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# *Semi-Infinite Programs and Conditional Gauss Variational Problems*

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## §1. Introduction

Let $X$ and $Y$ be real linear spaces which are in duality with respect to a bilinear functional $((\ ,\ ))_1$ and let $Z$ and $W$ be real linear spaces which are in duality with respect to a bilinear functional $((\ ,\ ))_2$. Denote by $w(X, Y)$ the weak topology on $X$. An infinite linear program for these paired spaces is a quintuple $(A, P, Q, y_0, z_0)$. In this quintuple, $A$ is a linear transformation from $X$ into $Z$ which is $w(X, Y)-w(Z, W)$ continuous, $P$ is a convex cone in $X$ which is $w(X, Y)$-closed, $Q$ is a convex cone in $Z$ which is $w(Z, W)$-closed, $y_0 \in Y$ and $z_0 \in Z$ are fixed elements. One of the basic problems in the theory of linear programming is to determine the value $M$ of the program defined by

$$M = \inf\{((x, y_0))_1;\ x \in S\},$$

where

$$S = \{x \in P;\ Ax - z_0 \in Q\}.$$

In this paper, we use the convention that the infimum and the supremum on the empty set $\phi$ are equal to $+\infty$ and $-\infty$ respectively.

The dual problem is to determine the value $M^*$ defined by

$$M^* = \sup\{((z_0, w))_2;\ w \in S^*\},$$

where

$$S^* = \{w \in Q^+;\ y_0 - A^*w \in P^+\}.$$

Here $A^*$ denotes the adjoint transformation of $A$, i.e., $A^*$ is the linear transformation from $W$ into $Y$ which is $w(W, Z)-w(Y, X)$ continuous and satisfies the relation

$$((Ax, w))_2 = ((x, A^*w))_1$$

for all $x \in X$ and $w \in W$ and $P^+$ and $Q^+$ are defined by

$$P^+ = \{y \in Y;\ ((x, y))_1 \geqq 0 \text{ for all } x \in P\},$$

$$Q^+ = \{w \in W;\ ((z, w))_2 \geqq 0 \text{ for all } z \in Q\}.$$

K. S. Kretschmer [8] investigated the following two problems:

(i) the existence of $x \in S$ or $w \in S^*$ which satisfies $M = ((x, y_0))_1$ or $M^* = ((z_0, w))_2$,

(ii) relations between values $M$ and $M^*$.

An snswer to problem (ii) is called a duality theorem. Some of the results in [8] have been further generalized by many mathematicians, for instance see R. Van Slyke and R. Wets [14] and M. Yamasaki [16; 17].

We say that an infinite linear program $(A, P, Q, y_0, z_0)$ is a *regular* semi-infinite linear program in the case where $Z$ and $W$ are $n$-dimensional Euclidean spaces, $((\ ,\ ))_2$ is defined by the usual inner product, $Q = \{0\}$, $z_0 = (c_1, \cdots, c_n) \in R^n$ and

$$Ax = (((x, y_1))_1, \cdots, ((x, y_n))_1)$$

for all $x \in X$, where $y_i$, $i = 1, \cdots, n$, are fixed elements of $Y$. We shall be concerned with problems (i) and (ii) for regular semi-infinite linear programs in this paper.

For later use, we shall consider in § 2 a slightly more generalized semi-infinite program than the regular semi-infinite linear program defined above. Several types of semi-infinite linear programs were discussed by A. Charnes, W.W. Cooper and K. O. Kortanek [4], R. J. Duffin and L. A. Karlovitz [5], R. J. Duffin [6] and K. Isii [7].

The conditional Gauss variational problem (= CGVP) investigated by M. Ohtsuka [9] may be regarded as a semi-infinite program with a nonlinear objective function. More precisely, let $K$ be a compact Hausdorff space, $\{g_k;\ k = 1, \cdots, n\}$ be a set of real-valued continuous functions on $K$, $\{c_k;\ k = 1, \cdots, n\}$ be a set of real numbers and let $G$ and $-f$ be lower semicontinuous functions on $K \times K$ and $K$ respectively which take values in $(-\infty, +\infty]$. Denote by $E_K$ the totality of non-negative Radon measures $\mu$ on $K$ such that

$$(\mu, \mu) = \iint G(u, v)\, d\mu(u)\, d\mu(v) < \infty.$$

CGVP is the problem to determine the value $V$ defined by

$$V = \inf\{(\mu, \mu) - 2\int f\, d\mu;\ \mu \in E_K(\{g_k\}, \{c_k\})\},$$

where

$$E_K(\{g_k\}, \{c_k\}) = \{\mu \in E_K;\ \int g_k\, d\mu = c_k \text{ for each } k\}.$$

Let $\mu^* \in E_K$. $\{g_k;\ k = 1, \cdots, n\}$ is called $\mu^*$-independent in [9] if there exists a set $\{\mu_k; k=1, \cdots, n\}$ of non-negative measures such that $\mu^*-\mu_k$ is a non-negative measure on $K$ for each $k$ and $\det(\int g_j d\mu_k)\neq 0$. One of our aims is to study the roles of this independence condition in the theory of semi-infinite programming. We shall discuss in § 4 Ohtsuka's independence condition in a more general form than the original one. By applying the results in § 4 to CGVP, we shall improve in § 8 some of the results in [9] relating to CGVP.

A superfeasibility condition will be introduced in § 5. This notion is closely related to the one investigated in [6]. A potential-theoretic semi-infinite linear program will be given in § 6 as an example of a regular semi-infinite linear program. Our aim in § 6 is to give an answer to the problem raised in connection with CGVP in [10] and remarked in [16], p. 354. Some gaps between equality constraints and inequality constraints will be clarified there. In § 7, we shall be interested in the problem how the values of regular semi-infinite linear programs change with $\{y_i\}$. Analogous problems were studied in [9] and [18].

We shall discuss CGVP in a slightly more generalized form than the above in § 8. The existence of optimal solutions for CGVP will be studied in § 9. We shall consider the value $V$ of CGVP as a function of $z=(c_1, \cdots, c_n)$ in § 10 and § 11. We shall examine the continuity of $V(z)$ and compute the directional derivatives of $V(z)$ by making use of Ohtsuka's independence condition. It must be noted that CGVP is a generalization of a classical quadratic program with linear constraints. Therefore some of the results in § 8 and § 10 may be regarded as new results for an indefinite quadratic program.

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## § 2. Semi-infinite programs

We begin with the definition of a semi-infinite program. Let $X$ be a real linear space and $R^n$ be the $n$-dimensional Euclidean space. Two $n$-dimensional Euclidean spaces are always considered to be in duality with respect to the bilinear functional $((\ ,\ ))_2$ defined by the usual inner product, i.e.,

$$((z^{(1)}, z^{(2)}))_2=\sum_{i=1}^{n} z_i^{(1)} z_i^{(2)} \quad \text{for } z^{(k)}=(z_1^{(k)}, \cdots, z_n^{(k)})$$

$(k=1, 2)$. Let $P$ be a convex cone in $X$, $\{f_i(x); i=1, \cdots, n\}$ be a set of finite real-valued additive and positively homogeneous functions on $P$, $f(x)$ be a finite real-valued function on $P$ which is convex and positively homogeneous, and let $z_0$ be a fixed element of $R^n$. Let $A$ be the transformation from $P$ into $R^n$ defined by

$$Ax=(f_1(x), \cdots, f_n(x)),$$

A semi-infinite program is defined as follows:

(I) Determine

$$M = \inf\{f(x);\ x \in S\},$$

where

$$S = \{x \in P;\ Ax = z_0\}.$$

As a dual program, we consider the following problem:

(II) Determine

$$M^* = \sup\{((z_0, w))_2;\ w \in S^*\},$$

where

$$S^* = \{w \in R^n;\ ((Ax, w))_2 \leqq f(x) \text{ for all } x \in P\}.$$

It is easily seen that problem (I) includes the regular semi-infinite linear program defined in § 1, by taking

$$f(x) = ((x, y_0))_1, \quad f_i(x) = ((x, y_i))_1 \quad (i = 1, \cdots, n).$$

For later use, we introduce some notations. Denote by $S_0$ and $S_0^*$ the sets of optimal solutions for problems (I) and (II):

$$S_0 = \{x \in S;\ M = f(x)\},$$

$$S_0^* = \{w \in S^*;\ M^* = ((z_0, w))_2\}.$$

Let $R_0$ be the set of non-negative real numbers and denote by $C^\circ$ the interior of a set $C$ in $R^n$ unless otherwise stated.

We shall utilize the following separation theorems:

PROPOSITION 1.[1)] *Let $C$ be a closed convex cone in $R^n$ and $v$ be an element of $R^n$ such that $v \notin C$. Then there exists $w \in R^n$ such that*

$$((v, w))_2 < 0 \leqq ((z, w))_2$$

*for all $z \in C$.*

PROPOSITION 2.[2)] *Let $C$ be a convex cone in $R^n$ such that $C \neq R^n$ and $v$ be a boundary point of $C$. Then there exists a non-zero $w \in R^n$ such that*

$$((v, w))_2 = 0 \leqq ((z, w))_2$$

1) [1], p. 73, Proposition 4.

2) [1], p. 77, Exercise 4.

*for all* $z \in C$.

## § 3. Duality theorems

We have

THEOREM 1. *It is always valid that* $M^* \leqq M$.

PROOF. By our convention, we may suppose that $S \neq \phi$ and $S^* \neq \phi$. Let $x \in S$ and $w \in S^*$. Then we have

$$f(x) \geqq ((Ax, w))_2 = ((z_0, w))_2,$$

and hence $M^* \leqq M$.

Let us define the set $H$ in $R^n \times R$ by

$$H = \{(Ax, r + f(x));\ x \in P \text{ and } r \in R_0\}.$$

We proved in [17]

THEOREM 2.[3] *Assume that the set $H$ is closed. If either $M$ or $M^*$ is finite, then $M = M^*$ and $S_0 \neq \phi$.*

We shall prepare

LEMMA 1. *Assume that the value $M$ is finite. Then there is a nonzero $(w, s) \in R^n \times R$ such that $s \geqq 0$ and*

$$Ms + ((z_0, w))_2 = 0 \leqq rs + ((z, w))_2$$

*for all $(z, r) \in H$. If $s > 0$, then it is valid that $M = M^*$ and $-w/s \in S_0^*$.*

PROOF. It is clear that $H$ is a convex cone in $R^{n+1}$, $H \neq R^{n+1}$ and $(z_0, M)$ is a boundary point of $H$. By means of Proposition 2, there exists a nonzero $(w, s) \in R^n \times R$ such that

$$Ms + ((z_0, w))_2 = 0 \leqq rs + ((z, w))_2$$

for all $(z, r) \in H$. Since $(0, r) \in H$ for all $r \in R_0$, we see that $s \geqq 0$. Let us consider the case where $s > 0$. Writing $\bar{w} = -w/s$, we have

$$M - ((z_0, \bar{w}))_2 = 0 \leqq r - ((z, \bar{w}))_2$$

for all $(z, r) \in H$. Since $(Ax, f(x)) \in H$ for all $x \in P$, we conclude that $\bar{w} \in S^*$ and

3) [17], Theorems 2 and 3. Cf. [16], p. 334, Lemma 3.

$$M^* \leqq M = ((z_0, \bar{w}))_2 \leqq M^*$$

by Theorem 1. Hence $M = M^*$ and $\bar{w} \in S_0^*$.

LEMMA 2. *Assume that the interior $A(P)^\circ$ of $A(P)$ is nonempty and $v \in A(P)^\circ$. If $((v, w))_2 = 0$ and $((Ax, w))_2 \geqq 0$ for all $x \in P$, then $w = 0$.*

PROOF. For any $z \in R^n$, there is $t > 0$ such that $v \pm tz \in A(P)^\circ$. Let $x_1$ and $x_2$ be elements of $P$ which satisfy $Ax_1 = v + tz$ and $Ax_2 = v - tz$. Then it follows that

$$0 \leqq ((Ax_1, w))_2 = t((z, w))_2,$$

$$0 \leqq ((Ax_2, w))_2 = -t((z, w))_2,$$

so that $((z, w))_2 = 0$. By the arbitrariness of $z$, we conclude that $w = 0$.

We have

THEOREM 3. *Assume that $z_0 \in A(P)^\circ$ and that the value $M$ is finite. Then it is valid that $M = M^*$ and $S_0^* \neq \phi$.*

PROOF. There exists a nonzero $(w, s) \in R^n \times R$ such that $s \geqq 0$ and $Ms + ((z_0, w))_2 = 0 \leqq rs + ((z, w))_2$ for all $(z, r) \in H$ by Lemma 1. If $s = 0$, then we have $((z_0, w))_2 = 0 \leqq ((Ax, w))_2$ for all $x \in P$, and hence $w = 0$ by Lemma 2. This is a contradiction. Thus we have $s > 0$. Our assertion follows from Lemma 1.

By applying Kretschmer's duality theorem to the regular semi-infinite linear program, we have

PROPOSITION 3.[4] *Let $(A, P, Q, y_0, z_0)$ be a regular semi-infinite linear program and denote by $s(Y, X)$ the Mackey topology on $Y$. If the $s(Y, X)$-interior $(P^+)^\circ$ of $P^+$ is nonempty and there is $w \in R^n$ such that $y_0 - A^*w \in (P^+)^\circ$, then the set $H$ is closed.*

Let $X \times R$ and $Y \times R$ be in duality with respect to the bilinear functional $((\ ,\ ))$ defined by

$$(((x, r), (y, s))) = ((x, y))_1 + rs$$

for all $(x, r) \in X \times R$ and $(y, s) \in Y \times R$ and let $G$ be the set in $Y \times R$ defined by

$$G = \{(A^*w + y, r - ((z_0, w))_2;\ w \in R^n,\ y \in P^+ \text{ and } r \in R_0\}.$$

The dual statement of Theorem 2 is as follows:

PROPOSITION 4.[5] *Let $(A, P, Q, y_0, z_0)$ be a regular semi-infinite linear*

4) [8], p. 236, Corollary 3.1.

5) [8], Theorem 3 and [16], Theorem 8*.

*program and assume that the set $G$ is $w(Y\times R, X\times R)$-closed. If either $M$ or $M^*$ is finite, then $M=M^*$ and $S_0^*\neq\phi$.*

We shall prove

PROPOSITION 5. *The following condition* (F) *implies that the set $G$ is $w(Y\times R, X\times R)$-closed*:

(F) *The relations $-A^*w\in P^+$ and $((z_0, w))_2\geqq 0$ imply that $w=0$.*

PROOF. Let $\{(y_\alpha, r_\alpha); \alpha\in D\}$ be any net in $G$ which $w(Y\times R, X\times R)$-converges to $(y, r)\in Y\times R$. Then there exists $w_\alpha\in R^n$ such that

$$y_\alpha-A^*w_\alpha\in P^+ \text{ and } r_\alpha\geqq-((z_0, w_\alpha))_2.$$

Let us put

$$|w|=(s_1^2+\cdots+s_n^2)^{1/2} \quad \text{for} \quad w=(s_1, \cdots, s_n)\in R^n.$$

Suppose that $\{|w_\alpha|; \alpha\in D\}$ is unbounded. Then there exists a subnet $\{w_\alpha; \alpha\in D_1\}$ of $\{w_\alpha; \alpha\in D\}$ such that $|w_\alpha|\to\infty$ along $D_1$. Writing $v_\alpha=w_\alpha/|w_\alpha|$, we can find a subnet $\{v_\alpha; \alpha\in D_2\}$ of $\{v_\alpha; \alpha\in D_1\}$ which converges to $\bar{v}\in R^n$, since $\{w\in R^n; |w|=1\}$ is compact. Then we have $|\bar{v}|=1$,

$$\begin{aligned}-((z_0, \bar{v}))_2&=\lim_{\alpha\in D_2}[-((z_0, v_\alpha))_2]=\lim_{\alpha\in D_2}[-((z_0, w_\alpha))_2]/|w_\alpha|\\&\leqq\lim_{\alpha\in D_2} r_\alpha/|w_\alpha|=0,\\((x, A^*\bar{v}))_1&=\lim_{\alpha\in D_2}((x, A^*v_\alpha))_1=\lim_{\alpha\in D_2}((x, A^*w_\alpha))_1/|w_\alpha|\\&\leqq\lim_{\alpha\in D_2}((x, y_\alpha))_1/|w_\alpha|=0\end{aligned}$$

for all $x\in P$. Thus we have

$$|\bar{v}|=1,\ ((z_0, \bar{v}))_2\geqq 0 \text{ and } -A^*\bar{v}\in P^+,$$

which contradicts condition (F). Therefore $\{|w_\alpha|; \alpha\in D\}$ is bounded and we may suppose that $\{w_\alpha; \alpha\in D\}$ converges to $\bar{w}$ by choosing a subnet if necessary. Then we see easily that

$$y-A^*\bar{w}\in P^+ \text{ and } r\geqq-((z_0, \bar{w}))_2,$$

and hence $(y, r)\in G$. Namely the set $G$ is $w(Y\times R, X\times R)$-closed.

PROPOSITION 6. *It is valid that $z_0\in A(P)^\circ$ if and only if $S\neq\phi$ and condition* (F) *is fulfilled.*

PROOF. Assume that $z_0\in A(P)^\circ$ and that $-A^*w\in P^+$ and $((z_0, w))_2\geqq 0$.

