HIROSHIMA MATH. J.
2 (1972), 205-214

Pseudo-coalescent Classes of Lie Algebras

Osamu MARUO

(Received February 25, 1972)

Introduction

In the study of infinite-dimensional Lie algebras the concepts of subideals and coalescency seem to play a central role. A subalgebra of a Lie algebra L is called a subideal of L it is a member of a finite series of subalgebras ending with L such that each member is an ideal of the following. A class $\mathfrak{X}$ of Lie algebras is called coalescent [4] if in any Lie algebra the join of any pair of subideals belonging to $\mathfrak{X}$ is always a subideal belonging to $\mathfrak{X}$. B. Hartley has shown in [1] that the class of finite-dimensional nilpotent Lie algebras and the class of finite-dimensional Lie algebras over a field of characteristic 0 are coalescent. Furthermore, S. Tôgô has shown in [5] that other eleven classes of Lie algebras, e.g., the class of finite-dimensional solvable Lie algebras over a field of characteristic 0, are coalescent.

We shall introduce the new concepts, weak ideals and pseudo-coalescency. We call a subalgebra H of a Lie algebra L to be a weak ideal of L if $L(\mathrm{ad}\,H)^n \subseteq H$ for some $n>0$. Then any subideal of L is a weak ideal but not conversely. We call a class $\mathfrak{X}$ of Lie algebras to be pseudo-coalescent if in any Lie algebra the join of any pair of subideal and weak ideal belonging to $\mathfrak{X}$ is always a weak ideal belonging to $\mathfrak{X}$. We may ask whether the results for subideals and coalescency hold analogously for weak ideals and pseudo-coalescency. The purpose of this paper is to investigate weak ideals and pseudo-coalescency.

Some properties of weak ideals are given in Section 2. For a weak ideal H of L, $H^{(\omega)}=\bigcap_{i=0}^{\infty} H^{(i)}$ and $H^{\omega}=\bigcap_{i=1}^{\infty} H^{i}$ are both characteristic ideals of L (Theorem 2.2), which generalizes the results of E. Schenkman. If H and K are weak ideals of L such that K idealizes H, then $H+K$ is also a weak ideal of L. In Section 3 we shall prove the pseudo-coalescency of the class of finite-dimensional nilpotent Lie algebras over a field of characteristic 0 (Theorem 3.5). In Section 4 we show the three results on pseudo-coalescency which are analogous to three general theorems on coalescency in [5]. We prove the pseudo-coalescency of all the classes of Lie algebras stated in [1, Theorems 2 and 5] and [5, Theorem 4.4] (Theorem 4.4). In Section 5 we show by example that a weak ideal is not necessarily a subideal.

The author would like to thank Professor Shigeaki Tôgô for his helpful suggestions.

1.

Throughout this paper we shall consider the Lie algebras over a field Φ which are not necessarily finite-dimensional and the characteristic of the basic field Φ will be arbitrary unless otherwise stated.

Let L be a Lie algebra over a field Φ. We write $H \leq L$ if H is a subalgebra of L and $H \lhd L$ if H is an ideal of L. We denote by $< S_1, \cdots, S_n >$ the subalgebra generated by subsets $S_1, \cdots, S_n$ of L. We recall the definitions of subideals and coalescency.

DEFINITION 1.1. *A subalgebra H of L is called an n-step subideal of L and written H n-si L if there is a finite series of subalgebras*

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = L$$

such that $H_i \lhd H_{i+1}$ $(0 \leq i < n)$. H is called a subideal of L and written H si L if it is an n-step subideal of L for some n.

DEFINITION 1.2. *A class $\mathfrak{X}$ of Lie algebras over a field Φ is called coalescent if H, K si L and $H, K \in \mathfrak{X}$ imply $<H, K>$ si L, $\in \mathfrak{X}$.*

We shall now introduce the new notions corresponding to subideals and coalescency, that is, weak ideals and pseudo-coalescency.

DEFINITION 1.3. *We call a subalgebra H of L an n-step weak ideal of L and write H n-wi L if $L(\mathrm{ad}\, H)^n \subseteq H$ with $n > 0$. We call H a weak ideal of L and write H wi L if it is an n-step weak ideal of L for some n.*

Here H 1-*wi* L is equivalent to each of H 1-*si* L and $H \lhd L$. For $n \geq 2$, if H n-*si* L, then H n-*wi* L. But the converse does not hold in general, which we shall show by example in Section 5.

