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Radicals of Infinite Dimensional Lie Algebras

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Introduction

Recently B. Hartley [3] and I. Stewart [9, 10] investigated the structure of infinite-dimensional Lie algebras in the spirit of infinite group theory. They considered subideals as the Lie analogue of subnormal subgroups of infinite groups and studied the connections between subideals of a Lie algebra and the structure of the algebra as a whole.

A subideal of a Lie algebra L is a member of a finite series of subalgebras ending with L such that each member is an ideal of the following. A class $\mathfrak{X}$ of Lie algebras is called coalescent [10] if in an arbitrary Lie algebra the join of any two subideals belonging to $\mathfrak{X}$ is always a subideal belonging to $\mathfrak{X}$. It has been shown in [3] that if the basic field is of characteristic 0, the class $\mathfrak{F}$ of finite-dimensional Lie algebras and the class $\mathfrak{N}\cap\mathfrak{F}$ with $\mathfrak{N}$ the class of nilpotent Lie algebras are coalescent. We may ask whether there exist coalescent classes besides $\mathfrak{F}$ and $\mathfrak{N}\cap\mathfrak{F}$.

A finite-dimensional Lie algebra has two kinds of radicals, the solvable radical and the nilpotent radical. As the Lie analogue of radicals of infinite groups, several radicals corresponding to the nilpotent radical in finite-dimensional case have been introduced for a Lie algebra L which is not necessarily of finite dimension [3, 10]. The Fitting radical $\nu(L)$ is the sum of all nilpotent ideals of L. The Hirsch-Plotkin radical $\rho(L)$ is the unique locally nilpotent ideal of L. If the basic field is of characteristic 0, on the base of the coalescency of $\mathfrak{N}\cap\mathfrak{F}$, the Baer radical $\beta(L)$ is defined as the subalgebra generated by all subideals of L belonging to $\mathfrak{N}\cap\mathfrak{F}$. As for the interrelation of these radicals it is shown [3, 10] that if the basic field is of characteristic 0, $\nu(L)\subseteq\beta(L)\subseteq\rho(L)$ and these are different in general, although these reduce to the nilpotent radical in the case where L is finite-dimensional. They are called locally nilpotent radicals. However, no study has been made about the ideals corresponding to the solvable radical in finite-dimensional case. We define local solvability of a subalgebra just as local nilpotency, that is, we call a subalgebra H of L locally solvable if every finite subset of H lies in a solvable subalgebra. Thus we may ask what can be said about locally solvable radicals of L.

The purpose of this paper is to investigate the structure of infinite-dimensional Lie algebras, especially to search for coalescent classes of Lie algebras and to study locally nilpotent and locally solvable radicals of a Lie algebra.

Part I will be devoted to the study of coalescent classes. For a class $\mathfrak{X}$ of Lie algebras, we define $\mathfrak{X}_{(\omega)}$ (resp. $\mathfrak{X}_{\omega}$) as the class of Lie algebras L such that $L/L^{(\omega)}$ (resp. L/L^{ω}) belongs to $\mathfrak{X}$, where $L^{(\omega)}=\bigcap_{n=0}^{\infty} L^{(n)}$ and $L^{\omega}=\bigcap_{n=1}^{\infty} L^{n}$. Denoting by $\mathfrak{S}$ the class of solvable Lie algebras, we consider the operations of getting, from $\mathfrak{X}$, other classes $\mathfrak{S}\cap\mathfrak{X}$, $\mathfrak{N}\cap\mathfrak{X}$, $\mathfrak{F}\cap\mathfrak{X}$, $\mathfrak{X}_{(\omega)}$ and $\mathfrak{X}_{\omega}$. By applying these operations to $\mathfrak{F}$ and $\mathfrak{N}_{\omega}\cap\mathfrak{G}$ where $\mathfrak{G}$ is the class of finitely generated Lie algebras, we obtain thirteen classes $\mathfrak{F}$, $\mathfrak{S}\cap\mathfrak{F}$, $\mathfrak{N}\cap\mathfrak{F}$, $\mathfrak{F}_{(\omega)}$, $\mathfrak{F}_{\omega}$, $(\mathfrak{N}\cap\mathfrak{F})_{(\omega)}$, $\mathfrak{S}\cap\mathfrak{F}_{\omega}$, $(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)}$, $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$, $\mathfrak{N}_{\omega}\cap\mathfrak{G}$, $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$, $(\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$ and $(\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$ (Theorem 3.7). We show that these classes are all coalescent if the basic field is of characteristic 0 (Theorem 4.4). We shall also show some general theorems on coalescent classes (Theorems 4.1, 4.2 and 4.3).

In Part II, we shall study the locally solvable and the locally nilpotent radicals of a Lie algebra. Denote by $\mathrm{L}\mathfrak{S}$ (resp. $\mathrm{L}\mathfrak{N}$) the class of all locally solvable (resp. locally nilpotent) Lie algebras. For a class $\mathfrak{X}$ such that either $\mathfrak{S}\cap\mathfrak{F}\subseteq\mathfrak{X}\subseteq\mathrm{L}\mathfrak{S}$ or $\mathfrak{N}\cap\mathfrak{F}\subseteq\mathfrak{X}\subseteq\mathrm{L}\mathfrak{N}$, we define the radical $\mathrm{Rad}_{\mathfrak{X}-si}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}}(L)$) as the subalgebra generated by all the subideals (resp. ideals) of L belonging to $\mathfrak{X}$. Then $\mathrm{Rad}_{\mathfrak{N}}(L)=\nu(L)$, $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}-si}(L)=\beta(L)$ and $\mathrm{Rad}_{\mathrm{L}\mathfrak{N}}(L)=\rho(L)$. We show that $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}}(L)$ is the union of all the ideals belonging to $\mathfrak{N}\cap\mathfrak{F}$, is a locally nilpotent characteristic ideal of L and is different from $\nu(L)$, $\beta(L)$ and $\rho(L)$ in general (Theorems 7.1 and 7.2). It of course reduces to the nilpotent radical when L is finite-dimensional. We show that if the basic field is of characteristic 0, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}}(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}}(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_{\omega}}(L)$, $\mathrm{Rad}_{\mathfrak{S}}(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-si}(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}-si}(L)$ and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_{\omega}-si}(L)$ are the union of all the ideals or subideals belonging to the corresponding classes, are locally solvable characteristic ideals of L and are different from each other in general (Theorems 8.1, 8.3 and 8.5), although these radicals reduce to the solvable radical when L is finite-dimensional. It is furthermore shown that for any one $\mathfrak{X}$ of the thirteen coalescent classes stated above, the subalgebra generated by all subideals (resp. ideals) of L belonging to $\mathfrak{X}$ is a characteristic ideal of L and every finite subset of the subalgebra lies in a subideal (resp. ideal) of L belonging to $\mathfrak{X}$ (Theorems 6.3 and 6.10).

PART I. COALESCENT CLASSES

§1. Definitions and lemmas

We shall be concerned with Lie algebras over a field Φ which is not necessarily finite-dimensional. Throughout this paper, the basic field Φ will be of arbitrary characteristic and L will be an arbitrary Lie algebra over a field Φ, unless otherwise specified.

We write $H\leq L$ when H is a subalgebra of L and $H\lhd L$ when H is an ideal of L. We denote by $<K_1,\cdots, K_n>$ the subalgebra generated by subsets $K_1, \cdots, K_n$ of L. The concepts of subideals and coalescency are fundamental in this paper. So we first recall their definitions.

DEFINITION 1.1. *A subalgebra H of L is called an n-step subideal of L if there is a finite series of subalgebras*

$$H=H_0\leq H_1\leq\cdots\leq H_n=L$$

such that $H_i \lhd H_{i+1} (0\leq i<n)$. We then write H n-si L. H is called a subideal of L if it is an n-step subideal of L for some $n>0$. We then write H si L.

DEFINITION 1.2. *A class $\mathfrak{X}$ of Lie algebras over a field Φ is called coalescent if H, K si L and $H, K \in \mathfrak{X}$ imply $<H, K>$ si L, $\in \mathfrak{X}$.*

We need the following classes of Lie algebras over a field Φ:

$\mathfrak{F}$ is the class of finite-dimensional Lie algebras.

$\mathfrak{G}$ is the class of finitely generated Lie algebras, that is, the class of Lie algebras L such that $L=<K>$ where K is a finite set.

$\mathfrak{A}$ is the class of abelian Lie algebras.

$\mathfrak{N}$ is the class of nilpotent Lie algebras.

$\mathfrak{S}$ is the class of solvable Lie algebras.

We furthermore introduce the following concepts.

DEFINITION 1.3. *Let $\mathfrak{X}$ be a class of Lie algebras.*

(1) *We denote by $\mathrm{L}\mathfrak{X}$ the class of locally $\mathfrak{X}$ Lie algebras, that is, the class of Lie algebras L such that every finite subset of L lies in a subalgebra of L belonging to $\mathfrak{X}$.*

(2) *We denote by $\mathfrak{X}_{(\omega)}$ the class of Lie algebras L such that $L/L^{(\omega)} \in \mathfrak{X}$ and by $\mathfrak{X}_\omega$ the class of Lie algebras L such that $L/L^\omega \in \mathfrak{X}$, where $L^{(\omega)}=\bigcap_{n=0}^{\infty} L^{(n)}$ and $L^\omega=\bigcap_{n=1}^{\infty} L^n$ as usual.*

We here state the following three fundamental lemmas, which are known and may be used without reference.

LEMMA 1.4. (1) *If H si L and $K\leq L$, then $H\cap K$ si K.*

(2) *If H si K and K si L, then H si L.*

(3) *If H si L and $K\lhd L$, then $H+K$ si L.*

(4) *Let f be a homomorphism of L onto a Lie algebra $\bar{L}$. If H si L, then $f(H)$ si $\bar{L}$. If $\bar{H}$ si $\bar{L}$, then $f^{-1}(\bar{H})$ si L.*

The proofs of these are all immediate ([3, Lemma 7] and [7, Theorem 1]).

LEMMA 1.5. $\mathfrak{N}\cap\mathfrak{G}\subseteq\mathfrak{F}$.

Proof. Let $L \in \mathfrak{N} \cap \mathfrak{G}$. Then $L = \langle x_1, \cdots, x_n \rangle$. L is spanned by all products $[\cdots[x_{i_1}, x_{i_2}], \cdots, x_{i_k}]$, where only finitely many of them are non-zero since $L \in \mathfrak{N}$. Hence $L \in \mathfrak{F}$.

Lemma 1.6. *If H si L, then H^{ω} and $H^{(\omega)}$ are characteristic ideals of L.*

Proof. Let $\mathfrak{D}(L)$ be the derivation algebra of L and let M be the semi-direct sum $L + \mathfrak{D}(L)$. Assume H n-*si* L. Then H $(n+1)$-*si* M, since L is an ideal of M. Hence for $k \geq 1$ we have

$$[M, H^{k+n}] \subseteq M(\mathrm{ad}_M H)^{k+n}$$
$$\subseteq H(\mathrm{ad}_M H)^{k-1} = H^k.$$

It follows that $[M, H^{\omega}] \subseteq H^{\omega}$, that is, $H^{\omega} \lhd M$. Therefore H^{ω} is a characteristic ideal of L. On the other hand, we can see by induction on k that

$$[M, H^{(k+n)}] \subseteq H^{(k)}, \qquad k \geq 0.$$

It follows that $[M, H^{(\omega)}] \subseteq H^{(\omega)}$, that is, $H^{(\omega)} \lhd M$. Thus $H^{(\omega)}$ is a characteristic ideal of L.

§ 2. Coalescency of $\mathfrak{N} \cap \mathfrak{F}$

Theorem 2.1. *For a field Φ of characteristic 0, $\mathfrak{N} \cap \mathfrak{F}$ is coalescent.*

This has been shown by B. Hartley in [3, Theorem 2]. But it will play a fundamental role for the development of our study in this paper. So in this section we shall give his proof with a slight modification. Φ is of characteristic 0 throughout this section.