There is $\bar{x} \in P$ such that $A\bar{x} = z_0$. It follows that

$$0 \leqq ((z_0, w))_2 = ((A\bar{x}, w))_2 = ((\bar{x}, A^*w))_1 \leqq 0.$$

Denoting $v = -w$, we have $((z_0, v))_2 = 0$ and $((Ax, v))_2 \geqq 0$ for all $x \in P$, so that $v = 0$ by Lemma 2. Therefore $w = 0$ and condition (F) is satisfied. Next assume that $S \neq \phi$ and condition (F) is satisfied. Suppose that $z_0$ is a boundary point of $A(P)$. Then there exists a nonzero $w \in R^n$ such that

$$((z_0, w))_2 = 0 \geqq ((Ax, w))_2$$

for all $x \in P$ by Proposition 2. It follows that $-A^*w \in P^+$ and $((z_0, w))_2 = 0$, so that $w = 0$ by condition (F). This is a contradiction. Therefore $z_0$ is not a boundary point of $A(P)$. Since $z_0 \in A(P)$, we conclude that $z_0 \in A(P)^\circ$.

COROLLARY. *If $z_0 \in A(P)^\circ$, then the set $G$ is $w(Y \times R, X \times R)$-closed.*

This is an improvement of Proposition 7 in [16].

## §4. An independence condition

We introduce

DEFINITION 1. Let $x \in P$. We say that $\{f_i; i=1, \cdots, n\}$ is $x$-independent if there exists a set $\{x_j; j=1, \cdots, n\}$ in $P$ called *a system of components* of $x$ such that $x - x_j \in P$ for each $j$ and

$$\det(f_i(x_j)) \neq 0,$$

where $\det(a_{ij})$ means the determinant of a matrix $(a_{ij})$.

We have

THEOREM 4. *Assume that $\bar{x} \in S$ and $\{f_i; i=1, \cdots, n\}$ is $\bar{x}$-independent. Then it is valid that $z_0 \in A(P)^\circ$.*

PROOF. Let $\{x_j; j=1, \cdots, n\}$ be a system of components of $\bar{x}$. Suppose that $z_0 \notin A(P)^\circ$. Then $z_0$ is a boundary point of $A(P)$, since $z_0 \in A(P)$. By means of Proposition 2, there is a nonzero $w = (w_1, \cdots, w_n) \in R^n$ such that

$$((z_0, w))_2 = 0 \leqq ((Ax, w))_2$$

for all $x \in P$. From $\bar{x} - x_j \in P$ and $x_j \in P$ for each $j$, it follows that

$$0 \leqq ((Ax_j, w))_2 \leqq ((A\bar{x}, w))_2 = ((z_0, w))_2 = 0.$$

Thus we have

$$0 = ((Ax_j, w))_2 = \sum_{i=1}^{n} w_i f_i(x_j)$$

for each $j$. Since $\det(f_i(x_j)) \neq 0$, we conclude that $w_i=0$ for each $i$, i.e., $w=0$. This is a contradiction. Therefore we have $z_0 \in A(P)^\circ$.

By Theorems 3 and 4, we have

COROLLARY. *Assume that* $\bar{x} \in S$ *and* $\{f_i; i=1, \cdots, n\}$ *is* $\bar{x}$*-independent. If the value* $M$ *is finite, then* $M=M^*$ *and* $S_0^* \neq \phi$.

We shall prove

THEOREM 5. *Assume that* $z_0 \in A(P)^\circ$. *Then there exists* $\bar{x} \in S$ *such that* $\{f_i; i=1, \cdots, n\}$ *is* $\bar{x}$*-independent.*

PROOF. Since $z_0 \in A(P)^\circ$, there exists a set $\{z_j; j=1, \cdots, n\}$ of points in $A(P)$ such that

$$z_0 = \sum_{j=1}^{n} a_j z_j \text{ with } a_j > 0 \text{ and } \sum_{j=1}^{n} a_j = 1,$$

and $\{z_j; j=1, \cdots, n\}$ is linearly independent. There is $x_j \in P$ such that $Ax_j = z_j$ for each $j$. Let us take

$$\bar{x} = \sum_{j=1}^{n} a_j x_j.$$

Then we see easily that $\bar{x} \in S$ and $\{a_j x_j; j=1, \cdots, n\}$ is a system of components of $\bar{x}$. Namely $\{f_i; i=1, \cdots, n\}$ is $\bar{x}$-independent.

An essential role of our independence condition in the theory of semi-infinite programming is given by

THEOREM 6. *Assume that* $\bar{x} \in S_0$ *and* $\{f_i; i=1, \cdots, n\}$ *is* $\bar{x}$*-independent and that* $f$ *is additive. Then the set* $S_0^*$ *consists of only one point* $w=(w_1, \cdots, w_n)$ *and* $\{w_i; i=1, \cdots, n\}$ *is the solution of the equations*

$$\sum_{i=1}^{n} w_i f_i(x_j) = f(x_j), \tag{1}$$

*where* $\{x_j; j=1, \cdots, n\}$ *is a system of components of* $\bar{x}$.

PROOF. It is clear that $S_0^* \neq \phi$ by the corollary of Theorem 4. Let $w=(w_1, \cdots, w_n)$ be any element of $S_0^*$. Then it is valid that

$$f(\bar{x}) = ((z_0, w))_2 = ((A\bar{x}, w))_2.$$

Taking a system of components $\{x_j; j=1, \cdots, n\}$ of $\bar{x}$, we have

$$((Ax_j, w))_2 \leqq f(x_j),$$

$$((A(\bar{x}-x_j), w))_2 \leqq f(\bar{x}-x_j) = f(\bar{x}) - f(x_j),$$

so that

$$f(x_j)=((Ax_j,\ w))_2=\sum_{i=1}^{n} w_i f_i(x_j).$$

Namely $\{w_i;\ i=1, \cdots, n\}$ is the solution of the equations (1). Since det $(f_i(x_j)) \neq 0$, the solution of the equations (1) is uniquely determined, so that $S_0^*$ consists of only one point.

REMARK 1. The set $S_0^*$ may contain more than one point if we change the condition $\bar{x} \in S_0$ for the condition $\bar{x} \in S$ in Theorem 6. This is shown by Example 5 in § 6 below.

## § 5. A superfeasibility condition

We are concerned with the regular semi-infinite linear program defined in § 1 in this section.

DEFINITION 2. We say that a regular semi-infinite linear program $(A, P, Q, y_0, z_0)$ is superfeasible if there exists $\bar{x} \in S$ such that $((\bar{x},\ y))_1 > 0$ for all $y \in P^+ \cap A^*(W)$, $y \neq 0$.

In connection with condition (F) in § 3, we consider the following condition (SF):

(SF) The relations $-A^*w \in P^+$ and $((z_0,\ w))_2 \geqq 0$ imply that $A^*w=0$.

It is obvious that condition (F) implies condition (SF). If $\{y_i;\ i=1, \cdots, n\}$ is linearly independent, then conditions (F) and (SF) are equivalent.

We shall prove

PROPOSITION 7. *A regular semi-infinite linear program* $(A, P, Q, y_0, z_0)$ *is superfeasible if and only if* $S \neq \phi$ *and condition* (SF) *is fulfilled.*

PROOF. Assume that $(A, P, Q, y_0, z_0)$ is superfeasible and that $-A^*w \in P^+$ and $((z_0,\ w))_2 \geqq 0$. There exists $\bar{x} \in P$ such that $A\bar{x}=z_0$ and $((\bar{x},\ y))_1 > 0$ for all $y \in P^+ \cap A^*(W)$, $y \neq 0$. If $A^*w \neq 0$, then it follows that

$$0<((\bar{x},\ -A^*w))_1=-((A\bar{x},\ w))_2=-((z_0,\ w))_2 \leqq 0.$$

This is a contradiction. Therefore $A^*w=0$ and condition (SF) is satisfied. On the other hand, assume that $S \neq \phi$ and condition (SF) is fulfilled. Suppose that $(A, P, Q, y_0, z_0)$ is not superfeasible. Then for any $\bar{x} \in S$ there exists $\bar{y} \in P^+ \cap A^*(W)$ such that $\bar{y} \neq 0$ and $((\bar{x},\ \bar{y}))_1=0$. Let us choose $w \in W$ satisfying $\bar{y}=-A^*w \in P^+$. It is valid that

$$((z_0,\ w))_2=((A\bar{x},\ w))_2=((\bar{x},\ A^*w))_1=-((\bar{x},\ \bar{y}))_1=0,$$

so that $A^*w=0$ by condition (SF). This is a contradiction. Therefore $(A, P, Q, y_0, z_0)$ is superfeasible.

$\{y_i; i=1, \cdots, n\}$ is called $\bar{x}$-independent for $\bar{x} \in P$ if the set $\{f_i; i=1, \cdots, n\}$ of functions defined by $f_i(x)=((x, y_i))_1$ for each $i$ is $\bar{x}$-independent. It is easily seen that $\{y_i; i=1, \cdots, n\}$ is linearly independent whenever there exists $\bar{x} \in P$ such that $\{y_i; i=1, \cdots, n\}$ is $\bar{x}$-independent.

By means of Theorems 4 and 5, Propositions 6 and 7 and the above observation, we have

THEOREM 7. *Consider a regular semi-infinite linear program* $(A, P, Q, y_0, z_0)$ *and assume that* $\{y_i; i=1, \cdots, n\}$ *is linearly independent. Then the following statements are equivalent*:

(a) $z_0 \in A(P)^\circ$.

(b) *There exists* $\bar{x} \in S$ *such that* $\{y_i; i=1, \cdots, n\}$ *is* $\bar{x}$*-independent.*

(c) $S \neq \phi$ *and condition* (F) *is satisfied.*

(d) $S \neq \phi$ *and condition* (SF) *is satisfied.*

(e) $(A, P, Q, y_0, z_0)$ *is superfeasible.*

We shall prove

THEOREM 8. *Assume that a regular semi-infinite linear program* $(A, P, Q, y_0, z_0)$ *is superfeasible and that the value* $M$ *is finite. Then it is valid that* $M=M^*$ *and* $S_0^* \neq \phi$.

PROOF. In the case where $\{y_i; i=1, \cdots, n\}$ is linearly independent, our assertion follows from Theorems 3 and 7. We consider the case where $\{y_i; i=1, \cdots, n\}$ is linearly dependent. We may assume that $\{y_i; i=1, \cdots, p\}$ $(1 \leqq p < n)$ is linearly independent and

$$y_{p+j} = \sum_{i=1}^{p} a_{ji} y_i \text{ with } a_{ji} \in R$$

for each $j$, $1 \leqq j \leqq n-p$. Then we have

$$c_{p+j} = \sum_{i=1}^{p} a_{ji} c_i \quad 1 \leqq j \leqq n-p,$$

so that

$$M = \inf\{((x, y_0))_1; x \in S'\},$$

where

$$S' = \{x \in P; ((x, y_i))_1 = c_i, 1 \leqq i \leqq p\}.$$

Let $T$ be the linear transformation from $X$ into $R^p$ defined by

$$Tx = (((x, y_1))_1, \cdots, ((x, y_p))_1)$$

and set $z_0' = (c_1, \cdots, c_p)$. Then we have $A^*(R^n) = T^*(R^p)$ by the above obseravation. It follows that the regular semi-infinite linear program $(T, P, \{0\}, y_0, z_0')$ is superfeasible. Therefore there exists $\bar{r}=(r_1, \cdots, r_p)\in R^p$ such that

$$M=\sum_{i=1}^{p} c_i r_i \quad \text{and} \quad y_0-\sum_{i=1}^{p} r_i y_i \in P^+.$$

Writing $\bar{w}=(r_1, \cdots, r_p, 0, \cdots, 0)\in R^n$, we see that $\bar{w}\in S^*$ and $M=((z_0, \bar{w}))_2$. This completes the proof.

## §6. Potential-theoretic semi-infinite linear programs

As an example of a regular semi-infinite linear program, we shall give a potential-theoretic linear program.

Let $K$ be a compact Hausdorff space, $M(K)$ be the totality of Radon measures on $K$ of any sign, $M^+(K)$ be the subset of $M(K)$ which consists of non-negative measures, $C(K)$ be the totality of finite real-valued continuous functions on $K$ and $C^+(K)$ be the subset of $C(K)$ which consists of non-negative functions. It is easily seen that $M(K)$ and $C(K)$ are real linear spaces which are in duality with respect to the bilinear functional $((\ ,\ ))_1$ defined by

$$((\nu, f))_1=\int f d\nu \quad \text{for } \nu\in M(K) \text{ and } f\in C(K).$$

Let us take

$$X=M(K),\ Y=C(K),\ P=M^+(K),\ y_i=g_i\in C(K)\ (i=0, 1, \cdots, n),$$

and call the regular semi-infinite linear program $(A, P, Q, y_0, z_0)$ the potential-theoretic semi-infinite linear program. We note that the notion of $\nu$-independence of $\{g_i; i=1, \cdots, n\}$ $(\nu\in M^+(K))$ coincides with the one introduced by Ohtsuka [9].

We shall discuss the question whether the condition that $M$ is finite and $z_0>0$, i.e., $c_i>0\,(i=1, \cdots, n)$, plays an essential role for problems (i) and (ii) in § 1 or not. This problem was raised in connection with the conditional Gauss variational problem in [10] and remarked in [16], p. 354.

We have examples which show respectively

1. $M$ is finite and $S_0=\phi$.
2. $M$ is finite and $S^*=\phi$.
3. $-\infty<M^*<M<\infty$.
4. $M^*$ is finite and $S_0^*=\phi$.

EXAMPLE 1. Let $K$ be the Alexandroff one point compactification $\{N, \alpha\}$ of the discrete space $N$ of all natural numbers. Let us take

$$c_1=1,$$

$$g_0(n)=1/n^2,\ g_0(\alpha)=0,$$

$$g_1(n)=1/n,\quad g_1(\alpha)=0.$$

It is easily seen that $M=0$ and $S_0=\phi$.

EXAMPLE 2. Let $K=\{N, \alpha\}$, $c_1=c_2=1$,

$$g_0(1)=1,\ g_0(n)=-1/n\quad (n\neq 1),\ g_0(\alpha)=0,$$

$$g_1(1)=1,\ g_1(n)=-1/n^2\quad (n\neq 1),\ g_0(\alpha)=0,$$

$$g_2(1)=1,\ g_2(n)=g_2(\alpha)=0\quad (n\neq 1).$$

It is easily verified that $M=1$. If $w=(r_1, r_2)\in S^*$, then

$$-1/n+r_1/n^2\geqq 0\quad (n\in N,\ n\neq 1)$$

so that $\infty > r_1\geqq n\,(n=2, 3, \cdots)$. This is a contradiction. Therefore $S^*=\phi$.

EXAMPLE 3. Let $K=\{N, \alpha\}$, $c_1=c_2=1$,

$$g_0(1)=1,\ g_0(n)=g_0(\alpha)=0\quad (n\neq 1),$$

$$g_1(n)=1/n,\ g_1(\alpha)=0,$$

$$g_2(1)=1,\ g_2(n)=1/(n+1)\quad (n\neq 1),\ g_2(\alpha)=0.$$

Then we have $M^*=0<1=M$. In fact, it follows from $\nu\in S$ that

$$\nu_1+\sum_{n=2}^{\infty}\nu_n/n=1\quad\text{and}\quad \nu_1+\sum_{n=2}^{\infty}\nu_n/(n+1)=1,$$

where $\nu_n=\nu(\{n\})\geqq 0$ and $\nu_\alpha=\nu(\{\alpha\})\geqq 0$. We have easily that $\nu_1=1$, $\nu_n=0$ $(n\neq 1)$ and $\nu_\alpha\geqq 0$, and hence $M=1$. On the other hand, we derive from $w=(r_1, r_2)\in S^*$ that

$$r_1+r_2\leqq 1\quad\text{and}\quad r_1/n+r_2/(n+1)\leqq 0\quad (n\neq 1).$$

It follows that $((z_0, w))_2=r_1+r_2\leqq 0$. Since $(0, 0)\in S^*$, we conclude that $M^*=0$.

EXAMPLE 4. Let $K=\{N, \alpha\}$, $c_1=c_2=1$,

$$g_0(n)=-1/n,\ g_0(\alpha)=0,$$

$$g_1(n)=g_1(\alpha)=1,$$

$$g_2(n)=(1+1/n^2)^{1/2},\ g_2(\alpha)=1.$$

First we show that $M^*=0$. If $w=(r_1, r_2)\in S^*$, then

$$r_1+r_2(1+1/n^2)^{1/2} \leqq -1/n,$$

$$r_1+r_2\leqq 0,$$

so that $M^*\leqq 0$. Define $w^{(k)}=(r_1^{(k)}, r_2^{(k)})\in R^2$ by

$$r_1^{(k)}=k \text{ and } r_2^{(k)}=-(1+k^2)^{1/2} \ (k\in N).$$

Then we have $w^{(k)}\in S^*$ and

$$((z_0, w^{(k)}))_2=r_1^{(k)}+r_2^{(k)}=-[k+(1+k^2)^{1/2}]^{-1}.$$

Letting $k\to\infty$, we conclude that $M^*=0$. Next we show that $S_0^*=\phi$. Supposing the contrary, we can find $w=(r_1, r_2)\in R^2$ such that

$$r_1+r_2=0 \text{ and } r_1+(1+1/n^2)^{1/2}r_2\leqq -1/n.$$

It follows that

$$[-1+(1+1/n^2)^{1/2}]r_2\leqq -1/n,$$

and hence

$$-\infty<r_2\leqq -n[1+(1+1/n^2)^{1/2}]$$

for all $n\in N$. This is a contradiction. Therefore $S_0^*=\phi$.

REMARK 2. In Examples 2, 3 and 4, $\{g_1, g_2\}$ is not $\nu$-independent for any $\nu\in S$.

