HIROSHIMA MATH. J.
1 (1971), 145-153

*On the Behavior of Solutions of the Cauchy Problem for Parabolic Equations with Unbounded Coefficients**)

Lu-San CHEN, Cheh-Chih YEH and Hung-Yih CHEN

(Received May 18, 1971)

Dedicated to President Y. K. Tai on his 70th birthday

1. Let $x=(x_1,\cdots, x_n)$ be a point of the n-dimensional Euclidean space R^n and let t be a non-negative number. The distance of the point $x \in R^n$ from the origin of R^n is denoted by $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. The $(n+1)$ -dimensional Euclidean half space $R^n\times(0, \infty)$ is the domain of interest.

Consider a parabolic differential equation

$$(1)\qquad L_0 u=\sum_{n=1}^{n}\frac{\partial^2 u}{\partial x_i^2}+(-k^2|x|^2+l)u-\frac{\partial u}{\partial t}=0,\ (k>0)$$

in $R^n\times(0, \infty)$. Krzyżański [4] proved the existence of the fundamental solution of this equation. By using this fundamental solution, we can see that the solution $u(x, t)$ of the above equation with Cauchy data $u(x, 0)=$ Mexp $(a|x|^2)$ $(2a<k)$ is given by

$$u(x, t)=M\left(\frac{k}{k\cosh 2kt-2a\sinh 2kt}\right)^{n/2}$$

$$\exp\left[\frac{k(2a\cosh 2kt-k\sinh 2kt)}{2(k\cosh 2kt-2a\sinh 2kt)}|x|^2+lt\right].$$

So, if $l-kn<0$, then $u(x, t)$ converges to zero uniformly on every compact set in R^n as $t\to\infty$, (cf. [7]). This fact leads us to the question whether the similar situation to the above holds or not for solutions of general parabolic equations of unbounded coefficients with suitable Cauchy data.

2. The following results, Theorem A and Theorem B, of Kusano [8] give us an answer to the question.

Let

$$(2)\qquad Lu=\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+cu-\frac{\partial u}{\partial t}=0$$

be a parabolic differential equation in $R^n\times(0, \infty)$, where the coefficients $a_{ij}(=a_{ji})$, b_i and c are functions defined in $R^n\times[0, \infty)$ and such that

*) This research was supported by the National Science Council.

$$\text{(3)}\quad \begin{cases} 0<\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqq k_1|\xi|^2 \text{ for any real vector } \xi=(\xi_1,\cdots,\xi_n)\neq 0, \\ |b_i|\leqq k_2(|x|^2+1)^{1/2},\ (1\leqq i\leqq n), \\ c\leqq -k_3|x|^2+k_4 \end{cases}$$

in $R^n\times[0,\infty)$ for some constants $k_1(>0)$, $k_2(\geqq 0)$, $k_3(>0)$ and k_4.

THEOREM A. Put

$$\text{(4)}\qquad \tilde{\alpha}=\min_{1\leqq i\leqq n}\left[\inf_{(x,t)\in R^n\times[0,\infty)} a_{ii}\right].$$

Let θ be the positive root of the equation $4k_1\theta^2+2k_2n\theta-k_3=0$ and let $u(x,t)$ continuous in $R^n\times[0,\infty)$ be a solution of (2) in $R^n\times(0,\infty)$ in the usual sense satisfying $|u(x,0)|\leqq M\exp(a|x|^2)$ in R^n for some positive constants M and a. Suppose that the following inequalities are satisfied:

$$4k_1a^2+2k_2na-k_3<0 \text{ and } k_4+2(k_2-\tilde{\alpha})n\theta<0.$$

Then $\lim_{t\to\infty} u(x,t)=0$, the convergence being of exponential order and uniform with respect to $x\in R^n$.

THEOREM B. Suppose that there exists a positive constant δ such that

$$\text{(5)}\qquad \sum_{i=1}^{n}(a_{ii}+b_ix_i)\geqq\delta$$

for $(x,t)\in R^n\times[0,\infty)$. Let $u=u(x,t)$ continuous in $R^n\times[0,\infty)$ be a solution of (2) in $R^n\times(0,\infty)$ in the usual sense satisfying $|u(x,0)|\leqq M\exp(a|x|^2)$ in R^n for some positive constants M and a. Assume the following inequalities are satisfied:

$$4k_1a^2+2k_2na-k_3<0 \text{ and } k_4-\delta\sqrt{\frac{k_3}{k_1}}<0.$$

Then $\lim_{t\to\infty} u(x,t)=0$, the convergence being of exponential order and uniform with respect to $x\in R^n$.

