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On Stable Homotopy Types of Stunted Lens Spaces

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§1. Introduction

The purpose of this note is to prove some results on the stable homotopy types of the stunted lens spaces analogous to those in [8], [9] and [6].

The $(2n+1)$-dimensional standard lens space mod k is the orbit space

$$L^n(k)=S^{2n+1}/Z_k,\ Z_k=\{e^{2\pi li/k}\mid l=0,1,\cdots,k-1\},\ (n>0),$$

where the action is given by $z(z_0,\cdots,z_n)=(zz_0,\cdots,zz_n)$. Let $[z_0,\cdots,z_n]\in L^n(k)$ denote the class of $(z_0,\cdots,z_n)\in S^{2n+1}$. Imbed naturally $L^m(k)\subset L^n(k)$ by identifying $[z_0,\cdots,z_m]=[z_0,\cdots,z_m,0,\cdots,0]$ for $m\leqq n$, and consider the subspace

$$L_0^m(k)=\{[z_0,\cdots,z_m]\mid z_m \text{ is real}\geqq 0\}\subset L^m(k)\subset L^n(k).$$

Then $L^m(k)-L_0^m(k)$ and $L_0^m(k)-L^{m-1}(k)$ $(m\leqq n)$ are $(2m+1)$- and $2m$-cells which make $L^n(k)$ a finite CW-complex. The stunted spaces

$$L^n(k)/L^{m-1}(k),\ L^n(k)/L_0^m(k),\ L_0^n(k)/L^{m-1}(k) \text{ and } L_0^n(k)/L_0^m(k),$$

for $k=p^r$ where p is a prime and $n>m$, will be studied in this note.

We say that two spaces X and Y are stably homotopy equivalent (S-equivalent), if the suspensions S^aX and S^bY are homotopy equivalent for some a and b.

We obtain the following theorem which is [8, Th. A] when $r=1$.

THEOREM 1.1. *Let p be a prime and r a positive integer such that $p^r\neq 2$. If the stunted lens space $L^n(p^r)/L^{m-1}(p^r)$ is S-equivalent to $L^{n+t}(p^r)/L^{m-1+t}(p^r)$ for $n>m$, then*

$$t\equiv 0 \bmod p^{[(n-m-1)/(p-1)]}.$$

The same is true for $L^n(p^r)/L_0^m(p^r)$, $L_0^n(p^r)/L^m(p^r)$ and $L_0^n(p^r)/L_0^m(p^r)$.

For the case $p^r=2$, we have the following theorem which is proved

THEOREM 1.2. *If the stunted projective space RP^n/RP^{m-1} is S-equivalent to RP^{n+t}/RP^{m-1+t} for $n>m$, then*

$$t\equiv 0 \bmod 2^{[\psi/2]+1} \text{ if } \psi<4, \qquad t\equiv 0 \bmod 2^{\max(\psi,\varphi)-1} \text{ if } \psi\geqq 4,$$

where

$$\psi=\begin{cases}[(n-m+1)/2] & \text{if } m \text{ is odd}\\ [(n-m)/2] & \text{if } m \text{ is even,}\end{cases} \qquad \varphi=\begin{cases}\varphi(n, m-1) & \text{if } m\not\equiv 0 \bmod 4\\ \varphi(n, m) & \text{if } m\equiv 0 \bmod 4,\end{cases}$$

and $\varphi(n, m)$ is the number of integers s such that $m<s\leqq n$ and $s\equiv 0$, 1, 2, or 4 mod 8.

For the converse of these results, it is known by [4, Prop. (2.6), Prop. (4.3)], [2-II, (6.3)] and [1, Th. 7.4] that

THEOREM. *If $t\equiv 0 \bmod 2^{\varphi(n-m,0)}$, then RP^n/RP^{m-1} is S-equivalent to RP^{n+t}/RP^{m-1+t}.*

Also, by [6, Th. 3],

THEOREM. *Suppose p is an odd prime. If $t\equiv 0 \bmod p^{[(n-m)/(p-1)]}$, then $L^n(p)/L^{m-1}(p)$ is S-equivelent to $L^{n+t}(p)/L^{m-1+t}(p)$. The same is true for $L^n(p)/L_0^m(p)$, $L_0^n(p)/L^{m-1}(p)$ and $L_0^n(p)/L_0^m(p)$.*

In addition, we have the corresponding results for $L^n(p^2)$.

THEOREM 1.3. *Let p be a prime, and $h=[(n-m)/(p-1)]$. Assume*

$$(1.4) \qquad t\equiv 0\begin{cases}\bmod p^h & \text{if } h\not\equiv 0 \bmod p \text{ or } h=0\\ \bmod p^{h+1} & \text{if } h\equiv 0 \bmod p \text{ and } h>0,\end{cases}$$

then $L^n(p^2)/L^{m-1}(p^2)$ is S-equivalent to $L^{n+t}(p^2)/L^{m-1+t}(p^2)$. The same is true for $L^n(p^2)/L_0^m(p^2)$.

For $L_0^n(p^2)/L^{m-1}(p^2)$ and $L_0^n(p^2)/L_0^m(p^2)$, the same is also true when p is an odd prime, and the same conclusion holds when $p=2$ under the assumption that

$$(1.5) \quad t\equiv 0 \bmod 2^{n-m+1} \text{ if } n-m\equiv 2 \bmod 4, \qquad t\equiv 0 \bmod 2^{n-m} \text{ otherwise.}$$

Also, corresponding to [6, Th. 4], we have

THEOREM 1.6. *Let p be a prime and $n>m$. $L^n(p^2)/L^{m-1}(p^2)$ (resp. $L^n(p^2)/L_0^m(p^2)$) is S-equivalent to $L^{n-m}(p^2)^+$ (resp. $L^{n-m}(p^2)$) if and only if (1.4) for $t=m$ holds. $L_0^n(p^2)/L^{m-1}(p^2)$ (resp. $L_0^n(p^2)/L_0^m(p^2)$) is S-equivalent to $L_0^{n-m}(p^2)^+$ (resp. $L_0^{n-m}(p^2)$) if and only if it holds (1.4) for $t=m$ when p is an odd prime and (1.5) for $t=m$ when $p=2$. Here, X^+ denotes the disjoint union of X and a point.*

We prepare some elements of the K-groups of the stunted lens spaces in §2 using the results on the K-group $K(L^n(p^r))$ [7, Th. 1.1 (i)], and prove Theorems 1.1 and 1.2 in §3 by the same way as in the proof of [8, Th. A], using the properties of the Adams operation Ψ^{p^r+1} on the K-groups [1].

Theorems 1.3 and 1.6 are proved in §§4 and 6 by the same methods of T. Kambe-H. Matsunaga-H. Toda [6], using the fact that the stunted lens space is homeomorphic to the Thom complex of some canonical bundle (cf. [6, Th. 1]), the results of M. F. Atiyah [4, §2] on the stable homotopy types of the Thom complexes, and the structures of the J-groups $J(L^n(p^2))$ and $J(L_0^n(p^2))$. They are determined in Theorems 4.5, 6.9 and 6.13, by making use of the J''-group of J. F. Adams [2-III], the results on the KO-groups $KO(L^n(4))$ and $KO(L_0^n(4))$ [10, Th. B, Th. 5.22] and those on $K(L^n(p^2))=K(L_0^n(p^2))$ [7, Th. 1.4].