Some of the examples in [4], [5], [6] and [14] show the duality gaps in semi-infinite linear programming problems. However none of them satisfy our assumption that $M$ is finite and $z_0>0$.

Next we are concerned with Remark 1 in § 4.

EXAMPLE 5. Let $K=\{N, \alpha\}$, $c_1=1/3$, $c_2=1/4$, and let $\{g_1, g_2\}$ be the same as in Example 3. Define $g_0$ by

$$g_0(1)=g_0(2)=1,\ g_0(n)=g_0(\alpha)=0 \quad (n\neq 1, 2).$$

If $\nu\in S$, then it is valid that

$$\nu_1+\nu_2/2+\nu_3/3+a=1/3,$$

$$\nu_1+\nu_2/3+\nu_3/4+b=1/4,$$

$$a=\sum_{n=4}^{\infty}\nu_n/n,\quad b=\sum_{n=4}^{\infty}\nu_n/(n+1),$$

where $\nu_n$ is the same as in Example 3. We have

$$M=\inf\{\nu_1+\nu_2;\ \nu\in S\}\geqq 0.$$

Denote by $\varepsilon_u$ the unit point measure at $u\in K$. Since $\varepsilon_3\in S$, we have $M=0$. If $\nu\in S_0$, then $\nu_1=\nu_2=0$, so that

$$\nu_3+3a=1 \text{ and } \nu_3+4b=1.$$

It follows that

$$0=4b-3a=\sum_{n=4}^{\infty}(n-3)\nu_n/n(n+1),$$

and hence $\nu_n=0\,(n\geqq 4)$. Therefore $S_0=\{\varepsilon_3+t\varepsilon_\alpha;\ t\in R_0\}$ and $\{g_1, g_2\}$ is not $\nu$-independent for any $\nu\in S_0$. Let us consider $\bar\nu=\varepsilon_1/12+\varepsilon_2/2$. Then it is valid that $\bar\nu\in S$ and $\{g_1, g_2\}$ is $\bar\nu$-independent. We can easily verify that

$$S_0^*=\{(w_1, w_2);\ 0\leqq w_1\leqq 18,\ 4w_1+3w_2=0\}.$$

Namely $S_0^*$ contains more than one point. We observe that

$$A(P)=\{(r_1, r_2);\ 0\leqq r_1,\ 2r_1/3\leqq r_2\leqq r_1\}.$$

Now we apply Theorem 2 and Proposition 3 to our problem. Since $s(C(K), M(K))$ coincides with the topology induced by the norm $\|f\|=\sup\{|f(u)|;\ u\in K\}$ on $C(K)$ (cf. [2]) and the $s(C(K), M(K))$-interior $(P^+)^\circ$ of $P^+=C^+(K)$ is equal to the set $\{f\in C(K);\ f>0 \text{ on } K\}$, we have

THEOREM 9. *If the value $M$ is finite and $g_i>0$ on $K$ for some $i\,(i=0, 1, \cdots, n)$, then $M=M^*$ and $S_0\neq\phi$.*

PROOF. It is enough to show that there exists $w=(w_1, \cdots, w_n)\in R^n$ such that

$$g_0-\sum_{i=1}^{n}w_i g_i>0 \quad \text{on } K.$$

This is easily verified if any one of $g_i$ belongs to $(P^+)^\circ$.

We remark that the assumptions in Theorem 9 do not always imply that $S_0^*\neq\phi$. This is shown by

EXAMPLE 6. Let $K=\{N, \alpha\}$, $c_1=c_2=1$,

$$g_0(n)=g_0(\alpha)=1,$$

$$g_1(n)=3/(2-1/n),\ g_1(\alpha)=3/2,$$

$$g_2(n)=[2+(1+1/n^2)^{1/2}]/(2-1/n),\ g_2(\alpha)=3/2.$$

It is easily seen that $S=S_0=\{(2/3)\varepsilon_\alpha\}$, since $g_1(n)<g_2(n)$ for all $n\in N$ and $g_1(\alpha)=g_2(\alpha)$. It follows from Theorem 9 that $M=M^*=2/3$. We show that $S_0^*=\phi$. Suppose that $w=(w_1, w_2)$ is an element of $S_0^*$. Then we must have

$$w_1+w_2=2/3,$$

$$3w_1+[2+(1+1/n^2)^{1/2}]w_2\leqq 2-1/n$$

for all $n\in N$. This is impossible, since

$$3(2/3-w_2)+[2+(1+1/n^2)^{1/2}]w_2-(2-1/n)$$

$$=[(1+1/n^2)^{1/2}-1]w_2+1/n$$

$$=[1+w_2/n(1+(1+1/n^2)^{1/2})]/n>0$$

for sufficiently large $n$. Therefore $S_0^*=\phi$.

REMARK 3. Example 6 shows a gap between equality constraints and inequality constraints in semi-infinite linear programs. Let us recall the potential-theoretic linear program in [16] which are concerned with the problems to determine the values $\tilde{M}$ and $\tilde{M}^*$ defined by

$$\tilde{M}=\inf\{((\mu, g_0))_1;\ \mu\in\tilde{S}\},\ \tilde{M}^*=\sup\{((z_0, w))_2;\ w\in\tilde{S}^*\},$$

where

$$\tilde{S}=\{\mu\in M^+(K);\ A\mu-z_0\in R_0^n\},\ \tilde{S}^*=\{w\in R_0^n;\ g_0-A^*w\in C^+(K)\}.$$

It was proved in [16] that $\tilde{M}=\tilde{M}^*$ and there exist $\bar{\mu}\in\tilde{S}$ and $\bar{w}\in\tilde{S}^*$ such that $\tilde{M}=((\bar{\mu}, g_0))_1$ and $\tilde{M}^*=((z_0, \bar{w}))_2$ whenever $z_0>0$, $g_i>0$ on $K$ for some $i$ $(i=0, 1, \cdots, n)$ and $M$ is finite.

## § 7. Change of extremal values

We shall be concerned with the problem how the value of a regular semi-infinite linear program $(A, P, Q, y_0, z_0)$ in § 1 changes as $\{y_i\}$ changes. Recall that $Q=\{0\}$ as agreed in § 1. Ohtsuka discussed an analogous problem related to the conditional Gauss variational problem ( [9], p. 228). We refer to [18] for another analogous problem.

Let $\{y_j^{(k)}\}$ be a sequence in $Y$ which $w(Y, X)$-converges to $y_i$ for each

$i$ $(i=0, 1, \cdots, n)$. Let us put

$$A_k x = (((x, y_1^{(k)}))_1, \cdots, ((x, y_n^{(k)}))_1),$$

$$M_k = \inf\{((x, y_0^{(k)}))_1;\ x \in S^{(k)}\},$$

$$M_k^* = \sup\{((z_0, w))_2;\ w \in S^{*(k)}\},$$

$$S^{(k)} = \{x \in P;\ A_k x = z_0\},$$

$$S^{*(k)} = \{w \in R^n;\ y_0^{(k)} - A_k^* w \in P^+\},$$

$$S_0^{(k)} = \{x \in S^{(k)};\ M_k = ((x, y_0^{(k)}))_1\},$$

$$S_0^{*(k)} = \{w \in S^{*(k)};\ M_k^* = ((z_0, w))_2\},$$

$$|w| = (w_1^2 + \cdots + w_n^2)^{1/2} \ \text{ for } \ w = (w_1, \cdots, w_n) \in R^n.$$

We shall prove

THEOREM 10. *Assume that $z_0 \in A(P)^\circ$ and that $M$ is finite. Then it is valid that $\overline{\lim_{k\to\infty}}\, M_k \leqq M$.*

PROOF. By choosing a subsequence if necessary, we may assume from the beginning that $M_k > -\infty$ for all $k$ and that $\lim_{k\to\infty} M_k$ exists and $\lim_{k\to\infty} M_k > -\infty$. First we observe that $M = M^*$ and $S_0^* \neq \phi$ by Theorem 3. We show that

$$A(P)^\circ \subset \bigcup_{k=m}^{\infty} A_k(P) \tag{2}$$

for all $m$. Let $z \notin \bigcup_{k=m}^{\infty} A_k(P)$. Then $z$ is a boundary point or an exterior point of $A_k(P)$ for each $k$, $m \leqq k < \infty$. By making use of Propositions 1 and 2, we can find $w^{(k)} \in R^n$ such that

$$|w^{(k)}| = 1 \ \text{ and } \ ((z, w^{(k)}))_2 \geqq ((A_k x, w^{(k)}))_2$$

for all $x \in P$. We may suppose that $\{w^{(k)}\}$ converges to $\bar{w}$. It follows that

$$|\bar{w}| = 1 \ \text{ and } \ ((z, \bar{w}))_2 \geqq ((Ax, \bar{w}))_2 = ((x, A^*\bar{w}))_1$$

for all $x \in P$. Since $P$ is a cone, we have

$$((z, \bar{w}))_2 \geqq 0 \ \text{ and } \ -A^*\bar{w} \in P^+.$$

Taking $v \in S^*$, we see that $v + t\bar{w} \in S^*$ for all $t \in R_0$. Suppose that $z \in A(P)^\circ$ and let us consider

$$M^*(z)=\sup\{((z, w))_2;\ w\in S^*\}.$$

Then we have by Theorem 1

$$\infty>M^*(z)\geqq((z, v+t\bar{w}))_2=((z, v))_2+t((z, \bar{w}))_2$$

for all $t\in R_0$, and hence $((z, \bar{w}))_2=0$. By means of Lemma 2 we conclude that $\bar{w}=0$, which is a contradiction. Therefore $z\notin A(P)^\circ$ and the relation (2) is established. Suppose that $A_k(P)^\circ=\phi$ for all $k$, $k\geqq m$. Then $A_k(P)$ is contained in a hyperplane[6] and $\mu(A_k(P))=0$ for each $k$, $k\geqq m$, where $\mu$ denotes Lebesgue's outer measure in $R^n$. It follows from (2) that

$$0=\sum_{k=m}^{\infty}\mu(A_k(P))\geqq\mu(A(P)^\circ)>0,$$

which is a contradiction. Therefore $A_k(P)^\circ\neq\phi$ for some $k\geqq m$. By the arbitrariness of $m$, we may suppose that $A_k(P)^\circ\neq\phi$ and $z_0\in A_k(P)^\circ$ for each $k$. There exists $w^{(k)}\in R^n$ by Theorem 3 such that

$$M_k=M_k^*=((z_0, w^{(k)}))_2 \text{ and } y_0^{(k)}-A_k^*w^{(k)}\in P^+.$$

We show that $\{|w^{(k)}|\}$ is bounded. Supposing the contrary, we may consider that $|w^{(k)}|\to\infty$ as $k\to\infty$. Writing $v^{(k)}=w^{(k)}/|w^{(k)}|$, we may assume that $\{v^{(k)}\}$ converges to $\bar{v}$. Then we have

$$((z_0, \bar{v}))_2=\lim_{k\to\infty}((z_0, v^{(k)}))_2=\lim_{k\to\infty}M_k/|w^{(k)}|\geqq 0,$$

$$((x, A^*\bar{v}))_1=\sum_{i=1}^{n}\bar{v}_i((x, y_i))_1=\lim_{k\to\infty}\sum_{i=1}^{n}v_i^{(k)}((x, y_i^{(k)}))_1$$

$$=\lim_{k\to\infty}((x, A_k^*v^{(k)}))_1\leqq\lim_{k\to\infty}((x, y_0^{(k)}))_1/|w^{(k)}|=0$$

for all $x\in P$, and hence $-A^*\bar{v}\in P^+$. Taking $w\in S^*$, we have $w+t\bar{v}\in S^*$ and

$$\infty>M^*\geqq((z_0, w))_2+t((z_0, \bar{v}))_2$$

for all $t\in R_0$. It follows that $((z_0, \bar{v}))_2=0$ and $\bar{v}=0$ by Lemma 2. This contradicts $|\bar{v}|=1$. Therefore $\{|w^{(k)}|\}$ is bounded and we may suppose that $\{w^{(k)}\}$ converges to $\bar{w}$ Then it is easily seen that $\bar{w}\in S^*$ and

$$\lim_{k\to\infty}M_k=\lim_{k\to\infty}M_k^*=((z_0, \bar{w}))_2\leqq M^*\leqq M.$$

This completes the proof.

The inequality $\overline{\lim}_{k\to\infty}M_k\leqq M$ is not always valid if we omit the condition $z_0\in A(P)^\circ$ in Theorem 10. This is shown by

6) Cf. [1], p. 54, Exercise 9, a).

EXAMPLE 7. Let $X=Y=R^2$, $P=R_0^2$ (the positive orthant of $R^2$), $z_0=(1, 1)\in R^2$,

$$y_0^{(k)}=y_0=(1, 1/4),$$

$$y_1^{(k)}=y_1=(1, 1/2),$$

$$y_2^{(k)}=(1, 1/2+1/k),\ y_2=(1, 1/2),$$

$$((x, y))_1=r_1s_1+r_2s_2 \quad \text{for } x=(r_1, r_2) \text{ and } y=(s_1, s_2).$$

Then we have

$$S^{(k)}=\{(1, 0)\},$$

$$S=\{(r_1, r_2);\ r_1\geqq 0,\ r_2\geqq 0,\ r_1+r_2/2=1\},$$

so that $M_k=1>1/2=M$ for all $k$.

The inequality $\varliminf_{k\to\infty} M_k\geqq M$ is not always valid even if $M$ is finite and $z_0\in A(P)^\circ$. This is verified by

EXAMPLE 8. Let $X$, $Y$, $P$ and $((\ ,\ ))_1$ be the same as in Example 7 and let $z_0=1\in R$,

$$y_0^{(k)}=y_0=(1, 0),$$

$$y_1^{(k)}=(1, 1/k),\ y_1=(1, 0).$$

Then we have

$$S^{(k)}=\{(r_1, r_2);\ r_1\geqq 0,\ r_2\geqq 0,\ r_1+r_2/k=1\},$$

$$S=\{(1, r_2);\ r_2\geqq 0\},$$

so that $M_k=0<1=M$ for all $k$.

We have

THEOREM 11. *Let $Y$ be a normed space with a norm $\|\ \|$ and $X$ be the strong dual of $Y$. Assume that $\|y_i^{(k)}-y_i\|\to 0$ as $k\to\infty$ for each $i$ $(i=0, 1, \cdots, n)$ and that $x_k\in S_0^{(k)}$ for each $k$. If $\{\|x_k\|\}$[7] is bounded, then $\varliminf_{k\to\infty} M_k\geqq M$ holds.*

PROOF. Since $\{x_k\}$ is relatively $w(X, Y)$-compact ([2], p. 112, Proposi-

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7) For $x\in X$, $\|x\|$ is defined by
$\|x\|=\sup\{|((x, y))_1|;\ y\in Y,\ \|y\|=1\}$.

tion 1 ), we may assume from the beginning that $\{x_k\}$ $w(X, Y)$-converges to $\bar{x}$ and $\lim_{k\to\infty} M_k$ exists by choosing a subsequence if necessary. We have

$$|((x_k, y_i^{(k)}))_1 - ((\bar{x}, y_i))_1|$$
$$\leqq |((x_k, y_i^{(k)} - y_i))_1| + |((x_k - \bar{x}, y_i))_1|$$
$$\leqq ||x_k|| ||y_i^{(k)} - y_i|| + |((x_k - x, y_i))_1|$$

for each $i$ and $k$. Since $\{||x_k||\}$ is bounded, we have

$$\overline{\lim_{k\to\infty}} |((x_k, y_i^{(k)}))_1 - ((\bar{x}, y_i))_1| = 0$$

for each $i$. Since $((x_k, y_i^{(k)}))_1 = c_i$ $(i=1, \cdots, n)$ for all $k$, we have $((\bar{x}, y_i))_1 = c_i$ for each $i$, i.e., $\bar{x} \in S$. For $i=0$, it is valid that

$$\lim_{k\to\infty} M_k = \lim_{k\to\infty} ((x_k, y_0^{(k)}))_1 = ((\bar{x}, y_0))_1 \geqq M.$$

This completes the proof.

## §8. Conditional Gauss variational problem

In the rest of this paper, we shall study the conditional Gauss variational problem which is slightly diffrent from the problem considered in [9]. Note that we shall change some notations in the preceding sections.

Let $\Omega$ be a locally compact Hausdorff space and $G$ be a Borel measurable function on $\Omega \times \Omega$ which takes values in $(-\infty, +\infty]$. We assume that $G(u, v) = G(v, u)$ for all $u, v \in \Omega$ and $G$ is bounded below on every compact set. Such a function $G$ is called a kernel. A non-negative Radon measure $\mu$ with compact support $S\mu$ will be called simply a measure hereafter. Denote by $M^+(\Omega)$ the totality of measures on $\Omega$. Given $\mu, \nu \in M^+(\Omega)$, we define $G(u, \mu)$ and $(\nu, \mu)$ by

$$G(u, \mu) = \int G(u, v) d\mu(v),$$
$$(\nu, \mu) = \int G(u, \mu) d\nu(u),$$

and call them the potential of $\mu$ and the mutual energy of $\mu$ and $\nu$ respectively. We call $(\mu, \mu)$ simply the energy of $\mu$. Denote by $E$ the set of measures with finite energy.

We shall say that a property holds n. e. (=nearly everywhere) on a set $B \subset \Omega$ if it holds on B$'$ such that $B' \subset B$, $\mu(K) = 0$ for all compact sets $K \subset B - B'$

and $\mu \in E$. A kernel $G$ is termed to be of positive type or positive semidefinite if

$$(\mu-\nu, \mu-\nu)=(\mu, \mu)+(\nu, \nu)-2(\nu, \mu) \geqq 0$$

for all $\mu, \nu \in E$. In case $g$ and $h$ are extended real-valued functions on $\Omega$ which are $\mu$-summable for all $\mu \in E$, we set

$$g(u)-h(u)=0$$

at points $u$ where $g(u)=h(u)=\infty$ or $g(u)=h(u)=-\infty$.