In this article we shall deal with the question stated in § 1 and extend Theorems A and B to the more general parabolic differential operator L of the form (2) whose coefficients satisfy the following conditions:

$$\text{(6)}\quad \begin{cases} 0<\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j\leqq k_1(|x|^2+1)^{1-\lambda}|\xi|^2 \text{ for any real vector } \xi=(\xi_1,\cdots,\xi_n)\neq 0, \\ |b_i|\leqq k_2(|x|^2+1)^{1/2}; \qquad (1\leqq i\leqq n) \\ c\leqq -k_3(|x|^2+1)^{\lambda}+k_4 \end{cases}$$

for some constants $k_1(>0)$, $k_2(\geqq 0)$, $k_3(>0)$ and k_4 in $R^n\times(0, \infty)$ for $\lambda\in[1, \infty)$.

3. In the later discussion, we shall need the following lemma which is a generalization of Krzyżański's theorem [4].

LEMMA. *Assume that the coefficients of L in (2) satisfy (6). Let $u=u(x, t)$ continuous in $R^n\times[0, \infty)$ satisfy $Lu=0$ and $|u(x, t)|\leqq M^*\exp[a^*(|x|^2+1)^\lambda]$ in $R^n\times(0, \infty)$ for some positive constants M^* and a^*. If there exists a positive constant M such that $|u(x, 0)|\leqq M$, then it holds that $|u(x, t)|\leqq M(t)\exp[-\alpha(|x|^2+1)^\lambda\tanh\beta t]$ in $R^n\times(0, \infty)$ for some positive constants α, β and for a positive continuous function $M(t)$ in $t>0$.*

PROOF: Consider a function

$$V(x, t)=M\exp[-\varphi(t)(|x|^2+1)^\lambda+\psi(t)],$$

where $\varphi(t)(>0)$ and $\psi(t)$ are differentiable once in $[0, \infty)$.

From (4) and (6) we see that

$$\begin{aligned}\frac{LV}{V}&=4\lambda^2\varphi^2(t)(|x|^2+1)^{2\lambda-2}\sum_{i,j=1}^n a_{ij}x_ix_j\\ &-4\lambda(\lambda-1)\varphi(t)(|x|^2+1)^{\lambda-2}\sum_{i,j=1}^n a_{ij}x_ix_j\\ &-2\lambda\varphi(t)(|x|^2+1)^{\lambda-1}\sum_{i=1}^n(a_{ii}+b_ix_i)+c\\ &+\varphi'(t)(|x|^2+1)^\lambda-\psi'(t)\\ &\leqq(|x|^2+1)^\lambda[\varphi'(t)+4k_1\lambda^2\varphi^2(t)+2k_2n\lambda\varphi(t)-k_3]\\ &+[-4k_1\lambda^2\varphi^2(t)-2\lambda\tilde{\alpha}n\varphi(t)+k_4-\psi'(t)].\end{aligned}\tag{7}$$

so, if

$$\varphi(t)=\alpha\tanh 4k_1\lambda^2\alpha t\tag{8}$$

where

$$\alpha=\frac{-k_2n+\sqrt{k_2^2n^2+4k_1k_3}}{4k_1\lambda}$$

is the positive root of the quadratic equation $4k_1\lambda^2X^2+2K_2n\lambda X-k_3=0$, then we easily see that

$$\varphi'(t)+4k_1\lambda^2\varphi^2(t)+2k_2n\lambda\varphi(t)-k_3\leqq 0.$$

Further, it is also easy to see that

$$(9)\qquad \psi(t)=(-4\lambda^2 k_1\alpha^2-2\lambda\alpha n\tilde{\alpha}+k_4)t+\frac{\tilde{\alpha}n}{2\lambda k_1}\log\frac{e^{8\lambda^2k_1\alpha t}}{e^{8\lambda^2k_1\alpha t}+1}$$

$$-\frac{2\alpha}{e^{8\lambda^2k_1\alpha t}+1}+\alpha+\frac{\tilde{\alpha}n}{2\lambda k_1}\log 2$$