If D is a nil derivation of L, then $\exp D = \sum_{n=0}^{\infty} D^n/n!$ is an automorphism of L.

Lemma 2.2. *Let D be a nil derivation of L and M be a subspace of L. Then*

$$MD \subseteq \sum_{n=1}^{\infty} M^{\exp(nD)}.$$

Proof. Let $x \in M$ and $xD^k = 0$. Then there exist $a_1, \cdots, a_k \in \Phi$ such that

$$\sum_{n=1}^{k} a_n n^i / i! = \delta_{1,i} \qquad (i = 0, \cdots, k-1).$$

Hence $\sum a_n x^{\exp(nD)} = xD$, from which the result follows.

Let M be a finite-dimensional subspace of L. For a subset S of the derivation algebra $\mathfrak{D}(L)$, the subspace M^S is defined by $M^S = \sum M D_1 \cdots D_k$ summed over all choices of $D_1, \cdots, D_k \in S$ for any $k \geq 0$. For a subset A of the automorphism group of L, $<A>$ is the subgroup generated by A and the subspace $M^{<A>}$ is defined by $M^{<A>}=\sum_{\alpha\in<A>} M^{\alpha}$. Then we have

LEMMA 2.3. *If M is a finite-dimensional subspace of L and S is a finite-dimensional nil subspace of $\mathfrak{D}(L)$, then*

$$M^S=M^{<\exp S>}=\sum_{i=1}^{n} M^{\alpha_i},\ \alpha_i\in<\exp S>.$$

PROOF. By Lemma 2.2, for any $D\in S$ we have

$$(M^{<\exp S>})D\subseteq \sum_{n=1}^{\infty}(M^{<\exp S>})^{\exp(nD)}\subseteq M^{<\exp S>}.$$

It follows that

$$M^S\subseteq(M^{<\exp S>})^S\subseteq M^{<\exp S>}\subseteq M^S$$

and therefore $M^S=M^{<\exp S>}$. Let $\{D_1, \cdots, D_n\}$ be a basis of S. Since S is nil and M is finite-dimensional, there exists $m>0$ such that $MD_{i_1}\cdots D_{i_m}=(0)$ for any $i_1, \cdots, i_m\in\{1, 2, \cdots, n\}$. Therefore M^S is finite-dimensional.

LEMMA 2.4. *If K si L, $\in\mathfrak{N}$, then* ad K *is nil.*

PROOF. Let K n-*si* L and let m be the class of nilpotency of K. Then

$$L(\operatorname{ad} K)^{n+m}\subseteq K(\operatorname{ad} K)^m=K^{m+1}=(0).$$

LEMMA 2.5. *If H, K si L and $H, K\in\mathfrak{N}\cap\mathfrak{F}$ and if $[H, K]\subseteq H$, then $H+K$ si L, $\in\mathfrak{N}\cap\mathfrak{F}$.*

PROOF. Let H n-*si* L. Then

$$H=H_0\lhd H_1\lhd\cdots\lhd H_n=L.$$

Let $A=<\exp(\operatorname{ad} K)>$ and put $\bar{H}_i=\bigcap_{\alpha\in A} H_i^{\alpha}$. Then $\bar{H}_0=H$, $\bar{H}_n=L$ and $\bar{H}_i\lhd\bar{H}_{i+1}$. By Lemma 2.2, $[\bar{H}_i, K]\subseteq\bar{H}_i$ and therefore $\bar{H}_i\lhd\bar{H}_{i+1}+K$. Since K *si* $\bar{H}_{i+1}+K$, $\bar{H}_i+K$ *si* $\bar{H}_{i+1}+K$. It follows that $H+K$ *si* L. If K k-*si* L and $K^{m+1}=(0)$, by Lemma 2.4 we have

$$(H+K)^{k+m+1}\subseteq H^2+(H+K)(\operatorname{ad} K)^{k+m}=H^2.$$

Thus $H\lhd H+K$, $H\in\mathfrak{N}$ and $H+K/H^2\in\mathfrak{N}$. Hence $H+K\in\mathfrak{N}$ (see [2]) and therefore $H+K\in\mathfrak{N}\cap\mathfrak{F}$, completing the proof.

We can now prove the theorem. Assume that H n-*si* L, K *si* L and $H, K\in\mathfrak{N}\cap\mathfrak{F}$. We must show that $J=<H, K>$ *si* L and $\in\mathfrak{N}\cap\mathfrak{F}$. We show it by induction on n. If $n=1$, then $H\lhd L$ and therefore $H+K$ *si* L, $\in\mathfrak{N}\cap\mathfrak{F}$ by Lemma 2.5. So we assume that $n>1$ and put $m=n-1$. Then

$$H=H_0\lhd H_1\lhd\cdots\lhd H_m\lhd H_n=L.$$

Since $\mathrm{ad}\,K$ is a finite-dimensional nil subspace of $\mathfrak{D}(L)$ by Lemma 2.4, it follows from Lemma 2.3 that

$$<H^{\mathrm{ad}K}> = <H^{\alpha_1}, \cdots, H^{\alpha_k}>, \qquad \alpha_i \in <\exp(\mathrm{ad}\,K)>.$$

Now $H_m^{\alpha_1}=H_m$, whence H^{α_i} *m-si* H_m and $H^{\alpha_i} \in \mathfrak{N}\cap\mathfrak{F}$ for each i. By induction hypothesis

$$<H^{\alpha_1}, \cdots, H^{\alpha_k}> si\ H_m, \ \in \mathfrak{N}\cap\mathfrak{F}.$$

Therefore $<H^{\mathrm{ad}K}> si\ L$, $\in \mathfrak{N}\cap\mathfrak{F}$. Obviously $J=K+<H^{\mathrm{ad}K}>$ and $[<H^{\mathrm{ad}K}>, K]\subseteq <H^{\mathrm{ad}K}>$. Therefore by Lemma 2.5 we see that $J\ si\ L$, $\in \mathfrak{N}\cap\mathfrak{F}$. Thus $\mathfrak{N}\cap\mathfrak{F}$ is coalescent. This completes the proof of Theorem 2.1.

§ 3. Thirteen classes of Lie algebras

First we give the following definition for our convenience.

Definition 3.1. *A class $\mathfrak{X}$ of Lie algebras is said to have the property* (P) *if $L \in \mathfrak{X}$ and $N \lhd L$ imply $L/N \in \mathfrak{X}$.*

Lemma 3.2. *If $\mathfrak{X}$ has the property* (P), *then $\mathfrak{X}_{(\omega)}$ and $\mathfrak{X}_\omega$ have the property* (P) *and $\mathfrak{X}\subseteq\mathfrak{X}_{(\omega)}\subseteq\mathfrak{X}_\omega$.*

Proof. Suppose that $L \in \mathfrak{X}_{(\omega)}$ and $N\lhd L$. By Lemma 1.6, $(L/N)^{(\omega)}\lhd L/N$ and therefore $(L/N)^{(\omega)}=M/N$ with $M\lhd L$. Hence $L^{(\omega)}\subseteq M$ and therefore $L/M \simeq(L/L^{(\omega)})/(M/L^{(\omega)})$. Since $L/L^{(\omega)} \in \mathfrak{X}$ and $\mathfrak{X}$ has the property (P), we see that $L/M \in \mathfrak{X}$. From the fact that

$$(L/N)/(L/N)^{(\omega)}=(L/N)/(M/N)\simeq L/M,$$

it follows that $L/N \in \mathfrak{X}_{(\omega)}$, that is, $\mathfrak{X}_{(\omega)}$ has the property (P).

The proof of the statement that $\mathfrak{X}_\omega$ has the property (P) is similar.

Now assume that $L \in \mathfrak{X}$. Since $L^{(\omega)}\lhd L$ by Lemma 1.6, $L/L^{(\omega)} \in \mathfrak{X}$ and therefore $L \in \mathfrak{X}_{(\omega)}$. Thus $\mathfrak{X}\subseteq\mathfrak{X}_{(\omega)}$. Next assume that $L \in \mathfrak{X}_{(\omega)}$. Since $L^{(\omega)}\lhd L$, $L/L^{(\omega)} \in \mathfrak{X}$. L/L^ω is the quotient algebra of $L/L^{(\omega)}$ by $L^\omega/L^{(\omega)}$. Hence $L/L^\omega \in \mathfrak{X}$, that is, $L \in \mathfrak{X}_\omega$. Thus $\mathfrak{X}_{(\omega)}\subseteq\mathfrak{X}_\omega$, completing the proof.

Lemma 3.3. $\mathfrak{S}\cap\mathfrak{X}_{(\omega)}=\mathfrak{S}\cap\mathfrak{X}$ *and*

$$\mathfrak{N}\cap\mathfrak{X}_\omega=\mathfrak{N}\cap\mathfrak{X}_{(\omega)}=\mathfrak{N}\cap\mathfrak{X}.$$

Proof. If $L \in \mathfrak{S}$, $L^{(\omega)}=(0)$. Therefore $L \in \mathfrak{X}_{(\omega)}$ if and only if $L \in \mathfrak{X}$. Hence $\mathfrak{S}\cap\mathfrak{X}_{(\omega)}=\mathfrak{S}\cap\mathfrak{X}$ and $\mathfrak{N}\cap\mathfrak{X}_{(\omega)}=\mathfrak{N}\cap\mathfrak{X}$. $\mathfrak{N}\cap\mathfrak{X}_\omega=\mathfrak{N}\cap\mathfrak{X}$ is similarly proved.

Lemma 3.4. $(\mathfrak{S}\cap\mathfrak{X})_{(\omega)}=\mathfrak{X}_{(\omega)}$ *for* $\mathfrak{X}\subseteq\mathfrak{S}_{(\omega)}$ *and*

$$(\mathfrak{N}\cap\mathfrak{X})_\omega=(\mathfrak{S}\cap\mathfrak{X})_\omega=\mathfrak{X}_\omega \quad \textit{for } \mathfrak{X}\subseteq\mathfrak{N}_\omega.$$

PROOF. If $L \in \mathfrak{X}_{(\omega)}$, $L/L^{(\omega)} \in \mathfrak{X}$. Since $\mathfrak{X} \subseteq \mathfrak{S}_{(\omega)}$, $(L/L^{(\omega)})^{(n)} = (L/L^{(\omega)})^{(\omega)} = (0)$ and therefore $L/L^{(\omega)} \in \mathfrak{S} \cap \mathfrak{X}$, that is, $L \in (\mathfrak{S} \cap \mathfrak{X})_{(\omega)}$. Hence $\mathfrak{X}_{(\omega)} = (\mathfrak{S} \cap \mathfrak{X})_{(\omega)}$.

The other formula is similarly proved.

LEMMA 3.5. *If* $\mathfrak{X} \subseteq \mathfrak{X}_{(\omega)} \subseteq \mathfrak{X}_{\omega}$, *then*

$$\mathfrak{X}_{\omega\omega} = \mathfrak{X}_{(\omega)\omega} = \mathfrak{X}_{\omega(\omega)} = \mathfrak{X}_{\omega} \text{ and } \mathfrak{X}_{(\omega)(\omega)} = \mathfrak{X}_{(\omega)}.$$

PROOF. It is evident that

$$\mathfrak{X}_{\omega} \subseteq \mathfrak{X}_{(\omega)\omega} \subseteq \mathfrak{X}_{\omega\omega} \text{ and } \mathfrak{X}_{(\omega)} \subseteq \mathfrak{X}_{(\omega)(\omega)}.$$

If $L \in \mathfrak{X}_{\omega\omega}$, then $L/L^{\omega} \in \mathfrak{X}_{\omega}$. Since $(L/L^{\omega})^{\omega} = (0)$, $L/L^{\omega} \in \mathfrak{X}$ and therefore $L \in \mathfrak{X}_{\omega}$. Thus $\mathfrak{X}_{\omega\omega} \subseteq \mathfrak{X}_{\omega}$. Similarly $\mathfrak{X}_{(\omega)(\omega)} \subseteq \mathfrak{X}_{(\omega)}$. Hence $\mathfrak{X}_{\omega} = \mathfrak{X}_{(\omega)\omega} = \mathfrak{X}_{\omega\omega}$ and $\mathfrak{X}_{(\omega)} = \mathfrak{X}_{(\omega)(\omega)}$. Finally, since $(L/L^{(\omega)})/(L/L^{(\omega)})^{\omega} \simeq L/L^{\omega}$, $L/L^{(\omega)} \in \mathfrak{X}_{\omega}$ if and only if $L/L^{\omega} \in \mathfrak{X}$. Hence $\mathfrak{X}_{\omega(\omega)} = \mathfrak{X}_{\omega}$, completing the proof.