Let $B$ be a set in $\Omega$ which is measurable with respect to every $\mu \in E$ and satisfies the condition that $E_B' \neq \{0\}$, where

$$E_B'=\{\mu \in E;\ \mu(\Omega-B)=0\}.$$

Let $f$ and $g_k$, $k=1, \cdots, n$, be real-valued functions on $B$ which are $\mu$-summable for every $\mu \in E_B'$ and let $\{c_k;\ k=1, \cdots, n\}$ be a set of real numbers. For simplicity, we shall write

$$\int f d\mu=\langle f, \mu\rangle.$$

A mutual energy $(\mu, \nu)$ can be written as $\langle G(\cdot, \mu), \nu\rangle$.

We shall consider the following class of measures:

$$S=E_B'(\{g_k\}, \{c_k\})=\{\mu \in E_B';\ \langle g_k, \mu\rangle=c_k \quad \text{for each } k\}.$$

We are interested in the problem of minimizing the expression (Gauss integral)

$$I(\mu)=(\mu, \mu)-2\langle f, \mu\rangle$$

for $\mu \in S$. Denote by $V$ the value of this preblem, i.e.,

$$V=\inf\{I(\mu);\ \mu \in S\}.$$

This is called the conditional Gauss variational problem.

In the case where $G=0$, $B$ is a compact set $K$, $f$ and $g_k$ are finite real-valued continuous functions, the above problem was discussed as a potential-theoretic semi-infinite linear program in § 6.

Ohtsuka [9] investigated the above problem in the case where $G$ is lower semicontinuous and $f$ is not necessarily $\mu$-summable for every $\mu \in E_B'$.

Let us define $S_0$ by

$$S_0=\{\mu \in S;\ V=I(\mu)\}.$$

In case $S_0 \neq \phi$, it is clear by our assumption for $f$ that $V$ is finite. We shall utilize the transformation $A$ from $E'_B$ into $R^n$ defined by

$$A\mu = (\langle g_1, \mu\rangle, \cdots, \langle g_n, \mu\rangle).$$

Writing $z_0 = (c_1, \cdots, c_n)$, we have

$$S = \{\mu \in E'_B;\ A\mu = z_0\}.$$

Thus the conditional Gauss variational problem may be regarded as a semi-infinite program with a nonlinear objective function $I(\mu)$. It must be observed that $E'_B$ is not necessarily a convex set and that $I(\mu)$ is not always a convex function on $E'_B$ even if $E'_B$ is a convex set.

We shall prove

THEOREM 12. *Assume that $E'_B$ is convex and that $\mu^* \in S_0$. If $z_0 \in A(E'_B)^\circ$, then there exists $\bar{w} = (r_1, \cdots, r_n) \in R^n$ such that*

(3) $$G(\cdot, \mu^*) - f \geqq \sum_{k=1}^{n} r_k g_k \quad \text{n.e. on } B,$$

(4) $$G(\cdot, \mu^*) - f \leqq \sum_{k=1}^{n} r_k g_k \quad \mu^*\text{-a.e.},$$

(5) $$\langle G(\cdot, \mu^*) - f, \mu^*\rangle = \sum_{k=1}^{n} r_k c_k.$$

*If, in addition, $G$ and $-f$ are lower semicontinuous, $f < \infty$ and each $g_k$ is finite valued and continuous on $B$, then*

(6) $$G(\cdot, \mu^*) - f \leqq \sum_{k=1}^{n} r_k g_k \quad \text{on } S\mu^* \cap B.$$

PROOF. From our assumption that $E'_B$ is convex, it follows that $S$ is convex. Let $\nu \in S$ and $t \in R_0$ with $0 < t < 1$. Then $t\nu + (1-t)\mu^* \in S$ and

$$\begin{aligned} I(\mu^*) &\leqq I(t\nu + (1-t)\mu^*) \\ &= t^2 I(\nu) + 2t(1-t)\,(\mu^*, \nu) + (1-t)^2 I(\mu^*) - 2t(1-t)\langle f, \mu^* + \nu\rangle, \end{aligned}$$

so that

$$t(2-t) I(\mu^*) \leqq 2t(1-t)\,(\mu^*, \nu) + t^2 I(\nu) - 2t(1-t)\langle f, \mu^* + \nu\rangle.$$

Dividing both sides by $t$ and letting $t \to 0$, we obtain

$$I(\mu^*) \leqq (\mu^*, \nu) - \langle f, \mu^* + \nu\rangle.$$

Thus we have

$$\langle G(\cdot, \mu^*)-f, \mu^*\rangle \leqq \langle G(\cdot, \mu^*)-f, \nu\rangle$$

for all $\nu \in S$. Writing $g_0=G(\cdot, \mu^*)-f$, we have

(7) $$M=\langle g_0, \mu^*\rangle=\min\{\langle g_0, \nu\rangle; \nu\in S\},$$

(8) $$V=\langle g_0, \mu^*\rangle-\langle f, \mu^*\rangle.$$

In order to apply Theorem 3 in § 3, let us choose

$$X=M^+(\Omega)-M^+(\Omega),\ P=E'_B,\ Z=W=R^n,$$

$$f_i(\nu)=\langle g_i, \nu\rangle\ (i=1, \cdots, n),$$

$$f(\nu)=\langle g_0, \nu\rangle.$$

Since $z_0\in A(E'_B)^\circ$ by our assumption, there exists $\bar{w}=(r_1, \cdots, r_n)$ such that

(9) $$((A\nu, \bar{w}))_2\leqq f(\nu)$$

for all $\nu\in E'_B$ and

(10) $$M=((z_0\ \bar{w}))_2$$

by Theorem 3. To prove the relation (3), we set

$$N=\{u\in B;\ g_0(u)-\sum_{k=1}^n r_k g_k(u)<0\}.$$

If we deny (3), then we can find a compact set $K_0$ and a measure $\nu_0\in E$ such that $K_0\subset N$ and $\nu_0(K_0)>0$. Let $\nu_1$ be the restriction of $\nu_0$ onto $K_0$. Then $\nu_1\in E'_B$ and

$$0>\langle g_0-\sum_{k=1}^n r_k g_k, \nu_1\rangle=f(\nu_1)-((A\nu_1, \bar{w}))_2,$$

which contradicts (9). Therefore the relation (3) is valid.

From (7) and (10), it follows that

(11) $$\langle g_0, \mu^*\rangle=\langle \sum_{k=1}^n r_k g_k, \mu^*\rangle.$$

We obtain (4) from (3) and (11).

Assume that $G$ and $-f$ are lower semicontinuous, $f<\infty$ and each $g_k$ is finite valued and continuous. Let us put

$$h=g_0-\sum_{k=1}^n r_k g_k.$$

We may suppose that $\mu^* \neq 0$. If there were a point $u_0 \in S\mu^* \cap B$ at which $h(u_0) > 0$, then $h(u) > 0$ on $B$ in a neighborhood of $u_0$ by the lower semicontinuity of $h$. Since $h \geqq 0$ n.e. on $B$ by (3) and $\mu^* \neq 0$, we have $\langle h, \mu^* \rangle > 0$, which contradicts (4). Therefore $h \leqq 0$ on $S\mu^* \cap B$. This completes the proof.

We have by (8) in the above proof

COROLLARY. *It is valid that*

$$V = I(\mu^*) = \sum_{k=1}^{n} c_k r_k - \langle f, \mu^* \rangle = 2 \sum_{k=1}^{n} c_k r_k - (\mu^*, \mu^*).$$

REMARK 4. $E'_B$ is convex if and only if $(\mu, \nu)$ is finite for all $\mu, \nu \in E'_B$. It is clear that $E'_B$ is convex whenever $G$ is of positive type or $G$ is bounded on $B \times B$.

For $\mu^* \in S_0$, we denote by $S_0^*(\mu^*)$ the set of points $\varpi$ which satisfy the relations (3) and (4). Note that the relation (5) follows from (3) and (4) and that the equality in the corollary of Theorem 12 holds for every $\varpi \in S_0^*(\mu^*)$ with $\varpi = (r_1, \cdots, r_n)$.

Let $\mu \in E'_B$. We say that $\{g_k; k=1, \cdots, n\}$ is $\mu$-independent if there exists a set $\{\mu_k; k=1, \cdots, n\}$ of measures in $M^+(S\mu)$ such that $\mu - \mu_k \in M^+(S\mu)$ for each $k$ and $\det(\langle g_j, \mu_k \rangle) \neq 0$, where

$$M^+(S\mu) = \{\nu \in M^+(\Omega);\ S\nu \subset S\mu\}.$$

The set $\{\mu_k; k=1, \cdots, n\}$ is called a system of components of $\mu$. A system of components $\{\mu_k; k=1, \cdots, n\}$ of $\mu$ is called to be *full* if $\mu = \sum_{k=1}^{n} \mu_k$. In this case we say that $\mu$ has a full system of components.

For $\mu \in E$, we define $C[\mu]$ by

$$C[\mu] = \{\nu \in M^+(S\mu);\ \mu - \nu \in M^+(S\mu)\}.$$

It is clear that $C[\mu]$ is convex and contains 0 and $\mu$.

We shall prove

LEMMA 3. *It is valid that* $C[\mu] \subset E$.

PROOF. Since $K = S\mu$ is compact, there is a number $b$ such that $G \geqq b$ on $K \times K$ by our assumption. Let $\nu \in C[\mu]$. Then we have $\mu - \nu \in M^+(K)$ and

$$G(\cdot, \nu) - b\nu(K) \leqq G(\cdot, \mu) - b\mu(K)$$

on $K$, so that

$$0 \leqq (\mu, \nu) - b\mu(K)\nu(K) \leqq (\mu, \mu) - b\mu(K)^2 < \infty.$$

Since $(\mu, \nu) = (\nu, \mu)$ by our assumption, we have

$$0 \leqq (\nu, \nu) - b\nu(K)^2 \leqq (\nu, \mu) - b\mu(K)\nu(K) < \infty,$$

and hence $\nu \in E$.

COROLLARY. *Let $\mu \in E'_B$ and $\{\mu_k ; k=1, \cdots, n\}$ be a system of components of $\mu$. Then $\mu_k \in E'_B$ and $\mu - \mu_k \in E'_B$ for each $k$.*

We have

PROPOSITION 8. *Assume that $E'_B$ is convex and that $\mu^* \in S_0$. If $\{g_k ; k=1, \cdots, n\}$ is $\mu^*$-independent, then $S_0^*(\mu^*)$ consists of only one point $\bar{w}(\mu^*)=(r_1, \cdots, r_n)$. If $\{\mu_k ; k=1, \cdots, n\}$ is a system of components of $\mu^*$, then $\{r_j ; j=1, \cdots, n\}$ is the solution of the equations*

$$\sum_{j=1}^{n} r_j \langle g_j, \mu_k \rangle = \langle G(\cdot, \mu^*) - f, \mu_k \rangle. \tag{12}$$

PROOF. Since $S_0^*(\mu^*)$ is the set of optimal solutions for the dual problem of the problem determining the value $M$ defined by (7) in Theorem 12, our assertion is an immediate consequence of Theorem 6 in § 4.

Without using a duality theorem, Ohtsuka proved the following result. Here we review his proof for completeness.

PROPOSITION 9.[8] *Assume that $\mu^* \in S_0$ and that $\{\mu_k ; k=1, \cdots, n\}$ is a system of components of $\mu^*$. Let $\{r_j ; j=1, \cdots, n\}$ be the solution of the equations*

$$\sum_{j=1}^{n} r_j \langle g_j, \mu_k \rangle = \langle G(\cdot, \mu^*) - f, \mu_k \rangle. \tag{13}$$

*Then it is valid that*

$$G(\cdot, \mu^*) \geqq f + \sum_{k=1}^{n} r_k g_k \quad \text{n.e. on } B, \tag{14}$$

$$G(\cdot, \mu^*) \leqq f + \sum_{k=1}^{n} r_k g_k \quad \mu_k\text{-a.e.} \tag{15}$$

*for each $k$.*

PROOF. Let $\nu$ be any measure of $E'_B$ and $\{t_k ; k=1, \cdots, n\}$ be the solution of the equations

$$\sum_{k=1}^{n} \langle g_j, \mu_k \rangle t_k = \langle g_j, \nu \rangle. \tag{16}$$

With a positive parameter $t$, we set

8) [9], p. 213, Theorem 2.1.

$$\mu(t)=\mu^*-t\sum_{k=1}^n t_k\mu_k+t\nu.$$

Since $(\mu^*-t\sum_{k=1}^n t_k\mu_k)-(1-t\sum_{k=1}^n |t_k|)\mu^*\in M^+(S\mu^*)$, $\mu(t)$ is a measure for sufficiently small $t$. In case $(\mu^*, \nu)$ is finite, we have $\mu(t)\in S$ by (16) and $I(\mu^*)=V\leqq I(\mu(t))$ for sufficiently small $t\geqq 0$. Thus we have

$$0\leqq \frac{dI(\mu(t))}{dt}\bigg|_{t=0}=2\langle G(\cdot, \mu^*), \nu-\sum_{k=1}^n t_k\mu_k\rangle-2\langle f, \nu-\sum_{k=1}^n t_k\mu_k\rangle.$$

Substituting (13) and (16), we obtain

$$\begin{aligned}\langle G(\cdot, \mu^*), \nu\rangle-\langle f, \nu\rangle &\geqq \sum_{k=1}^n t_k\langle G(\cdot, \mu^*)-f, \mu_k\rangle\\ &=\sum_{k=1}^n t_k\sum_{j=1}^n r_j\langle g_j, \mu_k\rangle\\ &=\sum_{j=1}^n r_j\sum_{k=1}^n\langle g_j, \mu_k\rangle t_k=\sum_{j=1}^n r_j\langle g_j, \nu\rangle.\end{aligned}$$

Namely we have

$$\langle G(\cdot, \mu^*), \nu\rangle\geqq\langle f+\sum_{j=1}^n r_j g_j, \nu\rangle. \tag{17}$$

This inequality is obvious in case $(\mu^*, \nu)=\infty$, so that (17) holds for all $\nu\in E_B'$. It is clear that (14) follows from (17).

Integrating (14) with respect to $\mu_k$, we obtain

$$\langle G(\cdot, \mu^*), \mu_k\rangle\geqq\langle f, \mu_k\rangle+\sum_{j=1}^n r_j\langle g_j, \mu_k\rangle.$$

By this relation and (13), we conclude (15).

In this result, $E_B'$ is not assumed to be convex. However $\{r_k\}$ seems to depend on both $\mu^*$ and $\{\mu_k\}$. Writing $\nu=\sum_{k=1}^n \mu_k$, we see that $\mu^*\neq\nu$ in general and that

$$G(\cdot, \mu^*)\leqq f+\sum_{k=1}^n r_k g_k \quad \nu-\text{a.e.}$$

by the relation (15). Therefore we may regard Theorem 12 and Proposition 8 as a partial generalization of Proposition 9. In order to generalize Propositions 8 and 9, we prepare

LEMMA 4. *Let $\mu^*\in S$ and assume that $\{g_k; k=1, \cdots, n\}$ is $\mu^*$-independent.*

*Then $\mu^*$ has a full system of components.*

Proof. Denote by $Q(\mu^*)$ the convex cone generated by $A(C[\mu^*])$, i.e., $z\in Q(\mu^*)$ if and only if there exist $\nu\in C[\mu^*]$ and $t\in R_0$ such that $z=tA\nu$. We first show that $z_0\in Q(\mu^*)^\circ$. Supposing the contrary, we see that $z_0$ is a boundary point of $Q(\mu^*)$, since $\mu^*\in C[\mu^*]$ and $A\mu^*=z_0$. There exists a non-zero $w=(w_1, \cdots, w_n)$ by Proposition 2 such that

$$((z_0, w))_2=0\leqq((z, w))_2$$

for all $z\in Q(\mu^*)$. Let $\{\mu_k; k=1, \cdots, n\}$ be a system of components of $\mu^*$. From $\mu_k\in C[\mu^*]$ and $\mu^*-\mu_k\in C[\mu^*]$ for each $k$, it follows that

$$0=((A\mu_k, w))_2=\sum_{j=1}^n w_j\langle g_j, \mu_k\rangle.$$

Since $\det(\langle g_j, \mu_k\rangle)\neq 0$, we conclude that $w_j=0$ for each $j$. This is a contradiction. Therefore $z_0\in Q(\mu^*)^\circ$. There exist a set $\{\nu_k; k=1, \cdots, n\}$ of measures of $C[\mu^*]$ and a set $\{s_k; k=1, \cdots, n\}$ of strictly positive numbers such that $\{A\nu_k; k=1, \cdots, n\}$ is linearly independent and

$$z_0=\sum_{k=1}^n s_k A\nu_k.$$

In the case where $s_0=\sum_{k=1}^n s_k\leqq 1$, we have

$$\bar\mu=\sum_{k=1}^n s_k\nu_k\in S \quad\text{and}\quad \nu=\mu^*-\bar\mu\in M^+(S\mu^*).$$

Choosing $\mu_k^*=s_k\nu_k+\nu/n$ for each $k$, we see that $\{\mu_k^*; k=1, \cdots, n\}$ is a full system of components of $\mu^*$. In the case where $s_0>1$, let us put $t_k=s_k/s_0$ for each $k$ and consider $\mu_0=\sum_{k=1}^n t_k\nu_k$. Then $\mu_0\in C[\mu^*]$ and $A\mu_0=z_0/s_0$. Taking $\nu_0=\mu^*-\mu_0\in M^+(S\mu^*)$ and $\mu_k^*=t_k\nu_k+\nu_0/n$, we have

$$\mu^*=\sum_{k=1}^n \mu_k^* \quad\text{and}\quad A\nu_0=(1-1/s_0)z_0.$$

In order to prove that $\{\mu_k^*; k=1,\cdots, n\}$ is a full system of components of $\mu^*$, it is enough to show that $\det(\langle g_j, \mu_k^*\rangle)\neq 0$, or equivalently, $\{A\mu_k^*; k=1, \cdots, n\}$ is linearly independent. Suppose that

$$\sum_{k=1}^n b_k A\mu_k^*=0.$$

Then it follows that

$$\begin{aligned}0&=\sum_{k=1}^n b_k t_k A\nu_k+(1-1/s_0)z_0\sum_{j=1}^n (b_j/n)\\ &=\sum_{k=1}^n b_k t_k A\nu_k+(1-1/s_0)\sum_{k=1}^n s_k A\nu_k\sum_{j=1}^n (b_j/n)\end{aligned}$$

$$=\sum_{k=1}^{n}[b_k t_k+(1-1/s_0)s_k\sum_{j=1}^{n}(b_j/n)]A\nu_k.$$

Since $\{A\nu_k;\, k=1, \cdots, n\}$ is linearly independent, we have

$$b_k t_k+(1-1/s_0)s_k\sum_{j=1}^{n}(b_j/n)=0,$$

or equivalently

$$b_k+(s_0-1)\sum_{j=1}^{n}(b_j/n)=0$$

for each $k$. We can easily conclude from this relation that $\sum_{k=1}^{n} b_k=0$ and hence $b_k=0$ for each $k$. Namely $\{A\mu_k^*;\, k=1, \cdots, n\}$ is linearly independent. This completes the proof.