satisfies

$$-4k_1\lambda^2\varphi^2-2\lambda\tilde{\alpha}n\varphi(t)+k_4-\psi'(t)=0$$

for $\varphi(t)$ given by (8). We have thus shown that the function

$$V(x,t)=M\Big(\frac{e^{8\lambda^2k_1\alpha t}}{e^{8\lambda^2k_1\alpha t}+1}\Big)^{\frac{\tilde{\alpha}n}{2\lambda k_1}}\exp\Big[-\frac{2\alpha}{e^{8\lambda^2k_1\alpha t}+1}+\alpha+\frac{\tilde{\alpha}n}{2\lambda k_1}\log 2\Big]\times$$

$$\exp\left[-\alpha(|x|^2+1)^\lambda\tanh 4k_1\lambda^2\alpha t+(-4\lambda^2k_1\alpha^2-2\lambda\tilde{\alpha}n\alpha+k_4)t\right]$$

satisfies the differential inequality $LV\leqq 0$ in $R^n\times(0,\infty)$. Consider the function $W_\pm(x,t)=V(x,t)\pm u(x,t)$ and apply the maximum principle of Bodanko [1] to $W_\pm(x,t)$. Then we have $W_\pm(x,t)\geqq 0$, i. e. $|u(x,t)|\leqq V(x,t)$ for $(x,t)\in R^n\times[0,\infty)$, thereby completing the proof of the lemma.

4. Now we assume that the coefficients of L in (2) satisfy the condition (6) in $R^n\times(0,\infty)$ for some constants $k_1(>0)$, $k_2(\geqq 0)$, $k_3(>0)$, k_4 and $\lambda\in[1,\infty)$. Let $u=u(x,t)$ continuous in $R^n\times[0,\infty)$ satisfy $Lu=0$ and $|u(x,t)|\leqq M^*\exp[a^*(|x|^2+1)^\lambda]$ in $R^n\times(0,\infty)$ and $|u(x,0)|\leqq M\exp[a(|x|^2+1)^\lambda]$ for positive constants M^*, M, a^* and a. Suppose that these constants fulfil the inequality

$$(10)\qquad 4a^2\lambda^2k_1+2a\lambda k_2n-k_3<0.$$

Now we use an idea presented [2], [6]. First we introduce a parameter $\rho(>1)$ and put

$$V(x,t)=M\exp\Big[a(|x|^2+1)^\lambda\rho^{-r_0t}+\frac{4\lambda^2k_1a+2\lambda k_1na+k_4}{r_0\log\rho}(1-\rho^{-r_0t})\Big],$$

where $r_0=(k_3a^{-1}-2\lambda k_2n-4a\lambda^2k_1)(\log\rho)^{-1}$. From (10) we see that $r_0>0$.

Since $\lambda\in[1,\infty)$, it is easy to see that $V(x,t)$ satisfies the differential inequality

$$\sum_{i,j=1}^n a_{ij}\frac{\partial^2 V}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i\frac{\partial V}{\partial x_i}+cV-\frac{\partial V}{\partial t}\leqq 0$$

in $R^n\times(0,r_0^{-1}]$. Putting $W_\pm(x,t)=V(x,t)\pm u(x,t)$ and applying the maximum principle of Bodanko [1] to $W_\pm(x,t)$ we have $W_\pm(x,t)\geqq 0$, i. e.,

$$|u(x,t)| \leqq M \exp\left[a(|x|^2+1)^\lambda \rho^{-r_0 t} + \frac{4\lambda^2 k_1 a + 2\lambda k_1 n a + k_4}{r_0 \log \rho}(1-\rho^{-r_0 t})\right]$$

in $R^n \times (0, r_0^{-1}]$. Hence we have

$$(11) \qquad |u(x, r_0^{-1})| \leqq M_1 \exp[a(|x|^2+1)^\lambda \rho^{-1}], \; x \in R^n,$$

where

$$M_1 = M \exp\left[\frac{4\lambda^2 k_1 a + 2\lambda k_1 n a + k_4}{\log \rho}(1-\rho^{-1}) r_0^{-1}\right].$$

we consider $t = r_0^{-1}$ as the initial time and (11) as the initial condition for u. Repeating the above procedure, we obtain

$$|u(x,t)| \leqq M_1 \exp\left[a\rho^{-1}(|x|^2+1)^\lambda \rho^{-r_1(t-r_0^{-1})} + \frac{4\lambda^2 k_1 a \rho^{-1} + 2\lambda n k_1 a \rho^{-1} + k_4}{r_1 \log \rho}(1-\rho^{-r_1(t-r_0^{-1})})\right]$$

