp}\phi\subset(0, \infty)$, we put $\rho = Y*\phi$, $\sigma = Y*\psi$ and $\chi=\rho-\sigma$. Then $\chi\in\mathcal{D}(R)$ and $\sigma_{(\varepsilon)}u=\rho_{(\varepsilon)}u-\chi_{(\varepsilon)}u$, where $\rho_{(\varepsilon)}u$ and $\chi_{(\varepsilon)}u$ converge in $\mathcal{D}'(R_{n+1})$ to u_+ and 0 respectively as $\varepsilon\downarrow 0$, and therefore $\sigma_{(\varepsilon)}u$ converges in $\mathcal{D}'(R_{n+1})$ to u_+. Thus we see that the product Yu exists and equals u_+.

PROPOSITION 3. *Let $u\in\mathring{\mathcal{D}}'(\bar{R}_{n+1}^+)$. Then the product Yu exists (or u is canonical) if and only if u has no mass on $t=0$.*

PROOF. Suppose the product Yu exists or u is canonical. Then u_+ exists. Owing to the preceding proposition, we see that u has no mass.

Conversely, suppose that u has no mass on $t=0$. Since $u=0$ for $t<0$, the canonical extension u_- exists. From Proposition 2 we see that the product $(1-Y)u$ exists and therefore Yu exists and equals u_+. Since $u-u_+$ has no mass and its support lies on $t=0$, we can conclude that $u=u_+$.

Finally we note that if $\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}u$ (resp. $\lim_{\varepsilon\downarrow 0}\chi_\varepsilon u$) exists for any $\chi\in\mathcal{D}(R^+)$ such that $\chi(t)\geqq 0$ and $\int\chi(t)dt=1$, where $\rho=Y*\chi$, then $\lim_{\varepsilon\downarrow 0} Y(u*_t\check{\chi}_\varepsilon)$ (resp. $\lim_{\varepsilon\downarrow 0}\delta(u*_t\check{\chi}_\varepsilon)$) exists and

$$\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}u=\lim_{\varepsilon\downarrow 0} Y(u*_t\check{\chi}_\varepsilon)\ (\text{resp. } \lim_{\varepsilon\downarrow 0}\chi_\varepsilon u=\lim_{\varepsilon\downarrow 0}\delta(u*_t\check{\chi}_\varepsilon)).$$

2. The trace and the section for distributions in the space H^{μ}

Let Ξ_N be the dual space of R_N. For any $x \in R_N$ and $\xi \in \Xi_N$, the scalar product $<x, \xi>$ is defined by $<x, \xi>=\sum_{j=1}^{N} x_j\xi_j$ and we put $|\xi|=(\sum_{j=1}^{N}|\xi_j|^2)^{\frac{1}{2}}$. For any $\phi \in \mathcal{S}$, its Fourier transform $\hat{\phi}$ is defined by the formula

$$\hat{\phi}(\xi)=\int_{R_N}\phi(x)e^{-i<x,\xi>}dx,$$

and for any $u \in \mathcal{S}'$, $\hat{u}$ is defined by

$$<\hat{u}, \phi>=<u, \hat{\phi}> \qquad \text{for any } \phi \in \mathcal{S}.$$

A positive-valued continuous function $\mu(\xi)$ defined on Ξ_N is called a temperate weight function if there exist positive constants C and k such that

$$\mu(\xi+\eta)\leqq C(1+|\xi|)^k\mu(\eta) \qquad \text{for any } \xi, \eta \in \Xi_N.$$

$\mu_1+\mu_2$, $\mu_1\mu_2$ and $\frac{1}{\mu_1}$ are temperate weight functions with μ_1 and μ_2.

Consider the space $H^{\mu}(R_N)$ (or simply H^{μ}) of $u \in \mathcal{S}'(R_N)$ such that $\hat{u}$ is a function satisfying

$$\|u\|_{\mu}^2=\int_{\Xi_N}|\hat{u}(\xi)|^2\mu^2(\xi)d\!\!\!{}^-\xi<+\infty, \qquad d\!\!\!{}^-\xi=\left(\frac{1}{2\pi}\right)^N d\xi.$$

$H^{\mu}(R_N)$ is a Hilbert space with the inner product

$$(u\,|\,v)=\int\hat{u}(\xi)\overline{\hat{v}(\xi)}\,\mu^2(\xi)\,d\!\!\!{}^-\xi.$$

Its strong dual space is $H^{\frac{1}{\mu}}(R_N)$ where we have

$$(w, u)=<w, \bar{u}>=\int\hat{w}(\xi)\bar{\hat{u}}(\xi)d\!\!\!{}^-\xi \qquad \text{for any } w \in H^{\frac{1}{\mu}}(R_N) \text{ and } u \in H^{\mu}(R_N).$$

Let $N=n+1$ and denote the points of Ξ_{n+1} by (τ, ξ), $\xi=(\xi_1, \cdots, \xi_n)$. For a polynomial $P(\tau, \xi)$ in (τ, ξ), we put $P(D)=P(D_t, D_x)$ and $\tilde{P}(\tau, \xi)=(\sum_{k+|\alpha|\geq 0}|D_{\tau}^k D_{\xi}^{\alpha}P(\tau, \xi)|^2)^{\frac{1}{2}}$.

Let us recall the notion of a trace mapping. For any $u(t, x) \in \mathcal{D}(R_{n+1})$, the trace $u(0, x)$ on R_n clearly belongs to $\mathcal{D}(R_n)$. $\mathcal{D}(R_{n+1})$ is dense in $H^{\mu}(R_{n+1})$. If the mapping $u\rightarrow u(0, x)$ can be continuously extended from $H^{\mu}(R_{n+1})$ into $\mathcal{D}'(R_n)$, then the extended mapping is called a trace mapping on the hyperplane and the image of $u \in H^{\frac{1}{\mu}}(R_{n+1})$ by this mapping is called the trace of u on $t=0$ [2, p. 13]. This means that $\delta \otimes \phi \in H^{\frac{1}{\mu}}(R_{n+1})$ for any $\phi \in \mathcal{D}(R_n)$. It is known [12, p. 36; 2, p. 14] that the trace mapping for $H^{\mu}(R_{n+1})$ is defined if and only if $\int \frac{1}{\mu^2(\tau, 0)} d\tau < +\infty$, that is, $\int \frac{1}{\mu^2(\tau, \xi)} d\tau < +\infty$ for every $\xi \in \Xi_n$. We can similarly consider the trace mapping on $t=t_0$ and the trace of u on $t=t_0$. We denote it by $u(t_0, \cdot)$ (or $\mathbf{u}(t_0)$).

Suppose $\int \frac{1}{\mu^2(\tau, 0)} d\tau < +\infty$ and put $\frac{1}{\nu^2(\xi)} = \int \frac{1}{\mu^2(\tau, \xi)} d\tau$. Consider the map: $\mathcal{D}(R_{n+1}) \ni u(t, x) \to \tau_{-t_0} u \in \mathcal{D}(R_{n+1})$ for any $t=t_0$. Since $(\tau_{-t_0} u)^{\wedge} = e^{it_0\tau} \hat{u}$ and $\|u(t_0, \cdot)\|_{\nu} \leqq \|u\|_{\mu}$ [2, p. 15], the mapping is unitary. Thus the trace $u(t_0, \cdot)$ for any $u \in H^{\mu}(R_{n+1})$ belongs to the space $H^{\nu}(R_n)$. For any $u \in H^{\mu}(R_{n+1})$ there exists a sequence $\{u_j\}$, $u_j \in \mathcal{D}(R_{n+1})$ such that $u = \lim_{j\to\infty} u_j$ in $H^{\mu}(R_{n+1})$. From the inequality

$$\|u_j(t_0, \cdot) - u(t_0, \cdot)\|_{\nu} \leqq \|u_j - u\|_{\mu}$$

we see that $u_j(t_0, \cdot)$ converges in $H^{\nu}(R_n)$ to $u(t_0, \cdot)$ uniformly with respect to t_0. Since $t \to u_j(t, \cdot)$ are $H^{\nu}(R_n)$-valued continuous functions, $u(t, \cdot)$ may be considered as an $H^{\nu}(R_n)$-valued continuous function of t. Then we have the following

PROPOSITION 4. *Suppose* $\int \frac{1}{\mu^2(\tau, 0)} d\tau < +\infty$. *Then every* $u \in H^{\mu}(R_{n+1})$ *is identified as a distribution with the* $H^{\nu}(R_n)$*-valued continuous function* $\mathbf{u}(t)$ *(that is, an element of* $C(H^{\nu})$*), where* $\frac{1}{\nu^2(\xi)} = \int \frac{1}{\mu^2(\tau, \xi)} d\tau$.