LEMMA 3.6. (1) $\mathfrak{F} \subseteq \mathfrak{S}_{(\omega)}$.

(2) $\mathfrak{F}, \mathfrak{F}_{\omega} \subseteq \mathfrak{N}_{\omega}$.

(3) $\mathfrak{F}$, $\mathfrak{N}$, $\mathfrak{N} \cap \mathfrak{F}$, $\mathfrak{F}_{\omega}$, $\mathfrak{S} \cap \mathfrak{F}_{\omega}$, $\mathfrak{N}_{\omega} \cap \mathfrak{G}$ *and* $\mathfrak{S} \cap \mathfrak{N}_{\omega} \cap \mathfrak{G}$ *have the property* (P).

PROOF. (1) and (2) are evident. $\mathfrak{F}$, $\mathfrak{N}$, $\mathfrak{N} \cap \mathfrak{F}$, $\mathfrak{S}$ and $\mathfrak{G}$ obviously have the property (P). By Lemma 3.2, $\mathfrak{F}_{\omega}$ and $\mathfrak{N}_{\omega}$ have the property (P) and therefore so do $\mathfrak{S} \cap \mathfrak{F}_{\omega}$, $\mathfrak{N}_{\omega} \cap \mathfrak{G}$ and $\mathfrak{S} \cap \mathfrak{N}_{\omega} \cap \mathfrak{G}$. Hence (3) is proved.

By making use of these lemmas we shall now prove the following

THEOREM 3.7. *We consider the operations of getting, from a class* $\mathfrak{X}$ *of Lie algebras, the classes*

$$\mathfrak{X},\ \mathfrak{S} \cap \mathfrak{X},\ \mathfrak{N} \cap \mathfrak{X},\ \mathfrak{F} \cap \mathfrak{X},\ \mathfrak{X}_{(\omega)},\ \mathfrak{X}_{\omega}.$$

(1) *By applying the above operations to* $\mathfrak{F}$, *we have*

$$\mathfrak{F},\ \mathfrak{S} \cap \mathfrak{F},\ \mathfrak{N} \cap \mathfrak{F},\ \mathfrak{F}_{(\omega)},\ \mathfrak{F}_{\omega},$$

$$(\mathfrak{N} \cap \mathfrak{F})_{(\omega)},\ \mathfrak{S} \cap \mathfrak{F}_{\omega},\ (\mathfrak{S} \cap \mathfrak{F}_{\omega})_{(\omega)},\ \mathfrak{N}_{(\omega)} \cap \mathfrak{F}.$$

(2) *By applying the above operations to* $\mathfrak{N}_{\omega} \cap \mathfrak{G}$, *we have the following classes besides the classes in* (1).

$$\mathfrak{N}_{\omega} \cap \mathfrak{G},\ \mathfrak{S} \cap \mathfrak{N}_{\omega} \cap \mathfrak{G},\ (\mathfrak{N}_{\omega} \cap \mathfrak{G})_{(\omega)},\ (\mathfrak{S} \cap \mathfrak{N}_{\omega} \cap \mathfrak{G})_{(\omega)}.$$

PROOF. (1) First we have

$$\mathfrak{F},\ \mathfrak{S} \cap \mathfrak{F},\ \mathfrak{N} \cap \mathfrak{F},\ \mathfrak{F}_{(\omega)},\ \mathfrak{F}_{\omega}.$$

Applying the operations to $\mathfrak{S} \cap \mathfrak{F}$ and $\mathfrak{N} \cap \mathfrak{F}$, we obtain $(\mathfrak{N} \cap \mathfrak{F})_{(\omega)}$. For, by Lemmas 3.4 and 3.6

$$(\mathfrak{S} \cap \mathfrak{F})_{(\omega)} = \mathfrak{F}_{(\omega)},$$

$$(\mathfrak{S}\cap\mathfrak{F})_\omega=(\mathfrak{N}\cap\mathfrak{F})_\omega=\mathfrak{F}_\omega.$$

Applying the operations to $\mathfrak{F}_{(\omega)}$ and $\mathfrak{F}_\omega$, we obtain $\mathfrak{S}\cap\mathfrak{F}_\omega$. For, by Lemma 3.3

$$\mathfrak{S}\cap\mathfrak{F}_{(\omega)}=\mathfrak{S}\cap\mathfrak{F},$$

$$\mathfrak{N}\cap\mathfrak{F}_{(\omega)}=\mathfrak{N}\cap\mathfrak{F}_\omega=\mathfrak{N}\cap\mathfrak{F};$$

by Lemmas 3.2 and 3.6

$$\mathfrak{F}\cap\mathfrak{F}_{(\omega)}=\mathfrak{F}\cap\mathfrak{F}_\omega=\mathfrak{F};$$

by Lemmas 3.2, 3.5 and 3.6

$$\mathfrak{F}_{(\omega)(\omega)}=\mathfrak{F}_{(\omega)},$$

$$\mathfrak{F}_{(\omega)\omega}=\mathfrak{F}_{\omega(\omega)}=\mathfrak{F}_{\omega\omega}=\mathfrak{F}_\omega.$$

Applying the operations to $(\mathfrak{N}\cap\mathfrak{F})_{(\omega)}$, we obtain $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$. For, by Lemma 3.3

$$\mathfrak{S}\cap(\mathfrak{N}\cap\mathfrak{F})_{(\omega)}=\mathfrak{S}\cap(\mathfrak{N}\cap\mathfrak{F})=\mathfrak{N}\cap\mathfrak{F},$$

$$\mathfrak{N}\cap(\mathfrak{N}\cap\mathfrak{F})_{(\omega)}=\mathfrak{N}\cap(\mathfrak{N}\cap\mathfrak{F})=\mathfrak{N}\cap\mathfrak{F};$$

by Lemmas 3.2 and 3.6

$$\mathfrak{F}\cap(\mathfrak{N}\cap\mathfrak{F})_{(\omega)}=\mathfrak{F}\cap\mathfrak{N}_{(\omega)}\cap\mathfrak{F}_{(\omega)}=\mathfrak{N}_{(\omega)}\cap\mathfrak{F};$$

by Lemmas 3.2, 3.5 and 3.6

$$(\mathfrak{N}\cap\mathfrak{F})_{(\omega)(\omega)}=(\mathfrak{N}\cap\mathfrak{F})_{(\omega)},$$

$$(\mathfrak{N}\cap\mathfrak{F})_{(\omega)\omega}=(\mathfrak{N}\cap\mathfrak{F})_\omega=\mathfrak{F}_\omega.$$

Applying the operations to $\mathfrak{S}\cap\mathfrak{F}_\omega$, we obtain $(\mathfrak{S}\cap\mathfrak{F}_\omega)_{(\omega)}$. For, by Lemma 3.3

$$\mathfrak{N}\cap(\mathfrak{S}\cap\mathfrak{F}_\omega)=\mathfrak{N}\cap\mathfrak{F};$$

by Lemmas 3.2 and 3.6

$$(\mathfrak{S}\cap\mathfrak{F}_\omega)\cap\mathfrak{F}=\mathfrak{S}\cap\mathfrak{F};$$

by Lemmas 3.4, 3.5 and 3.6

$$(\mathfrak{S}\cap\mathfrak{F}_\omega)_\omega=\mathfrak{F}_{\omega\omega}=\mathfrak{F}_\omega.$$

Applying the operations to $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$, we obtain no new classes. For, by Lemma 3.3

$$\mathfrak{S}\cap\mathfrak{N}_{(\omega)}\cap\mathfrak{F}=(\mathfrak{S}\cap\mathfrak{N})\cap\mathfrak{F}=\mathfrak{N}\cap\mathfrak{F},$$

$$\mathfrak{N}\cap\mathfrak{N}_{(\omega)}\cap\mathfrak{F}=\mathfrak{N}\cap\mathfrak{F};$$

by Lemmas 3.2, 3.5 and 3.6

$$(\mathfrak{N}_{(\omega)}\cap\mathfrak{F})_{(\omega)}=\mathfrak{N}_{(\omega)(\omega)}\cap\mathfrak{F}_{(\omega)}=\mathfrak{N}_{(\omega)}\cap\mathfrak{F}_{(\omega)}=(\mathfrak{N}\cap\mathfrak{F})_{(\omega)};$$

by Lemmas 3.2, 3.4, 3.5 and 3.6

$$(\mathfrak{N}_{(\omega)}\cap\mathfrak{F})_{\omega}=\mathfrak{N}_{(\omega)\omega}\cap\mathfrak{F}_{\omega}=\mathfrak{N}_{\omega}\cap\mathfrak{F}_{\omega}=(\mathfrak{N}\cap\mathfrak{F})_{\omega}=\mathfrak{F}_{\omega}.$$

Applying the operations to $(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)}$, we obtain no new classes. For, by Lemma 3.3

$$\mathfrak{S}\cap(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)}=\mathfrak{S}\cap(\mathfrak{S}\cap\mathfrak{F}_{\omega})=\mathfrak{S}\cap\mathfrak{F}_{\omega},$$

$$\mathfrak{N}\cap(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)}=\mathfrak{N}\cap(\mathfrak{S}\cap\mathfrak{F}_{\omega})=\mathfrak{N}\cap\mathfrak{F}_{\omega}=\mathfrak{N}\cap\mathfrak{F};$$

by Lemmas 3.2, 3.5 and 3.6

$$\mathfrak{F}\cap(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)}=\mathfrak{F}\cap\mathfrak{S}_{(\omega)}\cap\mathfrak{F}_{\omega(\omega)}=\mathfrak{F}\cap\mathfrak{S}_{(\omega)}\cap\mathfrak{F}_{\omega}=\mathfrak{F};$$

by Lemmas 3.2, 3.4, 3.5 and 3.6

$$(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)(\omega)}=(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)},$$

$$(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)\omega}=(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{\omega}=\mathfrak{F}_{\omega\omega}=\mathfrak{F}_{\omega}.$$

(2) By the first application of the operations we have

$$\mathfrak{N}_{\omega}\cap\mathfrak{G},\ \mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G},\ (\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}.$$

For, by Lemmas 1.5, 3.2 and 3.6

$$\mathfrak{N}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}=\mathfrak{N}\cap\mathfrak{G}=\mathfrak{N}\cap\mathfrak{F};$$

by Lemma 3.6

$$\mathfrak{F}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}=\mathfrak{F}\cap\mathfrak{N}_{\omega}=\mathfrak{F};$$

by Lemmas 1.5, 3.2, 3.4, 3.5 and 3.6

$$(\mathfrak{N}_{\omega}\cap\mathfrak{G})_{\omega}=\mathfrak{N}_{\omega\omega}\cap\mathfrak{G}_{\omega}=\mathfrak{N}_{\omega}\cap\mathfrak{G}_{\omega}=(\mathfrak{N}\cap\mathfrak{G})_{\omega}=(\mathfrak{N}\cap\mathfrak{F})_{\omega}=\mathfrak{F}_{\omega}.$$

Applying the operations to $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$, we obtain $(\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$. For by Lemma 1.5

$$\mathfrak{N}\cap(\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})=\mathfrak{N}\cap\mathfrak{G}=\mathfrak{N}\cap\mathfrak{F};$$

by Lemma 3.6

$$\mathfrak{F}\cap(\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})=\mathfrak{S}\cap\mathfrak{F};$$

by Lemma 3.4

$$(\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{\omega}=(\mathfrak{N}_{\omega}\cap\mathfrak{G})_{\omega}=\mathfrak{F}_{\omega}.$$