in $R^n \times (r_0^{-1}, r_0^{-1}+r_1^{-1}]$, where $r_1 = (k_3 a^{-1}\rho - 2\lambda k_2 n - 4a\lambda^2 k_1 \rho^{-1})(\log \rho)^{-1}$, so that

$$|u(x, r_0^{-1}+r_1^{-1})| \leqq M_2 \exp[a\rho^{-2}(|x|^2+1)^\lambda], \; x \in R^n,$$

where

$$M_2 = M \exp\left[\frac{4\lambda^2 k_1 a + 2\lambda k_1 n a}{\log \rho}(1-\rho^{-1})(r_0^{-1}+\rho^{-1} r_1^{-1}) + \frac{k_4}{\log \rho}(1-\rho^{-1})(r_0^{-1}+r_1^{-1})\right].$$

In general,

$$(12) \quad |u(x, r_0^{-1}+r_1^{-1}+\cdots+r_j^{-1})| \leqq M_{j+1} \exp[a\rho^{-j-1}(|x|^2+1)^\lambda], \; x \in \mathbf{R}^n,$$

where $r_j = (k_3 a^{-1}\rho^j - 2\lambda k_2 n - 4a\lambda^2 k_1 \rho^{-j})(\log \rho)^{-1}$ and

$$(13) \quad M_{j+1} = M \exp\left[\frac{4\lambda^2 k_1 a + 2\lambda k_1 n a}{\log \rho}(1-\rho^{-1})(r_0^{-1}+\rho^{-1} r_1^{-1}+\cdots+\rho^{-1} r_j^{-1}) + \frac{k_4}{\log \rho}(1-\rho^{-1})(r_0^{-1}+r_1^{-1}+\cdots+r_j^{-1})\right].$$

Let us consider the convergent series

$$f(\rho)=\sum_{i=0}^{\infty}\rho^{-i}r_i^{-1}=\sum_{i=0}^{\infty}\frac{\rho^{-i}\log\rho}{k_3a^{-1}\rho^i-2\lambda k_2n-4a\lambda^2k_1\rho^{-i}}$$

and

$$g(\rho)=\sum_{i=0}^{\infty}r_i^{-1}=\sum_{i=0}^{\infty}\frac{\log\rho}{k_3a^{-1}\rho^i-2\lambda k_2n-4a\lambda^2k_1\rho^{-i}}$$

It is a matter of simple calculation to derive the following:

$$f(\rho)\leq\frac{1}{k_3a^{-1}-2\lambda k_2n-4a\lambda^2k_1}\,\frac{\log\rho}{1-\rho^{-1}}, \tag{14}$$

and

$$\lim_{\rho\to1}g(\rho)=\lim_{\rho\to1}\int_0^{\infty}\frac{\log\rho}{k_3a^{-1}\rho^s-2\lambda k_2n-4a\lambda^2k_1\rho^{-s}}\,ds \tag{15}$$

$$=\frac{1}{2\lambda\sqrt{k_2^2n^2+4k_1k_3}}\log\frac{k_3a^{-1}-k_2n\lambda+\lambda\sqrt{k_2^2n^2+4k_1k_3}}{k_3a^{-1}-k_2n\lambda-\lambda\sqrt{k_2^2n^2+4k_1k_3}}$$

$$\equiv T_0,\ \text{say.}$$

From (13) and (14) it follows that

$$M_j\leq\tilde{M}\exp\left[\frac{k_4}{\log\rho}(1-\rho^{-1})\sum_{i=0}^{\infty}r_i^{-1}\right],\quad j=1,2,\cdots, \tag{16}$$

where we have set

$$\tilde{M}=M\exp\left[\frac{4\lambda^2k_1a+2\lambda k_1na}{k_3a^{-1}-2\lambda k_2n-4a\lambda^2k_1}\right],$$

and on account of (15) it is possible to choose $\rho_0(>1)$ so that the right-hand side of (16) does not exceed a constant, say $M_0=2\tilde{M}\exp(k_4T_0)$ provided $1<\rho<\rho_0$. Therefore it follows from (12) that

$$|u(x,\sum_{i=0}^{\infty}r_i^{-1})|\leq M_0\exp[a\rho^{-j-1}(|x|^2+1)^{\lambda}],\quad x\in R^n \tag{17}$$

provided that ρ is sufficiently near to 1. Let $x\in R^n$ be arbitrary but fixed. Given any positive number ε, we can find ρ_1 (>1) such that $|u(x,T_0)-u(x,f(\rho))|<\frac{\varepsilon}{2}$ for $1<\rho<\rho_1$, as can be seen from (15).