We shall prove

THEOREM 13. *Assume that $\mu^*\in S_0$ and that $\{g_k;\, k=1, \cdots, n\}$ is $\mu^*$-independent. Then $S_0^*(\mu^*)$ consists of only one point $\bar{w}(\mu^*)=(r_1, \cdots, r_n)$. If $\{\mu_k;\, k=1, \cdots, n\}$ is a system of components of $\mu^*$, then $\{r_j;\, j=1, \cdots, n\}$ is the solution of the equations* (12). *It is valid that*

$$V=I(\mu^*)=((z_0, \bar{w}(\mu^*)))_2-\langle f, \mu^*\rangle$$

$$=2((z_0, \bar{w}(\mu^*)))_2-(\mu^*, \mu^*).$$

PROOF. Let $\{\mu_k^*;\, k=1, \cdots, n\}$ be a full system of components of $\mu^*$ and define $\{r_j;\, j=1, \cdots, n\}$ by

$$\sum_{j=1}^{n} r_j\langle g_j, \mu_k^*\rangle=\langle G(\cdot, \mu^*)-f, \mu_k^*\rangle.$$

It follows from Proposition 9 that

$$G(\cdot, \mu^*)\geqq f+\sum_{j=1}^{n} r_j g_j \quad \text{n.e. on } B, \tag{18}$$

$$G(\cdot, \mu^*)\leqq f+\sum_{j=1}^{n} r_j g_j \quad \mu^*-\text{a.e.}, \tag{19}$$

since $\mu^*=\sum_{k=1}^{n}\mu_k^*$. Therefore $\bar{w}=(r_1, \cdots, r_n)\in S_0^*(\mu^*)$ and $V=I(\mu^*)=((z_0, \bar{w}))_2-\langle f, \mu^*\rangle=2((z_0, \bar{w}))_2-(\mu^*, \mu^*)$. If $w=(s_1, \cdots, s_n)\in S_0^*(\mu^*)$, then

$$\sum_{j=1}^{n} s_j\langle g_j, \mu_k^*\rangle=\langle G(\cdot, \mu^*)-f, \mu_k^*\rangle=\sum_{j=1}^{n} r_j\langle g_j, \mu_k^*\rangle,$$

so that

$$\sum_{j=1}^{n}(s_j-r_j)\langle g_j,\ \mu_k^*\rangle=0.$$

Since $\det(\langle g_j,\ \mu_k^*\rangle)\neq 0$, we have $w=\bar{w}$. Namely $S_0^*(\mu^*)$ consists of only one point. Let $\{\mu_k;\ k=1,\ \cdots,\ n\}$ be a system of components of $\mu^*$. Then we have (12) by (18) and (19). This completes the proof.

THEOREM 14. *Assume that $G$ is of positive type and that $\mu^*$ and $\nu^*$ are elements of $S_0$. Then it is valid that $S_0^*(\mu^*)=S_0^*(\nu^*)$.*

PROOF. From $I(\mu^*)=I(\nu^*)=V$ and $(\mu^*+\nu^*)/2\in S$, it follows that

$$I(\mu^*)\leqq I((\mu^*+\nu^*)/2)=I(\mu^*)-(\mu^*-\nu^*,\ \mu^*-\nu^*)/4,$$

and hence $(\mu^*-\nu^*,\ \mu^*-\nu^*)\leqq 0$. Since $G$ is of positive type, we have $(\mu^*-\nu^*,\ \mu^*-\nu^*)=0$ and $G(\cdot,\ \mu^*)=G(\cdot,\ \nu^*)$ n.e. in $\Omega$. Consequently $(\mu^*,\ \mu^*)=(\mu^*,\ \nu^*)=(\nu^*,\ \nu^*)$ and $\langle f,\ \mu^*\rangle=\langle f,\ \nu^*\rangle$. Assume that $\bar{w}=(r_1,\ \cdots,\ r_n)\in S_0^*(\mu^*)$. By the above observation, we see that

$$G(\cdot,\ \nu^*)-f=G(\cdot,\ \mu^*)-f\geqq\sum_{k=1}^{n}r_k g_k \quad \text{n.e. on } B,$$

$$\begin{aligned}\langle G(\cdot,\ \nu^*)-f,\ \nu^*\rangle&=\langle G(\cdot,\ \mu^*)-f,\ \mu^*\rangle\\&=\sum_{k=1}^{n}r_k c_k\\&=\sum_{k=1}^{n}r_k\langle g_k,\ \nu^*\rangle=\left\langle\sum_{k=1}^{n}r_k g_k,\ \nu^*\right\rangle,\end{aligned}$$

and hence

$$G(\cdot,\ \nu^*)-f=\sum_{k=1}^{n}r_k g_k \quad \nu^*-\text{a.e.},$$

so that $\bar{w}\in S_0^*(\nu^*)$. Therefore $S_0^*(\mu^*)\subset S_0^*(\nu^*)$. Since the discussion is symmetric, we have $S_0^*(\nu^*)\subset S_0^*(\mu^*)$ and hence $S_0^*(\mu^*)=S_0^*(\nu^*)$.

COROLLARY. *Assume that $G$ is of positive type and let $\mu^*$ and $\nu^*$ be elements of $S_0$. If $\{g_k;\ k=1,\ \cdots,\ n\}$ is $\mu^*$-independent and $\nu^*$-independent, then it is valid that $\bar{w}(\mu^*)=\bar{w}(\nu^*)$.*

This is an improvement of Theorem 2. 3 in [9]. We observe that Theorem 14 and its corollary are not always valid if $G$ is not of positive type. This is shown by

EXAMPLE 9.[9] Let $\Omega=B=\{u_1,\ u_2\}$, $g_1(u_1)=g_1(u_2)=1$, $c_1=2$, $f(u_1)=1$, $f(u_2)=2$ and $G$ be given by

9) [9], p. 254, Example 2.

$$G(u_1, u_1)=1,\ G(u_1, u_2)=G(u_2, u_1)=G(u_2, u_2)=2.$$

It is clear that $\mu\in E=E_B'$ if and only if $\mu=x_1\varepsilon_{u_1}+x_2\varepsilon_{u_2}$ with $0\leqq x_1,\ x_2<\infty$. Our problem is to minimize

$$I(\mu)=x_1^2+2x_2^2+4x_1x_2-2x_1-4x_2$$

subject to

$$x_1+x_2=2,$$

$$x_1\geqq 0,\ x_2\geqq 0,$$

or equivalently to minimize

$$I(\mu)=-(x_1-1)^2+1$$

subject to

$$0\leqq x_1\leqq 2.$$

It is easily seen that $S_0=\{2\varepsilon_{u_1}, 2\varepsilon_{u_2}\}$. Let us take $\mu^*=2\varepsilon_{u_1}$ and $\nu^*=2\varepsilon_{u_2}$. Obviously $\{g_1\}$ is $\mu^*$-independent and $\nu^*$-independent and $G$ is not of positive type. By Theorem 13, we have $\bar{w}(\mu^*)=1$ and $\bar{w}(\nu^*)=2$. Namely $S_0^*(\mu^*)\neq S_0^*(\nu^*)$.

We have

THEOREM 15. *Assume that $G$ is of positive type and that $\mu^*\in S$ and $w=(r_1, \cdots, r_n)\in R^n$ satisfy the relations*

$$G(\cdot, \mu)-f\geqq\sum_{k=1}^n r_k g_k \quad n.e. \quad on\ B,$$

$$\langle G(\cdot, \mu^*)-f, \mu^*\rangle=\sum_{k=1}^n r_k c_k.$$

*Then it is valid that $I(\mu^*)=V$, i.e., $\mu^*\in S_0$.*

PROOF. Let $\nu$ be any element of $S$. Then we have

$$\begin{aligned}2(\mu^*, \nu)=2\langle G(\cdot, \mu^*), \nu\rangle &\geqq 2\langle f, \nu\rangle+2\sum_{k=1}^n r_k c_k\\ &=2\langle f, \nu\rangle+2\langle G(\cdot, \mu^*)-f, \mu^*\rangle\\ &=2\langle f, \nu\rangle+I(\mu^*)+(\mu^*, \mu^*).\end{aligned}$$

Since $G$ is of positive type, we have

$$I(\mu^*) \leqq I(\mu^*) + (\mu^* - \nu,\ \mu^* - \nu) \leqq I(\nu).$$

This completes the proof.

## §9 Existence of optimal solutions

We shall discuss the existence of measures $\mu \in S$ such that $V = I(\mu)$. It is rather difficult to find conditions which ensure the existence under general circumstances as in § 8. So we shall limit ourselves to the special case in which $B = K$ is a compact set, $G$ is lower semicontinuous, $f$ is upper semicontinuous and $f < \infty$ on $K$ and each $g_k$ is finite valued and continuous on $K$. This restriction will be preserved in the rest of this paper.

The topology on $M^+(K)$ induced by the weak topology $w(M(K), C(K))$ is called the vague topology (cf. § 6 for the definition of $M(K)$, $M^+(K)$ and $C(K)$). We say that a set $H \subset M^+(K)$ is vaguely bounded if $\sup\{\mu(K);\ \mu \in H\} < \infty$.

We shall use the following two facts which are well-known in potential theory.

Proposition 10.[10] *Any vaguely bounded set $H$ is relatively compact in $M^+(K)$ with respect to the vague topology.*

Proposition 11.[11] *The mutual energy $(\mu, \nu)$ is lower semicontinuous on $M^+(K) \times M^+(K)$ with respect to the vague topology.*

We shall prove

Lemma 5.[12] *Assume that $V$ is finite. Let $\{\mu_m\}$ be a sequence in $S$ such that $I(\mu_m)$ tends to $V$ as $m \to \infty$. Then $\{\mu_m\}$ is vaguely bounded whenever any one of the following conditions is satisfied:*

(C. 1) $\quad g_0 = \sum_{k=1}^{n} g_k > 0$ *on $K$.*

(C. 2) $\quad g_k > 0$ *on $K$ for some $k$.*

(C. 3) $\quad G$ *is of positive type and $f > 0$.*

(C. 4) $\quad (\mu, \mu) \geqq 0$ *for all $\mu \in E'_K$ and $\beta = \sup\{f(u);\ u \in K\} < 0$.*

(C. 5) $\quad c(K) = \inf\{(\mu, \mu);\ S\mu \subset K,\ \mu(K) = 1\} > 0.$

Proof. Let us put $a_k = \min\{g_k(u);\ u \in K\}$ $(k = 0, 1, \dots, n)$. From condition (C. 1), it follows that $a_0 > 0$ and

$$a_0 \mu_m(K) \leqq \int g_0\, d\mu_m = \sum_{k=1}^{n} \int g_k\, d\mu_m = \sum_{k=1}^{n} c_k.$$

10) Cf. [9], p. 187, Proposition 3 and [3].

11) Cf. [9], p. 187, Proposition 4.

12) Cf. [15], Lemma 3.

From condition (C. 2), it follows that $a_k>0$ and

$$a_k\mu_m(K)\leqq\int g_k d\mu_m=c_k.$$

From condition (C. 4), it follows that

$$I(\mu_m)\geqq-2\beta\mu_m(K).$$

Assume condition (C. 5). For any $\mu_m\neq 0$, let us put $\nu_m=\mu_m/\mu_m(K)$. Then we have $\nu_m(K)=1$ and

$$c(K)\leqq(\nu_m, \nu_m)\leqq(\mu_m, \mu_m)/\mu_m(K)^2,$$

so that

$$I(\mu_m)\geqq c(K)\mu_m(K)^2-2\beta\mu_m(K).$$

Therefore $\{\mu_m(K)\}$ is bounded. Finally we assume condition (C. 3). Then $(\mu_m+\mu_p)/2\in S$ and

$$V\leqq I((\mu_m+\mu_p)/2)=I(\mu_m)/2+I(\mu_p)/2-(\mu_m-\mu_p, \mu_m-\mu_p)/4,$$

so that

$$(\mu_m-\mu_p, \mu_m-\mu_p)\leqq 2I(\mu_m)+2I(\mu_p)-4V.$$

Therefore $\{(\mu_m, \mu_m)\}$ is bounded, i.e., $0\leqq(\mu_m, \mu_m)\leqq b<\infty$. Suppose that $\{\mu_m(K)\}$ is not bounded. Then we may assume that $\mu_m(K)$ tends to $\infty$ with $m$. Writing $\nu_m=\mu_m/\mu_m(K)$, we can find a vaguely convergent subsequence of $\{\nu_m\}$ by Proposition 10. Denote it again by $\{\nu_m\}$ and let $\nu_0$ be the vague limit. Then we have

$$0\leqq(\nu_0, \nu_0)\leqq\varliminf_{m\to\infty}(\nu_m, \nu_m)\leqq\lim_{m\to\infty}b/\mu_m(K)^2=0.$$

Since $G$ is of positive type, we have

$$0\leqq(\nu_0\pm t\mu_m, \nu_0\pm t\mu_m)=\pm t(\nu_0, \mu_m)+t^2(\mu_m, \mu_m)$$

for all $t>0$. Dividing both sides by $t$ and letting $t\to 0$, we have $(\nu_0, \mu_m)=0$ for all $m$. Furthermore we have

$$\int g_k d\nu_0=\lim_{m\to\infty}\int g_k d\nu_m=\lim_{m\to\infty}c_k/\mu_m(K)=0$$

for each $k$. Therefore $\mu_m+m\nu_0\in S$ and

$$V\leqq I(\mu_m+m\nu_0)=I(\mu_m)-2m\int f d\nu_0.$$

Letting $m\to\infty$, we arrive at a contradiction, since $f>0$ on $K$ and $\nu_0(K)=1$. Thus $\{\mu_m(K)\}$ is bounded.

THEOREM 16. *Assume that $V$ is finite. If any one of conditions* (C. 1)–(C. 5) *is fulfilled, then there exists $\mu\in S$ such that $V=I(\mu)$.*

PROOF. Let $\{\mu_m\}$ be a sequence in $S$ for which $I(\mu_m)$ tends to $V$ as $m\to\infty$. Assume any one of conditions (C. 1)–(C. 5). Then $\{\mu_m\}$ is vaguely bounded by lemma 5 and contains a vaguely convergent subsequence by Proposition 10. Denote it again by $\{\mu_m\}$ and let $\mu_0$ be the vague limit. Then we have

$$\int g_k d\mu_0=\lim_{m\to\infty}\int g_k d\mu_m=c_k \quad \text{for each } k,$$

$$-\infty<(\mu_0,\ \mu_0)\leqq \varliminf_{m\to\infty}(\mu_m,\ \mu_m)=\varliminf_{m\to\infty}\left[I(\mu_m)+2\int f d\mu_m\right]$$

$$\leqq \varliminf_{m\to\infty}\left[I(\mu_m)+2\beta\mu_m(K)\right]=V+2\beta\mu_0(K)<\infty,$$

by Proposition 11, where $\beta=\sup\{f(u);\ u\in K\}$. Therefore $\mu_0\in S$. Since f is upper semicontinuous, we have

$$\varlimsup_{m\to\infty}\int f d\mu_m\leqq\int f d\mu_0,$$

so that

$$V=\lim_{m\to\infty} I(\mu_m)\geqq \varliminf_{m\to\infty}(\mu_m,\ \mu_m)\ -2\varlimsup_{m\to\infty}\int f\,d\mu_m$$

$$\geqq(\mu_0,\ \mu_0)-2\int f\,d\mu_0=I(\mu_0)\geqq V.$$

Thus we have $V=I(\mu_0)$. This completes the proof.

REMARK 5. Ohtsuka [9] called the conditional Gauss variational problem the $n$-dimensional problem in the case where $K$ consists of mutually disjoint compact sets $K_k$, $k=1,\cdots,n$, $g_k>0$ on $K_k$ and $g_k=0$ on $K_j (j\neq k)$.