Applying the operations to $(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}$, we obtain no new classes. For by Lemmas 1.5 and 3.3

$$\mathfrak{S}\cap(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}=\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G},$$

$$\mathfrak{N}\cap(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}=\mathfrak{N}\cap(\mathfrak{N}_\omega\cap\mathfrak{G})=\mathfrak{N}\cap\mathfrak{F};$$

by Lemmas 3.2, 3.5 and 3.6

$$\mathfrak{F}\cap(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}=\mathfrak{F}\cap\mathfrak{N}_{\omega(\omega)}\cap\mathfrak{G}_{(\omega)}=\mathfrak{F}\cap\mathfrak{N}_\omega\cap\mathfrak{G}_{(\omega)}=\mathfrak{F};$$

by Lemmas 3.2, 3.5 and 3.6

$$(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)(\omega)}=(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)},$$

$$(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)\omega}=(\mathfrak{N}_\omega\cap\mathfrak{G})_\omega=\mathfrak{F}_\omega.$$

Applying the operations to $(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}$, we obtain no new classes. For by Lemma 3.3

$$\mathfrak{S}\cap(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}=\mathfrak{S}\cap(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})=\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G},$$

$$\mathfrak{N}\cap(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}=\mathfrak{N}\cap(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})=\mathfrak{N}\cap\mathfrak{G}=\mathfrak{N}\cap\mathfrak{F};$$

by Lemma 3.6

$$\mathfrak{F}\cap(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}=\mathfrak{F}\cap\mathfrak{S}_{(\omega)}\cap(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}=\mathfrak{F};$$

by Lemmas 3.2, 3.5 and 3.6

$$(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)(\omega)}=(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)},$$

$$(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)\omega}=(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_\omega=\mathfrak{F}_\omega.$$

Thus the theorem is completely proved.

We shall here ask whether or not the classes in the theorem are different from each other.

EXAMPLE A. Let L be the 2-dimensional non-abelian Lie algebra, that is, $L=(x, y)$ with $[x, y]=y$. Then L does not belong to

$$\mathfrak{N}\cap\mathfrak{F},\ (\mathfrak{N}\cap\mathfrak{F})_{(\omega)},\ \mathfrak{N}_{(\omega)}\cap\mathfrak{F},$$

but L belongs to

$$\mathfrak{F},\ \mathfrak{S}\cap\mathfrak{F},\ \mathfrak{F}_{(\omega)},\ \mathfrak{F}_\omega,\ \mathfrak{S}\cap\mathfrak{F}_\omega,\ (\mathfrak{S}\cap\mathfrak{F}_\omega)_{(\omega)},$$

$$\mathfrak{N}_\omega\cap\mathfrak{G},\ \mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G},\ (\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)},\ (\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}.$$

EXAMPLE B. Let L be a finite-dimensional semisimple Lie algebra over a field Φ of characteristic 0. Then L does not belong to

$$\mathfrak{S}\cap\mathfrak{F},\ \mathfrak{N}\cap\mathfrak{F},\ \mathfrak{S}\cap\mathfrak{F}_\omega,\ \mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G},$$

but L belongs to

$$\mathfrak{F},\ \mathfrak{F}_{(\omega)},\ \mathfrak{F}_\omega,\ (\mathfrak{N}\cap\mathfrak{F})_{(\omega)},\ (\mathfrak{S}\cap\mathfrak{F}_\omega)_{(\omega)},\ \mathfrak{N}_{(\omega)}\cap\mathfrak{F},$$

$$\mathfrak{N}_\omega\cap\mathfrak{G},\ (\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)},\ (\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}.$$

EXAMPLE C. Let $A=(e_0, e_1, e_2,\cdots)$ be an infinite-dimensional abelian Lie algebra over a field Φ of characteristic 0. Let x, y and z be the following linear transformations of A:

$$x: e_i \longrightarrow e_{i+1} \qquad (i=0, 1, 2,\cdots)$$

$$y: e_0 \longrightarrow 0,\ e_i \longrightarrow ie_{i-1} \qquad (i=1, 2,\cdots)$$

$$z: e_i \longrightarrow e_i \qquad (i=0, 1, 2,\cdots).$$

Then $[x, y]=z$, $[x, z]=[y, z]=0$. Therefore (x, y, z) is a nilpotent Lie algebra over Φ. Let L be the semi-direct sum (see B. Hartley [3]):

$$L=A+(x, y, z).$$

Then $L^{(\omega)}=L^{(3)}=(0)$ and $L^\omega=L^3=A$. Hence L does not belong to

$$\mathfrak{F},\ \mathfrak{S}\cap\mathfrak{F},\ \mathfrak{F}_{(\omega)},$$

but L belongs to

$$\mathfrak{F}_\omega,\ \mathfrak{S}\cap\mathfrak{F}_\omega,\ (\mathfrak{S}\cap\mathfrak{F}_\omega)_{(\omega)},\ \mathfrak{N}_\omega\cap\mathfrak{G},$$

$$\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G},\ (\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)},\ (\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}.$$

EXAMPLE D. Let L be the following subalgebra of the Lie algebra in Example C:

$$L=A+(z).$$

Then $L^{(\omega)}=L^{(2)}=(0)$ and $L^\omega=L^2=A$. Hence L belongs to

$$\mathfrak{F}_\omega,\ \mathfrak{S}\cap\mathfrak{F}_\omega,\ (\mathfrak{S}\cap\mathfrak{F}_\omega)_{(\omega)},$$

but L does not belong to

$$\mathfrak{N}_\omega\cap\mathfrak{G},\ \mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G},\ (\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)},\ (\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}.$$

EXAMPLE E. Let L_i be the 3-dimensional simple Lie algebra over a field Φ of characteristic 0 $(i=1, 2,\cdots)$. Let L be the direct sum of all L_i. Then L does not belong to

$$\mathfrak{N}_\omega\cap\mathfrak{G},$$

but L belongs to

$$(\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)},\ (\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}.$$

Example F. Let $A=(e_0, e_1, e_2,\cdots)$ be an infinite-dimensional abelian Lie algebra over a field Φ of characteristic 0. Let x, y and z be the following linear transformations of A:

$$x: e_i\longrightarrow e_{i+1} \qquad (i=0, 1, 2,\cdots)$$

$$y: e_0\longrightarrow 0,\ e_i\longrightarrow i(i-1)e_{i-1} \qquad (i=1, 2,\cdots)$$

$$z: e_i\longrightarrow 2ie_i \qquad (i=0, 1, 2,\cdots).$$

Then (x, y, z) is the 3-dimensional simple Lie algebra over Φ such that

$$[x, z]=2x,\ [y, z]=-2y,\ [x, y]=z.$$

Now let L be the semi-direct sum:

$$L=A+(x, y, z).$$

Then L does not belong to

$$\mathfrak{F},\ \mathfrak{N}_{(\omega)}\cap\mathfrak{F},$$

but L belongs to

$$\mathfrak{F}_{(\omega)},\ (\mathfrak{N}\cap\mathfrak{F})_{(\omega)}.$$

Finally we have no examples to show

$$\mathfrak{F}_{\omega}\neq(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)} \text{ and } (\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}\neq(\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}.$$

§4. Coalescency of the classes obtained in the preceding section

In this section we show three general theorems on coalescency of classes of Lie algebras and also show the coalescency of the thirteen classes obtained in Theorem 3.7.

We begin with

Theorem 4.1. *Let $\mathfrak{X}$ be a class of Lie algebras over a field Φ having the property* (P). *If $\mathfrak{X}$ is coalescent, then so are $\mathfrak{X}_{(\omega)}$ and $\mathfrak{X}_{\omega}$.*

Proof. Assume that $\mathfrak{X}$ is coalescent and that H, K *si* L and $H, K\in\mathfrak{X}_{(\omega)}$ for any Lie algebra L. Put $J=\langle H, K\rangle$. By Lemma 1.6, $H^{(\omega)}\lhd L$ and $K^{(\omega)}\lhd L$ and therefore $I=H^{(\omega)}+K^{(\omega)}\lhd L$. Hence $(H+I)/I$, $(K+I)/I$ *si* L/I. We have

$$(H+I)/I\simeq H/(I\cap H)\simeq(H/H^{(\omega)})/((I\cap H)/H^{(\omega)})$$

and $H/H^{(\omega)} \in \mathfrak{X}$. Since $\mathfrak{X}$ has the property (P), it follows that $(H+I)/I \in \mathfrak{X}$. Similarly, $(K+I)/I \in \mathfrak{X}$. Since $\mathfrak{X}$ is coalescent,

$$J/I = \langle H/I, K/I \rangle \ si\ L/I, \ \in \mathfrak{X}.$$

Hence $J\ si\ L$. It is clear that $I \subseteq J^{(\omega)}$. Therefore

$$J/J^{(\omega)} \simeq (J/I)/(J^{(\omega)}/I).$$

Since $J/I \in \mathfrak{X}$ and $\mathfrak{X}$ has the property (P), it follows that $J/J^{(\omega)} \in \mathfrak{X}$, that is, $J \in \mathfrak{X}_{(\omega)}$. Thus $\mathfrak{X}_{(\omega)}$ is coalescent.

The coalescency of $\mathfrak{X}_\omega$ is similarly proved.

THEOREM 4.2. *Let $\mathfrak{X}$ be a class of Lie algebras over a field Φ contained in $\mathfrak{N}_\omega$ and having the property* (P). *If $\mathfrak{X}$ and $\mathfrak{N} \cap \mathfrak{X}$ are coalescent, then so is $\mathfrak{S} \cap \mathfrak{X}$.*

PROOF. Assume that $\mathfrak{X}$ and $\mathfrak{N} \cap \mathfrak{X}$ are coalescent and that $H, K\ si\ L, \in \mathfrak{S} \cap \mathfrak{X}$ for an arbitrary Lie algebra L. Put $J = \langle H, K \rangle$. Then $J\ si\ L, \in \mathfrak{X}$ since $\mathfrak{X}$ is coalescent. By Lemma 1.6 $I = H^\omega + K^\omega \lhd L$. Hence $(H+I)/I$, $(K+I)/I\ si\ L/I$. We have

$$(H+I)/I \simeq H/(I \cap H) \simeq (H/H^\omega)/((I \cap H)/H^\omega)$$

and $H/H^\omega \in \mathfrak{N}$ since $\mathfrak{X} \subseteq \mathfrak{N}_\omega$. It follows that $(H+I)/I \in \mathfrak{N}$. Since $\mathfrak{X}$ has the property (P), it follows that $(H+I)/I \in \mathfrak{X}$. Similarly, $(K+I)/I \in \mathfrak{N} \cap \mathfrak{X}$. We now use the coalescency of $\mathfrak{N} \cap \mathfrak{X}$ to see that $J/I \in \mathfrak{N} \cap \mathfrak{X}$. Combining with the fact that $I \in \mathfrak{S}$, we see that $J \in \mathfrak{S}$. Thus $\mathfrak{S} \cap \mathfrak{X}$ is coalescent, completing the proof.