On the other hand, for a fixed ρ with $1<\rho<\min(\rho_0,\rho_1)$ an integer N can be found such that

$$|u(x, f(\rho))-u(x, \sum_{i=0}^{j} r_i^{-1})|<\frac{\varepsilon}{2} \text{ for } \quad j>N.$$

Thus we obtain $|u(x, T_0)|<|u(x, \sum_{i=0}^{j} r_i^{-1})|+\varepsilon$ for $j>N$, whence in view of (17), $|u(x, T_0)|<M_0 \exp[a\rho^{-j-1}(|x|^2+1)^\lambda]+\varepsilon$ for $j>N$. This yields $|u(x, T_0)|\leqq M_0$ in the limit as $j\to\infty$ and $\varepsilon\to 0$. Since x is arbitrary, this inequality holds throughout R^n.

5. After these preparations, we can prove the following

THEOREM 1. *Let*

$$L=\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i\frac{\partial}{\partial x_i}+c-\frac{\partial}{\partial t}$$

be a parabolic differential operator in $R^n\times(0, \infty)$ *whose coefficients* $a_{ij}(=a_{ji})$, b_i *and* c *satisfy the condition* (6) *in* $R^n\times[0, \infty)$ *for some constants* $k_1(>0)$, k_2 $(\geqq 0)$, $k_3(>0)$, k_4 *and* $\lambda\in[1, \infty)$. *Let* $u=u(x, t)$ *continuous in* $R^n\times[0, \infty)$ *satisfy* $Lu=0$ *and* $|u(x, t)|\leqq M^* \exp[a^*(|x|^2+1)^\lambda]$ *in* $R^n\times(0, \infty)$ *for some positive constants* M^* *and* a^* *and* $|u(x, 0)|\leqq M \exp[a(|x|^2+1)^\lambda]$ *foe positive constants* M *and* a. *Assume that the inequalities* (10) *and*

$$-4\lambda^2 k_1\alpha^2-2\lambda\tilde{\alpha}n\alpha+k_4<0 \tag{18}$$

are valid. Then $\lim_{t\to\infty} u(x, t)=0$, *the convergence being of exponential order and uniform with respect to* $x\in R^n$.

PROOF. By the argument in § 4, we can find T_0 and M_0 such that $|u(x, T_0)|\leqq M_0$. Now, we discuss how $u(x, t)$ behaves for $t>T_0$. To make use of Lemma we introduce the function

$$\begin{aligned} W(x, t)=M_0 \exp[&-\alpha(|x|^2+1)^\lambda \tanh 4k_1\lambda^2\alpha(t-T_0) \\ &+(-4\lambda^2 k_1\alpha^2-2\lambda\tilde{\alpha}n\alpha+k_4)(t-T_0)]. \end{aligned} \tag{19}$$

Then we can verify that

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 W}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i\frac{\partial W}{\partial x_i}+cW-\frac{\partial W}{\partial t}\leqq 0$$

in $R^n\times(T_0, \infty)$. Thus, according to Bodanko's maximum principle, we conclude that $|u(x, t)|\leqq W(x, t)$ in $R^n\times(T_0, \infty)$. Now the assertion of the theorem follows from the observation that the asymptotic behavior of $W(x, t)$ as $t\to\infty$ is determined by the factor

$$e^{(-4\lambda^2 k_1\alpha^2-2\lambda\tilde{\alpha}n\alpha+k_4)t}$$

which decays exponentially to zero as $t\to\infty$ provided that (18) holds. This completes the proof.

By the quite similar method, we can prove the following. We may omit the proof of it.

THEOREM 2. *Let L be a parabolic differential operator of the form in* (2) *satisfying* (6) *in* $R^n\times[0, \infty)$ *for a number* $\lambda\in(0, 1]$. *Suppose that a continuous function* $u(x, t)$ *in* $R^n\times[0, \infty)$ *satisfy* $Lu=0$ *and* $|u(x, t)|\leqq M^*\times\exp[a^*(|x|^2+1)^\lambda]$ *in* $R^n\times(0, \infty)$ *for some positive constants* M^* *and* a^* *and* $|u(x, 0)|\leqq M\exp[a(|x|^2+1)^\lambda]$ *for positive constants* M *and* a. *Assume that the inequalities* (10) *and*

$$4\lambda(1-\lambda)\alpha-2\lambda n\alpha\tilde{\alpha}+k_4<0 \tag{20}$$

are valid. Then $\lim_{t\to\infty}u(x, t)=0$, *the convergence being of exponential order and uniform with respect to* $x\in R^n$.