PROOF. For any $\phi \in \mathcal{D}(R_{n+1})$ we put

$$<\mathbf{u}, \phi> = \int <\mathbf{u}(t), \phi(t, \cdot)> dt.$$

If we take $\mathbf{u}(t) = u(t, x) \in \mathcal{D}(R_{n+1})$, then we have $<\mathbf{u}, \phi> = <u, \phi>_{\mathcal{D}', \mathcal{D}}$. Since the space $\mathcal{D}(R_{n+1})$ is dense in $H^{\mu}(R_{n+1})$ we see that this relation holds true of any $u \in H^{\mu}(R_{n+1})$, completing the proof.

From the discussions given in the preceding section, we can show the following

THEOREM 1. *For the space* $H^{\mu}(R_{n+1})$ *the following statements are equivalent*:

(a) *The trace mapping is defined.*
(b) *The section exists for every* $u \in H^{\mu}$.
(b)′ *The condition* (b) *holds in the strict sense.*
(c) *The product* δu *exists for every* $u \in H^{\mu}$.

(c)′ *The condition* (c) *holds in the strict sense.*

(d) *The distributional limit* $\lim_{j\to\infty}(u*\rho_j)\delta$ *exists for a fixed* δ*-sequence* $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R_{n+1})$.

(e) *The distributional limit* $\lim_{j\to\infty}\rho_j u$ *exists for a fixed* δ*-sequence* $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R)$.

(f) *The boundary value* $\lim_{t\downarrow 0} u$ *exists for every* $u \in H^\mu$.

(f)′ *The condition* (f) *holds in the strict sense.*

(g) *The distributional limit* $\lim_{j\to\infty}\rho_j u$ *exists for a fixed* δ*-sequence* $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R)$ *with support* $\subset(0, \infty)$.

Proof. The equivalence of (b) and (c) (resp. (b)′ and (c)′) is shown in Section 1. Since the implications (c)′⇒(c), (d), (e) are trivial from the definition of the product between distributions, if we can show the implications (a)⇒(c)′, (d)⇒(a) and (e)⇒(a), then we see that the statements (a), (b)′, (c)′, (d) and (e) are equivalent and we can therefore conclude that the statements (a) through (g) are equivalent to each other.

(a)⇒(c)′. Suppose (a) holds. Then $\delta\otimes\psi \in H^{\frac{1}{\mu}}$ for any $\psi \in \mathcal{D}(R_n)$. Let $u \in H^\mu$ and let $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R_{n+1})$ be any δ-sequence. Then $u*\rho_j$ converges in H^μ to u. From the equation $<(u*\rho_j)\delta, \phi>_{\mathcal{D}',\mathcal{D}} = <u*\rho_j, \delta\otimes\phi(0, x)>_{H^\mu, H^{\frac{1}{\mu}}}$, $\phi \in \mathcal{D}(R_{n+1})$, we see that the distributional limit $\lim_{j\to\infty}(u*\rho_j)\delta$ exists for any $\{\rho_j\}$, that is, the product $\delta\cdot u$ exists.

(d)⇒(a). Suppose (d) holds for a fixed δ-sequence $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R_{n+1})$. Then the map $H^\mu \ni u \to (u*\rho_j)\delta = \delta\otimes(u*\rho_j)(0, x) \in \mathcal{D}'(R_{n+1})$ is continuous for each j. Since the space H^μ is barrelled, so the map $H^\mu \ni u \to \lim_{j\to\infty}(u*\rho_j)\delta \in \mathcal{D}'(R_{n+1})$ is continuous. For any $\phi \in \mathcal{D}(R_{n+1})$ there exists $w_\phi \in H^{\frac{1}{\mu}}$ such that

$$<\lim_{j\to\infty}(u*\rho_j)\delta, \phi>_{\mathcal{D}',\mathcal{D}} = <u, w_\phi>_{H^\mu, H^{\frac{1}{\mu}}}.$$

If we take $u=\alpha \in \mathcal{D}(R_{n+1})$, then $<\delta\alpha, \phi>_{\mathcal{D}',\mathcal{D}} = <\alpha, \delta\otimes\phi(0, x)> = <\alpha, w_\phi>$. $\mathcal{D}(R_{n+1})$ is dense in H^μ. Thus $\delta\otimes\phi(0, x) = w_\phi \in H^{\frac{1}{\mu}}$.

(e)⇒(a). Suppose (e) holds for a fixed δ-sequence $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R)$. The map $H^\mu \ni u \to \rho_j u \in \mathcal{D}'(R_{n+1})$ is continuous. From a theorem of Banach-Steinhaus the map $H^\mu \ni u \to \lim_{j\to\infty}\rho_j u \in \mathcal{D}'(R_{n+1})$ is continuous. Since $\lim_{j\to\infty}\rho_j u = \delta\otimes u(0, x)$ for any $u \in \mathcal{D}(R_{n+1})$, we see that the trace mapping is defined.

Thus we complete the proof of the equivalence of the statements (a) through (g).

Remark 1. Suppose $\lim_{t\downarrow 0} u = \alpha$ exists for every $u \in H^\mu$. Then for any δ-

sequence $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R)$, $\lim_{j\to\infty}\rho_j u$ converges in $H^{(1+|\tau|)^{-\sigma}\nu}$, $\sigma>1$ to $\delta\otimes\alpha$, where $\frac{1}{\nu^2(\xi)}=\int\frac{1}{\mu^2(\tau,\xi)}d\tau$. In fact, u may be considered as an $H^{\nu}(R_n)$-valued continuous function $\mathbf{u}(t)$. For any $\psi \in \mathcal{D}(R_{n+1})$ we have the estimates with a constant C

$$\begin{aligned}|(\rho_j u, \psi)| &= \left|\int_{-\infty}^{\infty}(\rho_j(t)\mathbf{u}(t), \psi(t, \cdot))dt\right| \\ &\leqq \int_{-\infty}^{\infty}\rho_j(t)\|\mathbf{u}(t)\|_{\nu}\|\psi(t, \cdot)\|_{\frac{1}{\nu}}dt \\ &\leqq \max\|\mathbf{u}(t)\|_{\nu}\max\|\psi(t, \cdot)\|_{\frac{1}{\nu}} \\ &\leqq C\|u\|_{\mu}\|\psi\|_{(1+|\tau|)^{\sigma}\frac{1}{\nu}},\end{aligned}$$

which means $\|\rho_j u\|_{(1+|\tau|)^{-\sigma}\nu}\leqq C\|u\|_{\mu}$. Consider u which can be written in the form $u=\phi_1(t)\otimes\phi_2(x)$ with $\phi_1 \in \mathcal{D}(R)$ and $\phi_2 \in \mathcal{D}(R_n)$. Observe that the set $\{\phi_1(t)\otimes\phi_2(x)\}$ is total in H^{μ}. Since $\|\rho_j(\phi_1\otimes\phi_2)-\phi_1(0)\delta\otimes\phi_2\|_{(1+|\tau|)^{-\sigma}\nu}=\|\rho_j\phi_1-\phi_1(0)\delta\|_{(1+|\tau|)^{-\sigma}}\|\phi_2\|_{\nu}$ and $\rho_j\phi_1$ converges in $H^{(1+|\tau|)^{-\sigma}}(R)$ to $\phi_1(0)\delta$, it follows from the Banach-Steinhaus theorem that $\lim_{j\to\infty}\rho_j u$ converges in $H^{(1+|\tau|)^{-\sigma}\nu}$ to $\delta\otimes a$ for every $u \in H^{\mu}$.