§2. Some results on K-rings of stunted lens spaces

The following results on the $\tilde{K}$-rings $\tilde{K}(L^n(k))$ and $\tilde{K}(L_0^n(k))$ are known (cf. [7, Lemmas 2.3–4]):

(2.1) *The induced homomorphism $\tilde{K}(L_0^n(k))\to\tilde{K}(L_0^{n-1}(k))$ of the inclusion is epimorphic.*

(2.2) *$\tilde{K}(L_0^n(k))$ contains exactly k^n elements, and $\tilde{K}^{\pm1}(L_0^n(k))=0$.*

(2.3) *$\tilde{K}(L^n(k))\cong\tilde{K}(L_0^n(k))$ by the induced homomorphism of the inclusion.*

Lemma 2.4. *We have the following exact and commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K}(L^n(k)/L_0^m(k)) & \xrightarrow{j^!} & \tilde{K}(L^n(k)) & \longrightarrow & \tilde{K}(L_0^m(k)) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \| & & \\
0 & \longrightarrow & \tilde{K}(L_0^n(k)/L_0^m(k)) & \xrightarrow{j^!} & \tilde{K}(L_0^n(k)) & \longrightarrow & \tilde{K}(L_0^m(k)) & \longrightarrow & 0,
\end{array}$$

where two j are the projections and others are induced by the inclusions.

Proof. By (2.1–3), it is easy to see that the two sequences are the Puppe exact sequences. Since the middle homomorphism is isomorphic by (2.3), the left is also so by Five Lemma. q.e.d.

Lemma 2.5. *We have the following split-exact and commutative diagram:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K}(L^n(k)/L_0^m(k)) & \longrightarrow & \tilde{K}(L^n(k)/L^{m-1}(k)) & \xrightarrow{i^!} & \tilde{K}(S^{2m}) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \| & & \\
0 & \longrightarrow & \tilde{K}(L_0^n(k)/L_0^m(k)) & \longrightarrow & \tilde{K}(L_0^n(k)/L^{m-1}(k)) & \longrightarrow & \tilde{K}(S^{2m}) & \longrightarrow & 0
\end{array}$$

where homomorphisms are induced by the appropriate inclusions and projections.

PROOF. We see that $\tilde{K}^1(L_0^n(k)/L_0^m(k))=0$ by the lower Puppe exact sequence of the above lemma and (2.2). Hence the desired lower sequence is the Puppe exact sequence, since $\tilde{K}^{-1}(S^{2m})=0$. The upper is also so, by the fact that $i^!$ is epimorphic. This follows from the commutativity of the diagram

$$\begin{array}{ccccc}\tilde{K}(L_0^{n+1}(k)/L^{m-1}(k)) & \longrightarrow & \tilde{K}(S^{2m}) & \longrightarrow & 0\\ \downarrow & & \| & & \\ \tilde{K}(L^n(k)/L^{m-1}(k)) & \xrightarrow{i!} & \tilde{K}(S^{2m}) & & \end{array}$$

where the upper sequence is exact as is seen above. The desired left homomorphism is isomorphic by the above lemma, and the middle is also so by Five Lemma. Finally, the two exact sequences are split since $\tilde{K}(S^{2m})\cong Z$.

q.e.d.

Now, let η be the canonical complex line bundle over $L^n(k)$ or $L_0^n(k)$, and put

$$\sigma=\eta-1\in\tilde{K}(L^n(k))=\tilde{K}(L_0^n(k)), \tag{2.6}$$

where the two rings are identified by (2.3). The following are known (cf. [7, Prop. 2.6]):

(2.7) *The ring* $\tilde{K}(L^n(k))=\tilde{K}(L_0^n(k))$ *is generated by* σ, *and*

$$(1+\sigma)^k-1=0, \qquad \sigma^{n+1}=0.$$

Consider the exact sequences of Lemma 2.4. Because $\sigma^{m+i}=0$ $(i>0)$ in $\tilde{K}(L_0^m(k))$ by (2.7), we can define

$$\sigma^{(m+i)}=j^{!-1}\sigma^{m+i}\in\tilde{K}(L^n(k)/L_0^m(k)), \qquad \text{for } i>0. \tag{2.8}$$

For the case $k=p^r$ and p is a prime, it is proved that the element $\sigma^i\in\tilde{K}(L^n(p^r))$ is of order p^{r+h}, $h=[(n-i)/(p-1)]$, in [7, Th. 1.1 (i)]. Since $j^!$ of Lemma 2.4 is monomorphic, we see the following

PROPOSITION 2.9. *For a prime* p, *the element* $\sigma^{(m+i)}$ $(i>0)$ *of* (2.8) *is of order* p^{r+h}, $h=[(n-m-i)/(p-1)]$, *in* $\tilde{K}(L^n(p^r)/L_0^m(p^r)))$.

§3. Proof of Theorems 1.1 and 1.2

We prepare the following lemma.

LEMMA 3.1. *Let* p *be a prime, and* $t=up^v$, $(u,p)=1$, *then*

$$(p^r\pm1)^t-(\pm1)^t\equiv up^{v+r} \bmod p^{v+r+1}, \qquad \textit{if } p^r\neq2.$$

Especially, let $t=u2^v$, $(u, 2)=1$, *then*

$$3^t-1\equiv 2^{v+2} \bmod 2^{v+3} \textit{ if } v\geqq 1, \quad \equiv 2 \bmod 8 \textit{ if } v=0.$$

PROOF. Let f be a positive integer, and x and y be integers such that $x-y\equiv p^f \bmod p^{f+1}$. Then clearly

$$x^p-y^p\equiv y^{p-1}p^{f+1} \bmod p^{f+2}, \qquad \text{if } p^f\neq 2. \tag{3.2}$$

$$x^n-y^n\equiv n\,y^{n-1}p^f \bmod p^{f+1}, \qquad \text{for any integer } n>0. \tag{3.3}$$

Since $(p^r\pm 1)-(\pm 1)=p^r$, the repeated applications of (3.2) show that

$$(p^r\pm 1)^{p^v}-(\pm 1)^{p^v}\equiv p^{v+r} \bmod p^{v+r+1}.$$

Then, for any integer $u>0$, we have

$$(p^r\pm 1)^{up^v}-(\pm 1)^{up^v}\equiv up^{v+r} \bmod p^{v+r+1}$$

by (3.3), as desired. Especially, for the case $p=r=2$, we have

$$3^t-(-1)^t\equiv 2^{v+2} \bmod 2^{v+3},$$

and so the desired result if $v\geqq 1$. Also, if $v=0$, then $3^t+1\equiv 4 \bmod 8$ and so $3^t-1\equiv 2 \bmod 8$, as claimed. q.e.d.