It is clear that condition (C. 1) is satisfied for the $n$-dimensional problem. An existence theorem for the $n$-dimensional problem was established in [9] (p. 219, Theorem 2. 6) without the assumption that $V$ is finite.

Let $\mu\in S$ and denote by $\mu_k$ the restriction of $\mu$ onto $K_k$. Then $\langle g_j,\ \mu_k\rangle=0$ if $j\neq k$ and $\langle g_k,\ \mu_k\rangle=c_k\geqq 0$. Thus we have

$$\det(\langle g_j,\ \mu_k\rangle)=c_1\ \cdots\ c_n.$$

In case $c_k>0$ for all $k$, $\{g_k;k=1,\cdots,n\}$ is $\mu$-independent and $\{\mu_k;\ k=1,\ \cdots,\ n\}$

is a full system of components of $\mu$.

## § 10. Some properties of $V(z)$

We shall study the change of the value $V$ of the conditional Gauss variational problem when $G$, $f$ and $\{g_k; k=1, \cdots, n\}$ are fixed but $z=(c_1, \cdots, c_n)\in R^n$ changes. Let us put

$$A\mu=(\langle g_1, \mu\rangle, \cdots, \langle g_n, \mu\rangle),$$

$$V(z)=\inf\{I(\mu); \mu\in S(z)\},$$

$$S(z)=\{\mu\in E_K; A\mu=z\},$$

$$S_0(z)=\{\mu\in S(z); V(z)=I(\mu)\},$$

where

$$E_K=\{\mu\in E; S\mu\subset K\}=E'_K.$$

We shall examine the continuity of $V(z)$ and compute the directional derivatives of $V(z)$.

In the case where $\{\mu_k^*; k=1, \cdots, n\}$ is a full system of components of $\mu^*\in S(z_0)$, we define $D(z_0)=D(z_0; \{\mu_k^*\})$ by

$$D(z_0)=D(z_0; \{\mu_k^*\})=\{\sum_{k=1}^{n} t_k A\mu_k^*; t_k\in R_0\ (k=1, \cdots, n)\},$$

which is the polyhedral cone generated by $\{A\mu_k^*; k=1, \cdots, n\}$. It is clear that $D(z_0)$ is a neighborhood of $z_0$ (cf. the proof of Lemma 4).

We shall prove

THEOREM 17. *Assume that $E_K$ is convex. Then $V(z)$ is upper semicontinuous in $A(E_K)^\circ$.*

PROOF. Let $z_0\in A(E_K)^\circ$. We show that $\overline{\lim}_{p\to\infty} V(z^{(p)})\leqq V(z_0)$ for any sequence $\{z^{(p)}\}$ of points in $A(E_K)^\circ$ which converges to $z_0$. For any number $a$ with $V(z_0)<a$, there exists $\mu\in S(z_0)$ such that $I(\mu)<a$. Since $z_0$ is an interior point of $A(E_K)$ and $E_K$ is convex, there exists $\bar{\mu}\in S(z_0)$ such that $\{g_k; k=1, \cdots, n\}$ is $\bar{\mu}$-independent by Theorem 5 in § 4. Writing $\mu^*=\varepsilon\bar{\mu}+(1-\varepsilon)\mu$ with $0<\varepsilon<1$, we see that $\{g_k; k=1, \cdots, n\}$ is $\mu^*$-independent, so that there exists a full system of components $\{\mu_k^*; k=1, \cdots, n\}$ of $\mu^*$ by Lemma 4. Let $\{z^{(p)}\}$ be any sequence of point in $A(E_K)^\circ$ which converges to $z_0$. We may suppose that $z^{(p)}\in D(z_0; \{\mu_k^*\})$ for all $p$. Namely we have

$$z^{(p)}=\sum_{k=1}^{n} t_k^{(p)} A\mu_k^*, \quad t_k^{(p)}\geqq 0.$$

It is valid that $\nu^{(p)}=\sum_{k=1}^{n} t_k^{(p)}\mu_k^* \in S(z^{(p)})$ and

$$V(z^{(p)})\leqq I(\nu^{(p)})=\sum_{j,k=1}^{n} t_j^{(p)}t_k^{(p)}(\mu_j^*, \mu_k^*)-2\sum_{k=1}^{n} t_k^{(p)}\langle f, \mu_k^*\rangle.$$

Since $\{A\mu_k^*; k=1, \cdots, n\}$ is linearly independent and

$$z_0=\sum_{k=1}^{n} A\mu_k^*,$$

we have $\lim_{p\to\infty} t_k^{(p)}=1$ for each $k$ and

$$\begin{aligned}\overline{\lim_{p\to\infty}}\ V(z^{(p)})&\leqq\lim_{p\to\infty} I(\nu^{(p)})=\sum_{j,k=1}^{n}(\mu_j^*, \mu_k^*)-2\sum_{k=1}^{n}\langle f, \mu_k^*\rangle\\ &=I(\mu^*)\ =I(\varepsilon\bar{\mu}+(1-\varepsilon)\mu)\\ &=\varepsilon I(\bar{\mu})+(1-\varepsilon)\, I(\mu)-\varepsilon(1-\varepsilon)\,(\mu-\bar{\mu}, \mu-\bar{\mu}).\end{aligned}$$

Letting $\varepsilon\to 0$, we have

$$\overline{\lim_{p\to\infty}}\ V(z^{(p)})\leqq I(\mu)<a.$$

By the arbitrariness of $a$, we obtain the desired inequality.

Similarly we can prove

PROPOSITION 12. *If there exists $\mu^*\in S_0(z_0)$ such that $\{g_k; k=1, \cdots, n\}$ is $\mu^*$-independent, then $V(z)$ is upper semicontinuous at $z_0$.*

We have

LEMMA 6. *Let $\{z^{(p)}\}$ be a sequence of points in $A(E_K)$ which converges to $z_0\in A(E_K)$ and let $\mu_p\in S_0(z^{(p)})$. If $\mu_p$ converges vaguely to $\mu^*$ and $\overline{\lim_{p\to\infty}}\ V(z^{(p)})\leqq V(z_0)$, then it is valid that $\mu^*\in S_0(z_0)$ and $\lim_{p\to\infty} V(z^{(p)})=V(z_0)$.*

PROOF. Let $z^{(p)}=(c_1^{(p)}, \cdots, c_n^{(p)})$ and $z_0=(c_1, \cdots, c_n)$. Then we have

$$\langle g_k, \mu^*\rangle=\underline{\lim_{p\to\infty}}\langle g_k, \mu^{(p)}\rangle=\underline{\lim_{p\to\infty}}\ c_k^{(p)}=c_k$$

for each $k$ and

$$\begin{aligned}-\infty<(\mu^*, \mu^*)&\leqq\underline{\lim_{p\to\infty}}\,(\mu^{(p)}, \mu^{(p)})=\underline{\lim_{p\to\infty}}\,[V(z^{(p)})+2\langle f, \mu_p\rangle]\\ &\leqq\overline{\lim_{p\to\infty}}\,[V(z^{(p)})+2\beta\mu_p(K)]\leqq V(z_0)+2\beta\mu^*(K)<\infty\end{aligned}$$

by Proposition 11 and our assumption, where $\beta=\sup\{f(u);u\in K\}$. Thus we have $\mu^*\in S(z_0)$ and

$$V(z_0)\geqq \overline{\lim_{p\to\infty}}\, V(z^{(p)})\geqq \underline{\lim_{p\to\infty}}\, V(z^{(p)})$$
$$=\underline{\lim_{p\to\infty}}\, I(\mu_p)\geqq I(\mu^*)\geqq V(z_0).$$

Hence $V(z_0)=I(\mu^*)$ and $\mu^*\in S_0(z_0)$.

THEOREM 18. *Let $z_0\in A(E_K)$ and assume that for any sequence $\{z^{(p)}\}$ of points in $A(E_K)$ which converges to $z_0$ there exists a sequence $\{\mu_p\}$ of measures such that $\mu_p\in S_0(z^{(p)})$ and $\{\mu_p\}$ is vaguely bounded. Then $V(z)$ is lower semicontinuous at $z_0$.*

PROOF. Suppose that $V(z)$ is not lower semicontinuous at $z_0$. Then there exists a sequence $\{z^{(p)}\}$ of points in $A(E_K)$ such that $z^{(p)}\to z_0$ as $p\to\infty$ and

$$\lim_{p\to\infty} V(z^{(p)})<V(z_0). \tag{20}$$

There exists a sequence $\{\mu_p\}$ of measures such that $\mu_p\in S_0(z^{(p)})$ and $\{\mu_p\}$ is vaguely bounded by our assumption. To save notation, we assume that $\mu_p$ converges vaguely to $\mu^*$. It follows from Lemma 6 that $\lim_{p\to\infty} V(z^{(p)})=V(z_0)$, which contradicts (20). Therefore $V(z)$ is lower semicontinuous at $z_0$.

Let us consider the following two conditions:

(H. 1) $V(z)$ is finite whenever $S(z)\neq\phi$.

(H. 2) $S(0)$ is vaguely bounded.

It is clear that condition (H. 2) is equivalent to condition $S(0)=\{0\}$.

We have

LEMMA 7. *Assume condition* (H. 2). *Let $\{z^{(p)}\}$ be a sequence of points in $A(E_K)$ which converges to some $z_0\in R^n$ and let $\mu_p\in S(z^{(p)})$. Then $\{\mu_p\}$ is vaguely bounded.*

PROOF. Supposing the contrary, we may assume that $\mu_p(K)\to\infty$ as $p\to\infty$ by choosing a subsequence if necessary. Writing $\nu_p=\mu_p/\mu_p(K)$, we can find a vaguely convergent subsequence of $\{\nu_p\}$. Denote it again by $\{\nu_p\}$ and let $\bar\nu$ be its limit. Then we have $\bar\nu(K)=1$ and

$$\langle g_k, \bar\nu\rangle=\lim_{p\to\infty}\langle g_k, \mu_p\rangle/\mu_p(K)=\lim_{p\to\infty} c_k^{(p)}/\mu_p(K)=0$$

for each $k$, where $z^{(p)}=(c_1^{(p)}, \cdots, c_1^{(p)})$. Namely $A\bar\nu=0$ and $t\bar\nu\in S(0)$ for all $t\in R_0$, which contradicts condition (H. 2). Therefore $\{\mu_p\}$ is vaguely bounded.

COROLLARY 1. *If condition* (H. 2) *is fulfilled, then $S(z)$ is vaguely bounded* *for every* $z \in A(E_K)$ *and* $A(E_K)$ *is closed.*

It is clear that any one of conditions (C. 1) and (C. 2) in § 9 implies condition (H. 2).

By the same argument as in the proof of Theorem 16, we have

COROLLARY 2. *Assume condition* (H. 2). *Then* $S_0(z) \neq \phi$ *for every* $z \in A(E_K)$.

Noting that $f$ is $\mu$-summable for all $\mu \in E_K$ by our assumption, we see that condition (H. 2) implies condition (H. 1).

We have

PROPOSITION 13. *Assume condition* (H. 2). *Then* $V(z)$ *is lower semicontinuous in* $R^n$.

PROOF. Observing that $A(E_K)$ is closed and $V(z)=\infty$ for all $z \notin A(E_K)$, our assertion follows from Theorem 18 and Lemma 7.

PROPOSITION 14. *Let* $z_0 \in A(E_K)$. *Assume that* $V(z)$ *is upper semicontinuous at* $z_0$ *and condition* (H. 1) *is satisfied. If any one of conditions* (C. 4) *and* (C. 5) *is fulfilled, then* $V(z)$ *is continuous at* $z_0$.

PROOF. Let $\{z^{(p)}\}$ be any sequence of points in $A(E_K)$ which converges to $z_0$. We see that $S_0(z^{(p)}) \neq \phi$ by Theorem 16. Taking $\mu_p \in S_0(z^{(p)})$, we have either

$$V(z^{(p)}) = I(\mu_p) \geqq -2\beta\mu_p(K)$$

or

$$V(z^{(p)}) = I(\mu_p) \geqq c(K)[\mu_p(K)]^2 - 2\beta\mu_p(K)$$

by conditions (C. 4) or (C. 5) (cf. the proof of Lemma 5). If $\{\mu_p\}$ is not vaguely bounded, then we have

$$\infty = \overline{\lim_{p\to\infty}}\, V(z^{(p)}) \leqq V(z_0) < \infty$$

by the above observation and our assumption. This is absurd. Therefore $\{\mu_p\}$ is vaguely bounded and $V(z)$ is lower semicontinuous at $z_0$ by Theorem 18, so that $V(z)$ is continuous at $z_0$.

Summing up the above results, we have

THEOREM 19. *Let* $z_0 \in A(E_K)^\circ$. *Assume either that* $E_K$ *is convex or that there exists* $\mu^* \in S_0(z_0)$ *such that* $\{g_k;\, k=1, \cdots, n\}$ *is* $\mu^*$*-independent. If condition* (H. 1) *and any one of conditions* (H. 2), (C. 4) *and* (C. 5) *are fulfilled, then* $V(z)$ *is finite-valued and continuous at* $z_0$.

We shall prove

THEOREM 20. *Let $z_0 \in A(E_K)^\circ$ and assume that there exists $\mu^* \in S_0(z_0)$ such that $\{g_k; k=1, \dots, n\}$ is $\mu^*$-independent. Then it is valid that*

$$\varlimsup_{\varepsilon\to+0} \frac{V(z_0+\varepsilon x)-V(z_0)}{\varepsilon} \leqq 2((x, \bar{w}(\mu^*)))_2$$

*for every $x \in R^n$.*

PROOF. Let $\{\mu_k^*; k=1, \dots, n\}$ be a full system of components of $\mu^*$. There exists $\varepsilon_0$ such that $z_0+\varepsilon x \in D(z_0; \{\mu_k^*\})$ for all $\varepsilon$, $0<\varepsilon<\varepsilon_0$. It is valid that

$$z_0+\varepsilon x = \sum_{k=1}^{n} t_k^{(\varepsilon)} A\mu_k^*, \quad t_k^{(\varepsilon)} \geqq 0,$$

$$\nu^{(\varepsilon)} = \sum_{k=1}^{n} t_k^{(\varepsilon)} \mu_k^* \in S(z_0+\varepsilon x),$$

so that

$$V(z_0+\varepsilon x)-V(z_0) \leqq I(\nu^{(\varepsilon)})-I(\mu^*)$$

$$= \sum_{j,k=1}^{n} (t_j^{(\varepsilon)} t_k^{(\varepsilon)}-1)(\mu_j^*, \mu_k^*) - 2\sum_{k=1}^{n} (t_k^{(\varepsilon)}-1)\langle f, \mu_k^*\rangle.$$

From the relation

$$\varepsilon x = \sum_{k=1}^{n} (t_k^{(\varepsilon)}-1) A\mu_k^*,$$

we see that $\lim_{\varepsilon\to+0} t_k^{(\varepsilon)}=1$ and $\lim_{\varepsilon\to+0} (t_k^{(\varepsilon)}-1)/\varepsilon = y_k$ for each $k$, where $\{y_k\}$ are defined by

$$\sum_{k=1}^{n} y_k A\mu_k^* = x.$$

It follows that

$$\varlimsup_{\varepsilon\to+0} \frac{V(z_0+\varepsilon x)-V(z_0)}{\varepsilon}$$

$$\leqq \sum_{j,k=1}^{n} (y_j+y_k)(\mu_j^*, \mu_k^*) - 2\sum_{k=1}^{n} y_k \langle f, \mu_k^*\rangle$$

$$= 2\sum_{k=1}^{n} y_k \langle G(\cdot, \mu^*)-f, \mu_k^*\rangle$$

$$= 2\sum_{k=1}^{n} y_k \sum_{j=1}^{n} \bar{w}_j(\mu^*) \langle g_j, \mu_k^*\rangle$$

$$= 2 \sum_{j=1}^{n} x_j \bar{w}_j(\mu^*) = 2((x, \bar{w}(\mu^*)))_2.$$

This completes the proof.