Now let $P(D)$ be a differential polynomial such that $P(\tau, \xi)=\sum_{j=0}^{m}\tau^j\gamma_j(\xi)$, $m\geqq 0$ and $\gamma_m(\xi)\neq 0$. We have shown in [2, p. 14] that the trace $(P(D)u)(0, \cdot)$ exists for every $u \in H^{\mu}$ if and only if

$$\int\frac{\tilde{P}^2(\tau, \xi)}{\mu^2(\tau, \xi)}d\tau<+\infty \qquad \text{for some } \xi \in \Xi_n \tag{5}$$

or equivalently

$$\int\frac{|P(\tau, \xi)|^2}{\mu^2(\tau, \xi)}d\tau<+\infty \qquad \text{for every } \xi \in \Xi_n. \tag{6}$$

If $\lim_{t\downarrow 0} P(D)u$ exists for every $u \in H^{\mu}$, then in the same way as in the proof of the implication (e)⇒(a) in Theorem 1 we see that the trace $(P(D)u)(0, \cdot)$ exists for every $u \in H^{\mu}$. We note that u, $D_t u$, $\cdots$, $D_t^m u$ have the sections for $t=0$ if and only if $\int\frac{\tau^{2m}}{\mu^2(\tau, 0)}d\tau<+\infty$.

COROLLARY 1. *$\lim_{t\downarrow 0}P(D)u$ exists for every $u \in H^{\mu}$ if and only if $\int\frac{\tau^{2m}}{\mu^2(\tau, 0)}d\tau<+\infty$, that is, $\lim_{t\downarrow 0}u$, $\lim_{t\downarrow 0}D_t u$, $\cdots$, $\lim_{t\downarrow 0}D_t^m u$ exist for every $u \in H^{\mu}$.*

PROOF. Suppose $\lim_{t\downarrow 0}P(D)u$ exists for every $u \in H^{\mu}$. Then the trace

$(P(D)u)(0, \cdot)$ exists. Taking $\xi=\xi_0$ such that $\gamma_m(\xi_0)\neq 0$, we see from (6) that $\int\frac{\tau^{2m}}{\mu^2(\tau, 0)}d\tau<+\infty$.

Conversely, suppose $\int\frac{\tau^{2m}}{\mu^2(\tau, 0)}d\tau<+\infty$. Then u, $D_t u$, $\cdots$, $D_t^m u$ have the traces on $t=0$ for every $u \in H^{\mu}$ and therefore have the boundary values. Thus $\lim_{t\downarrow 0} P(D)u$ exists.

3. The canonical extension for distributions in the space H^{μ}

PROPOSITION 5. *The canonical extension u_+ exists for every $u \in H^{\mu}(R_{n+1})$ if and only if* $\int\frac{1}{(1+\tau^2)\mu^2(\tau, 0)}d\tau<+\infty$.

PROOF. Let v be such that $u=D_t v-iv$. Then u_+ exists if and only if $\lim_{t\downarrow 0} v$ exists. Since the mapping $v\to(D_t-i)v$ is an isomorphism of $H^{(1+\tau^2)^{\frac{1}{2}}\mu}$ onto H^{μ}, Proposition 5 follows from Theorem 1.

COROLLARY 2. *Let $P(D)$ be a differential polynomial considered in Corollary 1. $(P(D)u)_+$ exists for every $u \in H^{\mu}(R_{n+1})$ if and only if* $\int\frac{\tau^{2m}}{(1+\tau^2)\mu^2(\tau, 0)}d\tau<+\infty$, *that is, u_+, $(D_t u)_+$, $\cdots$, $(D_t^m u)_+$ exist for every $u \in H^{\mu}$.*

PROOF. Suppose $(P(D)u)_+$ exists for every $u \in H^{\mu}$. As in the proof of Proposition 5 $\lim_{t\downarrow 0} P(D)v$ exists for every $v \in H^{(1+\tau^2)^{\frac{1}{2}}\mu}$ and therefore $\int\frac{\tau^{2m}}{(1+\tau^2)\mu^2(\tau, 0)}d\tau<+\infty$.

The converse follows from Corollary 1 and Proposition 5 since $\lim_{t\downarrow 0} u$, $\lim_{t\downarrow 0} D_t u$, $\cdots$, $\lim_{t\downarrow 0} D_t^{m-1}u$($u_+$ if $m=0$) and, a fortiori, u_+, $(D_t u)_+$, $\cdots$, $(D_t^m u)_+$ exist, which implies the existence of $(P(D)u)_+$ for every $u \in H^{\mu}$.

If the map $\tilde{Y}\colon \mathcal{D}(R_{n+1}) \ni u\to u_+ \in \mathcal{D}'(R_{n+1})$ can be continuously extended from $H^{\mu}(R_{n+1})$ into $\mathcal{D}'(R_{n+1})$, then we shall denote this extended mapping by the same symbol $\tilde{Y}$. We note that $\phi_+ \in H^{\frac{1}{\mu}}$ for any $\phi \in \mathcal{D}(R_{n+1})$ if $\tilde{Y}$ is defined, since the map $\mathcal{D}(R_{n+1}) \ni \psi\to(\psi, \phi_+)=(\psi_+, \phi)$ can be extended to a continuous linear form on H^{μ}.

We shall show the following theorem which is an analogue of Theorem 1.

THEOREM 2. *For the space $H^{\mu}(R_{n+1})$ the following statements are equivalent:*

(a) *The map $\tilde{Y}$ is defined.*

(b) *The product* Yu *exists for every* $u \in H^{\mu}$.

(b)′ *The condition* (b) *holds in the strict sense.*

(c) *The distributional limit* $\lim_{j\to\infty}(u*\rho_j)Y$ *exists for a fixed* δ*-sequence* $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R_{n+1})$.

(d) *The distributional limit* $\lim_{j\to\infty}(Y*\rho_j)u$ *exists for a fixed* δ*-sequence* $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R)$.

(e) *The canonical extension* u_+ *exists for every* $u \in H^{\mu}$.

(e)′ *The condition* (e) *holds in the strict sense.*

(f) *The distributional limit* $\lim_{j\to\infty}(Y*\rho_j)u$ *exists for a fixed* δ*-sequence* $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R)$ *with support* $\subset (0, \infty)$.

PROOF. Since the implications (b)′⇒(b), (c) and (d) are trivial, if we can show the implications (a)⇒(b)′, (c)⇒(a) and (d)⇒(a), then we see that the statements (a), (b)′, (c) and (d) are equivalent and we can therefore conclude that the statements (a) through (f) are equivalent to each other.

(a)⇒(b)′. Suppose (a) holds. Then $\phi Y = \phi_+ \in H^{\frac{1}{\mu}}$ for any $\phi \in \mathcal{D}(R_{n+1})$. Let $u \in H^{\mu}$ and $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R_{n+1})$ be any δ-sequence. Then $u*\rho_j$ converges in H^{μ} to u. From the equation $<(u*\rho_j)Y, \phi>_{\mathcal{D}',\mathcal{D}} = <u*\rho_j, \phi_+>_{H^{\mu},H^{\frac{1}{\mu}}}$, we see that the distributional limit $\lim_{j\to\infty}(u*\rho_j)Y$ exists for any $\{\rho_j\}$, that is, the product $Y\cdot u$ exists.