To prove Theorem 1.1, we use some results on the Adams operations:

$$\Psi^j\colon K(X)\to K(X),$$

which enjoy the following properties [1, Th. 5.1]:

(3.4) *Ψ^j is natural for maps, and is a ring homomorphism.*

(3.5) *If ξ is a complex line bundle over X, then $\Psi^j\xi=\xi^j$.*

For the element $\sigma\in\tilde{K}(L^n(k))$ of (2.6), these show that

$$\Psi^j\sigma^i=((\sigma+1)^j-1)^i \quad \text{in } \tilde{K}(L^n(k)). \tag{3.6}$$

Now, consider the following diagram:

$$\begin{array}{ccc}
\tilde{K}(L^n(k)/L_0^m(k)) & \xrightarrow[\cong]{I^t} & \tilde{K}(S^{2t}(L^n(k)/L_0^m(k)) \\
\Psi^j\downarrow & & \Psi^j\downarrow \\
\tilde{K}(L^n(k)/L_0^m(k)) & \xrightarrow[\cong]{I^t} & \tilde{K}(S^{2t}(L^n(k)/L_0^m(k))
\end{array}$$

where I denotes the isomorphism defined by the Bott periodicity [5, Th. 1]. This diagram is not commutative, and

$$\Psi^j I^t = j^t I^t \Psi^j$$

by [1, Cor. 5.3].

For the case $j=k+1$, we see that the left Ψ^{k+1} is the identity by (2.7), (3.6) and Lemma 2.4. Therefore, we have

$$\Psi^{k+1} = (k+1)^t \text{ on } \tilde{K}(S^{2t}(L^n(k)/L_0^m(k))). \tag{3.7}$$

PROOF OF THEOREM 1.1. In the first place, we shall prove the theorem for $L^n(k)/L_0^m(k)$, where $k=p^r \neq 2$. Suppose that $L^n(k)/L_0^m(k)$ is S-equivalent to $L^{n+t}(k)/L_0^{m+t}(k)$, then there is a homotopy equivalence $g\colon S^{2t+2s}(L^n(k)/L_0^m(k)) \to S^{2s}(L^{n+t}(k)/L_0^{m+t}(k))$ for some integers s and t.

The map g induces isomorphisms of $\tilde{K}$-rings, and the following commutative diagram by (3.4):

$$\begin{array}{ccc}
\tilde{K}(S^{2s}(L^{n+t}(k)/L_0^{m+t}(k))) & \xrightarrow{g^!} & \tilde{K}(S^{2t+2s}(L^n(k)/L_0^m(k))) \\
\Big\downarrow{\scriptstyle \Psi^{k+1}} & & \Big\downarrow{\scriptstyle \Psi^{k+1}} \\
\tilde{K}(S^{2s}(L^{n+t}(k)/L_0^{m+t}(k))) & \xrightarrow{g^!} & \tilde{K}(S^{2t+2s}(L^n(k)/L_0^m(k))).
\end{array}$$

Hence (3.7) implies that

$$(k+1)^{t+s} g^! = g^!(k+1)^s = (k+1)^s g^!.$$

On the other hand, $\tilde{K}(S^{2t+2s}(L^n(k)/L_0^m(k))) (\simeq \tilde{K}(L^n(k)/L_0^m(k)))$ for $k=p^r$ contains the element $I^{t+s}\sigma^{(m+1)}$ of order $p^{r+[(n-m-1)/(p-1)]}$ by Proposition 2.9. Since $g^!$ is an isomorphism, these facts imply that

$$(p^r+1)^{t+s} - (p^r+1)^s \equiv 0 \bmod p^{r+[(n-m-1)/(p-1)]}.$$

Because $p^r+1 \not\equiv 0 \bmod p$, it follows that

$$(p^r+1)^t - 1 \equiv 0 \bmod p^{r+[(n-m-1)/(p-1)]}. \tag{3.8}$$

Therefore, we have $t \equiv 0 \bmod p^{[(n-m-1)/(p-1)]}$ by Lemma 3.1, as claimed.

The theorem for $L_0^n(k)/L_0^m(k)$ is proved in the same way since $\tilde{K}(L_0^n(k)/L_0^m(k)) \cong \tilde{K}(L^n(k)/L_0^m(k))$ by Lemma 2.4. For $L^n(k)/L^{m-1}(k)$, a cellular homotopy equivalence $g'\colon S^{2t+2s}(L^n(k)/L^{m-1}(k)) \to S^{2s}(L^{n+t}(k)/L^{m-1+t}(k))$ defines a map $g\colon S^{2t+2s}(L^n(k)/L_0^m(k)) \to S^{2s}(L^{n+t}(k)/L_0^{m+t}(k))$, and it is easy to see that g induces an isomorphism of $\tilde{K}$-rings by the direct sum decomposition of Lemma 2.5. Thus we have the desired results for $L^n(k)/L^{m-1}(k)$ by the above proofs, and in the same way for $L_0^n(k)/L^{m-1}(k)$. q.e.d.

PROOF OF THEOREM 1.2. Assume that there is a homotopy equivalence $g: S^{t+2s}(RP^n/RP^{m-1}) \to S^{2s}(RP^{n+t}/RP^{m-1+t})$ for $n>m$. Then we see that t is even by their homology groups. By [1, Th. 7.3],

$$\tilde{K}(RP^n/RP^{m-1}) \cong Z_{2^\psi} \text{ if } m \text{ is odd}, \qquad \cong Z_{2^\psi} \oplus Z \quad \text{if } m \text{ is even},$$

where ψ is the number of the theorem. Therefore, we have

$$3^{t/2}-1 \equiv 0 \bmod 2^\psi$$

by the similar way to (3.8). This shows that $t \equiv 0 \bmod 4$ if $\psi=2$ and

$$t \equiv 0 \bmod 2^{\psi-1} \quad \text{if } \psi \geqq 3,$$

by the latter half of Lemma 3.1. Thus, $t \equiv 0 \bmod 8$ if $\psi \geqq 4$.

On the other hand, under the assumption that $t \equiv 0 \bmod 8$, it is proved in [9, Lemma (4.2)] that $t \equiv 0 \bmod 2^{\varphi-1}$ if RP^n/RP^{m-1} and RP^{n+t}/RP^{m-1+t} are mod 2 S-related, where φ is the number of the theorem, using the Adams operation Ψ^3 on $\widetilde{KO}$-rings. It is clear that two spaces are mod 2 S-related if they are S-equivalent, and so we have the theorem. q.e.d.

REMARK. For the numbers ψ and φ in Theorem 1.2, it holds that $\psi-\varphi=0, \pm 1$.

§4. *J*-groups of $L^n(4)$ and $L_0^n(4)$

Let $J(X)$ be the J-group of a finite CW-complex X and J: $KO(X) \to J(X)$ the projection (J-homomorphism). Then, J. F. Adams [2–III, Th. (1.1)] has proved that the diagram

$$\begin{array}{c} KO(X) \xrightarrow{J} J(X) \\ {\scriptstyle J''}\downarrow \qquad\quad \downarrow{\scriptstyle \rho} \\ J''(X) = J'(X) \end{array} \tag{4.1}$$

is commutative, where

$$J''(X) = KO(X)/\sum_k \left(\bigcap_e k^e(\Psi^k-1)KO(X)\right) \tag{4.2}$$

and J'' is the natural projection and ρ is an epimorphism.