In order to compute the directional derivatives of $V(z)$, we prepare

LEMMA 8. *Let $\{z^{(p)}\}$ be a sequence of points in $A(E_K)^\circ$ which converges to $z_0$ and let $\mu^{(p)} \in S_0(z^{(p)})$. Assume that $\lim_{p\to\infty} V(z^{(p)}) = V(z_0)$ and that $\mu^{(p)}$ has a full system of components $\{\mu_k^{(p)}; k=1, \cdots, n\}$ such that $\mu_k^{(p)}$ converges vaguely to $\mu_k^*$ as $p \to \infty$ for every $k$ and $\det(\langle g_j, \mu_k^*\rangle) \neq 0$. Then it is valid that*

$$\mu^* = \sum_{k=1}^{n} \mu_k^* \in S_0(z_0),$$

$$\lim_{p\to\infty} \bar{w}(\mu^{(p)}) = \bar{w}(\mu^*),$$

$$\lim_{p\to\infty} (\mu^{(p)}, \mu^{(p)}) = (\mu^*, \mu^*),$$

$$\lim_{p\to\infty} \langle f, \mu^{(p)}\rangle = \langle f, \mu^*\rangle.$$

PROOF. It follows from Lemma 6 that $\mu^* \in S_0(z_0)$. It is valid that

$$\varliminf_{p\to\infty} \langle G(\cdot, \mu^{(p)}) - f, \mu_k^{(p)}\rangle \geqq \langle G(\cdot, \mu^*) - f, \mu_k^*\rangle$$

by Proposition 11 and that

$$\begin{aligned}\varlimsup_{p\to\infty} \sum_{k=1}^{n} \langle G(\cdot, \mu^{(p)}) - f, \mu_k^{(p)}\rangle &= \varlimsup_{p\to\infty} \langle G(\cdot, \mu^{(p)}) - f, \mu^{(p)}\rangle \\ &= \varlimsup_{p\to\infty} \{V(z^{(p)}) + \langle f, \mu^{(p)}\rangle\} \\ &\leqq V(z_0) + \langle f, \mu^*\rangle \\ &= \sum_{k=1}^{n} \langle G(\cdot, \mu^*) - f, \mu_k^*\rangle\end{aligned}$$

by our assumption that $\lim_{p\to\infty} V(z^{(p)}) = V(z_0)$. Thus we have

$$\lim_{p\to\infty} \langle G(\cdot, \mu^{(p)}) - f, \mu_k^{(p)}\rangle = \langle G(\cdot, \mu^*) - f, \mu_k^*\rangle$$

for each $k$. From the relation

$$\sum_{j=1}^{n} \bar{w}_j(\mu^{(p)}) \langle g_j, \mu_k^{(p)}\rangle = \langle G(\cdot, \mu^{(p)}) - f, \mu_k^{(p)}\rangle$$

and our assumption that $\mu_k^{(p)}$ converges vaguely to $\mu_k^*$ for each $k$ and $\det(\langle g_j, \mu_k^*\rangle) \neq 0$, it follows that $\lim_{p\to\infty} \bar{w}_j(\mu^{(p)}) = r_j$ exists for each $j$ and $\{r_j; j=1, \cdots, n\}$ is the solution of the equations

$$\sum_{j=1}^{n} r_j\langle g_j, \mu_k^*\rangle = \langle G(\cdot, \mu^*)-f, \mu_k^*\rangle,$$

and hence $r_j=\bar{w}_j(\mu^*)$ by Theorem 13. Therefore $\lim_{p\to\infty} \bar{w}(\mu^{(p)})=\bar{w}(\mu^*)$. It is valid that

$$\begin{aligned}\lim_{p\to\infty} (\mu^{(p)}, \mu^{(p)}) &= \lim_{p\to\infty} [2((z^{(p)}, \bar{w}(\mu^{(p)}))_2 - V(z^{(p)})] \\ &= 2((z_0, \bar{w}(\mu^*)))_2 - V(z_0) = (\mu^*, \mu^*), \\ \lim_{p\to\infty} \langle f, \mu^{(p)}\rangle &= \lim_{p\to\infty} [((z^{(p)}, \bar{w}(\mu^{(p)})))_2 - V(z^{(p)})] \\ &= ((z_0, \bar{w}(\mu^*)))_2 - V(z_0) = \langle f, \mu^*\rangle.\end{aligned}$$

This completes the proof.

COROLLARY. *Under the same assumptions as in Lemma* 8, *we have*

$$\lim_{p\to\infty} (\mu_j^{(p)}, \mu_k^{(p)}) = (\mu_j^*, \mu_k^*),$$

$$\lim_{p\to\infty} \langle f, \mu_k^{(p)}\rangle = \langle f, \mu_k^*\rangle$$

*for every* $j$ *and* $k$.

Let $x\in R^n$, $x\neq 0$. We say that $z_0\in R^n$ is an *x-regular* point of $V(z)$ if the following properties (D. 1) and (D. 2) are fulfilled:

(D. 1) There exists $\mu^*\in S_0(z_0)$ such that $\{g_k; k=1, \dots, n\}$ is $\mu^*$-independent.

(D. 2) For any sequence $\{z^{(p)}\}$ of points in the segment $L(z_0; x)=\{z_0+\varepsilon x; 0<\varepsilon<\varepsilon_0\}$ contained in $A(E_K)^\circ$ which converges to $z_0$, there exist a measure $\mu^{(p)}\in S_0(z^{(p)})$ and a full system of components $\{\mu_k^{(p)}; k=1, \dots, n\}$ of $\mu^{(p)}$ for every $p$ such that

$$\varliminf_{p\to\infty} |\det(\langle g_j, \mu_k^{(p)}\rangle)| > 0$$

and $\{\mu^{(p)}\}$ is vaguely bounded.

Denote by $S_{0i}(z)$ the set of measures $\mu\in S_0(z)$ such that $\{g_k; k=1, \dots, n\}$ is $\mu$-independent and put

$$\underline{\alpha}(z; x) = \inf\{((x, \bar{w}(\mu)))_2; \mu\in S_{0i}(z)\}.$$

We shall prove

THEOREM 21. *Assume that* $z_0$ *is an x-regular point of* $V(z)$. *Then it is valid that*

$$\varliminf_{\varepsilon\to+0} \frac{V(z_0+\varepsilon x)-V(z_0)}{\varepsilon} \geqq 2\,\underline{\alpha}(z_0; x). \tag{21}$$

PROOF. Let $\{\varepsilon_p\}$ be any sequence of numbers such that $\varepsilon_0 > \varepsilon_p > \varepsilon_{p+1} > 0$ and $\lim_{p\to\infty} \varepsilon_p = 0$. Let us put $z^{(p)} = z_0 + \varepsilon_p x$. There exist $\mu^{(p)} \in S_0(z^{(p)})$ and a full system of components $\{\mu_k^{(p)}; k=1, \dots, n\}$ of $\mu^{(p)}$ such that $\underline{\lim}_{p\to\infty} |\det(\langle g_j, \mu_k^{(p)}\rangle)| > 0$ and $\{\mu^{(p)}\}$ is vaguely bounded. In order to establish (21), we may assume that $\mu_k^{(p)}$ converges vaguely to $\mu_k^*$ for each $k$. Then $\mu^{(p)}$ converges vaguely to $\mu^* = \sum_{k=1}^n \mu_k^*$. Since $S_{0i}(z_0) \neq \phi$ by our assumption, it is valid that $\overline{\lim}_{p\to\infty} V(z^{(p)}) \leqq V(z_0)$ by Proposition 12. Thus we have $\mu^* \in S_0(z_0)$ and $\lim_{p\to\infty} V(z^{(p)}) = V(z_0)$ by Lemma 6. $\{\mu_k^*; k=1, \dots, n\}$ is a full system of components of $\mu^*$, since we have

$$|\det(\langle g_j, \mu_k^*\rangle)| = \lim_{p\to\infty} |\det(\langle g_j, \mu_k^{(p)}\rangle)| > 0.$$

Denote by $D_p$ the set $D(z^{(p)}; \{\mu_k^{(p)}\})$. We show that there exists $p_0$ such that $z_0 \in D_p$ for all $p$, $p \geqq p_0$. If we suppose the contrary, we see that $z_0 \notin D_p$ for infinitely many $p$. In case $z_0 \notin D_p$, there exists $v^{(p)} \in R^n$ such that $|v^{(p)}| = 1$ and

$$((z_0, v^{(p)}))_2 < 0 \leqq ((z, v^{(p)}))_2 \tag{22}$$

for all $z \in D_p$ by Proposition 1. It follows from (22) that

$$0 \leqq ((A\mu_k^{(p)}, v^{(p)}))_2 = \sum_{j=1}^n v_j^{(p)} \langle g_j, \mu_k^{(p)}\rangle \tag{23}$$

for each $k$. By choosing a subsequence if necessary, we may assume that $v^{(p)}$ converges to $\bar{v}$. We have by (23) that

$$\sum_{j=1}^n \bar{v}_j \langle g_j, \mu_k^*\rangle \geqq 0.$$

It follows from (22) that

$$0 \geqq \lim_{p\to\infty} ((z_0, v^{(p)}))_2 = ((z_0, \bar{v}))_2 = \sum_{j=1}^n \bar{v}_j \langle g_j, \mu^*\rangle$$
$$= \sum_{k=1}^n \sum_{j=1}^n \bar{v}_j \langle g_j, \mu_k^*\rangle,$$

so that

$$\sum_{j=1}^n \bar{v}_j \langle g_j, \mu_k^*\rangle = 0$$

for each $k$. Since $\det(\langle g_j, \mu_k^*\rangle) \neq 0$, we conclude that $\bar{v} = 0$, which contradicts $|\bar{v}| = 1$. Consequently $z_0 \in D_p$ for all $p \geqq p_0$. It follows that

$$z^{(p)} = z_0 + \varepsilon_p x = \sum_{k=1}^n A\mu_k^{(p)}$$

and

$$z_0=\sum_{k=1}^{n} t_k^{(p)} A\mu_k^{(p)}, \quad t_k^{(p)} \geqq 0.$$

The measure $\nu^{(p)}=\sum_{k=1}^{n} t_k^{(p)}\mu_k^{(p)}$ belongs to $S(z_0)$, so that

$$V(z^{(p)})-V(z_0) \geqq I(\mu^{(p)})-I(\nu^{(p)})$$

$$=\sum_{j,k=1}^{n}(1-t_j^{(p)}t_k^{(p)})(\mu_j^{(p)}, \mu_k^{(p)})-2\sum_{k=1}^{n}(1-t_k^{(p)})\langle f, \mu_k^{(p)}\rangle.$$

Since $\lim_{p\to\infty} A\mu_k^{(p)}=A\mu_k^*$ for each $k$ and $z_0=\sum_{k=1}^{n} A\mu_k^*$, we see that $\lim_{p\to\infty} t_k^{(p)}=1$ and $\lim_{p\to\infty}[1-t_k^{(p)}]/\varepsilon_p=y_k$ for each $k$, where $\{y_k\}$ is defined by $\sum_{k=1}^{n} y_k A\mu_k^*=x$. By means of the corollary of Lemma 8, we have

$$\begin{aligned}&\lim_{p\to\infty}[I(\mu^{(p)})-I(\nu^{(p)})]/\varepsilon_p\\ &=\sum_{j,k=1}^{n}(y_j+y_k)(\mu_j^*, \mu_k^*)-2\sum_{k=1}^{n} y_k\langle f, \mu_k^*\rangle\\ &=2\sum_{k=1}^{n}\langle G(\cdot, \mu^*)-f, \mu_k^*\rangle y_k=2((x, \bar{w}(\mu^*)))_2\\ &\geqq 2\underline{\alpha}(z_0; x).\end{aligned}$$

By the arbitrariness in choosing subsequences of $\{\mu_k^{(p)}\}$, we have

$$\underline{\lim}_{p\to\infty}\frac{V(z_0+\varepsilon_p x)-V(z_0)}{\varepsilon_p}\geqq 2\underline{\alpha}(z_0; x).$$

By the arbitrariness of $\{\varepsilon_p\}$, we complete the proof.

By Theorems 20 and 21, we have

THEOREM 22. *Assume that $z_0$ is an $x$-regular point of $V(z)$. Then it is valid that*

$$\lim_{\varepsilon\to+0}\frac{V(z_0+\varepsilon x)-V(z_0)}{\varepsilon}=2\underline{\alpha}(z_0; x).$$

In the case where the set $\{\bar{w}(\mu);\ \mu\in S_{0i}(z_0)\}$ consists of only one point, we can compute the partial derivatives of $V(z)$ at $z_0$ by Theorem 22.

We shall study the notion that $z_0$ is an $x$-regular point of $V(z)$. We can easily verify that every $z_0\in A(E_K)^\circ$ is an $x$-regular point of $V(z)$ for all $x$, $x\neq 0$, in the case where the problem is $n$-dimensional in Ohtsuka's sense (cf. Remark 5 in § 9).

We shall prove

PROPOSITION 15. *Assume that* $g_k \geqq 0$ *on* $K$ *for each* $k$, *that condition* (H. 2) *is fulfilled and that there is a neighborhood* $U$ of $z_0$ *such that* $U \subset A(E_K)$ *and* $S_{0i}(z) = S_0(z)$ *for all* $z \in U$. *Then* $z_0$ *is an* $x$-*regular point of* $V(z)$ *for all* $x$, $x \neq 0$.

We need some preparations for the proof of this proposition. The assumptions in Proposition 15 persist in the rest of this section except in Proposition 16 below.

It is valid that $A(E_K) \subset R_0^n$ and that $z = (z_1, \cdots, z_n) \in A(E_K)^\circ$ implies $z_k > 0$ for each $k$, since $g_k \geqq 0$ on $K$ for each $k$. Let $\mu^* \in S_{0i}(z)$ and denote by $Q(\mu^*)$ the convex cone generated by $A(C[\mu^*])$ (cf. the proof of Lemma 4) and by $e_k$ the point of $R^n$ whose $j$-th coordinate is equal to 0 if $j \neq k$ and 1 if $j = k$. Define $d(\mu^*; z)$ by

$$d(\mu^*; z) = \sup\{r > 0;\ z + re_k \in Q(\mu^*)\ (k = 1, \cdots, n)\}.$$

Since $z$ is an interior point of $Q(\mu^*)$, $d(\mu^*; z) > 0$.

LEMMA 9. *Let* $F$ *be a nonempty compact set contained in* $U$. *Then it is valid that*

$$d(F) = \inf\{\inf\{d(\mu; z);\ \mu \in S_0(z)\};\ z \in F\} > 0.$$

PROOF. Suppose that $d(F) = 0$. Then there exist $z^{(p)} \in F$ and $\mu^{(p)} \in S_0(z^{(p)})$ such that $d(\mu^{(p)}; z^{(p)}) < 1/p$. We may suppose that $z^{(p)}$ converges to $\bar{z} \in F$ and that there is $i$ such that $z^{(p)} + (1/p)e_i \notin Q(\mu^{(p)})$ for all $p$. There exists $v^{(p)} \in R^n$ such that $|v^{(p)}| = 1$ and

$$((z^{(p)} + (1/p)e_i,\ v^{(p)}))_2 < 0 \leqq ((u,\ v^{(p)}))_2$$

for all $u \in Q(\mu^{(p)})$ by Proposition 1. It follows that

$$\sum_{k=1}^{n} v_k^{(p)} g_k \geqq 0 \quad \text{on } S\mu^{(p)}. \tag{24}$$

We may assume that $\mu^{(p)}$ converges vaguely to $\mu^*$ (cf. Lemma 7) and that $v^{(p)}$ converges to $\bar{v}$. It is valid that $\mu^* \in S_0(\bar{z})$ by Theorem 19 and Lemma 6 and that $|\bar{v}| = 1$ and

$$\left\langle \sum_{k=1}^{n} \bar{v}_k g_k,\ \mu^* \right\rangle = ((\bar{z}, \bar{v}))_2 = \lim_{p \to \infty} ((z^{(p)} + (1/p)e_i,\ v^{(p)}))_2 \leqq 0.$$

It follows from (24) that

$$\sum_{k=1}^{n} \bar{v}_k g_k \geqq 0 \ \text{ on } S\mu^*.$$

Thus we have $\sum_{k=1}^{n} \bar{v}_k g_k = 0$ on $S\mu^*$. Since $\mu^* \in S_0(\bar{z}) = S_{0i}(\bar{z})$, there exists a system of components $\{\mu_k^*;\ k = 1, \cdots, n\}$ of $\mu^*$. We have

$$\sum_{j=1}^{n} \bar{v}_j \langle g_j, \mu_k^* \rangle = 0,$$

and hence $\bar{v}_k = 0$ for each $k$, since $\det(\langle g_j, \mu_k^* \rangle) \neq 0$. This contradicts $|\bar{v}| = 1$. Therefore $d(F) > 0$.

Let $F$ be a compact neighborhood of $z_0$ such that $F \subset U$ and let $r_0$ be a number such that $0 < r_0 < d(F)$. For $z \in F$ and $\mu^* \in S_0(z)$, there exist a set $\{\nu_k;\, k=1, \cdots, n\}$ of measures in $C[\mu^*]$ and a set $\{r_k;\, k=1, \cdots, n\}$ of strictly positive numbers such that

$$r_k A\nu_k = z + r_0 e_k \quad \text{and} \quad \nu_k(K) = \mu^*(K)/n$$

for each $k$. Define $\{s_k\}$ by

$$z = \sum_{k=1}^{n} s_k(z + r_0 e_k).$$

It is clear that

$$s_k = z_k/(r_0 + \sum_{j=1}^{n} z_j) > 0$$

for each $k$ with $z = (z_1, \cdots, z_n)$. Let us put $a_k = r_k s_k$ and $a_0 = \sum_{k=1}^{n} a_k$. Since $g_k \geqq 0$ for each $k$ and $\mu^* - \nu_k \in M^+(K)$, we have $A\mu^* - A\nu_k \in R_0^n$ and

$$\sum_{k=1}^{n} a_k(A\mu^* - A\nu_k) = \sum_{k=1}^{n} a_k A\mu^* - \sum_{k=1}^{n} a_k A\nu_k$$
$$= (a_0 - 1)z \in R_0^n,$$

and hence $a_0 \geqq 1$. Let us define $\mu_k^*$ by

$$\mu_k^* = (a_k/a_0)\nu_k + [\mu^* - \sum_{j=1}^{n} (a_j/a_0)\nu_j]/n.$$

Since $\{A\nu_k;\, k=1, \cdots, n\}$ is linearly independent, we see that $\{\mu_k^*;\, k=1, \cdots, n\}$ is a full system of components of $\mu^*$ (cf. the proof of Lemma 4). We call this a normalized full system of components of $\mu^*$.