(c)⇒(a). Suppose (c) holds for a fixed δ-sequence $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R_{n+1})$. Then the map $H^{\mu} \ni u \to (u*\rho_j)Y \in \mathcal{D}'(R_{n+1})$ is continuous and the Banach-Steinhaus theorem implies that the map $H^{\mu} \ni u \to \lim_{j\to\infty}(u*\rho_j)Y$ is continuous. For any $\phi \in \mathcal{D}(R_{n+1})$ there exists $w_\phi \in H^{\frac{1}{\mu}}$ such that $<\lim_{j\to\infty}(u*\rho_j)Y, \phi>_{\mathcal{D}',\mathcal{D}} = <u, w_\phi>_{H^{\mu},H^{\frac{1}{\mu}}}$. If we take $u=\alpha \in \mathcal{D}(R_{n+1})$, then $<\alpha Y, \phi> = <\alpha, \phi_+> = <\alpha, w_\phi>$. Thus we have $\phi_+ = w_\phi \in H^{\frac{1}{\mu}}$.

(d)⇒(a). In the same way as in the proof of the implication (c)⇒(a) we can prove that map $H^{\mu} \ni u \to \lim_{j\to\infty} u(Y*\rho_j) \in \mathcal{D}'(R_{n+1})$ is continuous, where $\{\rho_j\}$ is a fixed δ-sequence with $\rho_j \in \mathcal{D}(R)$ and we can write $\lim_{j\to\infty} u(Y*\rho_j) = u_+$ for any $u \in \mathcal{D}(R_{n+1})$. This means that $\tilde{Y}$ is defined.

Thus the proof is complete.

REMARK 2. Suppose u_+ exists for every $u \in H^{\mu}$. Then for any δ-sequence $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R)$, $(Y*\rho_j)u$ converges in $\mathcal{D}'_{L^2}$ to u_+. In fact, let $v \in H^{(1+\tau^2)^{\frac{1}{2}}\mu}$ be such that $D_t v - iv = u$ and put $\mu_1 = (1+\tau^2)^{\frac{1}{2}}\mu$ and $\frac{1}{\nu_1^2(\xi)} = \int \frac{1}{\mu_1^2(\tau, \xi)}\, d\tau$. Then $(Y*\rho_j)u = (D_t - i)((Y*\rho_j)v) + i\rho_j v$ and $\rho_j v$ converges in $H^{(1+\tau^2)^{-\frac{1}{2}}\nu_1}$ to $\delta \otimes (\lim_{t\downarrow 0} v)$.

Given $\varepsilon>0$, there exists a j_0 such that $Y*(\rho_j-\rho_{j'})$ vanishes in $|t|\geqq\varepsilon$ for every j, j' with $j, j'\geqq j_0$. For any $\psi\in\mathcal{D}(R_{n+1})$ there exists a constant C such that for every $j, j'\geqq j_0$

$$
\begin{aligned}
&|((Y*\rho_j)v-(Y*\rho_{j'})v, \psi)| \\
&\leqq\int_{-\varepsilon}^{\varepsilon}|Y*(\rho_j-\rho_{j'})|\,\|\mathbf{v}(t)\|_{\nu_1}\|\psi(t, \cdot)\|_{\frac{1}{\nu_1}}dt \\
&\leqq C\varepsilon^{\frac{1}{2}}\max\|\mathbf{v}(t)\|_{\nu_1}\left\{\int_{-\infty}^{\infty}\|\psi(t, \cdot)\|_{\frac{1}{\nu_1}}^2 dt\right\} \\
&\leqq C\varepsilon^{\frac{1}{2}}\|v\|_{\mu_1}\|\psi\|_{\frac{1}{\nu_1}}.
\end{aligned}
$$

Consequently we can confer that $(Y*\rho_j)v-(Y*\rho_1)v$ is a Cauchy sequence in H^{ν_1}, and therefore $(D_t-i)((Y*\rho_j)v-(Y*\rho_1)v)$ is so in $H^{(1+\tau^2)^{-\frac{1}{2}}\nu_1}$. Thus $(Y*\rho_j)u-(Y*\rho_1)u$ is a Cauchy sequence in $H^{(1+\tau^2)^{-\frac{1}{2}}\nu_1}$. Let σ be such that $H^\mu\subset\mathcal{H}_{(\sigma)}(R_{n+1})$. Then $(Y*\rho_j)u$ belongs to the space $\mathcal{H}_{(\sigma)}(R_{n+1})$. If we put $\kappa(\tau, \xi)=\min((1+\tau^2)^{-\frac{1}{2}}\nu_1, (1+\tau^2+|\xi|^2)^{\frac{\sigma}{2}})$, then $(Y*\rho_j)u$ converges in H^κ to u_+.

Assume that the product Yu exists for every $u\in H^\mu$ and, in addition, that $Yu\in H^\mu$. Then Y is a continuous linear map of H^μ into itself. If we consider the adjoint map of Y, we see that Yv exists in $H^{\frac{1}{\mu}}$ for every $v\in H^{\frac{1}{\mu}}$ and, a fortiori, $\int\frac{\mu^2(\tau, 0)}{1+\tau^2}d\tau<+\infty$. We denote by H_+^μ (resp. H_-^μ) the subspace consisting of the elements of $H^\mu(R_{n+1})$ with support in $\bar{R}_{n+1}^+$ (resp. support in $\bar{R}_{n+1}^-$). Then we have

Proposition 6. *Suppose Yu exists for every $u\in H^\mu(R_{n+1})$. Then the following conditions are equivalent*:

(a) *$Yu\in H^\mu$ for every $u\in H^\mu$.*

(b) *$Yu\in H^{\frac{1}{\mu}}$ for every $u\in H^{\frac{1}{\mu}}$.*

(c) *$\phi Y\in H^{\frac{1}{\mu}}$ for every $\phi\in\mathcal{D}(R_{n+1})$ and $H^\mu=H_+^\mu+H_-^\mu$ (topological sum).*

(d) *$\phi Y\in H^\mu$ for every $\phi\in\mathcal{D}(R_{n+1})$ and $H^{\frac{1}{\mu}}=H_+^{\frac{1}{\mu}}+H_-^{\frac{1}{\mu}}$ (topological sum).*

Proof. It suffices to prove the equivalence between (a) and (c).

Suppose (a) holds. Then $Yu=u_+\in H_+^\mu$ and $(1-Y)u=u_-\in H_-^\mu$. Thus $u=u_++u_-\in H_+^\mu+H_-^\mu$. Owing to Proposition 3 we see that every $u\in H_+^\mu$ has no mass on $t=0$ and so does for every $u\in H_-^\mu$, and therefore we can conclude that $H_+^\mu\cap H_-^\mu=\{0\}$.

Conversely, suppose (c) holds. The spaces $\mathcal{D}(R_{n+1}^+)$ and $\mathcal{D}(R_{n+1}^-)$ are dense in H_+^μ and H_-^μ respectively. Consider the map $l:\mathcal{D}(R_{n+1}^+)+\mathcal{D}(R_{n+1}^-)\ni \phi_1+\phi_2\to\phi_1\in\mathcal{D}(R_{n+1}^+)$. Then we have the estimate with a constant C

$$\|l(\phi_1+\phi_2)\|_\mu=\|\phi_1\|_\mu\leqq C\|\phi_1+\phi_2\|_\mu,$$

and therefore l can be continuously extended from H^μ into H_+^μ. It is sufficient to show that $l(\phi)=\phi_+$ for every $\phi\in\mathcal{D}(R_{n+1}^+)+\mathcal{D}(R_{n+1}^-)$ which is dense in H^μ. This is an immediate consequence of the definition of the map l. Thus the proof is complete.

With the aid of a theorem due to E. M. Stein [10] we can prove the following

PROPOSITION 7. *Suppose that μ satisfies the inequality*:

$$\left|1-\frac{\mu(\tau,\xi)}{\mu(\tau',\xi)}\right|\leqq C\left|1-\left|\frac{\tau}{\tau'}\right|^\beta\right|,\qquad |\beta|<\frac{1}{2}. \tag{7}$$

Then $Yu\in H^\mu(R_{n+1})$ for every $u\in H^\mu(R_{n+1})$ and $(Y*\rho_j)u$ converges in $H^\mu(R_{n+1})$ to Yu for any δ-sequence $\{\rho_j\}$, $\rho_j\in\mathcal{D}(R)$.