The KO-groups of $L^n(4)$ and $L_0^n(4)$ are given as follows [10, Th. B, Th. 5.22]:

$$\widetilde{KO}(L^n(4)) \cong \begin{cases} Z_{2^{n+1}} \oplus Z_{2^{n/2}} & \textit{for even } n>0 \\ Z_{2^n} \oplus Z_{2^{[n/2]+1}} & \textit{for } n \equiv 1 \bmod 4 \\ Z_{2^n} \oplus Z_{2^{[n/2]}} & \textit{for } n \equiv 3 \bmod 4, \end{cases} \tag{4.3}$$

$$\widetilde{KO}(L_0^n(4)) \cong \begin{cases} Z_{2^n} \oplus Z_{2^{n/2}} & \text{for } n\equiv 0 \bmod 4,\ n>0 \\ \widetilde{KO}(L^n(4)) & \text{for } n\not\equiv 0 \bmod 4, \end{cases}$$

and the first summand is generated by $r\sigma$ and the second by $\kappa+2^{[n/2]}r\sigma$, where $r\sigma$ is the real restriction of $\sigma=\eta-1$ of (2.6) and $\kappa=\rho-1$ is the stable class of the non-trivial real line bundle ρ over $L^n(4)$ or $L_0^n(4)$.

LEMMA 4.4. *The Adams operation Ψ^j on $\widetilde{KO}(L^n(4))$ or $\widetilde{KO}(L_0^n(4))$ is given by*

$$\Psi^j r\sigma = \begin{cases} r\sigma & \text{for odd } j \\ 2\kappa & \text{for } j\equiv 2 \bmod 4 \\ 0 & \text{for } j\equiv 0 \bmod 4, \end{cases} \qquad \Psi^j\kappa = \begin{cases} \kappa & \text{for odd } j \\ 0 & \text{for even } j. \end{cases}$$

PROOF. Since $\kappa+1=\rho$ is a real line bundle, $\Psi^j\kappa=(1+\kappa)^j-1$ by [1, Th. 5.1]. Hence, we have the second equality using $(1+\kappa)^2=1$ of [10, (6.3)].

To prove the first equality, it is sufficient to show it in $\widetilde{KO}(L^n(4))$ for the case $n\equiv 3 \bmod 4$, by the naturality. Consider the complexification

$$c\colon \widetilde{KO}(L^n(4)) \to \widetilde{K}(L^n(4)).$$

Then $c\Psi^j=\Psi^j c$ by [1, Th. 4.1], and $cr\sigma=\sigma^2/(1+\sigma)$ and $c\kappa=\sigma(1)=(1+\sigma)^2-1$ by [10, Lemmas 3.10–11]. Therefore, we have

$$\begin{aligned} c\Psi^j r\sigma &= \Psi^j cr\sigma = \Psi^j(\sigma^2/(1+\sigma)) = ((1+\sigma)^j-1)^2/(1+\sigma)^j \\ &= \begin{cases} \sigma^2/(1+\sigma)=cr\sigma & \text{if } j \text{ is odd} \\ 2(1+\sigma)^2-2=2\sigma(1)=2c\kappa & \text{if } j\equiv 2 \bmod 4 \\ 0 & \text{if } j\equiv 0 \bmod 4, \end{cases} \end{aligned}$$

by (3.4), (3.6) and $(1+\sigma)^4=1$ of (2.7). Because c is monomorphic if $n\equiv 3 \bmod 4$ [10, Cor. 5.4], this equality implies the desired result. q.e.d.

By this lemma, (4.3) and (4.2), we see that $\operatorname{Ker} J''=0$ for $X=L^n(4)$ or $L_0^n(4)$. Therefore, we have the following theorem by the commutativity of (4.1).

THEOREM 4.5. *The J-homomorphisms*

$$J\colon KO(L^n(4)) \cong J(L^n(4)), \qquad J\colon KO(L_0^n(4)) \cong J(L_0^n(4))$$

are isomorphic, and their reduced groups are given by (4.3).

COROLLARY 4.6. *The order of the element $Jr\sigma$ is equal to*

2^{n+1} *in $J(L^n(4))$ for even n, or in $J(L_0^n(4))$ for $n\equiv 2 \bmod 4$,*

2^n *in* $J(L^n(4))$ *for odd* n, *or in* $J(L_0^n(4))$ *for* $n\not\equiv 2 \bmod 4$.

The following results are used to prove Theorems 1.3 and 1.6.

For a real s-vector bundle α over a finite CW-complex X, X^α will denote the Thom complex of α, i.e., the mapping cone of the $(s-1)$-sphere bundle p: $E\to X$ associated with α. A cellular decomposition $X=\cup e_i^{n_i}$ of X gives naturally a cellular decomposition of $X^\alpha=e^0\cup\cup e_i^{s+n_i}$ of X^α.

THEOREM 4.7. *There exists a cellular homeomorphism between the stunted lens space* $L^n(k)/L^{m-1}(k)$ *and the Thom complex* $(L^{n-m}(k))^{mr\eta}$, *where* $\eta=\sigma+1$ *is the canonical complex line bundle and* $r\eta$ *is its real restriction.*

This theorem is proved by the same proofs of [6, Th. 1], which is the theorem for the case $k=p$.

COROLLARY 4.8. *We have the following cellular homeomorphisms*:

$$L^n(k)/L_0^m(k)\approx(L^{n-m}(k))^{mr\eta}/S^{2m},$$

$$L_0^n(k)/L^{m-1}(k)\approx(L_0^{n-m}(k))^{mr\eta},$$

$$L_0^n(k)/L_0^m(k)\approx(L_0^{n-m}(k))^{mr\eta}/S^{2m}.$$

PROOF OF THEOREMS 1.3 AND 1.6 FOR $p=2$. Assume (1.4) for $p=2$, then $tJr\sigma=0$ in $J(L^{n-m}(4))$ by Corollary 4.6, and so

$$J(mr\eta)=J(mr\eta+2t+tr\sigma)=J((m+t)r\eta)\quad\text{in } J(L^{n-m}(4)),$$

since $1+\sigma=\eta$. Therefore $(L^{n-m}(4))^{mr\eta}$ and $(L^{n-m}(4))^{(m+t)r\eta}$ are S-equivalent by [4, Prop. (2.6)]. Then, Theorem 1.3 for $L^n(4)/L^{m-1}(4)$ follows from Theorem 4.7. In the same way, we have the desired results for the other cases using Corollaries 4.6 and 4.8.

Similarly, Theorem 1.6 is proved by use of [4, Prop. (2.9)]. q.e.d.

§5. *J*-homomorphism for $L_0^n(p^2)$, p odd prime

Now, the rest of this note is devoted mostly to the J-group $J(L_0^n(p^2))$ for an odd prime p, which is determined in Theorem 6.9.