DEFINITION 1.4. *A class $\mathfrak{X}$ of Lie algebras over a field Φ is called pseudo-coalescent if H si L, K wi L and $H, K \in \mathfrak{X}$ imply $<H, K>$ wi L, $\in \mathfrak{X}$.*

We need the following classes of Lie algebras over Φ.

$\mathfrak{F}$: the class of finite-dimensional Lie algebras.

$\mathfrak{G}$: the class of finitely generated Lie algebras.

$\mathfrak{A}$: the class of abelian Lie algebras.

$\mathfrak{N}$: the class of nilpotent Lie algebras.

$\mathfrak{S}$: the class of solvable Lie algebras.

$L \in \mathfrak{D}$ if and only if $H \leq L$ implies H *si* L.

$L \in \mathfrak{I}$ if and only if $H < L$ implies $H < I_L(H)$, where $I_L(H)$ is the idealizer of H in L.

2.

In this section we shall show several results on weak ideals. As an easy consequence of Definition 1.3, we have

LEMMA 2.1. (1) *If H wi L and $K \leq L$, then $H \cap K$ wi K.*

(2) *If H wi K and K wi L, then H wi L.*

(3) *If H wi L and $K \lhd L$, then $H+K$ wi L.*

(4) *Let f be a homomorphism of L onto a Lie algebra $\bar{L}$. If H wi L, then $f(H)$ wi $\bar{L}$. If $\bar{H}$ wi $\bar{L}$, then $f^{-1}(\bar{H})$ wi L.*

PROOF. (1), (2) and (4) are obvious. If H n-*wi* L and $K \lhd L$, then

$$L(\mathrm{ad}\,(H+K))^n \subseteq L(\mathrm{ad}\,H)^n + K \subseteq H+K.$$

Hence $H+K$ n-*wi* L and (3) is proved.

If H *si* L, then it is known [2, 3] that $H^{(\omega)} = \bigcap_{i=0}^{\infty} H^{(i)}$ and $H^{\omega} = \bigcap_{i=1}^{\infty} H^{i}$ are characteristic ideals of L. We generalize this in the following

THEOREM 2.2. *If H wi L, then $H^{(\omega)}$ and H^{ω} are characteristic ideals of L.*

PROOF. Let M be the semi-direct sum $L+\mathfrak{D}(L)$, where $\mathfrak{D}(L)$ is the derivation algebra of L. Assume that H n-*wi* L. Then, since $L \lhd M$, H $(n+1)$-*wi* M. By induction we see that $[M, H^k] \subseteq M(\mathrm{ad}_M H)^k$ for $k \geq 1$. Hence

$$[M, H^{(n)}] \subseteq [M, H^{n+1}] \subseteq M(\mathrm{ad}_M H)^{n+1} \subseteq H.$$

Therefore, by induction on k we have

$$[M, H^{(k+n)}] \subseteq H^{(k)}, \qquad k \geq 0.$$

It follows that $[M, H^{(\omega)}] \subseteq H^{(\omega)}$, that is, $H^{(\omega)} \lhd M$. Thus $H^{(\omega)}$ is a characteristic ideal of L. On the other hand, we can see by induction on k that

$$[M, H^{k+n}] \subseteq H^k, \qquad k \geq 1.$$

Hence H^{ω} is characteristic in L. This completes the proof.

LEMMA 2.3. *If H, K wi L and $[H, K] \subseteq H$, then $H+K$ wi L.*

PROOF. Let H n-*wi* L and K m-*wi* L for some n and m. If $m=1$, $K \lhd L$ and therefore $H+K$ *wi* L by Lemma 2.1. So we may assume that $m>1$. Put $l=n(m-1)+1$. Then

$$L(\mathrm{ad}\,(H+K))^l = \sum L(\mathrm{ad}\,N_1)\cdots(\mathrm{ad}\,N_l)$$

where N_i is either H or K for $i=1, 2, \cdots, l$. We shall consider

$$M = L(\mathrm{ad}\, N_1)\cdots(\mathrm{ad}\, N_l).$$

Let k be the number of N_i which equals H and consider the two cases $k \geq n$ and $k < n$.