PROOF. By a simple calculation we obtain $\dfrac{1}{\mu(\tau,0)}\leqq C'(1+|\tau|)^{|\beta|}$ for some constant C', which implies $\displaystyle\int\frac{1}{(1+\tau^2)\mu^2(\tau,0)}\,d\!\!{}^-\tau<+\infty$, so the product Yu exists for every $u\in H^\mu$.

First we observe that $\left\|\dfrac{1}{i\tau}*_\tau\hat{w}\right\|_{L^2}\leqq 3\pi\|\hat{w}\|_{L^2}$ for any $\hat{w}\in L^2(\Xi_{n+1})$, the space of square integrable function with respect to $d\!\!{}^-\xi=\left(\dfrac{1}{2\pi}\right)^{n+1}d\xi$. In fact, owing to the equality $\dfrac{1}{i\tau}=\hat{Y}-\pi\delta$, we can write

$$\left\|\frac{1}{i\tau}*_\tau\hat{w}\right\|_{L^2}=\|\hat{Y}*_\tau\hat{w}-\pi\hat{w}\|_{L^2}\leqq 2\pi\|(Yw)^\wedge\|_{L^2}+\pi\|\hat{w}\|_{L^2}\leqq 3\pi\|\hat{w}\|_{L^2}.$$

Now for any $\phi\in\mathcal{D}(R_{n+1})$

$$\mu\left(\frac{1}{\tau}*_\tau\hat{\phi}\right)=\frac{1}{\tau}*_\tau(\hat{\phi}\mu)+\int\frac{1}{\tau-\tau'}\hat{\phi}(\tau',\xi)\,\mu(\tau',\xi)\left(\frac{\mu(\tau,\xi)}{\mu(\tau',\xi)}-1\right)d\tau'.$$

Here

$$\left\|\frac{1}{\tau}*_\tau(\hat{\phi}\mu)\right\|_{L^2}\leqq 3\pi\|\hat{\phi}\mu\|_{L^2}=3\pi\|\phi\|_\mu$$

and

$$\left|\int \frac{1}{\tau-\tau'} \hat{\phi}(\tau', \xi) \mu(\tau', \xi) \left(\frac{\mu(\tau, \xi)}{\mu(\tau', \xi)} - 1 \right) d\tau' \right|$$

$$\leqq C \int \frac{\left| 1 - \left| \frac{\tau}{\tau'} \right|^\beta \right|}{|\tau-\tau'|} |\hat{\phi}(\tau', \xi)| \mu(\tau', \xi) \, d\tau'.$$

In virtue of Lemma 1 in [10, p. 250], we obtain with a constant C_1

$$\left\| \int \frac{1}{\tau-\tau'} \hat{\phi}(\tau', \xi) \mu(\tau', \xi) \left(\frac{\mu(\tau, \xi)}{\mu(\tau', \xi)} - 1 \right) d\tau' \right\|_{L^2} \leqq C_1 \|\phi\|_\mu.$$

Combining these inequalities, we obtain with a constant C_2

$$\| Y\phi \|_\mu \leqq C_2 \|\phi\|_\mu,$$

which implies that Yu belongs to the space H^μ for every $u \in H^\mu$.

Let $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R)$ be any δ-sequence. Since we can write $(Y*\rho_j)\hat{}$ $= \frac{1}{i\tau} \hat{\rho}_j + \pi\delta$, we obtain, as before $\left\| \frac{\hat{\rho}_j}{\tau} *_\tau \hat{w} \right\|_{L^2} \leqq 3\pi \|\hat{w}\|_{L^2}$ for any $\hat{w} \in L^2(\Xi_{n+1})$. Then proceeding along the same line as in the above proof, we can conclude that for any $\phi \in \mathcal{D}(R_{n+1})$ there exists a constant C_3 independent of j such that

$$\|(Y*\rho_j)\phi\|_\mu \leqq C_3 \|\phi\|_\mu,$$

and therefore $(Y*\rho_j)u \in H^\mu$ for every $u \in H^\mu$.

We shall now show that $u(Y*\rho_j)$ converges in H^μ to Yu as $j \to \infty$ for any $u \in H^\mu$. To do this, owing to the Banach-Steinhaus theorem it suffices to confine ourselves to the case where $u \in \mathcal{S}(R_{n+1})$ and $\hat{u}(\tau', \xi) = 0$ for $|\tau'| \geqq M$, M being a positive constant. For any $\hat{w} \in L^2(\Xi_{n+1})$ we can write $\frac{\hat{\rho}_j - 1}{i\tau} *_\tau \hat{w}$ $= 2\pi((Y*\rho_j)w - Yw)\hat{}$, so that we have $\left\| \frac{\hat{\rho}_j - 1}{\tau} *_\tau \hat{w} \right\|_{L^2} = 2\pi \|(Y*\rho_j - Y)w\|_{L^2(R_{n+1})}$, which converges to 0 as $j \to \infty$. Put

$$\int \frac{\hat{\rho}_j(\tau-\tau') - 1}{\tau-\tau'} \hat{u}(\tau', \xi) \mu(\tau', \xi) \left(\frac{\mu(\tau, \xi)}{\mu(\tau', \xi)} - 1 \right) d\tau' = v_j^{(1)}(\tau, \xi) + v_j^{(2)}(\tau, \xi),$$

where

$$v_j^{(1)} = \int_{|\tau-\tau'| \leq k} \frac{\hat{\rho}_j(\tau-\tau') - 1}{\tau-\tau'} \hat{u}(\tau', \xi) \mu(\tau', \xi) \left(\frac{\mu(\tau, \xi)}{\mu(\tau', \xi)} - 1 \right) d\tau',$$

$$v_j^{(2)} = \int_{|\tau-\tau'| \geq k} \frac{\hat{\rho}_j(\tau-\tau') - 1}{\tau-\tau'} \hat{u}(\tau', \xi) \mu(\tau', \xi) \left(\frac{\mu(\tau, \xi)}{\mu(\tau', \xi)} - 1 \right) d\tau'.$$

Let k be a positive constant which will be determined later on. For any $\varepsilon>0$, we can choose a j_0 such that for every $j\geqq j_0$ we have $|\hat{\rho}_j(\tau)-1|<\varepsilon$ for $|\tau|\leqq k$. Then from the inequality

$$|v_j^{(1)}(\tau,\xi)|\leqq C\varepsilon\int_{-\infty}^{\infty}\frac{\left|1-\left|\frac{\tau}{\tau'}\right|^{\beta}\right|}{|\tau-\tau'|}|\hat{u}(\tau',\xi)|\mu(\tau',\xi)d\tau',$$

we obtain with a constant C_4

$$\|v_j^{(1)}\|_{L^2}\leqq C_4\varepsilon\|u\|_{\mu}.$$

Consider the case $\beta\geqq 0$. Using the inequalities

$$\frac{\left|1-\left|\frac{\tau}{\tau'}\right|^{\beta}\right|}{|\tau-\tau'|}\leqq\frac{1}{|\tau'|^{\beta}|\tau-\tau'|^{1-\beta}},$$

and

$$|v_j^{(2)}(\tau,\xi)|\leqq 2\int\frac{1}{|\tau'|^{\beta}|\tau-\tau'|^{1-\beta}}|\hat{u}(\tau',\xi)|\mu(\tau',\xi)d\tau',$$

we obtain with constants C_5, C_6

$$\begin{aligned}&\iint|v_j^{(2)}(\tau,\xi)|^2d\tau d\xi\\&\leqq 4\int\left\{\int\left(\int_{|\tau-\tau'|\geq k}\frac{1}{|\tau-\tau'|^{2(1-\beta)}}d\tau\right)^{\frac{1}{2}}\frac{1}{|\tau'|^{\beta}}|\hat{u}(\tau',\xi)|\mu(\tau',\xi)d\tau'\right\}^2d\xi\\&\leqq C_5\frac{1}{k^{1-2\beta}}\int\left\{\int_{|\tau'|\leq M}\frac{1}{|\tau'|^{\beta}}|\hat{u}(\tau',\xi)|\mu(\tau',\xi)d\tau'\right\}^2d\xi\\&\leqq C_6\frac{1}{k^{1-2\beta}}\|u\|_{\mu}^2.\end{aligned}$$

Let k be chosen so large that we have $\frac{C_6}{k^{1-2\beta}}<\varepsilon^2$. From these estimates we can see that $(Y*\rho_j)u$ converges in H^{μ} to Yu.