Consider the real restriction r and the projection J'' of (4.1):

$$K(L_0^n(p^2))\xrightarrow{r}KO(L_0^n(p^2))\xrightarrow{J''}J''(L_0^n(p^2)).$$

LEMMA 5.1. *For an odd prime* p, r *is an epimorphism, and* Ker r *is generated additively by the elements*

$$(5.2)\qquad (1+\sigma)^j-(1+\sigma)^{p^2-j}\quad(0<j<p^2).$$

PROOF. The first half is proved in the proof of [7, Prop. 2.11 (i)]. Let t be the conjugation, then $1+t=cr$, $r=rt$ and t is a ring homomorphism (cf. [1]). By use of $(1+\sigma)^{p^2}=1$ of (2.7), we have $t((1+\sigma)^j)=(t(1+\sigma))^j=(1+\sigma)^{-j}=(1+\sigma)^{p^2-j}$, and so $r((1+\sigma)^j-(1+\sigma)^{p^2-j})=0$.

Conversely, assume $\beta \in \operatorname{Ker} r$, then $\beta \in \tilde{K}(L_0^n(p^2))$ and so $\alpha=\beta/2$ exists and $r\alpha=0$ by (2.2). Also, $\beta=\alpha+\alpha=\alpha-t\alpha$ since $\alpha+t\alpha=cr\alpha=0$. Then β is a linear combination of the elements of (5.2), because α is a linear combination of $(1+\sigma)^j$, $0\leqq j<p^2$ by (2.7). q.e.d.

LEMMA 5.3. *The kernel of the epimorphism $J''r$ is generated additively by the elements*

$$\sigma^{j-1}+\sigma^j \ (1<j<p), \quad \sigma(1)\sigma^{j-1}+\sigma(1)\sigma^j \ (1\leqq j<p^2-p), \tag{5.4}$$

where $\sigma(1)=(1+\sigma)^p-1$.

PROOF. Since $r\Psi^k=\Psi^k r$ [3, Lemma A2], Ker $J''r$ is generated by the elements of (5.2) and $\bigcap_e k^e(\Psi^k-1)K(L_0^n(p^2))$, by (4.2) and the above lemma. Since $\Psi^k(1+\sigma)^j=(1+\sigma)^{kj}$ by (3.4–5), it follows from (2.7) and (2.2) that $\bigcap_e k^e(\Psi^k-1)K(L_0^n(p^2))$ is 0 if $k\equiv 0 \bmod p$ and is generated by $(1+\sigma)^{kj}-(1+\sigma)^j$ if $k\not\equiv 0 \bmod p$. Thus, Ker $J''r$ is generated additively by

$$\left\{\begin{array}{ll} \alpha(i,j)=(1+\sigma)^{ip+j}-(1+\sigma) & (0\leqq i<p,\quad 1\leqq j<p), \\ \beta(i)=(1+\sigma)^{ip}-(1+\sigma)^p & (1\leqq i<p), \end{array}\right. \tag{5.5}$$

where $\alpha(0,1)=\beta(1)=0$. Considering the elements $\sigma(1)=(1+\sigma)^p-1$, we have

$$\begin{array}{ll} \alpha(0,j)-\alpha(0,j-1)=\sigma(1+\sigma)^{j-1} & (1<j<p), \\ \alpha(i,j)-\alpha(i-1,j)=\sigma(1)(1+\sigma)^{(i-1)p+j} & (1\leqq i<p,\ 1\leqq j<p), \\ \beta(i)-\beta(i-1)=\sigma(1)(1+\sigma)^{(i-1)p} & (1<i<p). \end{array}$$

Therefore, we see that Ker $J''r$ is generated additively by the elements

$$\sigma(1+\sigma)^{j-1} \ (1<j<p), \quad \sigma(1)(1+\sigma)^j \quad (1\leqq j<p^2-p).$$

It is easy to see that the elements of the lemma are linear combinations of these elements and the inverse is also true. q.e.d.

LEMMA 5.6. Ker $J=$Ker J'' *in* (4.1) *for* $X=L_0^n(p^2)$, *and so* $J(L_0^n(p^2))=J''(L_0^n(p^2))$.

PROOF. It is proved in [2-I, Th. (1.3)] that, if $\alpha \in KO(L_0^n(p^2))$ is a linear combination of $O(1)$- and $O(2)$-bundles, then, for each k, there is an integer $e>0$ such that $J(k^e(\Psi^k-1)\alpha)=0$.

This is true for $\alpha=r((1+\sigma)^j)=r(\eta^j)$ and we have

$$Jr(k^e(\Psi^k-1)(1+\sigma)^j)=k^e Jr((1+\sigma)^{kj}-(1+\sigma)^j)=0.$$

This implies that $Jr((1+\sigma)^{kj}-(1+\sigma)^j)=0$ if $k\not\equiv 0 \bmod p$, since the order of $\tilde{K}(L_0^n(p^2))$ is p^{2n} by (2.2). Thus the elements of (5.5) vanish under Jr, and we have the desired results by the commutativity of (4.1). q.e.d.

Combining these lemmas with (2.7), we have

Proposition 5.7. *For an odd prime p, the composition*

$$Jr\colon K(L_0^n(p^2))\xrightarrow{r}KO(L_0^n(p^2))\xrightarrow{J}J(L_0^n(p^2))$$

of the real restriction r and J-homomorphism is an epimorphism, and its kernel is generated additively by the elements (5.4). *Furthermore, $J(L_0^n(p^2))\cong Z\oplus\tilde{J}(L_0^n(p^2))$ and $\tilde{J}(L_0^n(p^2))$ is generated additively by the elements*

$$\text{(5.8)}\qquad \begin{cases} \alpha_0=Jr\sigma=(-1)^{j-1}Jr(\sigma^j) & (1\leqq j<p),\\ \alpha_1=Jr\sigma(1)=(-1)^j Jr(\sigma(1)\sigma^j) & (0\leqq j<p^2-p),\end{cases}$$

where $\sigma=\eta-1$ is the element of (2.6) *and $\sigma(1)=(1+\sigma)^p-1=\eta^p-1$.*

Furthermore, we have

Lemma 5.9. *$Jr(\sigma^j)=(-1)^{j-1}\theta(j)(\alpha_0-\alpha_1)+(-1)^{j-1}\alpha_1$ for $1\leqq j<p^2$ in $J(L_0^n(p^2))$, where $\theta(j)$ is the integer defined by*

$$\text{(5.10)}\qquad \theta(j)=\sum_{i=0}^{\infty}(-1)^i\binom{j}{ip}.$$

To prove this lemma, we use the following lemmas.

Lemma 5.11. $\sum_{i=0}^{a}(-1)^i\binom{a}{i}\binom{b+i}{c}=(-1)^a\binom{b}{c-a}$.

Proof. These are the coefficients of x^c on both sides of the equality $(1-(x+1))^a(x+1)^b=(-1)^a x^a(x+1)^b$. q.e.d.

Lemma 5.12. *For odd p and the integer $\theta(j)$ of* (5.10),

$$\sum_{i=1}^{p}(-1)^i\binom{p}{i}\theta(j+i)=0.$$

Proof. The left hand side is equal to

$$\sum_{i=1}^{p}(-1)^i\binom{p}{i}\sum_{k=0}^{\infty}(-1)^k\binom{j+i}{kp}=\sum_{k=0}^{\infty}(-1)^k\sum_{i=1}^{p}(-1)^i\binom{p}{i}\binom{j+i}{kp}$$
$$=\sum_{k=0}^{\infty}(-1)^k\left\{-\binom{j}{(k-1)p}-\binom{j}{kp}\right\}=0,$$

as desired, using the above lemma. q.e.d.