PROOF OF PROPOSITION 15: There exists $\varepsilon_0$ such that $L(z_0; x)$ is contained in the above $F$. Let $\{z^{(p)}\}$ be any sequence of points in $L(z_0; x)$ which converges to $z_0$. We take $\mu^{(p)} \in S_0(z^{(p)})$ and a normalized full system of components $\{\mu_k^{(p)};\, k=1, \cdots, n\}$ of $\mu^{(p)}$, i.e.,

$$\mu_k^{(p)} = (a_k^{(p)}/a_0^{(p)})\nu_k^{(p)} + [\mu^{(p)} - \sum_{j=1}^{n} (a_j^{(p)}/a_0^{(p)})\nu_j^{(p)}]/n,$$

$$a_k^{(p)} = r_k^{(p)} s_k^{(p)},\ a_0^{(p)} = \sum_{k=1}^{n} a_k^{(p)},\ \nu_k^{(p)}(K) = \mu^{(p)}(K)/n,$$

$$r_k^{(p)} A\nu_k^{(p)} = z^{(p)} + r_0 e_k,\ \nu_k^{(p)} \in C[\mu^{(p)}],\ r_k^{(p)} > 0,$$

$$z^{(p)} = \sum_{k=1}^{n} s_k^{(p)} (z^{(p)} + r_0 e_k).$$

We show first that $r^{(p)} = \sum_{k=1}^{n} r_k^{(p)}$ is bounded. Supposing the contrary, we may assume that $r_i^{(p)} \to \infty$ as $p \to \infty$ for some $i$. Since $\{\mu^{(p)}\}$ is vaguely bounded by Lemma 7, we may also assume that $\mu^{(p)}$ and $\nu_i^{(p)}$ converge vaguely to $\mu^*$ and $\nu_i^*$ respectively. Then it is valid that $\mu^* \in S_0(z_0)$ and $\mu^*(K) > 0$, so that

$$\nu_i^*(K) = \lim_{p\to\infty} \nu_i^{(p)}(K) = \lim_{p\to\infty} \mu^{(p)}(K)/n = \mu^*(K)/n > 0.$$

We have

$$A\nu_i^* = \lim_{p\to\infty} (z^{(p)} + r_0 e_i)/r_i^{(p)} = 0.$$

This contradicts condition (H. 2). Therefore $\{r^{(p)}\}$ is bounded. Now we prove that $\varliminf_{p\to\infty} |\det(\langle g_j, \mu_k^{(p)}\rangle)| > 0$. Supposing the contrary, we may consider that $\mu_k^{(p)}$ converges vaguely to $\mu_k^*$ for each $k$ and $\lim_{p\to\infty} \det(\langle g_j, \mu_k^{(p)}\rangle) = 0$, by choosing subsequences if necessary. We may also assume that $\nu_k^{(p)}$ converges vaguely to $\nu_k^*$ and that $r_k^{(p)}$ converges to $r_k$ for each $k$. It is valid that

$$\lim_{p\to\infty} s_k^{(p)} = s_k = c_k/(r_0 + \sum_{j=1}^{n} c_j),\ z_0 = (c_1, \cdots, c_n)$$

$$r_k A\nu_k^* = z_0 + r_0 e_k,\ \nu_k^*(K) = \mu^*(K)/n.$$

It is clear that $r_k > 0$ and $s_k > 0$ for each $k$. We have

$$\lim_{p\to\infty} a_k^{(p)} = r_k s_k = a_k,\ \lim_{p\to\infty} a_0^{(p)} = \sum_{k=1}^{n} a_k = a_0,$$

$$\mu_k^* = (a_k/a_0)\nu_k^* + [\mu^* - \sum_{j=1}^{n} (a_j/a_0)\nu_j^*]/n,$$

$$\mu^* = \sum_{k=1}^{n} \mu_k^* \in S_0(z_0).$$

Namely $\{\mu_k^*;\ k=1, \cdots, n\}$ is a normalized full system of components of $\mu^*$, so that $\det(\langle g_j, \mu_k^*\rangle) \neq 0$. This contradicts our assumption that

$$\det(\langle g_j, \mu_k^*\rangle) = \lim_{p\to\infty} \det(\langle g_j, \mu_k^{(p)}\rangle) = 0.$$

This completes the proof.

In the case where the problem is $n$-dimensional in Ohtsuka's sense, it is valid that $S_0(z) = S_{0i}(z)$ for all $z \in A(E_K)^\circ$. If $S(z)$ consists of only one point for all $z \in A(E_K)$, then $S_0(z) = S_{0i}(z)$ for all $z \in A(E_K)^\circ$ by Theorem 5 in § 4.

We have

PROPOSITION 16. *Let $z_0 \in A(E_K)^\circ$. Assume that $g_k \geqq 0$ on $K$ for each $k$, that condition* (H. 2) *is fulfilled and that $S_{0i}(z_0)=S_0(z_0)$. Then $S_0^*(z_0)=\{\bar{w}(\mu);\ \mu \in S_0(z_0)\}$ is a compact set. If we further assume that $S_0(z_0)$ is convex, then $S_0^*(z_0)$ is connected.*

PROOF. Since $d(\{z_0\})>0$ by Lemma 9, we can define for $\mu \in S_0(z_0)$ a normalized full system of components of $\mu$ with a number $r_0$ such that $0<r_0 <d(\{z_0\})$ (cf. the proof of Proposition 15). Let $\{\mu^{(p)}\}$ be any sequence of measures in $S_0(z_0)$ and $\{\mu_k^{(p)};\ k=1, \cdots, n\}$ be a normalized full system of components of $\mu^{(p)}$. If $\mu_k^{(p)}$ converges vaguely to $\mu_k^*$ for each $k$, then we see by the same argument as in the proof of Proposition 15 that $\det(\langle g_j, \mu_k^*\rangle)\neq 0$. By this fact and Lemma 8, it can be shown that $S_0^*(z_0)$ is closed. We show that $S_0^*(z_0)$ is bounded. Supposing the contrary, we can find a sequence $\{\mu^{(p)}\}$ of measures in $S_0(z_0)$ such that $|\bar{w}(\mu^{(p)})|\to\infty$ as $p\to\infty$ and

$$\sum_{j=1}^{n} \bar{w}_j(\mu^{(p)}) \langle g_j, \mu_k^{(p)}\rangle = \langle G(\cdot, \mu^{(p)})-f, \mu_k^{(p)}\rangle,$$

where $\{\mu_k^{(p)};\ k=1, \cdots, n\}$ is a normalized full system of components of $\mu^{(p)}$. Writing $v^{(p)}=\bar{w}(\mu^{(p)})/|\bar{w}(\mu^{(p)})|$, we may assume that $v^{(p)}$ converges to $\bar{v}$ and that $\mu_k^{(p)}$ converges vaguely to $\mu_k^*$ by Lemma 7. We have

$$b\mu^{(p)}(K)\mu_k^{(p)}(K)-\beta\mu_k^{(p)}(K)\leqq\langle G(\cdot, \mu^{(p)})-f, \mu_k^{(p)}\rangle,$$

$$\begin{aligned}\sum_{k=1}^{n}\langle G(\cdot, \mu^{(p)})-f, \mu_k^{(p)}\rangle &= V(z_0)+\langle f, \mu^{(p)}\rangle \\ &\leqq V(z_0)+\beta\mu^{(p)}(K),\end{aligned}$$

where $\beta=\sup\{f(u);\ u\in K\}$ and $b=\inf\{G(u, u');\ (u, u')\in K\times K\}$. Since $S_0(z_0)$ is vaguely bounded, it is easily seen by the above relations that $\{\langle G(\cdot, \mu^{(p)})-f, \mu_k^{(p)}\rangle\}$ is bounded. Thus we have

$$\sum_{j=1}^{n}\bar{v}_j\langle g_j, \mu_k^*\rangle = 0,$$

and hence $\bar{v}=0$, since $\det(\langle g_j, \mu_k^*\rangle)\neq 0$. This contradicts $|\bar{v}|=1$. Therefore $S_0^*(z_0)$ is bounded.

Next we show that $S_0^*(z_0)$ is connected whenever $S_0(z_0)$ is convex. For any $\mu^{(p)}\in S_0(z_0)(p=1, 2)$ and $t\in R$ with $0\leqq t\leqq 1$, we have $\mu^*=t\mu^{(1)}+(1-t)\mu^{(2)} \in S_0(z_0)$ by our assumption. Let $\{\mu_k^{(p)};\ k=1, \cdots, n\}$ be a normalized full system of components of $\mu^{(p)}$ for $p=1, 2$ and set $\mu_k^*=t\mu_k^{(1)}+(1-t)\mu_k^{(2)}$ for each $k$. Then $\{\mu_k^*;\ k=1, \cdots, n\}$ is a full system of components of $\mu^*$. We have

$$\sum_{j=1}^{n}\bar{w}_j(\mu^{(p)})\langle g_j, \mu_k^{(p)}\rangle = \langle G(\cdot, \mu^{(p)})-f, \mu_k^{(p)}\rangle$$

for $p=1, 2$ and

$$\sum_{j=1}^{n} \bar{w}_j(\mu^*)\langle g_j, \mu_k^*\rangle = \langle G(\cdot, \mu^*)-f, \mu_k^*\rangle.$$

It is valid that

$$\begin{aligned}
&\langle G(\cdot, \mu^*)-f, \mu_k^*\rangle \\
&= t^2\langle G(\cdot, \mu^{(1)})-f, \mu_k^{(1)}\rangle + (1-t)^2\langle G(\cdot, \mu^{(2)})-f, \mu_k^{(2)}\rangle \\
&\quad + t(1-t)[\langle G(\cdot, \mu^{(1)})-f, \mu_k^{(2)}\rangle + \langle G(\cdot, \mu^{(2)})-f, \mu_k^{(1)}\rangle] \\
&= \sum_{j=1}^{n} [t\bar{w}_j(\mu^{(1)}) + (1-t)\bar{w}_j(\mu^{(2)})]\langle g_j, \mu_k^*\rangle + t(1-t)a_k,
\end{aligned}$$

where

$$\begin{aligned}
a_k &= \langle G(\cdot, \mu^{(1)})-f, \mu_k^{(2)}\rangle + \langle G(\cdot, \mu^{(2)})-f, \mu_k^{(1)}\rangle \\
&\quad - \sum_{j=1}^{n} \bar{w}_j(\mu^{(1)})\langle g_j, \mu_k^{(2)}\rangle - \sum_{j=1}^{n} \bar{w}_j(\mu^{(2)})\langle g_j, \mu_k^{(1)}\rangle.
\end{aligned}$$

Namely we have

$$\sum_{j=1}^{n} [\bar{w}_j(\mu^*) - t\bar{w}_j(\mu^{(1)}) - (1-t)\bar{w}_j(\mu^{(2)})]\langle g_j, \mu_k^*\rangle = t(1-t)a_k.$$

Define $q=(q_1, \cdots, q_n)$ by

$$\sum_{j=1}^{n} q_j\langle g_j, \mu_k^*\rangle = a_k.$$

It follows that

$$\bar{w}(t\mu^{(1)}+(1-t)\mu^{(2)}) = t\bar{w}(\mu^{(1)}) + (1-t)\bar{w}(\mu^{(2)}) + t(1-t)q \tag{25}$$

for all $t$, $0 \leqq t \leqq 1$. Since $\bar{w}(t\mu^{(1)}+(1-t)\mu^{(2)}) \in S_0^*(z_0)$ and the right side of (25) represents a curve connecting $\bar{w}(\mu^{(1)})$ and $\bar{w}(\mu^{(2)})$, we conclude that $S_0^*(z_0)$ is connected.

This is a generalization of Theorem 2. 9 in [9].

## § 11. The case where $G$ is of positive type

We continue the study of $V(z)$ under an additional condition on the kernel $G$. We shall prove

THEOREM 23. *Assume that $G$ is of positive type and that condition* (H. 1) *is fulfilled. Then $V(z)$ is finite-valued and convex on $A(E_K)$ and $V(z)=\infty$ outside $A(E_K)$.*

PROOF. Let $z^{(k)} \in A(E_K)$ $(k=1, 2)$ and $t \in R$, $0 \leqq t \leqq 1$. For any $\varepsilon > 0$, there is $\mu_k \in S(z^{(k)})$ such that $V(z^{(k)}) > I(\mu_k) - \varepsilon$ $(k=1, 2)$. It follows that

$$t\mu_1 + (1-t)\mu_2 \in S(tz^{(1)} + (1-t)z^{(2)}),$$

$$\begin{aligned} tV(z^{(1)}) + (1-t)V(z^{(2)}) &> tI(\mu_1) + (1-t)I(\mu_2) - \varepsilon \\ &= I(t\mu_1 + (1-t)\mu_2) + t(1-t)(\mu_1 - \mu_2, \mu_1 - \mu_2) - \varepsilon \\ &\geqq V(tz^{(1)} + (1-t)z^{(2)}) - \varepsilon. \end{aligned}$$

By the arbitrariness of $\varepsilon$, we have

$$V(tz^{(1)} + (1-t)z^{(2)}) \leqq tV(z^{(1)}) + (1-t)V(z^{(2)}).$$

It is clear that $S(z) = \phi$ if and only if $z \notin A(E_K)$. In this case we have $V(z) = \infty$ by our convention.

On account of this fact, we can apply the general results concerning convex functions in [1], [12] and [13] to the study of $V(z)$. In the rest of this section, we always assume that $G$ is of positive type and that condition (H. 1) is fulfilled.

By a well-known result ([1], p. 92, Corollaire 2 of Proposition 2), we see that $V(z)$ is continuous in $A(E_K)^\circ$. However $V(z)$ is not always continuous at boundary points of $A(E_K)$. This is easily verified by Example 3 in § 6.

Let $x \in R^n$, $x \neq 0$ and $z_0 \in A(E_K)^\circ$. Define $DV(z_0)$ and $V'(z_0; x)$ as follows:

$$DV(z_0) = \{w \in R^n;\ V(z) - V(z_0) \geqq ((z - z_0, w))_2 \text{ for all } z \in A(E_K)\},$$

$$V'(z_0; x) = \lim_{\varepsilon \to +0} \frac{V(z_0 + \varepsilon x) - V(z_0)}{\varepsilon}.$$

The existence of $V'(z_0; x)$ follows from the convexity of $V(z)$. It is valid that

$$V'(z_0; x) = \sup \{((x, w))_2;\ w \in DV(z_0)\}$$

(cf. [12], Theorem 3).

We shall determine $DV(z_0)$ explicitly. For $\mu^* \in S_0(z_0)$, let $S_0^*(\mu^*; z_0)$ be the set of $w \in R^n$ which satisfies the relations (3) and (4) in Theorem 12 with $z_0 = (c_1, \cdots, c_n)$. It is valid that $S_0^*(\mu^*; z_0) = S_0^*(\nu^*; z_0)$ for $\mu^*, \nu^* \in S_0(z_0)$ by Theorem 14.

We have

THEOREM 24. *Assume that $z_0 \in A(E_K)^\circ$ and $S_0(z_0) \neq \phi$. Then it is valid* *that*

$$DV(z_0)=\{2w;\ w\in S_0^*(\mu^*;\ z_0)\}$$

*for any* $\mu^*\in S_0(z_0)$.

Proof. We first assume that $\mu^*\in S_0(z_0)$ and $w\in S_0^*(\mu^*;\ z_0)$. Let $z\in A(E_K)$. For any $\varepsilon>0$, there is $\nu\in S(z)$ such that $V(z)+\varepsilon>I(\nu)$. Integrating both sides of (3) by $\nu$, we obtain

$$(\nu,\ \mu^*)-\langle f,\ \nu\rangle\geqq((z,\ w))_2.$$

Since $(\mu^*,\ \mu^*)=\langle f,\ \mu^*\rangle+((z_0,\ w))_2$, we have

$$\begin{aligned}V(z)+\varepsilon-V(z_0)&>I(\nu)-I(\mu^*)\\&=(\nu,\ \nu)-(\mu^*,\ \mu^*)+2\langle f,\ \mu^*\rangle-2\langle f,\ \nu\rangle\\&\geqq(\mu^*-\nu,\ \mu^*-\nu)+2((z-z_0,\ w))_2\\&\geqq((z-z_0,\ 2w))_2.\end{aligned}$$

By the arbitrariness of $\varepsilon$, we have

$$V(z)-V(z_0)\geqq((z-z_0,\ 2w))_2$$

for all $z\in A(E_K)$, and hence $2w\in DV(z_0)$. Next we assume that $2w\in DV(z_0)$ and $\mu^*\in S_0(z_0)$. Let $\nu$ be any element of $E_K$ and set $z=A\nu$. For any $t\in R$ with $0<t<1$, we have

$$\begin{aligned}((t(z-z_0),\ 2w))_2&\leqq V(tz+(1-t)z_0)-V(z_0)\\&\leqq I(t\nu+(1-t)\mu^*)-I(\mu^*)\\&=t^2(\nu,\ \nu)+2t(1-t)\,(\nu,\ \mu^*)-t(2-t)\,(\mu^*,\ \mu^*)-2t\langle f,\ \nu\rangle+2t\langle f,\ \mu^*\rangle.\end{aligned}$$

Dividing both sides by $t$ and letting $t\to 0$, we obtain

$$((z-z_0,\ w))_2\leqq\langle G(\cdot,\ \mu^*)-f,\ \nu\rangle-\langle G(\cdot,\ \mu^*)-f,\ \mu^*\rangle,$$

or equivalently

$$\langle G(\cdot,\ \mu^*)-f-\sum_{k=1}^{n}w_k g_k,\ \nu\rangle\geqq\langle G(\cdot,\ \mu^*)-f-\sum_{k=1}^{n}w_k g_k,\ \mu^*\rangle,$$

where $w=(w_1,\ \cdots,\ w_n)$. From this relation, we can easily conclude that $w\in S_0^*(\mu^*;\ z_0)$. This completes the proof.

COROLLARY. $V'(z_0; x) = \sup\{2((x, w))_2; w \in S_0^*(\mu^*; z_0)\}$.

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