For the case $\beta<0$ the proof will be carried out in a similar way. Thus the proof is complete.

REMARK 3. Let μ be a temperate weight function written in the form

$$\mu=(\tau^2+\lambda^2(\xi))^{\frac{\sigma}{2}}\gamma(\xi),\qquad|\sigma|<\frac{1}{2}.$$

Then the condition (7) is clearly satisfied.

We shall denote by $\lambda(D_x)$ the operator with symbol $\lambda(\xi)$.

THEOREM 3. *Let μ be a temperate weight function such that for every $u \in H^{\mu}(R_{n+1})$ the product Yu exists and belongs to the same space $H^{\mu}(R_{n+1})$, and put $\mu_k=(\tau^2+\lambda^2(\xi))^{\frac{k}{2}}\mu$ for any positive integer k, where $\lambda(\xi)$ is a temperate weight function such that $\tau^2+\lambda^2(\xi)$ is also a temperate weight function. Then we have*

$$H_+^{\mu_k}+H_-^{\mu_k}=\{u \in H^{\mu_k}: \lim_{t\downarrow 0} u=\lim_{t\downarrow 0} D_t u=\cdots=\lim_{t\downarrow 0} D_t^{k-1}u=0\}.$$

In other words, for any $u \in H^{\mu_k}$ the product Yu belongs to the space H^{μ_k} if and only if $\lim_{t\downarrow 0} u=\lim_{t\downarrow 0} D_t u=\cdots=\lim_{t\downarrow 0} D_t^{k-1}u=0$.

PROOF. Since $\int \frac{\tau^{2(k-1)}}{\mu_k^2(\tau,0)}d\tau<+\infty$, for every $u \in H^{\mu_k}$ $u, D_t u, \cdots, D_t^{k-1}u$ have the boundary values (or the sections) on $t=0$. It follows then that $H_+^{\mu_k}+H_-^{\mu_k}\subset\{u \in H^{\mu_k}: \lim_{t\downarrow 0} u=\lim_{t\downarrow 0} D_t u=\cdots=\lim_{t\downarrow 0} D_t^{k-1}u=0\}$.

Conversely, if $u \in H^{\mu_k}$ and $\lim_{t\downarrow 0} u=\lim_{t\downarrow 0} D_t u=\cdots=\lim_{t\downarrow 0} D_t^{k-1}u=0$, then $(D_t-i\lambda(D_x))^k Yu=Y(D_t-i\lambda(D_x))^k u$, where $(D_t-i\lambda(D_x))^k u \in H^{\mu}$. Our assumption then implies $Y(D_t-i\lambda(D_x))^k u \in H^{\mu}$. Thus we obtain $Yu \in H_+^{\mu_k}$. Similarly we have $(1-Y)u \in H_-^{\mu_k}$. Consequently we can write $u=Yu+(1-Y)u \in H_+^{\mu_k}+H_-^{\mu_k}$.

As an immediate consequence of Theorem 3 we have

COROLLARY 3. *Let μ, μ_k be temperate weight functions considered in Theorem 3, k being a positive integer. Let $u \in H^{\mu_k}$. Then, according to the cases (1) $\lim_{t\downarrow 0} u\neq 0$, (2) $\lim_{t\downarrow 0} u=0$, (3) $\lim_{t\downarrow 0} u=\lim_{t\downarrow 0} D_t u=0$, $\cdots$, (k+1) $\lim_{t\downarrow 0} u=\lim_{t\downarrow 0} D_t u=\cdots=\lim_{t\downarrow 0} D_t^{k-1}u=0$, Yu belongs to the spaces H^{μ}, H^{μ_1}, $\cdots$, H^{μ_k} respectively.*

EXAMPLE 1. Let us consider the space $\mathscr{H}_{(\sigma,s)}(R_{n+1})$ [1, p. 51], where $|\sigma-k|<\frac{1}{2}$, k is a positive integer and σ, s are real numbers. The space $\mathscr{H}_{(\sigma-k,s)}(R_{n+1})$ satisfies the assumption of the preceding theorem. As a result, for any $u \in \mathscr{H}_{(\sigma,s)}(R_{n+1})$, $Yu \in \mathscr{H}_{(\sigma,s)}(R_{n+1})$ if and only if $\lim_{t\downarrow 0} u=\lim_{t\downarrow 0} D_t u=\cdots=\lim_{t\downarrow 0} D_t^{k-1}u=0$. This statement is an extension of Proposition 6 in [5, p. 16].

EXAMPLE 2. Every $u \in \mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}_{n+1}^+)$ does not have $\lim_{t\downarrow 0} u$ in general. In fact, since $\mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}_{n+1}^-)$ is a closed subspace of $\mathscr{H}_{(\frac{1}{2},s)}(R_{n+1})$, we can write

$$\mathscr{H}_{(\frac{1}{2},s)}(R_{n+1})=\mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}_{n+1}^-)\oplus\mathfrak{M},$$

where $\mathfrak{M}$ is an orthocomplement of $\mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}^-_{n+1})$. For any $u \in \mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}^-_{n+1})$ and $v \in \mathfrak{M}$ we have $(u\,|\,v)=0$ and therefore $(\hat{u}\,|\,\hat{v}(1+\tau^2+|\xi|^2)^{\frac{1}{2}}(1+|\xi|^2)^s)_{L^2}=0$. If we take $u \in \mathscr{D}(R^-_{n+1})$, then we see that $(D_t+i\lambda(D_x))^{\frac{1}{2}}(D_t-i\lambda(D_x))^{\frac{1}{2}}\lambda(D_x)^{2s}v$ vanishes in R^-_{n+1} and therefore we obtain [1, p. 53; 11, p. 45]

$$\left(\frac{D_t+i\lambda(D_x)}{D_t-i\lambda(D_x)}\right)^{\frac{1}{2}} v \in \mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}^+_{n+1}),$$

where $\lambda(\xi)=(1+|\xi|^2)^{\frac{1}{2}}$. If we put $g=v-\left(\frac{D_t+i\lambda(D_x)}{D_t-i\lambda(D_x)}\right)^{\frac{1}{2}}v$, then we can show that $g \in \mathscr{H}_{(\frac{3}{2},s-1)}(R_{n+1})$, and so we obtain

$$\mathscr{H}_{(\frac{1}{2},s)}(R_{n+1})=\mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}^+_{n+1})+\mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}^-_{n+1})+\mathscr{H}_{(\frac{3}{2},s-1)}(R_{n+1}). \tag{8}$$

If $\lim_{t\downarrow 0} u$ existed for every $u \in \mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}^+_{n+1})$, then the equation (8) implies that $\lim_{t\downarrow 0} u$ would exist for every $u \in \mathscr{H}_{(\frac{1}{2},s)}(\bar{R}_{n+1})$, which is a contradiction.

EXAMPLE 3. Every $u \in \mathring{\mathscr{H}}_{(-\frac{1}{2},s)}(\bar{R}^+_{n+1})$ does not have the canonical extension u_+ in general. In fact, the map $v \to D_t v - i\lambda(D_x)v$ is an isomorphism from $\mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}^+_{n+1})$ onto $\mathring{\mathscr{H}}_{(-\frac{1}{2},s)}(\bar{R}^+_{n+1})$. If u_+ existed for every $u \in \mathring{\mathscr{H}}_{(-\frac{1}{2},s)}(\bar{R}^+_{n+1})$, then, as shown in the following section, $\lim_{t\downarrow 0} v$ would exist for every $v \in \mathring{\mathscr{H}}_{(\frac{1}{2},s)}(\bar{R}^+_{n+1})$, which is a contradiction as seen from Example 2.