THEOREM 4.3. *Let $\mathfrak{X}$ be a class of Lie algebras over a field Φ having the property* (P). *If $\mathfrak{X}$ and $\mathfrak{N} \cap \mathfrak{X}$ are coalescent, then so is $\mathfrak{N}_{(\omega)} \cap \mathfrak{X}$.*

PROOF. Assume that $\mathfrak{X}$ and $\mathfrak{N} \cap \mathfrak{X}$ are coalescent and that $H, K\ si\ L, \in \mathfrak{N}_{(\omega)} \cap \mathfrak{X}$ for an arbitrary Lie algebra L. Then $J = \langle H, K \rangle\ si\ L, \in \mathfrak{X}$ since $\mathfrak{X}$ is coalescent. Put $I = H^{(\omega)} + K^{(\omega)}$. Then $I \lhd L$ by Lemma 1.6. Hence $(H+I)/I$, $(K+I)/I\ si\ L/I$. We have

$$(H+I)/I \simeq H/(I \cap H) \simeq (H/H^{(\omega)})/((I \cap H)/H^{(\omega)})$$

and $H/H^{(\omega)} \in \mathfrak{N}$ since $H \in \mathfrak{N}_{(\omega)}$. It follows that $(H+I)/I \in \mathfrak{N}$. Since $H \in \mathfrak{X}$ and $\mathfrak{X}$ has the property (P), it follows that $(H+I)/I \in \mathfrak{X}$. Similarly $(K+I)/I \in \mathfrak{N} \cap \mathfrak{X}$. Since $\mathfrak{N} \cap \mathfrak{X}$ is coalescent, $J/I \in \mathfrak{N} \cap \mathfrak{X}$. But then $J^{(n)} \subseteq I \subseteq J^{(\omega)}$ for some n and therefore $I = J^{(\omega)}$. It follows that $J \in \mathfrak{N}_{(\omega)}$. Thus $\mathfrak{N}_{(\omega)} \cap \mathfrak{X}$ is coalescent, completing the proof.

Now we are in a position to show the main theorem of Part I which contains as part the results of B. Hartley [3, Theorems 2 and 5].

THEOREM 4.4. *If Φ is of characteristic* 0, *then the thirteen classes*

$$\mathfrak{F},\ \mathfrak{S}\cap\mathfrak{F},\ \mathfrak{N}\cap\mathfrak{F},\ \mathfrak{F}_{(\omega)},\ \mathfrak{F}_{\omega},$$

$$(\mathfrak{N}\cap\mathfrak{F})_{(\omega)},\ \mathfrak{S}\cap\mathfrak{F}_{\omega},\ (\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)},\ \mathfrak{N}_{(\omega)}\cap\mathfrak{F},$$

$$\mathfrak{N}_{\omega}\cap\mathfrak{G},\ \mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G},\ (\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)},\ (\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$$

are all coalescent.

If Φ is of arbitrary characteristic, any classes containing $\mathfrak{A}$, e.g. $\mathfrak{S}$, $\mathfrak{N}$, $\mathrm{L}\mathfrak{S}$, $\mathrm{L}\mathfrak{N}$, are not coalescent.

PROOF. (1) $\mathfrak{N}\cap\mathfrak{F}$ is coalescent by Theorem 2.1.

(2) Coalescency of $\mathfrak{F}$ and $\mathfrak{N}_{\omega}\cap\mathfrak{G}$: Assume that $H, K\ si\ L$ and $H, K \in \mathfrak{N}_{\omega}\cap\mathfrak{G}$ (resp. $\mathfrak{F}$) for an arbitrary Lie algebra L. Put $J=\langle H, K\rangle$. We have $H^{\omega}=H^{p}\lhd L$ and $K^{\omega}=K^{q}\lhd L$ by Lemma 1.6 and therefore $I=H^{\omega}+K^{\omega}\lhd L$. Hence we have $(H+I)/I\ si\ L/I$, $(K+I)/I\ si\ L/I$. We also have

$$(H+I)/I\simeq H/(I\cap H)\simeq(H/H^{p})/((I\cap H)/H^{p}) \in \mathfrak{N}\cap\mathfrak{G}=\mathfrak{N}\cap\mathfrak{F}.$$

Similarly $(K+I)/I \in \mathfrak{N}\cap\mathfrak{F}$. Since $\mathfrak{N}\cap\mathfrak{F}$ is coalescent by (1),

$$J/I=\langle(H+I)/I, (K+I)/I\rangle si\, L/I,\ \in \mathfrak{N}\cap\mathfrak{F}.$$

Hence $J\ si\ L$. Since $J/I \in \mathfrak{N}$, $J^{m}\subseteq I\subseteq J$ for some m and therefore $I=J^{\omega}$. Hence $J\in\mathfrak{N}_{\omega}\cap\mathfrak{G}$. (resp. Since J/I and $I\in\mathfrak{F}$, we have $J\in\mathfrak{F}$.) Thus $\mathfrak{N}_{\omega}\cap\mathfrak{G}$ (resp. $\mathfrak{F}$) is coalescent.

(3) $\mathfrak{F}$, $\mathfrak{N}\cap\mathfrak{F}$ and $\mathfrak{N}_{\omega}\cap\mathfrak{G}$ have the property (P) by Lemma 3.6 and are coalescent by the first part and Theorem 2.1. Hence by Theorem 4.1 $\mathfrak{F}_{(\omega)}$, $\mathfrak{F}_{\omega}$, $(\mathfrak{N}\cap\mathfrak{F})_{(\omega)}$ and $(\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$ are coalescent.

By Lemma 3.6 $\mathfrak{F}\subseteq\mathfrak{N}_{\omega}$ and $\mathfrak{F}$ has the property (P). Since $\mathfrak{F}$ and $\mathfrak{N}\cap\mathfrak{F}$ are coalescent, by Theorem 4.2 $\mathfrak{S}\cap\mathfrak{F}$ is coalescent.

By Lemma 3.6 $\mathfrak{F}_{\omega}\subseteq\mathfrak{N}_{\omega}$ and $\mathfrak{F}_{\omega}$ has the property (P). Since $\mathfrak{F}_{\omega}$ and $\mathfrak{N}\cap\mathfrak{F}_{\omega}=\mathfrak{N}\cap\mathfrak{F}$ are coalescent, so is $\mathfrak{S}\cap\mathfrak{F}_{\omega}$ by Theorem 4.2. It follows from Lemma 3.6 and Theorem 4.1 that $(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)}$ is coalescent.

Since $\mathfrak{F}$ has the property (P) and $\mathfrak{F}$, $\mathfrak{N}\cap\mathfrak{F}$ are coalescent, by Theorem 4.3 we see that $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$ is coalescent.

$\mathfrak{N}_{\omega}\cap\mathfrak{G}$ has the property (P) by Lemma 3.6 and $\mathfrak{N}_{\omega}\cap\mathfrak{G}$, $\mathfrak{N}\cap(\mathfrak{N}_{\omega}\cap\mathfrak{G})=\mathfrak{N}\cap\mathfrak{F}$ are coalescent. Hence by Theorem 4.2 $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ is coalescent.

If follows from Lemma 3.6 and Theorem 4.1 that $(\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$ is coalescent.

(4) It has been shown by I. Stewart [10, Theorem 12.1] that there exists a Lie algebra L over an arbitrary field Φ having the following properties: 1) $L=V+J$ is the semi-direct sum with $V\lhd L$ and $\in\mathfrak{A}$; 2) $J=\langle H, K\rangle$ where $H, K\leq L$, $H, K\in\mathfrak{A}$, K is 1-dimensional and H is infinite-dimensional; 3) H 5-si L, K 5-si L; 4) $J=I_{L}(J)$, whence J is not a subideal of L. This

example shows that any class containing $\mathfrak{A}$ is not coalescent.

Thus the theorem is completely proved.

It should be noted that the assumption on the characteristic of Φ is essential in the first part of Theorem 4.4. If we drop it, the theorem does not hold. In fact, the example 7.2 in [3] shows that if Φ is not of characteristic 0, any class containing $\mathfrak{A}\cap\mathfrak{F}$ is not coalescent. Therefore any class of Theorem 4.4 is not coalescent.

PART II. RADICALS

§ 5. Definitions

When we are concerned with a finite-dimensional Lie algebra L, we have two kinds of radicals, the solvable radical and the nilpotent radical. Furthermore, if we restrict the basic field Φ to be of characteristic 0, it is known [7] that the subalgebra generated by all the solvable (resp. nilpotent) subideals of L coincides with the solvable radical (resp. the nilpotent radical) of L.

However, if we take off the restriction of finite-dimensionality, we are in a different situation. As the Lie analogues of the radicals introduced in [1, 4, 6] in the study of infinite groups, several radicals corresponding to the nilpotent radical were introduced to a Lie algebra L over a field Φ which is not necessarily of finite dimension ([3, 10]). The Fitting radical $\nu(L)$ is the sum of all nilpotent ideals of L and the Hirsch-Plotkin radical $\rho(L)$ is the unique maximal locally nilpotent ideal of L. If Φ is of characteristic 0, the Baer radical $\beta(L)$ is the subalgebra generated by all finite-dimensional nilpotent subideals of L. If Φ is of characteristic 0, $\nu(L)\subseteq\beta(L)\subseteq\rho(L)$ and these are different in general ([3, Section 7.1] and [10, Corollary to Theorem 12.1]).

However, no study has been made in [3, 10] about the ideals which correspond to the solvable radical of a finite-dimensional Lie algebra. Thus in this part of the paper, we shall give the general definition of radicals which correspond to the solvable radical of a finite-dimensional Lie algebra and make use of the coalescent classes of Lie algebras found in Part I to introduce the seven kinds of such locally solvable radicals. We shall furthermore introduce one more locally nilpotent radical which together with $\nu(L)$, $\beta(L)$ and $\rho(L)$ reduces to the nilpotent radical in finite-dimensional case.

DEFINITION 5.1. *Let $\mathfrak{X}$ be a class of Lie algebras over a field Φ. We call a subideal (resp. ideal) of a Lie algebra L over Φ an $\mathfrak{X}$ subideal (resp. ideal) of L if it belongs to $\mathfrak{X}$.*

DEFINITION 5.2. *Let L be a Lie algebra over a field Φ and let $\mathfrak{X}$ be a class of Lie algebras over Φ such that*

$$\mathfrak{S}\cap\mathfrak{F}\subseteq\mathfrak{X}\subseteq \mathrm{L}\mathfrak{S}\ (\textit{resp. } \mathfrak{N}\cap\mathfrak{F}\subseteq\mathfrak{X}\subseteq \mathrm{L}\mathfrak{N}).$$

We call the sum of all the $\mathfrak{X}$ ideals of L the $\mathfrak{X}$ radical of L and denote it by $\mathrm{Rad}_{\mathfrak{X}}(L)$. *We call the subalgebra of L generated by all the $\mathfrak{X}$ subideals of L the $\mathfrak{X}$-si radical of L and denote it by* $\mathrm{Rad}_{\mathfrak{X}-si}(L)$.

The existence of these radicals are known by Zorn's lemma. According to this definition, $\mathrm{Rad}_{\mathfrak{N}}(L)=\nu(L)$ and $\mathrm{Rad}_{\mathrm{L}\mathfrak{N}}(L)=\rho(L)$. If Φ is of characteristic 0, $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}-si}(L)=\beta(L)$ and B. Hartley has shown that $\mathrm{Rad}_{\mathfrak{N}-si}(L)$ exists and is equal to $\beta(L)$ (see [10, Theorem 10.4]). $\mathrm{Rad}_{\mathfrak{X}}(L)$ for $L\in\mathfrak{F}$ and $\mathrm{Rad}_{\mathfrak{X}-si}(L)$ for $L\in\mathfrak{F}$ and Φ of characteristic 0 reduce to the solvable or the nilpotent radical of L.

§6. Characteristic ideals

Before we begin the discussion on radicals, in this section we shall derive several general theorems connected with radicals from the results in Part I. We shall show that if Φ is of characteristic 0 and if $\mathfrak{X}$ is any class of Lie algebras stated in Theorem 4.4, then the subalgebra generated by all $\mathfrak{X}$ subideals (resp. ideals) of L is a characteristic ideal of L belonging to $\mathrm{L}\mathfrak{X}$.