The case $k \geq n$: First we show that

$$L(\mathrm{ad}\, K)^i(\mathrm{ad}\, H)^j(\mathrm{ad}\, K) \subseteq L(\mathrm{ad}\, H)^j(\mathrm{ad}\, K) \subseteq L(\mathrm{ad}\, H)^j \quad \text{for} \quad j \geq 1.$$

The first inclusion is obvious and the second inclusion follows by induction on j, since we have

$$\begin{aligned} L(\mathrm{ad}\, H)^j(\mathrm{ad}\, K) &\subseteq L(\mathrm{ad}\, H)^{j-1}(\mathrm{ad}\,[H, K]) + L(\mathrm{ad}\, H)^{j-1}(\mathrm{ad}\, K)(\mathrm{ad}\, H) \\ &\subseteq L(\mathrm{ad}\, H)^j + L(\mathrm{ad}\, H)^{j-1}(\mathrm{ad}\, H) \\ &= L(\mathrm{ad}\, H)^j. \end{aligned}$$

Now owing to this formula we have

$$M \subseteq L(\mathrm{ad}\, H)^k \subseteq H(\mathrm{ad}\, H)^{k-n} \subseteq H.$$

The case $k < n$: We then have either

$$M = L(\mathrm{ad}\, N_1)\cdots(\mathrm{ad}\, N_{l-m})(\mathrm{ad}\, K)^m \quad \text{or}$$

$$M = L(\mathrm{ad}\, N_1)\cdots(\mathrm{ad}\, N_k)(\mathrm{ad}\, K)^m(\mathrm{ad}\, H)(\mathrm{ad}\, N_{k+m+2})\cdots(\mathrm{ad}\, N_l).$$

Since K m-*wi* L, in the first case we have

$$M \subseteq L(\mathrm{ad}\, K)^m \subseteq K$$

and in the second case we have

$$M \subseteq K(\mathrm{ad}\, H)(\mathrm{ad}\, N_{k+m+2})\cdots(\mathrm{ad}\, N_l) \subseteq H.$$

Thus we conclude that $L(\mathrm{ad}\,(H+K))^l \subseteq H \cup K \subseteq H+K$ and $H+K$ *wi* L, completing the proof.

3.

In this section we shall show the pseudo-coalescency of $\mathfrak{N} \cap \mathfrak{F}$ for a field Φ of characteristic 0. This will be fundamental for showing the pseudo-coalescency of other classes in Section 4.

We begin with

Lemma 3.1. *If $H, K \leq L$ and $[H, K] \subseteq H$, then*

$$(H+K)^n \subseteq H^2 + (H+K)(\mathrm{ad}\, K)^{n-1}, \qquad n = 1, 2, 3, \cdots.$$

Proof. We can prove this by induction on n. If $n=1$, the statement is obvious. Assume the case $n=k-1$, $k\geq 2$. Then

$$\begin{aligned}(H+K)^k &\subseteq [H^2+(H+K)(\mathrm{ad}\,K)^{k-2}, H+K] \\ &\subseteq H^2+(H+K)(\mathrm{ad}\,K)^{k-1}+K(\mathrm{ad}\,K)^{k-2}(\mathrm{ad}\,H) \\ &\subseteq H^2+(H+K)(\mathrm{ad}\,K)^{k-1}+H(\mathrm{ad}\,K)^{k-1} \\ &= H^2+(H+K)(\mathrm{ad}\,K)^{k-1}.\end{aligned}$$

Hence we have the case $n=k$ and the statement is proved.

LEMMA 3.2. (1) *If K wi L and $K \in \mathfrak{N}$, then* $\mathrm{ad}\,K$ *is a nil set of derivations of L.*

(2) *If $H\leq L$, K wi L, $K\in\mathfrak{N}$ and $[H, K]\subseteq H$, then $(H+K)/H^2 \in \mathfrak{N}$.*

PROOF. (1) Let K n-*wi* L and $K^m=(0)$. Then

$$L(\mathrm{ad}\,K)^{n+m-1}\subseteq K(\mathrm{ad}\,K)^{m-1}=K^m=(0).$$

(2) Assume that $H\leq L$, $H(\mathrm{ad}\,K)^n\subseteq K$, $K^m=(0)$ and $[H, K]\subseteq H$. Then by Lemma 3.1, we have

$$\begin{aligned}(H+K)^{n+m} &\subseteq H^2+(H+K)(\mathrm{ad}\,K)^{n+m-1} \\ &\subseteq H^2+K(\mathrm{ad}\,K)^{m-1}+K^{n+m} \\ &= H^2.\end{aligned}$$

Since $H^2\lhd H+K$, it follows that $((H+K)/H^2)^{n+m}=(0)$ and therefore $(H+K)/H^2 \in \mathfrak{N}$, completing the proof.