4. The canonical extension and the boundary value for distributions in the space $\mathscr{S}'(R_{n+1})$

Let $u \in \mathscr{S}'(R_{n+1})$ and let ϕ be an arbitrary element of $\mathscr{D}(R)$ such that $\phi(t) \geqq 0$, $\int \phi(t)dt=1$ and $\operatorname{supp}\phi \subset (0, \infty)$. We shall say that $\mathscr{S}'$-$\lim_{t\downarrow 0} u$ exists if $\{\phi_\varepsilon u\}$ converges in $\mathscr{S}'(R_{n+1})$ as $\varepsilon \downarrow 0$ and also say that u has the $\mathscr{S}'$-canonical extension u_+ if $\{\rho_{(\varepsilon)}u\}$ converges in $\mathscr{S}'(R_{n+1})$, where $\rho=Y*\phi$. As for $u \in \mathscr{S}'(R^+_{n+1})$, a similar terminology will be applied.

LEMMA 1. *If u is $\mathscr{S}'$-canonical, then* $\mathscr{S}'$-$\lim_{t\downarrow 0}(Y*_t u)=0$.

PROOF. There exists a positive integer m and a constant C such that

$$|\langle u, \psi\rangle| \leqq C \sup_{(t,x)\in R_{n+1}} \sup_{|p|<m} |(1+t^2)^{\frac{m}{2}}(1+|x|^2)^{\frac{m}{2}}(D^p\psi)(t,x)|,\ \psi \in \mathscr{S}(R_{n+1}). \tag{9}$$

Let ϕ be an arbitrary element of $\mathscr{D}(R)$ such that $\phi \geqq 0$, $\int \phi(t)dt=1$ and $\operatorname{supp}\phi$

$\subset(0, \infty)$ and take $\alpha \in \mathcal{D}(R)$ such that $\alpha=1$ in a 0-neighbourhood. Then we can write

$$<\phi_\varepsilon(Y *_t u), \psi> = \mathrm{I}_\varepsilon^{(1)} + \mathrm{I}_\varepsilon^{(2)} + \mathrm{I}_\varepsilon^{(3)},$$

where

$$\mathrm{I}_\varepsilon^{(1)} = <\alpha u, \int \phi_\varepsilon(s)\ \psi(s, x) ds>$$

$$\mathrm{I}_\varepsilon^{(2)} = -<\alpha u,\ Y *_t (\phi_\varepsilon \psi)>$$

$$\mathrm{I}_\varepsilon^{(3)} = <(1-\alpha) u,\ \check{Y} *_t (\phi_\varepsilon \psi)>.$$

Clearly $\lim_{\varepsilon \downarrow 0} \mathrm{I}_\varepsilon^{(3)} = 0$. We shall show that $\lim_{\varepsilon \downarrow 0} \mathrm{I}_\varepsilon^{(1)} = <u, \alpha(t)\psi(0, x)>$ and $\lim_{\varepsilon \downarrow 0} \mathrm{I}_\varepsilon^{(2)} = -<u, \alpha(t)\psi(0, x)>$. To do this, we write

$$\psi(t, x) = \psi(0, x) + t\frac{\partial \psi}{\partial t}(0, x) + \cdots + \frac{t^{m-1}}{(m-1)!}\frac{\partial^{m-1}\psi}{\partial t^{m-1}}(0, x) + r_m(t, x),$$

$$r_m(t, x) = \frac{t^m}{(m-1)!}\int_0^1 (1-\theta)^{m-1}\frac{\partial^m \psi}{\partial t^m}(\theta t, x) d\theta.$$

Then we have

$$\mathrm{I}_\varepsilon^{(1)} = <u, \alpha(t)\psi(0, x)> + <\alpha u, \int \phi_\varepsilon(s) r_1(s, x)\, ds>$$

$$= <u, \alpha(t)\psi(0, x)> + \varepsilon <u, \alpha(t)\beta_\varepsilon(x)>,$$

where

$$\beta_\varepsilon(x) = \int s\phi(s)\Big(\int \frac{\partial \psi}{\partial t}(\theta\varepsilon s, x) d\theta\Big) ds.$$

From (9) we obtain an estimate

$$\sup_{|p|<m} |(1+t^2)^{\frac{m}{2}}(1+|x|^2)^{\frac{m}{2}} D^p(\alpha(t)\beta_\varepsilon(x))| \leqq M,$$

where M is a constant independent of ε. Thus we see that $\lim_{\varepsilon \downarrow 0} \mathrm{I}_\varepsilon^{(1)} = <u, \alpha(t)\psi(0, x)>$.

We can write $\mathrm{I}_\varepsilon^{(2)}$ as follows:

$$-\mathrm{I}_\varepsilon^{(2)} = \sum_{j=0}^{m-1}\frac{1}{j!}<\alpha u,\ Y *_t t^j \phi_\varepsilon \frac{\partial^j \psi}{\partial t^j}(0, x)> + <\alpha u, Y *_t (\phi_\varepsilon r_m)>$$

$$=\sum_{j=0}^{m-1}\frac{\varepsilon^j}{j!}<(Y*_t(t^j\phi)_\varepsilon)u, \alpha(t)\frac{\partial^j\psi}{\partial t^j}(0, x)>+$$

$$+\frac{\varepsilon^m}{(m-1)!}<u, \alpha(t)\int_0^{\frac{t}{\varepsilon}}s^m\phi(s)\int_0^1(1-\theta)^{m-1}\frac{\partial^m\psi}{\partial t^m}(\theta\varepsilon s, x)d\theta\, ds>,$$

where $\varepsilon^j<(Y*_t(t^j\phi)_\varepsilon)u, \alpha(t)\frac{\partial^j\psi}{\partial t^j}(0, x)>$ tends to 0 when $j>0$ and to $<u, \alpha(t)\psi(0, x)>$ when $j=0$ as $\varepsilon\downarrow 0$, since u is $\mathscr{S}'$-canonical. Put

$$\gamma_\varepsilon(t, x)=\varepsilon^{m-1}\int_0^{\frac{t}{\varepsilon}}s^m\phi(s)\int_0^1(1-\theta)^{m-1}\frac{\partial^m\psi}{\partial t^m}(\theta\varepsilon s, x)\, d\theta\, ds.$$

Then from (9) we have an estimate

$$\sup_{|p|<m}|(1+t^2)^{\frac{m}{2}}(1+|x|^2)^{\frac{m}{2}}D^p(\alpha(t)\gamma_\varepsilon(t, x))|\leqq M',$$

where M' is a constant independent of ε. Thus $\lim_{\varepsilon\downarrow 0}\mathrm{I}_\varepsilon^{(2)}=-<u, \alpha(t)\psi(0, x)>$. The proof is complete.

Owing to Theorem VI in [7, p. 239], for a given $u\in\mathscr{S}'(R_{n+1})$ we can find a positive integer k such that $Y_k *_t u$ is an $\mathscr{S}'$-valued continuous function of t vanishing for $t\leqq 0$, where Y_k stands for $\frac{t_+^{k-1}}{(k-1)!}$.

PROPOSITION 8. *Let $u, v\in\mathscr{S}'(\bar{R}_{n+1}^+)$ satisfy the equation:*

$$D_t u-i\lambda(D_x)u=v,$$

where $\lambda(\xi)\in\mathcal{O}_M$. Then the following conditions are equivalent:

(a) *v has the $\mathscr{S}'$-canonical extension $v_\sim$.*

(b) *$\mathscr{S}'$-$\lim_{t\downarrow 0} u$ exists.*

PROOF. Let $\rho=Y*\phi$, where $\phi\in\mathscr{D}(R)$ is chosen as in the proof of Lemma 1. The implication (b)$\Rightarrow$(a) is trivial from the equation $\rho_{(\varepsilon)}(D_t u-i\lambda(D_x)u)=D_t(\rho_{(\varepsilon)}u)+i\phi_\varepsilon u-i\lambda(D_x)\rho_{(\varepsilon)}u$.