Lemma 6.1. *Let $\mathfrak{X}$ be a class of Lie algebras having the property* (P). *Assume that the sum of two $\mathfrak{X}$ ideals of any Lie algebra is an $\mathfrak{X}$ ideal. Then the sum of two $\mathfrak{X}_{(\omega)}$ (resp. $\mathfrak{X}_{\omega}$) ideals of any Lie algebra is an $\mathfrak{X}_{(\omega)}$ (resp. $\mathfrak{X}_{\omega}$) ideal.*

Proof. Let H and K be $\mathfrak{X}_{\omega}$ ideals of a Lie algebra L. Then H^{ω}, K^{ω} and $(H+K)^{\omega}$ are characteristic ideals of L. Hence H/H^{ω} and K/K^{ω} are $\mathfrak{X}$ ideals of L/H^{ω} and L/K^{ω} respectively. $(H+(H+K)^{\omega})/(H+K)^{\omega}$ and $(K+(H+K)^{\omega})/(H+K)^{\omega}$ are $\mathfrak{X}$ ideals of $L/(H+K)^{\omega}$, since they are respectively isomorphic to

$$(H/H^{\omega})/((H\cap(H+K)^{\omega})/H^{\omega}),\ (K/K^{\omega})/((K\cap(H+K)^{\omega})/K^{\omega})$$

and $\mathfrak{X}$ has the property (P). By our assumption, it follows that $(H+K)/(H+K)^{\omega}$ is an $\mathfrak{X}$ ideal of $L/(H+K)^{\omega}$. Hence $H+K$ is an $\mathfrak{X}_{\omega}$ ideal of L.

The statement on $\mathfrak{X}_{(\omega)}$ is similarly proved.

Lemma 6.2. *Let $\mathfrak{X}$ be any one of the classes*

$$\mathfrak{S},\ \mathfrak{F},\ \mathfrak{S}\cap\mathfrak{F},\ \mathfrak{N}\cap\mathfrak{F},\ \mathfrak{F}_{(\omega)},\ \mathfrak{F}_{\omega},\ (\mathfrak{N}\cap\mathfrak{F})_{(\omega)},\ \mathfrak{S}\cap\mathfrak{F}_{\omega},\ (\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)},$$

$$\mathfrak{N}_{(\omega)}\cap\mathfrak{F},\ \mathfrak{N}_{\omega}\cap\mathfrak{G},\ \mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G},\ (\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)},\ (\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}.$$

Then the sum of two $\mathfrak{X}$ ideals of any Lie algebra L is an $\mathfrak{X}$ ideal of L.

Proof. The statement is immediate for $\mathfrak{X}=\mathfrak{S},\ \mathfrak{F},\ \mathfrak{S}\cap\mathfrak{F},\ \mathfrak{N}\cap\mathfrak{F},\ \mathfrak{N},\ \mathfrak{G}$. By Lemmas 3.6 and 6.1, it holds for $\mathfrak{X}=\mathfrak{F}_{(\omega)},\ \mathfrak{F}_{\omega},\ (\mathfrak{N}\cap\mathfrak{F})_{(\omega)},\ \mathfrak{N}_{(\omega)},\ \mathfrak{N}_{\omega}$ and therefore for $\mathfrak{S}\cap\mathfrak{F}_\omega$, $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$, $\mathfrak{N}_\omega\cap\mathfrak{G}$, $\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}$. It follows from Lemmas 3.6 and 6.1 that the statement holds for $\mathfrak{X}=(\mathfrak{S}\cap\mathfrak{F}_\omega)_{(\omega)}$, $(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}$, $(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}$, completing the proof.

THEOREM 6.3. (1) *Let* $\mathfrak{X}$ *be any one of the classes*

$$\mathfrak{S},\ \mathfrak{F},\ \mathfrak{S}\cap\mathfrak{F},\ \mathfrak{N}\cap\mathfrak{F},\ \mathfrak{F}_{(\omega)},\ \mathfrak{F}_\omega,\ (\mathfrak{N}\cap\mathfrak{F})_{(\omega)},\ \mathfrak{S}\cap\mathfrak{F}_\omega,\ (\mathfrak{S}\cap\mathfrak{F}_\omega)_{(\omega)},$$

$$\mathfrak{N}_{(\omega)}\cap\mathfrak{F},\ \mathfrak{N}_\omega\cap\mathfrak{G},\ \mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G},\ (\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)},\ (\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}.$$

Let M be the sum of all the $\mathfrak{X}$ *ideals of any Lie algebra L. Then every finite subset of M lies in an* $\mathfrak{X}$ *ideal of L. Especially, M belongs to* L$\mathfrak{X}$.

(2) *Let* $\mathfrak{X}$ *be any one of the above classes except* $\mathfrak{S}$. *Let M be the subalgebra generated by all the* $\mathfrak{X}$ *subideals of any Lie algebra L. If* Φ *is of characteristic* 0, *every finite subset of M lies in an* $\mathfrak{X}$ *subideal of L. Especially, M belongs to* L$\mathfrak{X}$.

PROOF. (1) Assume that $x_1,\cdots,x_n \in M$. Then each x_i belongs to the sum of a finite number of $\mathfrak{X}$ ideals of L. Hence $\langle x_1,\cdots,x_n\rangle$ is contained in the sum of a finite number of $\mathfrak{X}$ ideals of L, which is an $\mathfrak{X}$ ideal of L by Lemma 6.2.

(2) Assume that $x_1,\cdots,x_n \in M$. Then each x_i belongs to $\langle x_{i1},\cdots,x_{im_i}\rangle$ with $x_{ij}\in N_{ij}$, where all N_{ij} are $\mathfrak{X}$ subideals of L. Hence $\langle x_1,\cdots,x_n\rangle \subseteq \langle N_{11},\cdots,N_{nm_n}\rangle$. Since $\mathfrak{X}$ is coalescent, $\langle N_{11},\cdots,N_{nm_n}\rangle$ is an $\mathfrak{X}$ subideal of L.

Thus the theorem is proved.

To show that the ideals and subalgebras generated respectively by all the $\mathfrak{X}$ ideals and subideals of L in Theorem 6.3 are all characteristic ideals of L, we first employ the method of constructing Lie algebras of formal power series that B. Hartley used in [3].

Let L be a Lie algebra over a field Φ of characteristic 0. Let Φ_0 be the field of formal power series $a=\sum_{\nu=n}^{\infty} a_\nu t^\nu$, $a_\nu\in\Phi$, and L_0 be the set of all formal power series $x=\sum_{\nu=n}^{\infty} x_\nu t^\nu$, $x_\nu\in L$. L_0 is a Lie algebra over a field Φ_0 as follows: For $y=\sum y_\nu t^\nu$, $y_\nu\in L$,

$$x+y=\sum(x_\nu+y_\nu)t^\nu$$

$$[x,\ y]=\sum z_\nu t^\nu,\ z_\nu=\sum_{i+j=\nu}[x_i,\ y_j]$$

$$ax=\sum u_\nu t^\nu,\ u_\nu=\sum_{i+j=\nu} a_i x_j.$$

For any $D\in\mathfrak{D}(L)$, the automorphism $\exp(tD)$ of L_0 is defined by

$$(\sum x_\nu t^\nu)^{\exp(tD)}=\sum w_\nu t^\nu,\ w_\nu=\sum_{i+j=\nu} x_i D^j/j!.$$

For $M\leq L$, we denote by $M^{\div}$ the set of all elements $x\in L_0$ with $x_\nu\in M$ for all ν. Then we have

Lemma 6.4. (1) *If* $M \lhd N \leq L$, *then* $M^{\#} \lhd N^{\#} \leq L^{\#} = L_0$.

(2) *If* $M^n = (0)$, *then* $M^{\#n} = (0)$.

(3) *If* $M^{(n)} = (0)$, *then* $M^{\#(n)} = (0)$.

(4) *If M is finite-dimensional over* Φ, *then* $M^{\#}$ *is finite-dimensional over* Φ_0.

Proof. We can show by induction on n that $M^{\#(n)} \subseteq M^{\#(n)}$. (3) follows from this. (1), (2) and (4) are proved in [3, Section 4.2].

For $K_0 \leq L_0$, we denote by $K_0^{\flat}$ the set of all leading coefficients of elements of K_0, together with 0. Then

Lemma 6.5. (1) *If* $K_0 \lhd M_0 \leq L_0$, *then* $K_0^{\flat} \lhd M_0^{\flat} \leq L_0^{\flat} = L$.

(2) *If* $K_0^n = (0)$, *then* $K_0^{\flat n} = (0)$.

(3) *If* $K_0^{(n)} = (0)$, *then* $K_0^{\flat(n)} = (0)$.

Proof. We can show by induction on n that $K_0^{\flat(n)} \subseteq K_0^{(n)\flat}$. (3) follows from this. (1) and (2) are proved in [3, Section 4.2].

Lemma 6.6. *Let* Φ *be of characteristic* 0. *Then every derivation of a Lie algebra L maps any* $\mathfrak{N} \cap \mathfrak{F}$ *subideal (resp. ideal) of L into an* $\mathfrak{N} \cap \mathfrak{F}$ *subideal (resp. ideal) of L.*

Proof. Let D be a derivation of L and put $\alpha = \exp(tD)$.

(1) Let H be an $\mathfrak{N} \cap \mathfrak{F}$ subideal of L. Then by Lemma 6.4 $H^{\#}$ is an $\mathfrak{N} \cap \mathfrak{F}$ subideal of L_0 and therefore so is $H^{\#\alpha}$. Since $\mathfrak{N} \cap \mathfrak{F}$ is coalescent by Theorem 4.4, it follows that $K_0 = \langle H^{\#}, H^{\#\alpha} \rangle$ is an $\mathfrak{N} \cap \mathfrak{F}$ subideal of L_0. Putting $N = K_0^{\flat}$, we see by Lemma 6.5 that N is an $\mathfrak{N}$ subideal of L. For any $x \in H$, $x \in H^{\#}$ and therefore $x^{\alpha} \in K_0$. Hence $x^{\alpha} - x = txD + \cdots \in K_0$. It follows that $xD \in N$. Therefore $HD \subseteq N$. As a finitely generated subalgebra of a nilpotent algebra N, $\langle HD \rangle$ is an $\mathfrak{N} \cap \mathfrak{F}$ subideal of N and therefore of L.

(2) Let N be an $\mathfrak{N} \cap \mathfrak{F}$ ideal of L. By the first part (1), $\langle ND \rangle$ is an $\mathfrak{N} \cap \mathfrak{F}$ subideal of L. Since $\mathfrak{N} \cap \mathfrak{F}$ is coalescent, $M = N + \langle ND \rangle$ is an $\mathfrak{N} \cap \mathfrak{F}$ subideal of L. For any $x \in L$,

$$[x, ND] \subseteq [x, N]D + [xD, N] \subseteq ND + N \subseteq M.$$

Hence M is an ideal of L. Thus ND is contained in an $\mathfrak{N} \cap \mathfrak{F}$ ideal M of L.

This completes the proof.

Lemma 6.7. *Let* $\mathfrak{X}$ *be a coalescent class of Lie algebras having the property* (P). *Assume that every derivation of any Lie algebra maps any* $\mathfrak{X}$ *subideal (resp. ideal) into an* $\mathfrak{X}$ *subideal (resp. ideal). Then every derivation of any Lie algebra L maps any* $\mathfrak{X}_{(\omega)}$ *subideal (resp. ideal) and any* $\mathfrak{X}_{\omega}$ *subideal (resp. ideal) of L into an* $\mathfrak{X}_{(\omega)}$ *subideal (resp. ideal) and an* $\mathfrak{X}_{\omega}$ *subideal (resp. ideal) of L respectively.*

Proof. Let H be an $\mathfrak{X}_{(\omega)}$ subideal (resp. ideal) of L. Then $H^{(\omega)}$ is a characteristic ideal of L by Lemma 1.6. Hence $H/H^{(\omega)}$ is an $\mathfrak{X}$ subideal (resp. ideal) of $L/H^{(\omega)}$ and a derivation D of L induces a derivation of $L/H^{(\omega)}$. Therefore by assumption $(HD+H^{(\omega)})/H^{(\omega)}$ is contained in an $\mathfrak{X}$ subideal (resp. ideal) $K/H^{(\omega)}$ of $L/H^{(\omega)}$. Put $J=<H, K>$. Then $J/H^{(\omega)}$ is an $\mathfrak{X}$ subideal (resp. ideal) of $L/H^{(\omega)}$ since $\mathfrak{X}$ is coalescent. Therefore J is a subideal (resp. an ideal) of L. Since

$$J/J^{(\omega)} \simeq (J/H^{(\omega)})/(J^{(\omega)}/H^{(\omega)})$$

and $\mathfrak{X}$ has the property (P), we see that $J/J^{(\omega)} \in \mathfrak{X}$, that is, $J \in \mathfrak{X}_{(\omega)}$. Thus HD is contained in an $\mathfrak{X}_{(\omega)}$ subideal (resp. ideal) of L.