LEMMA 3.3. *If $H\leq L$, K wi L, $H, K\in\mathfrak{N}$ and $[H, K]\subseteq H$, then $H+K\in\mathfrak{N}$.*

PROOF. By assumption $H\lhd H+K$ and $H\in\mathfrak{N}$. Furthermore Lemma 3.2 tells us that $(H+K)/H^2\in\mathfrak{N}$. Therefore we conclude that $H+K\in\mathfrak{N}$.

If D is a nil derivation of L over a field of characteristic 0, then $\exp D=\sum_{n=0}^{\infty} D^n/n!$ is an automorphism of L. Let M be a subspace of L and S a subset of the derivation algebra of L. We shall denote by M^S the smallest subspace of L containing M and invariant under S.

LEMMA 3.4. *Let Φ be of characteristic* 0. *If M is a finite-dimensional subspace of L and S is a finite-dimensional nil subspace of the derivation algebra of L, then there exist automorphisms $\alpha_1, \alpha_2, \cdots, \alpha_n$ which are products of finite number of elements* $\exp D(D\in S)$, *such that*

$$M^S=\sum_{i=1}^{n} M^{\alpha_i}.$$

For the proof, see B. Hartley [1, Corollary to Theorem 3]. We can now show the following

THEOREM 3.5. *Let Φ be of characteristic* 0. *Then $\mathfrak{N}\cap\mathfrak{F}$ is pseudo-coalescent.*

PROOF. Assume that H n-*si* L, K *wi* L and $H, K \in \mathfrak{N}\cap\mathfrak{F}$ for an arbitrary Lie algebra L. We must show that $J=<H, K>$ *wi* L, $\in \mathfrak{N}\cap\mathfrak{F}$. If $n=1$, then $H\triangleleft L$ and therefore $H+K$ *wi* L, $\in \mathfrak{N}\cap\mathfrak{F}$ by Lemmas 2.3 and 3.3. So we may assume that $n>1$. Since $\operatorname{ad}K$ is a finite-dimensional nil subspace of $\mathfrak{D}(L)$ by Lemma 3.2, it follows from Lemma 3.4 that

$$<H^{\operatorname{ad}K}>=<H^{\alpha_1}, \cdots, H^{\alpha_k}>,$$

where α_i is a product of finite number of elements $\exp(\operatorname{ad}x)(x \in K)$. Evidently H^{α_i} *si* L and $H^{\alpha_i} \in \mathfrak{N}\cap\mathfrak{F}$ for each i. Hence by the coalescency of $\mathfrak{N}\cap\mathfrak{F}$ ([1, Theorem 2]) we have

$$<H^{\operatorname{ad}K}> \textit{ si } L, \ \in \mathfrak{N}\cap\mathfrak{F}\,.$$

Now $J=K+<H^{\operatorname{ad}K}>$ and $[<H^{\operatorname{ad}K}>, K]\subseteq<H^{\operatorname{ad}K}>$. Hence we can use Lemmas 2.3 and 3.3 to see that J *wi* L, $\in \mathfrak{N}\cap\mathfrak{F}$.

Thus the theorem is proved.

COROLLARY 3.6. *Let Φ be of characteristic* 0. *Then $\mathfrak{N}\cap\mathfrak{F}$, $\mathfrak{N}\cap\mathfrak{G}$, $\mathfrak{D}\cap\mathfrak{F}$, $\mathfrak{D}\cap\mathfrak{G}$, $\mathfrak{I}\cap\mathfrak{F}$ and $\mathfrak{I}\cap\mathfrak{G}$ are all equal. Therefore these classes are pseudo-coalescent.*