Next we shall prove the following

THEOREM 3. *Let L be a parabolic differential operator of the form* (2) *with coefficients satisfying* (6) *for some* $\lambda\in(0, 1]$ *and let* $u=u(x, t)$ *continuous in* $R^n\times[0, \infty)$ *satisfy* $Lu=0$ *and* $|u(x, t)|\leqq M^*\exp[a^*(|x|^2+1)^\lambda]$ *in* $R^n\times(0, \infty)$ *for positive constants* M^* *and* a^* *and* $|u(x, 0)|\leqq M\exp[a(|x|^2+1)^\lambda]$ *for positive constants M and a. Assume that the inequalities* (10) *and*

$$k_4+2(1-\lambda)\sqrt{k_1k_3}-\delta\sqrt{\frac{k_3}{k_1}}<0 \tag{21}$$

are valid. Then $\lim_{t\to\infty}u(x, t)=0$, *the convergence being of exponential order and uniform with respect to* $x\in R^n$.

To see this, it is enough to introduce the function

$$W(x, t)=M_0[\cosh 2\lambda\sqrt{k_1k_3}(t-T_0)]^{\frac{2k_1(1-\lambda)-\delta}{2\lambda k_1}}$$
$$\times\exp\left[-(|x|^2+1)^\lambda\sqrt{\frac{k_3}{4\lambda^2k_1}}\tanh 2\lambda\sqrt{k_1k_3}(t-T_0)+k_4(t-T_0)\right]$$

and to proceed exactly as in the proof of Theorem 1. we may omit the details.

REMARK 1. Our Theorem 1 corresponds to Theorem 2 of [5]. If we take $a=0$ in the Cauchy data $|u(x, 0)|\leqq M\exp[a(|x|^2+1)^\lambda]$ in our theorem, then we get the result due to Kuroda [5].

REMARK 2. In our Theorem 1, consider the case $\lambda=1$. If we put $k_1=1$,

$k_2=0$, $\tilde{\alpha}=1$, $k_3=k^2$ and $k_4=k^2+l$, our theorem can be applied to $L_0u=0$ stated in § 1. In this case, α is the positive root of $4x^2-k_3=0$ and $\alpha=\frac{k}{2}$, hence the condition (18) is equivalent to $l<kn$. Thus Theorem 1 gives us, as a special case, Krzyżański's result stated in § 1.

REMARK 3. If we take $a=0$ in the Cauchy data $|u(x, 0)|\leqq M \exp [a(|x|^2+1)^{\lambda}]$ in Theorem 2, then we get the result stated in [3].

REMARK 4. In the case $\lambda=1$, Theorem 2 and Theorem 3 coincide with results due to Kusano [8] (Theorems A and B stated in § 2).

References

[1] W. Bodanko; Sur le problème de Cauchy et les problèmes de Fourier pour les equations paraboliques dans un domaine non borné, Ann. Polon. Math., **18** (1966), 79-94.

[2] Lu-San Chen; On the behavior of solutions for large $|x|$ of parabolic equations with unbounded coefficients, Tohoku Math. J., **20** (1968), 589-595.

[3] Lu-San Chen; Note on the behavior of solutions of parabolic equations with unbounded coefficients, Nagoya Math. J., **37** (1970), 1-4.

[4] M. Krzyżański; Evaluations des solutions de l'equation lineaire du type parabolique à coefficients non borné, Ann. Polon. Math., **11** (1962), 253-260.

[5] T. Kuroda; Asymptotic behavior of solutions of parabolic equations with unbounded coefficients, Nagoya Math. J., **37** (1970), 5-12.

[6] T. Kuroda and Lu-San Chen; On the behavior of solutions of parabolic equations with unbounded coefficients, Ann. Polon. Math., **23** (1970), 57-64.

[7] T. Kusano; On the decay for large $|x|$ of solutions of parabolic equations with unbounded coefficients, Publ. Res. Inst. Math. Sci., Kyoto Univ., Ser. A, **3** (1967), 203-210.

[8] T. Kusano; Asymptotic behavior of solutions of parabolic differential equations with unbounded coefficients, J. Sci. Hiroshima Univ., Ser. A-1, **33** (1969), 151-159.

Department of Mathematics
National Central University
Chung-Li, Taiwan
and
Institute of Mathematics
Academia Sinica, Taipei.