The statement for $\mathfrak{X}_\omega$ is similarly proved.

LEMMA 6.8. *Let $\mathfrak{X}$ be a class of Lie algebras such that $H \in \mathfrak{X}$ if and only if H^ω, $H/H^\omega \in \mathfrak{X}$. Assume that $\mathfrak{N}\cap\mathfrak{X}$ is coalescent and every derivation of any Lie algebra maps any $\mathfrak{N}\cap\mathfrak{X}$ subideal (resp. ideal) into an $\mathfrak{N}\cap\mathfrak{X}$ subideal (resp. ideal). Then every derivation of any Lie algebra L maps any $\mathfrak{N}_\omega\cap\mathfrak{X}$ subideal (resp. ideal) of L into an $\mathfrak{N}_\omega\cap\mathfrak{X}$ subideal (resp. ideal) of L.*

The statement holds with $\mathfrak{N}$ replaced by $\mathfrak{N}\cap\mathfrak{F}$ and also with H^ω, $\mathfrak{N}_\omega$ replaced by $H^{(\omega)}$, $\mathfrak{N}_{(\omega)}$.

PROOF. Let H be an $\mathfrak{N}_\omega\cap\mathfrak{X}$ subideal (resp. ideal) of L. Since H^ω is a characteristic ideal of L, H/H^ω is an $\mathfrak{N}\cap\mathfrak{X}$ subideal (resp. ideal) of L/H^ω and every derivation D of L induces a derivation of L/H^ω. Hence $(HD+H^\omega)/H^\omega$ is contained in an $\mathfrak{N}\cap\mathfrak{X}$ subideal (resp. ideal) K/H^ω of L/H^ω. Put $J=<H, K>$. Then J/H^ω is an $\mathfrak{N}\cap\mathfrak{X}$ subideal (resp. ideal) of L/H^ω since $\mathfrak{N}\cap\mathfrak{X}$ is coalescent. Hence J is a subideal (resp. ideal) of L. Since $J^n \subseteq H^\omega$ for some n, it follows that $J^\omega = H^\omega$. Since $J^\omega \in \mathfrak{X}$, we have $J \in \mathfrak{X}$ by our assumption on $\mathfrak{X}$. Thus J is an $\mathfrak{N}_\omega\cap\mathfrak{X}$ subideal (resp. ideal) of L containing HD.

The other parts are similarly proved.

THEOREM 6.9. *Let L be a Lie algebra over a field Φ of characteristic 0. Let $\mathfrak{X}$ be any one of the classes*

$$\mathfrak{F},\ \mathfrak{S}\cap\mathfrak{F},\ \mathfrak{N}\cap\mathfrak{F},\ \mathfrak{F}_{(\omega)},\ \mathfrak{F}_\omega,\ (\mathfrak{N}\cap\mathfrak{F})_{(\omega)},\ \mathfrak{S}\cap\mathfrak{F}_\omega,\ (\mathfrak{S}\cap\mathfrak{F}_\omega)_{(\omega)},$$

$$\mathfrak{N}_{(\omega)}\cap\mathfrak{F},\ \mathfrak{N}_\omega\cap\mathfrak{G},\ \mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G},\ (\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)},\ (\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}.$$

Then every derivation of L maps any $\mathfrak{X}$ subideal (resp. ideal) of L into an $\mathfrak{X}$ subideal (resp. ideal) of L. Furthermore, it maps any $\mathfrak{S}$ (resp. $\mathfrak{N}$) ideal of L into an $\mathfrak{S}$ (resp. $\mathfrak{N}$) ideal of L.

PROOF. (1) The case $\mathfrak{X}=\mathfrak{N}\cap\mathfrak{F}$ is proved in Lemma 6.6. It is evident that $\mathfrak{F}$, $\mathfrak{S}$ and $\mathfrak{S}\cap\mathfrak{F}$ satisfy the first assumption of Lemma 6.8. Since $\mathfrak{F}\subseteq\mathfrak{N}_\omega$, $\mathfrak{S}\cap\mathfrak{F}\subseteq\mathfrak{N}_\omega$ and $(\mathfrak{N}\cap\mathfrak{F})_\omega=\mathfrak{F}_\omega$ by Lemmas 3.4 and 3.6, the cases $\mathfrak{X}=\mathfrak{F}$, $\mathfrak{S}\cap\mathfrak{F}$, $\mathfrak{S}\cap\mathfrak{F}_\omega$ and $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$ of the theorem are immediate from Lemma 6.8.

(2) $\mathfrak{X}=\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$: Let H be an $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ subideal (resp. ideal) of L. Then H/H^{ω} is an $\mathfrak{N}\cap\mathfrak{F}$ subideal (resp. ideal) of L/H^{ω} and every derivation D of L induces a derivation of L/H^{ω}. Hence by (1) $(HD+H^{\omega})/H^{\omega}$ is contained in an $\mathfrak{N}\cap\mathfrak{F}$ subideal (resp. ideal) K/H^{ω} of L/H^{ω}. Put $J=\langle H, K\rangle$. Then J/H^{ω} is an $\mathfrak{N}\cap\mathfrak{F}$ subideal (resp. ideal) of L/H^{ω} since $\mathfrak{N}\cap\mathfrak{F}$ is coalescent. Hence J is a subideal (resp. ideal) of L. It follows that $J^{\omega}=H^{\omega}$. Hence $J\in\mathfrak{N}_{\omega}$. Since $H\in\mathfrak{S}$, $H^{\omega}\in\mathfrak{S}$ and therefore $J\in\mathfrak{S}$. Since $H\in\mathfrak{G}$ and $K/H^{\omega}\in\mathfrak{F}$, $H=\langle x_1,\cdots,x_m\rangle$ and $K/H^{\omega}=(y_1+H^{\omega},\cdots, y_n+H^{\omega})$ with $y_j\in K$. Then it follows that $J=\langle x_1,\cdots,x_m, y_1,\cdots,y_n\rangle$. Thus J is an $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ subideal (resp. ideal) of L containing HD.

The case $\mathfrak{X}=\mathfrak{N}_{\omega}\cap\mathfrak{G}$ is similarly proved.

(3) The statement for the cases $\mathfrak{X}=\mathfrak{F}_{(\omega)}, \mathfrak{F}_{\omega}, (\mathfrak{N}\cap\mathfrak{F})_{(\omega)}, (\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)}, (\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}, (\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$ now follows from (1) and (2) by Lemmas 3.6 and 6.7.

(4) Let D be a derivation of L and put $\alpha=\exp(tD)$. Let H be an $\mathfrak{S}$ (resp. $\mathfrak{N}$) ideal of L. By Lemma 6.4 $H^{\#}$ is an $\mathfrak{S}$ (resp. $\mathfrak{N}$) ideal of L_0 and therefore so is $H^{\#\alpha}$. Put $K_0=H^{\#}+H^{\#\alpha}$. Then K_0 is an $\mathfrak{S}$ (resp. $\mathfrak{N}$) ideal of L_0. Denote $N=K_0^{\natural}$. It follows from Lemma 6.5 that N is an $\mathfrak{S}$ (resp. $\mathfrak{N}$) ideal of L. As in the first part of the proof of Lemma 6.6, for any $x\in H$ we have $xD\in N$. Therefore HD is contained in an $\mathfrak{S}$ (resp. $\mathfrak{N}$) ideal N of L.

Thus the theorem is completely proved.

As an immediate consequence of Theorem 6.9 we have the following

THEOREM 6.10. *Let L be a Lie algebra over a field Φ of characteristic* 0. *Let $\mathfrak{X}$ be any class of Lie algebras stated in the preceding theorem. Then the subalgebra generated by all the $\mathfrak{X}$ subideals (resp. ideals) of L and the sum of all the $\mathfrak{S}$ (resp. $\mathfrak{N}$) ideals of L are characteristic ideals of L.*

We here note that the parts on $\mathfrak{N}$ ideals, $\mathfrak{N}\cap\mathfrak{F}$ subideals and $\mathfrak{F}$ subideals of Theorems 6.9 and 6.10 are Theorem 1, Theorem 3 and its corollary, and Theorem 5 in [3].

§7. Locally nilpotent radicals

We know three locally nilpotent radicals $\mathrm{Rad}_{\mathfrak{N}}(L)$, $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}-si}(L)$ and $\mathrm{Rad}_{\mathrm{L}\mathfrak{N}}(L)$. For any class $\mathfrak{X}$ such that $\mathfrak{N}\cap\mathfrak{F}\subseteq\mathfrak{X}\subseteq\mathrm{L}\mathfrak{N}$, we have

$$\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}}(L)\subseteq\mathrm{Rad}_{\mathfrak{X}}(L)\subseteq\mathrm{Rad}_{\mathrm{L}\mathfrak{N}}(L)$$

and therefore $\mathrm{Rad}_{\mathfrak{X}}(L)$ is a locally nilpotent ideal of L. If Φ is of characteristic 0, for any class $\mathfrak{X}$ such that $\mathfrak{N}\cap\mathfrak{F}\subseteq\mathfrak{X}\subseteq\mathfrak{N}$ we have

$$\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}-si}(L)=\mathrm{Rad}_{\mathfrak{X}-si}(L)=\mathrm{Rad}_{\mathfrak{N}-si}(L).$$

We shall here examine the properties of $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}}(L)$.

THEOREM 7.1. *Let L be a Lie algebra over a field Φ.*

(1) $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}}(L)$ *is the union of all the* $\mathfrak{N}\cap\mathfrak{F}$ *ideals of* L.

(2) $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}}(L)$ *is locally nilpotent.*

(3) *If Φ is of characteristic* 0, $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}}(L)$ *is a characteristic ideal of* L.

PROOF. (1) and (2) are consequences of Theorem 6.3 and (3) follows from Theorem 6.10.

THEOREM 7.2. *Let L be a Lie algebra over a field Φ of characteristic* 0. *Then* $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}}(L)$ *is generally different from* $\mathrm{Rad}_{\mathfrak{N}}(L)$, $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}\text{-}si}(L)$ *and* $\mathrm{Rad}_{\mathrm{L}\mathfrak{N}}(L)$.

To see the theorem it suffices to show that $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}}(L)$ need not equal $\mathrm{Rad}_{\mathfrak{N}}(L)$. We need the following

LEMMA 7.3. *Let L be the Lie algebra in Example C in Section* 3. *Then*

(1) *Every non-zero ideal of L contains A.*

(2) *There exist no non-zero $\mathfrak{F}$ ideals of L.*

(3) *Every $\mathfrak{F}$ subideal of L is contained in A.*

PROOF. (1) Let N be a non-zero ideal of L. Then N contains a non-zero element $u=\sum_{i=0}^{n}a_ie_i+bx+cy+dz$.

In the case $\sum_{i=0}^{n}a_ie_i\neq 0$, we may suppose $a_n\neq 0$. Since

$$[u, z](\mathrm{ad}\,y)^n=(\sum_{i=0}^{n}a_ie_i)(\mathrm{ad}\,y)^n=n!\,a_ne_0,$$

we have $e_0\in N$. It follows that $e_k=e_0(\mathrm{ad}x)^k\in N$. Therefore $A\subseteq N$.