PROOF. By Lemma 1 in [1] any subalgebra of a nilpotent Lie algebra is its subideal, which shows that $\mathfrak{N}\subseteq\mathfrak{D}$. Obviously $\mathfrak{D}\subseteq\mathfrak{I}$. Therefore we have $\mathfrak{N}\cap\mathfrak{F}\subseteq\mathfrak{D}\cap\mathfrak{F}\subseteq\mathfrak{I}\cap\mathfrak{F}$ and $\mathfrak{N}\cap\mathfrak{G}\subseteq\mathfrak{D}\cap\mathfrak{G}\subseteq\mathfrak{I}\cap\mathfrak{G}$. Since $\mathfrak{N}\cap\mathfrak{G}\subseteq\mathfrak{F}$ by Lemma 1 in [1], we have $\mathfrak{N}\cap\mathfrak{F}=\mathfrak{N}\cap\mathfrak{G}$. It is known [1, Corollary to Theorem 4] that $\mathfrak{I}$ is a class of locally nilpotent Lie algebras. Therefore $\mathfrak{I}\cap\mathfrak{G}\subseteq\mathfrak{N}\cap\mathfrak{G}$, whence $\mathfrak{I}\cap\mathfrak{G}=\mathfrak{N}\cap\mathfrak{G}$. This completes the proof.

4.

Let $\mathfrak{X}$ be any class of Lie algebras. Following the notations in [5] we denote by $\mathfrak{X}_{(\omega)}$ the class of Lie algebras L such that $L/L^{(\omega)} \in \mathfrak{X}$ and by $\mathfrak{X}_{\omega}$ the class of Lie algebras L such that $L/L^{\omega} \in \mathfrak{X}$. Consider the operations getting from $\mathfrak{X}$ another classes $\mathfrak{X}$, $\mathfrak{S}\cap\mathfrak{X}$, $\mathfrak{N}\cap\mathfrak{X}$, $\mathfrak{F}\cap\mathfrak{X}$, $\mathfrak{X}_{(\omega)}$ and $\mathfrak{X}_{\omega}$. Then S. Tôgô has shown in [5] that the application of the above operations to $\mathfrak{F}$ and $\mathfrak{N}_{\omega}\cap\mathfrak{G}$ produces the classes $\mathfrak{F}$, $\mathfrak{S}\cap\mathfrak{F}$, $\mathfrak{N}\cap\mathfrak{F}$, $\mathfrak{F}_{(\omega)}$, $\mathfrak{F}_{\omega}$, $(\mathfrak{N}\cap\mathfrak{F})_{(\omega)}$, $\mathfrak{S}\cap\mathfrak{F}_{\omega}$, $(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)}$, $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$, $\mathfrak{N}_{\omega}\cap\mathfrak{G}$, $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$, $(\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$ and $(\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$. We shall show at the end of this section that these classes are pseudo-coalescent.

S. Tôgô [5] has shown three general theorems on coalescency. We here show similar results on pseudo-coalescency. We say that a class $\mathfrak{X}$ of Lie algebras has the property (P) if $L \in \mathfrak{X}$ and $N \lhd L$ imply $L/N \in \mathfrak{X}$.

THEOREM 4.1. *Let $\mathfrak{X}$ be a class of Lie algebras over a field Φ having the property* (P). *If $\mathfrak{X}$ is pseudo-coalescent, then so are $\mathfrak{X}_{(\omega)}$ and $\mathfrak{X}_\omega$.*

PROOF. Assume that H *si* L, K *wi* L and $H, K \in \mathfrak{X}_{(\omega)}$. Put $J=<H, K>$. By Theorem 2.2, $H^{(\omega)} \lhd L$ and $K^{(\omega)} \lhd L$. Hence $I=H^{(\omega)}+K^{(\omega)} \lhd L$. By Lemma 2.1 we have $(H+I)/I$ *si* L/I and $(K+I)/I$ *wi* L/I. Since $H/H^{(\omega)} \in \mathfrak{X}$ and $\mathfrak{X}$ has the property (P),

$$(H+I)/I \simeq H/(I \cap H) \simeq (H/H^{(\omega)})/((I \cap H)/H^{(\omega)}) \in \mathfrak{X}.$$

Similarly $(K+I)/I \in \mathfrak{X}$. Since $\mathfrak{X}$ is pseudo-coalescent, J/I *wi* L/I, $\in \mathfrak{X}$. It follows from Lemma 2.1 that J *wi* L. Since $I \leq J^{(\omega)}$, $J/J^{(\omega)} \simeq (J/I)/(J^{(\omega)}/I)$. But $J/I \in \mathfrak{X}$ and $\mathfrak{X}$ has the property (P). Therefore $J/J^{(\omega)} \in \mathfrak{X}$, that is, $J \in \mathfrak{X}_{(\omega)}$. Thus $\mathfrak{X}_{(\omega)}$ is pseudo-coalescent.

The pseudo-coalescency of $\mathfrak{X}_\omega$ is similarly proved.