PROOF OF LEMMA 5.9. If $j<p$, the desired equality is the first equality of (5.8) since $\theta(j)=1$. For $j\geqq p$, it is proved by the induction on j as follows:

$$\begin{aligned}
&Jr(\sigma^j)-(-1)^{j-1}\alpha_1 \\
&=Jr\left(\sigma(1)\sigma^{j-p}-\sum_{i=1}^{p-1}\binom{p}{i}\sigma^{j-p+i}\right)-(-1)^{j-1}\alpha_1 \qquad (\text{by } \sigma(1)=(1+\sigma)^p-1) \\
&=-(-1)^j\sum_{i=1}^{p-1}(-1)^i\binom{p}{i}\theta(j-p+i)(\alpha_0-\alpha_1)-(-1)^j\sum_{i=1}^{p-1}(-1)^i\binom{p}{i}\alpha_1 \\
&\qquad\qquad (\text{by (5.8) and the inductive assumptions}) \\
&=(-1)^{j-1}\theta(j)(\alpha_0-\alpha_1) \qquad \left(\text{by Lemma 5.12 and } \sum_{i=0}^{p}(-1)^i\binom{p}{i}=0\right).
\end{aligned}$$

q.e.d.

The following properties of $\theta(j)$ are used in the next section.

LEMMA 5.13. *Let* $j-1=a(p-1)+b$, $0\leqq b<p-1$, *then*

(5.14) $\theta(j)\equiv 0 \bmod p^a$ *for any* $j>0$,

(5.15) $\theta(j)\equiv(-1)^a p^a \bmod p^{a+1}$ *for* $b=p-2$ *or* $a=p$.

PROOF. Consider the integer $\theta(j, k)=\sum_{i=0}^{\infty}(-1)^i\binom{j}{ip+k}$ for $0\leqq k<p$, then it is clear that

$$\theta(j)=\theta(j, 0)=\theta(j-1, 0)-\theta(j-1, p-1).$$

Also, because $(1+x)^{j-1}\equiv\sum_{k=0}^{p-1}\theta(j-1, k)x^k \bmod x^p+1$, we have

$$(1+x)^{j-1}\equiv\sum_{k=0}^{p-2}\{\theta(j-1, k)-(-1)^k\theta(j-1, p-1)\}x^k \bmod P(x), \tag{5.16}$$

where $P(x)=(x^p+1)/(x+1)=\sum_{i=0}^{p-1}(-1)^i x^i$, and the right hand side of (5.16) has the constant term $\theta(j)$ by the above equality.

On the other hand, there is an integral polynomial $Q(x)$ such that

$$(1+x)^{p-1}=pQ(x)+P(x),$$

since $\binom{p-1}{i}\equiv(-1)^i \bmod p$ for $0\leqq i\leqq p-1$. Therefore, we have

$$(1+x)^{j-1}=((1+x)^{p-1})^a(1+x)^b\equiv p^aQ(x)^a(1+x)^b \bmod P(x).$$

This equality and (5.16) show the first desired result.

Since $Q(-1)=-1$ by the definition, there is an integral polynomial $Q'(x)$ such that $Q(x)=(1+x)Q'(x)-1$. Therefore, if $b=p-2$,

$$\begin{aligned}(1+x)^{j-1}&\equiv p^aQ(x)^a(1+x)^{p-2}=p^a((1+x)Q'(x)-1)^a(1+x)^{p-2}\\&=(-1)^ap^a(1+x)^{p-2}+p^a(1+x)^{p-1}R_1(x)\\&\equiv(-1)^ap^a(1+x)^{p-2}+p^{a+1}Q(x)R_1(x) \bmod P(x),\end{aligned}$$

for some integral polynomial $R_1(x)$. Also, if $a=p$,

$$\begin{aligned}(1+x)^{j-1}&\equiv p^p((1+x)Q'(x)-1)^p(1+x)^b\\&=(-1)^pp^p(1+x)^b+p^{p+1}R_2(x)+p^p(1+x)^pQ'(x)^p\\&\equiv(-1)^pp^p(1+x)^b+p^{p+1}R_3(x) \bmod P(x),\end{aligned}$$

for some integral polynomial $R_i(x)$. These and (5.16) show the second desired property. q.e.d.

§6. *J*-group of $L_0^n(p^2)$ and $L^n(p^2)$ for odd prime p

The reduced K-group $\tilde{K}(L_0^n(p^2))$, which is isomorphic to $\tilde{K}(L^n(p^2))$ by (2.3), is given as follows [7, Th. 1.4]: Let

(6.1) $\qquad n-p^i+1=a_i(p^{i+1}-p^i)+b_i \ (0\leqq b_i<p^{i+1}-p^i) \quad$ for $i=0, 1,$

and consider the following elements of $\tilde{K}(L_0^n(p^2))$:

(6.2) $$\sigma(1, j)=\begin{cases}\sigma(1)\sigma^j+p^{a_1(p-1)}\sigma^{p+j} & (\text{if } b_1\leqq j<b_1+p-1)\\ \sigma(1)\sigma^j+p^{(a_1+1)(p-1)}\sigma^{p+j} & (\text{if } j<b_1-(p-1)^2)\\ \sigma(1)\sigma^j & (\text{otherwise}),\end{cases}$$

for $0\leqq j\leqq \min(p^2-p-1, n-p)$, where σ is the element of (2.6) and $\sigma(1)=(1+\sigma)^p-1$. Then,

(6.3) *For an odd prime p and $n>0$,*

$$\tilde{K}(L_0^n(p^2))\cong\sum_{j=1}^m Z_{t_j},\ m=\min(p^2-1, n),\quad (\textit{direct sum})$$

where Z_t indicates a cyclic group of order t and

$$t_j=\begin{cases}p^{2-i+a_i} & (\textit{if } p^i\leqq j<p^i+b_i \ (i=0, 1))\\ p^{1-i+a_i} & (\textit{if } p^i+b_i\leqq j<p^{i+1} \ (i=0, 1)),\end{cases}$$

and the j-th direct summand Z_{t_j} *is generated by*

$$\sigma^j \ (if\ 1 \leqq j < p), \qquad \sigma(1, j-p) \ \ (if\ p \leqq j < p^2).$$

In connection to (6.1), we see easily the following

LEMMA 6.4. *Let* $c_1 = [b_1/(p-1)]$, *then*

$$a_0 = [n/(p-1)] = a_1 p + 1 + c_1, \qquad b_1 = c_1(p-1) + b_0.$$

Hence, the condition $a_0 \equiv 0 \bmod p$ *is equivalent to* $c_1 = p-1$, *and so to* $p^2 - 2p < b_1$.

By the results of the last sections, we have the following lemmas in $J(L_0^n(p^2))$.