Suppose (a) holds. Let $U\in\mathscr{S}'(R_{n+1})$ be such that $U=u$ for $t>0$. Then U is written in the form $U=\sum D_t^k D_x^\beta f_{k,\beta}(t, x)$, where $f_{k,\beta}$ are slowly increasing continuous functions defined on R_{n+1}. Put $u_1=\sum D_t^k D_x^\beta(f_{k,\beta})_+$. Then the support of $D_t u_1-i\lambda(D_x)u_1-v_\sim$ lies in the hyperplane $t=0$ and therefore there exist $\alpha_j\in\mathscr{S}'(R_n)$, $j=0, 1, \cdots, m$ such that

$$D_t u_1-i\lambda(D_x)u_1=v_\sim+\sum_{j=0}^m D_t^j\delta\otimes\alpha_j.$$

Here we may take $m=0$. Indeed, suppose $m>0$. Putting $u_2=u_1-D_t^{m-1}\delta\otimes\alpha_m$, we can write after simple calculation

$$D_t u_2 - i\lambda(D_x) u_2 = v_{\smile} + \sum_{j=0}^{m-1} D_t^j \delta \otimes \beta_j.$$

Let k be the least positive integer such that $\mathscr{S}'\text{-}\lim_{t \downarrow 0} (Y_k *_t u_1) = 0$. Then we can write

$$(10) \qquad Y_{k-1} *_t u_1 + \lambda(D_x)(Y_k *_t u_1) = i Y_k *_t v_{\smile} + i Y_k \otimes \alpha_0.$$

Suppose $k \geqq 2$. Applying Lemma 1 together with the relation (10), we obtain $\mathscr{S}'\text{-}\lim_{t \downarrow 0} (Y_{k-1} *_t u_1) = 0$, which is a contradiction. Thus we see that $k=1$, which implies that $\mathscr{S}'\text{-}\lim_{t \downarrow 0} u = i\alpha_0$. The proof is complete.

By making use of the preceding proposition we show the following

PROPOSITION 9. *Let $\lambda(\xi)$ be a temperate weight function on Ξ_n such that $\lambda(\xi) \geqq c > 0$, c being a positive constant, and such that $\tau^2 + \lambda^2(\xi)$ is also a temperate weight function on Ξ_{n+1}. If $u, v \in \mathscr{S}'(R_{n+1})$ are related by the equation*

$$(11) \qquad (D_t^2 + \lambda(D_x)^2)^{\frac{1}{2}} u = v,$$

then the following conditions are equivalent:

(a) *v has the $\mathscr{S}'$-canonical extension v_+.*

(b) *$\mathscr{S}'\text{-}\lim_{t \downarrow 0} u$ exists.*

PROOF. Suppose (a) holds. Setting $f = \left(\frac{D_t + i\lambda(D_x)}{D_t - i\lambda(D_x)}\right)^{\frac{1}{2}} u$, we obtain from (11)

$$(D_t - i\lambda(D_x)) f = v.$$

Applying Proposition 8, we can see that $\mathscr{S}'\text{-}\lim_{t \downarrow 0} f$ exists. Let us write

$$(12) \qquad \hat{u} = \left(\frac{\tau - i\lambda(\xi)}{\tau + i\lambda(\xi)}\right)^{\frac{1}{2}} \hat{f}$$

$$= \hat{f} - \frac{1}{2}\left(\frac{2i\lambda(\xi)}{\tau + i\lambda(\xi)}\right)\hat{f} + \cdots + (-1)^k \binom{\frac{1}{2}}{k} \left(\frac{2i\lambda(\xi)}{\tau + i\lambda(\xi)}\right)^k \hat{f} + r_k(\tau, \xi) \hat{f}.$$

Then we can find a constant C such that

$$(13) \qquad |r_k(\tau, \xi)| \leqq C \frac{\lambda(\xi)^{k+1}}{(\tau^2 + \lambda^2(\xi))^{\frac{k+1}{2}}}.$$

In fact, we put $z=\frac{2i\lambda(\xi)}{\tau+i\lambda(\xi)}$. Clearly $|z|\leqq 2$. For $|z|<\frac{1}{2}$, $r_k(\tau, \xi)$ being considered as the remainder term of Taylor's expansion of $(1-z)^{\frac{1}{2}}$ at $z=0$ of order k, we can find a constant C with the required property. For $\frac{1}{2}\leqq|z|\leqq 2$, $r_k(\tau, \xi)=(1-z)^{\frac{1}{2}}-\left\{1-\frac{1}{2}z+\cdots+(-1)^k\binom{\frac{1}{2}}{k}z^k\right\}$ is bounded since $|1-z|=1$, so we can also find a constant C as desired.

Putting $\hat{w}=\frac{2i\lambda(\xi)}{\tau+i\lambda(\xi)}\hat{f}$, we show that $\mathscr{S}'\text{-}\lim_{t\downarrow 0} w$ exists. To this end, we write

$$D_t w+i\lambda(D_x)w=2i\lambda(D_x)f,$$

then, since $\lambda(D_x)f$ has the $\mathscr{S}'$-canonical extension, it follows from Proposition 8 that $\mathscr{S}'\text{-}\lim_{t\downarrow 0} w$ exists. Repeated use of this process allows us to infer the $\mathscr{S}'\text{-}\lim_{t\downarrow 0}\left(\frac{2i\lambda(D_x)}{D_t+i\lambda(D_x)}\right)^j f$ exists for each j, $j=0, 1, \cdots, k$.

Thus if we can show that $\mathscr{S}'\text{-}\lim_{t\downarrow 0} r_k(D_t, D_x)f$ exists, we can conclude from (12) that $\mathscr{S}'\text{-}\lim_{t\downarrow 0} u$ exists.

Note that f can be written in the form

$$f=(1+t^2+|x|^2)^m w,$$

where m is a positive integer and $w\in\mathscr{H}_{(-m)}(R_{n+1})$. By a simple calculation we can verify that for $p+|\beta|\leqq 2m$ we have with a constant C' and a positive integer k'

$$|D_\tau^p D_\xi^\beta r_k(\tau, \xi)|\leqq C'\frac{(1+|\xi|^2)^{\frac{k'}{2}}}{(\tau^2+\lambda^2(\xi))^{\frac{k+1}{2}}}.$$

From these considerations it will be not difficult to see that $r_k(D_t, D_x)f$ can be written in the form:

$$r_k(D_t, D_x)f=\sum_{p+|\beta|\leq 2m} t^p x^\beta f_{p,\beta}, \tag{14}$$

where $f_{p,\beta}\in\mathscr{H}_{(-m+k+1,-k')}(R_{n+1})$. Consequently if we take $k=m$, it follows from (14) that $\mathscr{S}'\text{-}\lim_{t\downarrow 0} r_k(D_t, D_x)f$ exists. Thus we have shown the implication (a)⇒(b).

Conversely, let us suppose (b) holds. Put $f=\left(\frac{D_t+i\lambda(D_x)}{D_t-i\lambda(D_x)}\right)^{\frac{1}{2}}u$. Then $(D_t-i\lambda(D_x))f=v$ and u can be written in the form $u=(1+t^2+|x|^2)^m w$, where m is a positive integer and $w\in\mathscr{H}_{(-m)}(R_{n+1})$. In the same way as in the proof of the implication (a)$\Rightarrow$(b) we can show that $\mathscr{S}'\text{-}\lim_{t\downarrow 0} f$ exists. Then it follows from Proposition 8 that v has the $\mathscr{S}'$-canonical extension v_{+}.

Thus the proof is complete.

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