THEOREM 4.2. *Let $\mathfrak{X}$ be a class of Lie algebras over a field Φ contained in $\mathfrak{N}_\omega$ and having the property* (P). *If $\mathfrak{X}$ and $\mathfrak{N} \cap \mathfrak{X}$ are pseudo-coalescent, then so is $\mathfrak{S} \cap \mathfrak{X}$.*

PROOF. Let H *si* L, K *wi* L and $H, K \in \mathfrak{S} \cap \mathfrak{X}$. Since $\mathfrak{X}$ is pseudo-coalescent, $J=<H, K>$ *wi* L, $\in \mathfrak{X}$. To see the pseudo-coalescency of $\mathfrak{S} \cap \mathfrak{X}$, it suffices to show that $J \in \mathfrak{S}$. By Theorem 2.2 $I=H^\omega+K^\omega \lhd L$. It follows that $(H+I)/I$ *si* L/I and $(K+I)/I$ *wi* L/I. We have

$$(H+I)/I \simeq H/(I \cap H) \simeq (H/H^\omega)/((I \cap H)/H^\omega).$$

Since $\mathfrak{X} \subseteq \mathfrak{N}_\omega$, $H \in \mathfrak{N}_\omega$ and therefore $H/H^\omega \in \mathfrak{N}$. Hence $(H+I)/I \in \mathfrak{N}$. Since $\mathfrak{X}$ has the property (P), it follows that $(H+I)/I \in \mathfrak{X}$. Similarly, $(K+I)/I \in \mathfrak{N} \cap \mathfrak{X}$. Since $\mathfrak{N} \cap \mathfrak{X}$ is pseudo-coalescent, $J/I \in \mathfrak{N} \cap \mathfrak{X}$. But $I \in \mathfrak{S}$. Hence $J \in \mathfrak{S}$. This completes the proof.

THEOREM 4.3. *Let $\mathfrak{X}$ be a class of Lie algebras over a field Φ having the property* (P). *If $\mathfrak{X}$ and $\mathfrak{N} \cap \mathfrak{X}$ are pseudo-coalescent, then so is $\mathfrak{N}_{(\omega)} \cap \mathfrak{X}$.*

PROOF. Let H *si* L, K *wi* L and $H, K \in \mathfrak{N}_{(\omega)} \cap \mathfrak{X}$. Then $J=<H, K>$ *wi* L, $\in \mathfrak{X}$ since $\mathfrak{X}$ is pseudo-coalescent. It suffices to show that $J \in \mathfrak{N}_{(\omega)}$. Since $I=H^{(\omega)}+K^{(\omega)} \lhd L$, we have $(H+I)/I$ *si* L/I and $(K+I)/I$ *wi* L/I. Since $H, K \in \mathfrak{N}_{(\omega)} \cap \mathfrak{X}$ and $\mathfrak{X}$ has the property (P), it follows that $(H+I)/I$, $(K+I)/I \in \mathfrak{N} \cap \mathfrak{X}$. Since $\mathfrak{N} \cap \mathfrak{X}$ is pseudo-coalescent, $J/I \in \mathfrak{N} \cap \mathfrak{X}$. But then $J^{(n)} \leq I \leq J^{(\omega)}$ for some n and therefore $I=J^{(\omega)}$. It follows that $J \in \mathfrak{N}_{(\omega)}$, completing the proof.

By making use of these three theorems, we shall show the following theorem which is the analogue of Theorem 4.4 in [5] for the pseudo-coalescency case.

THEOREM 4.4. *If Φ is of characteristic* 0, *then the classes*

$$\mathfrak{F},\quad \mathfrak{S}\cap\mathfrak{F},\quad \mathfrak{N}\cap\mathfrak{F},\quad \mathfrak{F}_{(\omega)},\quad \mathfrak{F}_{\omega},$$

$$(\mathfrak{N}\cap\mathfrak{F})_{(\omega)},\quad \mathfrak{S}\cap\mathfrak{F}_{\omega},\quad (\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)},\quad \mathfrak{N}_{(\omega)}\cap\mathfrak{F},$$

$$\mathfrak{N}_{\omega}\cap\mathfrak{G},\quad \mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G},\quad (\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)},\quad (\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$$

are all pseudo-coalescent.

If Φ is of arbitrary characteristic, any classes containing $\mathfrak{A}$, e.g., $\mathfrak{N}$, $\mathfrak{S}$, $\mathfrak{D}$ and $\mathfrak{F}$, are not pseudo-coalescent.