LEMMA 6.5. *For the generators* α_0 *and* α_1 *of* (5.8),

$$\begin{cases} p^{1+a_0}\alpha_0 = 0 \ \ if\ n \geqq p-1 \\ \qquad \alpha_0 = 0 \ \ if\ n < p-1, \end{cases} \qquad \begin{cases} p^{a_1}\alpha_1 = 0 \ \ if\ b_1 \leqq p^2 - 2p \\ p^{a_1+1}\alpha_1 = 0 \ \ if\ b_1 > p^2 - 2p. \end{cases}$$

PROOF. We see that $p^{1+a_0}\sigma^{p-1} = 0$ if $n \geqq p-1$ by (6.3) and $1 + b_0 \leqq p-1$, $\sigma^{p-1} = 0$ if $n < p-1$ by (2.7), and $p^{[n/p(p-1)]}\sigma(1)\sigma^{p^2-p-1} = 0$ by [7. Prop. 4.13]. These show the above results by (5.8) and (6.1). q.e.d.

By (5.14), there is an integer $\theta'(j)$ such that

$$\theta(j) = p^a \theta'(j), \qquad a = [(j-1)/(p-1)]. \tag{6.6}$$

LEMMA 6.7. *For the elements of* (6.2) *and Jr of Proposition* 5.7,

$$Jr\sigma(1, j) = \begin{cases} (-1)^j \alpha_1 + (-1)^j \theta(p+j) p^{a_1(p-1)} \alpha_0 & (if\ b_1 \leqq j < b_1 + p - 1) \\ (-1)^j \alpha_1 + (-1)^j \theta(p+j) p^{(a_1+1)(p-1)} \alpha_0 & (if\ j < b_1 - (p-1)^2) \end{cases}$$

$$= (-1)^j \alpha_1 + (-1)^j \theta'(p+j) p^{a_0 - a_1} \alpha_0 \quad (if\ one\ of\ the\ above\ holds).$$

PROOF. The first equality follows from (6.2), (5.8) and Lemmas 5.9 and 6.5. The second follows from Lemma 6.4, (6.6) and the fact that

$[(p+j-1)/(p-1)] = c_1 + 1$ if $b_1 \leqq j < b_1 + p - 1$, $=1$ if $0 \leqq j < b_1 - (p-1)^2$. q.e.d.

LEMMA 6.8. $p^{a_0}\alpha_0 = \begin{cases} 0 & if\ b_1 \leqq p^2 - 2p\ or\ n < p-1 \\ p^{a_1}\alpha_1 & if\ b_1 > p^2 - 2p\ and\ n \geqq p-1. \end{cases}$

PROOF. Let j_0 be the integer such that

(*) $\qquad j_0=a(p-1)+p-2$ and $b_1\leqq j_0\leqq \min(b_1+p-2,\ p^2-p-1)$,

then $\theta'(p+j_0)\equiv(-1)^{a+1} \bmod p$ by (6.6) and (5.15).

If $n\geqq p^2-1$, $\sigma(1, j_0)$ is of order p^{a_1} by (6.3), and so

$$0=(-1)^{j_0}p^{a_1}Jr\sigma(1, j_0)=p^{a_1}\alpha_1+\theta'(p+j_0)p^{a_0}\alpha_0=p^{a_1}\alpha_1+(-1)^{a+1}p^{a_0}\alpha_0$$

by the above two lemmas. This implies the lemma, because $p^{a_1}\alpha_1=0$ if $b_1\leqq p^2-2p$ by Lemma 6.5, and $a=p-1$ if $b_1>p^2-2p$.

If $p-1\leqq n<p^2-1$, then $a_1=0$ and $b_1=n-p+1=(a_0-1)(p-1)+b_0$, and so $a=a_0-1$ and $p^2>p+j_0\geqq n+1$ by (*). Therefore, we have $\sigma^{p+j_0}=0$ by (2.7), and

$$0=(-1)^{j_0}Jr(\sigma^{p+j_0})=\theta(p+j_0)(\alpha_0-\alpha_1)+\alpha_1=(-1)^{a_0}p^{a_0}\alpha_0+\alpha_1$$

by Lemma 5.9, (5.15) and Lemma 6.5. This shows the lemma as above.

If $n<p-1$, then $a_0=0$ and $\alpha_0=0$ by Lemma 6.5. q.e.d.

Now, the group structure of $J(L_0^n(p^2))$ is determined by the above considerations.

THEOREM 6.9. *Let p be an odd prime, and*

$$a_0=[n/(p-1)], \quad a_1=[(n-p+1)/(p^2-p)]$$

be the integers of (6.1) *for $n>0$. Then, the J-group $J(L_0^n(p^2))\cong Z\oplus \tilde{J}(L_0^n(p^2))$ is given by*

$$\tilde{J}(L_0^n(p^2))\cong\begin{cases} 0 & \text{if } a_0=0\\ Z_{p^{a_0}}\oplus Z_{p^{a_1}} & \text{if } a_0\not\equiv 0 \bmod p\\ Z_{p^{a_0+1}}\oplus Z_{p^{a_1}} & \text{if } a_0\equiv 0 \bmod p \text{ and } a_0>0,\end{cases}$$

and the first summand is generated by α_0 and the second by $\alpha_1-p^{a_0-a_1}\alpha_0$ which can be replaced by α_1 for the second case. Here, $\alpha_0=Jr\sigma$ and $\alpha_1=Jr\sigma(1)$ are the elements of (5.8).

PROOF. For the case $a_0=0$, we have $n<p-1$ and $\alpha_0=\alpha_1=0$ by Lemma 6.5, and so the desired result by Proposition 5.7.

For the case $n\geqq p-1$, we consider the abelian group

$$G=\begin{cases} Z_{p^{a_0}}\oplus Z_{p^{a_1}} & \text{if } a_0\not\equiv 0 \bmod p\\ Z_{p^{a_0+1}}\oplus Z_{p^{a_1}} & \text{if } a_0\equiv 0 \bmod p,\end{cases} \tag{6.10}$$

whose summands are generated by β_0 and β_1' respectively, and put

$$\beta_1=\beta_1' \text{ (if } a_0\not\equiv 0 \bmod p), \quad =\beta_1'+p^{a_0-a_1}\beta_0 \text{ (otherwise)}.$$

Then, by Lemmas 6.5, 6.8, 6.4 and Proposition 5.7, we see that the homomorphism

$$h\colon G\to \tilde{J}(L_0^n(p^2)), \quad h\beta_0=\alpha_0, \quad h\beta_1=\alpha_1,$$

is well-defined and epimorphic. To prove that h is isomorphic as claimed, we consider the diagram

$$\begin{array}{ccc} \tilde{K}(L_0^n(p^2)) & \xrightarrow{g} & G \\ \| & & \downarrow h \\ \tilde{K}(L_0^n(p^2)) & \xrightarrow{Jr} & \tilde{J}(L_0^n(p^2)), \end{array}$$

where the homomorphism g is defined for the generators of (6.3) by

$$g(\sigma^j)=(-1)^{j-1}\beta_0 \qquad \text{if } 1\leqq j<p,$$

$$g\sigma(1,j)=\begin{cases} (-1)^j\beta_1+(-1)^j\theta'(p+j)p^{a_0-a_1}\beta_0 & \\ \qquad\qquad (\text{if } b_1\leqq j<b_1+p-1 \text{ or } j<b_1-(p-1)^2) & \\ (-1)^j\beta_1 \quad (\text{otherwise}). & \end{cases} \tag{6.11}$$

If it is proved that g is well-defined and

$$g(\sigma^j)=(-1)^{j-1}\theta(j)(\beta_0-\beta_1)+(-1)^{j-1}\beta_1 \text{ for } 1\leqq j<p^2, \tag{6.12}$$

then the theorem is proved as follows: According to (5.8), Lemma 6.7 and the definition of h, we see that the above diagram is commutative and so Ker $g\subset$ Ker Jr. On the other hand, for $0\leqq j<p^2-p$, we have

$$g(\sigma(1)\sigma^j)=\sum_{i=1}^{p}(-1)^{i+j-1}\binom{p}{i}\theta(i+j)(\beta_0-\beta_1)+\sum_{i=1}^{p}(-1)^{i+j-1}\binom{p}{i}\beta_1=(-1)^j\beta_1$$

by $\sigma(1)=(1+\sigma)^p-1$, (6.12) and Lemma 5.12. This and the first equality of (6.11) show that $g(\text{Ker } Jr)=0$ by Proposition 5.7. Thus we see that Ker $g=$ Ker Jr and h is isomorphic since g is epimorphic, and the theorem is proved.