In the case $\sum_{i=0}^{n}a_ie_i=0$, if $b\neq 0$, then $[u, y]=bz$. If $c\neq 0$, then $[u, x]=-cz$. If $b=c=0$, then $d\neq 0$ and $dz\in N$. Thus in this case we have $z\in N$. It follows that $e_k=[e_k, z]\in N$. Therefore $A\subseteq N$.

(2) is an immediate consequence of (1).

(3) Let H be a non-zero $\mathfrak{F}$ subideal of L. Then H m-si L for some m. Now assume that $H\not\subseteq A$. Then there exists a non-zero element $u=\sum_{i=0}^{n}a_ie_i+bx+cy+dz$ in $H\setminus A$. If $b\neq 0$,

$$e_k(\mathrm{ad}u)^m=b^me_{m+k}+\sum_{i=0}^{m+k-1}f_ie_i\in H,\ k=0, 1, 2, \cdots.$$

Hence $H\notin\mathfrak{F}$. If $b=0$ and $d\neq 0$,

$$e_k(\mathrm{ad}u)^m=d^me_k+\sum_{i=0}^{k-1}f_ie_i\in H,\ k=0, 1, 2, \cdots.$$

Therefore $H \supseteq A$, whence $H \notin \mathfrak{F}$. If $b=d=0$ and $c \neq 0$,

$$e_{k+m}(\mathrm{ad}\,u)^m = (k+m)!c^m e_k/k!,\ k=0, 1, 2, \cdots,$$

whence $H \supseteq A$ and therefore $H \notin \mathfrak{F}$. Thus in any case we have a contradiction. Therefore we conclude that $H \subseteq A$.

Thus the proof of the lemma is completed.

PROOF OF THEOREM 7.2.

Let L be a Lie algebra in the above lemma. Then by the second part of the lemma, $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}}(L)=(0)$. By the first part of the lemma, $\mathrm{Rad}_{\mathfrak{N}}(L) \supseteq A$. As a matter of fact, it is immediate that $\mathrm{Rad}_{\mathfrak{N}}(L)=A$. The theorem is proved.

§8. Locally solvable radicals

In this section we shall study several locally solvable radicals.

THEOREM 8.1. *Let L be a Lie algebra over a field Φ. Then the radical $\mathrm{Rad}_{\mathfrak{X}}(L)$ for each $\mathfrak{X}$ of $\mathfrak{S}\cap\mathfrak{F}$, $\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}$, $\mathfrak{S}\cap\mathfrak{F}_\omega$ and $\mathfrak{S}$ is the union of all the $\mathfrak{X}$ ideals of L. If Φ is of characteristic* 0, *the radical $\mathrm{Rad}_{\mathfrak{X}\text{-}si}(L)$ for each $\mathfrak{X}$ of $\mathfrak{S}\cap\mathfrak{F}$, $\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}$ and $\mathfrak{S}\cap\mathfrak{F}_\omega$ is the union of all the $\mathfrak{X}$ subideals of L.*

PROOF. The statement follows from Theorem 6.3.

We have the following inclusion:

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}}(L) \subseteq \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}}(L) \subseteq \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega}(L) \subseteq \mathrm{Rad}_{\mathfrak{S}}(L).$$

The relation follows from

LEMMA 8.2. $\mathfrak{S}\cap\mathfrak{F} \subseteq \mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G} \subseteq \mathfrak{S}\cap\mathfrak{F}_\omega \subseteq \mathfrak{S}$.

PROOF. If $L \in \mathfrak{N}_\omega\cap\mathfrak{G}$, then $L/L^\omega \in \mathfrak{N}\cap\mathfrak{G} \subseteq \mathfrak{F}$ by Lemma 1.5. Hence $L \in \mathfrak{F}_\omega$. Therefore $\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G} \subseteq \mathfrak{S}\cap\mathfrak{F}_\omega$. The other parts are immediate.

When Φ is of characteristic 0, we have the following inclusion:

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-}si}(L) \subseteq \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}\text{-}si}(L) \subseteq \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega\text{-}si}(L).$$

THEOREM 8.3. *Let L be a Lie algebra over a field Φ.*

(1) *The radical $\mathrm{Rad}_{\mathfrak{X}}(L)$ for each $\mathfrak{X}$ of $\mathfrak{S}\cap\mathfrak{F}$, $\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}$, $\mathfrak{S}\cap\mathfrak{F}_\omega$ aud $\mathfrak{S}$ is locally solvable. If Φ is of characteristic* 0, *the radical $\mathrm{Rad}_{\mathfrak{X}\text{-}si}(L)$ for each $\mathfrak{X}$ of $\mathfrak{S}\cap\mathfrak{F}$, $\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}$ and $\mathfrak{S}\cap\mathfrak{F}_\omega$ is locally solvable.*

(2) *If Φ is of characteristic* 0, *the seven radicals of L stated in* (1) *are characteristic ideals of L.*

PROOF. (1) is a special case of Theorem 6.3 and (2) follows from Theorem 6.10.

COROLLARY 8.4. *For any class* $\mathfrak{X}$ *such that* $\mathfrak{S}\cap\mathfrak{F}\subseteq\mathfrak{X}\subseteq\mathfrak{S}$ $\mathrm{Rad}_{\mathfrak{X}}(L)$ *is a locally solvable ideal of* L. *If* Φ *is of characteristic* 0, *for any class* $\mathfrak{X}$ *such that* $\mathfrak{S}\cap\mathfrak{F}\subseteq\mathfrak{X}\subseteq\mathfrak{S}\cap\mathfrak{F}_{\omega}$ $\mathrm{Rad}_{\mathfrak{X}-si}(L)$ *is locally solvable.*

We finally show the following

THEOREM 8.5. *Let* L *be a Lie algebra over a field* Φ *of characteristic* 0. *Then the radicals*

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}}(L),\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}}(L),\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_{\omega}}(L),\ \mathrm{Rad}_{\mathfrak{S}}(L),$$

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-si}(L),\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}-si}(L),\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_{\omega}-si}(L)$$

are different from each other in general.

PROOF. (1) Let L be the Lie algebra in Example C in Section 3. By Lemma 7.3 (2), only (0) is an $\mathfrak{S}\cap\mathfrak{F}$ ideal of L. Hence $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}}(L)=(0)$. Since the $\mathfrak{S}\cap\mathfrak{F}$ subideals of L are the finite-dimensional subspaces of A by Lemma 7.3 (3), we have $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-si}(L)=A$. It is obvious that $L^{(3)}=(0)$ and $L^{\omega}=L^{3}=A$. Hence $L\in\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ and therefore $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}}(L)=L$. Thus we have

$$\begin{array}{ccc}\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}}(L) & \neq & \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}}(L)\\ \not\parallel & \not\nearrow & \parallel\\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-si}(L) & \neq & \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}-si}(L).\end{array}$$

(2) Let L be the Lie algebra in Example F in Section 3. Then the $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ subideals of L are the finite-dimensional subspaces of A. In fact, if H is a finite-dimensional subspace of A, then $H\lhd A\lhd L$ and therefore H *si* L. Conversely, let H be an $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ subideal of L. If $H\not\subseteq A$, then $(H+A)/A$ is a non-zero solvable subideal of L/A, which contradicts the fact that L/A is a three-dimensional simple algebra. Therefore $H\subseteq A$. Since $H\in\mathfrak{G}$, H is a finite-dimensional subspace of A. Thus we have $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}-si}(L)=A$.

Next we see that only (0) is an $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ ideal of L. In fact, if H is such an ideal, then H is a finite-dimensional subspace of A. If $H\neq(0)$, H contains $u=\sum_{i=0}^{n}a_ie_i$ with $a_n\neq 0$, and it follows that

$$u(\mathrm{ad}\,x)^k=\sum_{i=0}^{n}a_ie_{i+k}\in H,\ k=0,1,2,\cdots.$$

This shows that $H\notin\mathfrak{F}$, which is a contradiction. Hence $H=(0)$. Therefore we have $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}}(L)=(0)$. Thus

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}}(L)\neq\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}-si}(L).$$

(3) Let L be the Lie algebra in Example D in Section 3. Then the $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ subideals and the $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ ideals of L are both the finite-dimensional subspaces of A. In fact, it is immediate that any finite-dimensional subspace of A is an $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ ideal of L. So conversely let H be an $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$ sub-

ideal of L. If $H \not\subseteq A$, then there exists $u=\sum a_i e_i + bz$ with $b \neq 0$ in H. H n-si L for some n. It follows that

$$e_i(\mathrm{ad}\, u)^n = b^n e_i \in H,\ i=0, 1, 2, \cdots$$

and therefore $H \supseteq A$, which contradicts $H \in \mathfrak{G}$. Therefore $H \subseteq A$. Since $H \in \mathfrak{G}$, H is a finite-dimensional subspace of A. Thus

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}}(L) = \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}-si}(L) = A.$$

Furthermore, $L^\omega = L^2 = A$ and therefore $L \in \mathfrak{S}\cap\mathfrak{F}_\omega$. Hence we have $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega}(L)=L$. Thus we have

$$\begin{array}{ccc} \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}}(L) & \neq & \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega}(L) \\ \| & \not\nearrow & \| \\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}-si}(L) & \neq & \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega-si}(L). \end{array}$$

(4) Let L be the subalgebra $A+(x)$ of the Lie algebra given in Example C in Section 3. Then the $\mathfrak{S}\cap\mathfrak{F}_\omega$ subideals of L are the finite-dimensional subspaces of A. In fact, it is immediate that a finite-dimensional subspace of A is an $\mathfrak{S}\cap\mathfrak{F}_\omega$ subideal of L. Conversely, let H be an $\mathfrak{S}\cap\mathfrak{F}_\omega$ subideal of L. H n-si L for some n. If $H \not\subseteq A$, H contains $u=\sum a_i e_i + bx$ with $b \neq 0$. It follows that

$$e_i(\mathrm{ad}\, u)^n = b^n e_{n+i} \in H,\ i=0, 1, 2, \cdots.$$

Hence $H \supseteq (e_n, e_{n+1}, \cdots)$. Since $L^\omega=(0)$, $H^\omega=(0)$. It follows that $H \notin \mathfrak{F}_\omega$, which is a contradiction. Therefore $H \subseteq A$. Since $H \in \mathfrak{F}_\omega$, it follows that H is a finite-dimensional subspace of A. Thus $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega-si}(L)=A$.

It is immediate that only (0) is an $\mathfrak{F}_\omega$ ideal of L. Therefore $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega}(L) = (0)$. Since $\mathrm{Rad}_{\mathfrak{S}}(L)=L$, we have

$$\begin{array}{ccc} \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega}(L) & \neq & \mathrm{Rad}_{\mathfrak{S}}(L) \\ \nparallel & \not\nearrow & \\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega-si}(L). & & \end{array}$$

Thus the theorem is completely proved.

References

[1] R. Baer, Nilgruppen, Math. Z., **62** (1955), 402-437.
[2] C.-Y. Chao, Some characterizations of nilpotent Lie algebras, Math. Z., **103** (1968), 40-42.
[3] B. Hartley, Locally nilpotent ideals of a Lie algebra, Proc. Camb. Phil. Soc., **63** (1967), 257-272.
[4] K. A. Hirsch, Uber lokal-nilpotente Gruppen, Math. Z., **63** (1955), 290-294.
[5] N. Jacobson, Lie Algebras, Interscience, New York, 1962.
[6] B. I. Plotkin, Algebraic sets in groups and Lie algebras, Uspekhi Mat. Nauk., **13** (1958), no. 6 (84), 133-139 (Russian).
[7] E. Schenkman, A theory of subinvariant Lie algebras, Amer. J. Math., **73** (1951), 453-474.
[8] E. Schenkman, Infinite Lie algebras, Duke Math. J., **19** (1952), 529-535.

[9] I. Stewart, The minimal condition for subideals of Lie algebras, Math. Z., **111** (1969), 301-310.
[10] I. Stewart, Lie Algebras, Lecture Notes in Mathematics 127, Springer, Berlin · Heidelberg · New York, 1970.

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