PROOF. Let $H\ si\ L$, $K\ wi\ L$, $H, K\in\mathfrak{F}$ (resp. $\mathfrak{N}_{\omega}\cap\mathfrak{G}$) and $J=<H, K>$. By Theorem 2.2 we have $H^{\omega}=H^{p}\lhd L$ and $K^{\omega}=K^{q}\lhd L$ for some p and q. Therefore $I=H^{\omega}+K^{\omega}\lhd L$. We have $(H+I)/I\ si\ L/I$, $(K+I)/I\ wi\ L/I$ and $(H+I)/I$, $(K+I)/I\in\mathfrak{N}\cap\mathfrak{F}$. Hence by Theorem 3.5, $J/I\ wi\ L/I$, $\in\mathfrak{N}\cap\mathfrak{F}$. Therefore $J\ wi\ L$. Since $I\in\mathfrak{F}$, we have $J\in\mathfrak{F}$ (resp. Since $J/I\in\mathfrak{N}$, $J^{m}\leq I\leq J^{\omega}$ for some m and therefore $I=J^{\omega}$. Hence $J\in\mathfrak{N}_{\omega}$, whence $J\in\mathfrak{N}_{\omega}\cap\mathfrak{G}$.). Thus $\mathfrak{F}$ (resp. $\mathfrak{N}_{\omega}\cap\mathfrak{G}$) is pseudo-coalescent.

$\mathfrak{F}$ and $\mathfrak{N}\cap\mathfrak{F}$ have obviously the property (P) and are pseudo-coalescent by Theorem 3.5 and the first part of the proof. Hence by Theorem 4.1 $\mathfrak{F}_{(\omega)}$, $\mathfrak{F}_{\omega}$ and $(\mathfrak{N}\cap\mathfrak{F})_{(\omega)}$ are pseudo-coalescent, and by Theorem 4.3 $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$ is pseudo-coalescent. $\mathfrak{F}\subseteq\mathfrak{N}_{\omega}$ and $\mathfrak{F}$ has the property (P). Hence by Theorem 4.2, $\mathfrak{S}\cap\mathfrak{F}$ is pseudo-coalescent.

Now we see that $\mathfrak{F}_{\omega}$ (resp. $\mathfrak{N}_{\omega}\cap\mathfrak{G}$) has the property (P). In fact, let $L\in\mathfrak{F}_{\omega}$ (resp. $\mathfrak{N}_{\omega}\cap\mathfrak{G}$) and $N\lhd L$. Then $(L/N)^{\omega}\lhd L/N$ by Theorem 2.2. Therefore $(L/N)^{\omega}=M/N$ with $M\lhd L$. Since $L/L^{\omega}\in\mathfrak{F}$ (resp. $\mathfrak{N}$),

$$(L/N)/(L/N)^{\omega}\simeq L/M\simeq(L/L^{\omega})/(M/L^{\omega})\in\mathfrak{F}\text{ (resp. }\mathfrak{N}),$$

that is, $L/N\in\mathfrak{F}_{\omega}$ (resp. $\mathfrak{N}_{\omega}$). It follows that $\mathfrak{F}_{\omega}$ (sesp. $\mathfrak{N}_{\omega}\cap\mathfrak{G}$) has the property (P), as desired.

Observing the facts that $\mathfrak{F}_{\omega}\subseteq\mathfrak{N}_{\omega}$ and $\mathfrak{N}\cap\mathfrak{F}_{\omega}=\mathfrak{N}\cap\mathfrak{F}$, it now follows from Theorem 4.2 that $\mathfrak{S}\cap\mathfrak{F}_{\omega}$ is pseudo-coalecent. It is immediate that $\mathfrak{S}\cap\mathfrak{F}_{\omega}$ has the property (P). Therefore by Theorem 4.1 $(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)}$ is pseudo-coalescent. $\mathfrak{N}_{\omega}\cap\mathfrak{G}$ is pseudo-coalescent and has the property (P). Hence by Theorem 4.1, $(\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$ is pseudo-coalescent. Since $\mathfrak{N}_{\omega}\cap\mathfrak{G}$ and $\mathfrak{N}\cap(\mathfrak{N}_{\omega}\cap\mathfrak{G})$ $=\mathfrak{N}\cap\mathfrak{F}$ are pseudo-coalescent, so is $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ by Theorem 4.2. It follows from Theorem 4.2 that $(\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$ is pseudo-coalescent.