Proof that g is well-defined. For the case $b_1\leqq j<b_1+p-1$, the order of $\sigma(1,j)$ is p^{a_1} by (6.3), and it is clear that $p^{a_1}(\beta_1+\theta'(p+j)p^{a_0-a_1}\beta_0)=0$ if $a_0\not\equiv 0$ mod p by (6.10). If $a_0\equiv 0$ mod p, then $b_1>p^2-2p$ by Lemma 6.4, and so $[(p+j-1)/(p-1)]=p$ and $\theta'(p+j)\equiv -1$ mod p by (5.15) and (6.6). Thus

$$p^{a_1}(\beta_1+\theta'(p+j)p^{a_0-a_1}\beta_0)=p^{a_1}\beta_1-p^{a_0}\beta_0=0$$

by (6.10). These show that g is well-defined for $\sigma(1, j)$ if $b_1 \leqq j < b_1 + p - 1$. The proofs for the other generators are easier.

Proof of (6.12). Suppose $p-1 \leqq n < p^2 - 1$ and $n < j < p^2$, then $\sigma^j = 0$ by (2.7). Also, $a_1 = 0$ and $1 \leqq a_0 \leqq [(j-1)/(p-1)] \leqq p$. If $a_0 < p$, then $\beta_1 = 0$ and $p^{a_0}\beta_0 = 0$ by (6.10), and $\theta(j) \equiv 0 \bmod p^{a_0}$ by (5.14). If $a_0 = p$, then $\beta_1 = p^{a_0}\beta_0$ by (6.10), and $\theta(j) \equiv -p^{a_0} \bmod p^{a_0+1}$ by (5.15). These show that the right hand side of (6.12) is 0, and we obtain (6.12).

For $n \geqq p^2 - 1 \geqq j$ or $p^2 - 1 > n \geqq j$, (6.12) is proved by the induction on j. If $b_1 \leqq j - p < b_1 + p - 1$, then we have $\sigma(1)\sigma^{j-p} = \sigma(1, j-p) - p^{a_1(p-1)}\sigma^j$ by (6.2), and so

$$\begin{aligned}(1+p^{a_1(p-1)})g(\sigma^j) &= g\Big(\sigma(1,j-p) - \sum_{i=1}^{p-1}\binom{p}{i}\sigma^{i+j-p}\Big) \\ &= (-1)^{j-1}\beta_1 + (-1)^{j-1}\theta(j)p^{a_1(p-1)}\beta_0 + (-1)^{j-1}\theta(j)(\beta_0 - \beta_1)\end{aligned}$$

inductively, using (6.11) and (6.6), by the same way as in the proof of Lemma 5.9. Also, the last is equal to

$$(1+p^{a_1(p-1)})((-1)^{j-1}\theta(j)(\beta_0-\beta_1) + (-1)^{j-1}\beta_1),$$

and so we have (6.12) since the order of G is a power of p. We can prove (6.12) similarly for the other j. q.e.d.

For the J-group $J(L^n(p^2))$ of the lens space mod p^2, we have the following theorem, which is proved by the same proofs of [6, Prop. 2] using the split-exact and commutative diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & K(L^n(p^2)) & \longrightarrow & K(L_0^n(p^2)) & \longrightarrow & 0 \\ & & & & \Big\downarrow r & & \Big\downarrow r & & \\ 0 & \longrightarrow & \widetilde{KO}(S^{2n+1}) & \longrightarrow & KO(L^n(p^2)) & \longrightarrow & KO(L_0^n(p^2)) & \longrightarrow & 0 \end{array}$$

(cf. [7, Lemma 2.4 (ii)]) and the fact that $\widetilde{KO}(S^{2n+1}) = \tilde{J}''(S^{2n+1}) = \tilde{J}(S^{2n+1})$ [2-II, (3.5)].

THEOREM 6.13. *For an odd prime p,* $J(L^n(p^2)) \cong J(L_0^n(p^2))$ *if* $n \not\equiv 0 \bmod 4$, $\cong J(L_0^n(p^2)) \oplus Z_2$ *if* $n \equiv 0 \bmod 4$.

COROLLARY 6.14. *For an odd prime p, the order of the element $Jr\sigma$ of* $J(L^n(p^2))$ *or* $J(L_0^n(p^2))$ *is equal to*

$$p^{a_0} \text{ if } a_0 \not\equiv 0 \bmod p \text{ or } a_0 = 0, \quad p^{a_0+1} \text{ if } a_0 \equiv 0 \bmod p \text{ and } a_0 > 0,$$

where $a_0 = [n/(p-1)]$.

PROOF OF THEOREMS 1.3 AND 1.6 FOR ODD PRIME p. We can prove them for an odd prime p using the above corollary, by the same way as in the proofs for $p=2$ in §4. q.e.d.

References

[1] J. F. Adams: *Vector fields on spheres*, Ann. of Math., **75** (1962), 603-632.
[2] ———: *On the groups $J(X)$*, I-III, Topology **2** (1963), 181-195, **3** (1965), 137-171, 193-222.
[3] ——— and G. Walker: *Complex Stiefel manifolds*, Proc. Camb. Phil. Soc., **61** (1965), 81-103.
[4] M. F. Atiyah: *Thom complexes*, Proc. London Math. Soc. (3), **11** (1961), 291-310.
[5] R. Bott: *Quelques remarques sur les théorèmes de périodicité*, Bull. Soc. Math. France, **87** (1959), 293-310.
[6] T. Kambe, H. Matsunaga and H. Toda: *A note on stunted lens space*, J. Math. Kyoto Univ., **5** (1966), 143-149.
[7] T. Kawaguchi and M. Sugawara: *K- and KO-rings of the lens space $L^n(p^2)$ for odd prime p*, this journal, 273-286.
[8] T. Kobayashi: *Non-immersion theorems for lens spaces*, II, J. Sci. Hiroshima Univ., Ser. A-I, **32** (1968), 285-292.
[9] ———: *On the odd order non-singular immersions of real projective spaces*, ibid., **33** (1969), 197-207.
[10] ——— and M. Sugawara: *K_Λ-rings of lens spaces $L^n(4)$*, this journal, 253-271.

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