It has been shown by I. Stewart [4, Theorem 12.1] that there exists a Lie algebra L over any field Φ such that 1) L is the semi-direct sum $V+J$, $V\lhd L$, $V\cap J=(0)$; 2) $V\in\mathfrak{A}$, $J=<H, K>$ where H, K are abelian subalgebras of L, H is infinite-dimensional and K is 1-dimensional; 3) $H, K\ si\ L$ and

$J=I_L(J)$. Then H *si* L, K *wi* L and $H, K \in \mathfrak{A}$. Suppose that J *wi* L. Then $L(\mathrm{ad}\,J)^n \subseteq J$ for some n. It follows that $L(\mathrm{ad}\,J)^{n-1} \subseteq I_L(J)=J$ by 3). Continuing this procedure, we have $L \subseteq I_L(J)=J$, which is a contradiction. Hence J is not a weak ideal of L. Thus this example shows that any class containing $\mathfrak{A}$ is not pseudo-coalescent.

Thus the theorem is completely proved.

We remark that if Φ is of characteristic p, then any classes containing $\mathfrak{A} \cap \mathfrak{F}$, e.g., all the classes in Theorem 4.4, are not pseudo-coalescent. In fact, let A be a p-dimensional abelian Lie algebra over a field Φ of characteristic p with a basis $e_0, e_1, \cdots, e_{p-1}$. We consider linear transformations of A:

$$x: e_i \longrightarrow e_{i+1},\ e_{p-1} \longrightarrow 0 \qquad (i=0, 1, \cdots, p-2)$$

$$y: e_0 \longrightarrow 0, \quad e_i \longrightarrow ie_{i-1} \qquad (i=1, 2, \cdots, p-1)$$

$$z: e_i \longrightarrow e_i \qquad (i=0, 1, \cdots, p-1).$$

Then $Q=(x, y, z)$ is a nilpotent Lie algebra over Φ. Let L be the semi-direct sum $A+Q$ (see B. Hartley [1]). Since

$$(x) \lhd (e_{p-1}, x) \lhd (e_{p-2}, e_{p-1}, x) \lhd \cdots \lhd A+(x) \lhd A+(x, z) \lhd L,$$

$H=(x)$ *si* L. Since $L(\mathrm{ad}\,y)^{p+2}=(0)$, $K=(y)$ *wi* L. However, their join $<H, K>=Q$, which contains z, is its own idealizer in L. Therefore $<H, K>$ is not a weak ideal of L. Thus any class containing $\mathfrak{A} \cap \mathfrak{F}$ is not pseudo-coalescent.

5.

In Section 1, we noted that a weak ideal is not necessarily a subideal. We show it by example. Let L be a 3-dimensional simple Lie algebra over a field of characteristic $\neq 2$, with a basis x, y, z such that

$$[x, z]=2x, \quad [y, z]=-2y, \quad [x, y]=z\,.$$

Let $H=(x)$. Then it is immediate that H 2-*wi* L. But H is not a subideal of L, since L has no non-zero proper ideals.

Another kind of coalescency of a class $\mathfrak{X}$ of Lie algebras might be defined by the condition that in any Lie algebra the join of a pair of weak ideals belonging to $\mathfrak{X}$ is always a weak ideal belonging to $\mathfrak{X}$. However, this is not interesting for us. Because the join of two 1-dimensional weak ideals may be even simple which is shown as follows. Let L be the Lie algebra stated above. Then $H=(x)$ and $K=(y)$ are both abelian weak ideals of L. Since $<H, K>=L$, $<H, K>$ *wi* L and $<H, K>$ is simple.

References

[1] B. Hartley, Locally nilpotent ideals of a Lie algebra, Proc. Camb. Phil. Soc., **63** (1967), 257-272.
[2] E. Schenkman, A theory of subinvariant Lie algebras, Amer. J. Math., **73** (1951), 453-474.
[3] E. Schenkman, Infinite Lie algebras, Duke Math. J., **19** (1952), 529-535.
[4] I. Stewart, Lie Algebras, Lecture Notes in Mathematics 127, Springer, Berlin・Heidelberg・New York, 1970.
[5] S. Tôgô, Radicals of infinite dimensional Lie algebras, this journal.

Department of Mathematics,
Faculty of Science,
Hiroshima University