HIROSHIMA MATH. J.
2 (1972), 71-98

On the Irreducibility of Some Series of Representations

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(Received February 25, 1972)

1. Introduction

Let G be a connected semi-simple Lie group with finite center, and $\mathfrak{g}_0$ its Lie algebra. Let $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$ fixed throughout. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$, and let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\mathfrak{a}$ denote complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$ and $\mathfrak{a}_0$ respectively. Let Λ be the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. As an element of Λ takes real values on $\mathfrak{a}_0$, the set Λ can be regarded as a subset of $\mathfrak{a}_0'=\mathrm{Hom}_{R}(\mathfrak{a}_0, \boldsymbol{R})$. We fix a lexicographical order in $\mathfrak{a}_0'$ and let Λ_+ be the set of all positive roots in Λ. For each $\alpha\in\Lambda$, we define the root space $\mathfrak{g}^{\alpha}$ by

$$\mathfrak{g}^{\alpha}=\{X\in\mathfrak{g};\ [H, X]=\alpha(H)X \text{ for every } H\in\mathfrak{a}\}.$$

For the later use, we put $\rho=\frac{1}{2}\sum_{\alpha\in\Lambda_+}(\dim_{C}\mathfrak{g}^{\alpha})\alpha$. And we define nilpotent subalgebras $\mathfrak{n}$ of $\mathfrak{g}$ and $\mathfrak{n}_0$ of $\mathfrak{g}_0$ by

$$\mathfrak{n}=\sum_{\alpha\in\Lambda_+}\mathfrak{g}^{\alpha},\qquad \mathfrak{n}_0=\mathfrak{g}_0\cap\mathfrak{n}.$$

Then $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{a}_0+\mathfrak{n}_0$ is an Iwasawa decomposition of $\mathfrak{g}_0$. If K, A_+ and N denote subgroups of G generated by $\mathfrak{k}_0$, $\mathfrak{a}_0$ and $\mathfrak{n}_0$, $G=KA_+N$ gives an Iwasawa decomposition of G. Let M be the centralizer of $\mathfrak{a}_0$ in K and $\mathfrak{m}_0$ the subalgebra of $\mathfrak{g}_0$ corresponding to M. We define the subgroup B of G by $B=MA_+N$. The irreducibility of the representation of the group G induced from a finite-dimensional irreducible representation ξ of B has long been studied. First Bruhat [1] proved sufficient conditions for the irreducibility of such representations using an analytic method. In certain algebraic way, necessary and sufficient conditions have been given; (1) by Parthasarathy, Ranga Rao and Varadarajan [7] in case that G is a complex semi-simple Lie group and ξ is trivial on M (2) by Želobenko [3] in case of complex semi-simple Lie groups, and recently (3) by Kostant [4] in case that G is a real semi-simple Lie group and ξ is trivial on M. We shall attempt here an extension of the Kostant's method.

Let $\lambda=(\varepsilon, \mu)$ be a pair of a character ε of M and an element $\mu\in\mathfrak{a}'=\mathrm{Hom}_{C}(\mathfrak{a}, \boldsymbol{C})$, and we define the character ξ_{λ} of B by

$$\xi_{\lambda}(m\exp H\cdot n)=\varepsilon(m)e^{\mu(H)}$$

where $m \in M$, $H \in \mathfrak{a}_0$ and $n \in N$. Let $X^{(\lambda)}$ be the space of all $\boldsymbol{C}$-valued real analytic functions f on G such that $f(xb)=\xi_\lambda(b^{-1})f(x)$ for every $x \in G$ and $b \in B$. We define a G-module structure π^λ on $X^{(\lambda)}$ by setting $(\pi^\lambda(x)f)(y)=f(x^{-1}y)$ for x, $y \in G$ and $f \in X^{(\lambda)}$. The representation π^λ determines the infinitesimal representation π_*^λ of $\mathfrak{g}_0$ on $X^{(\lambda)}$, which can be extended to the representation of the universal enveloping algebra $\mathfrak{U}=\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. For the sake of simplicity we write xf and uf instead of $\pi^\lambda(x)f$ and $\pi_*^\lambda(u)f$, where $x \in G$, $u \in \mathfrak{U}$ and $f \in X^{(\lambda)}$. Let X^λ be the subspace of $X^{(\lambda)}$ consisting of all $f \in X^{(\lambda)}$ which are K-finite. The space X^λ is not G-stable, but it is stable under K-action and $\mathfrak{U}$-action, and so the space X^λ has a structure of K-module and U-module.

Now we suppose that the K-module X^λ admits a one-dimensional K-invariant subspace throughout this paper, and we consider the irreducibility of $(\pi^\lambda, X^{(\lambda)})$ under this assumption. Our method relies heavily on the way of Kostant's constructions [4]. Our results, however, not only give informations about the irreducibility of representations, but also make it possible to find out irreducible components of reducible representations in special cases. As an example, we shall describe in this paper how the representations of the discrete series of a real 2×2 unimodular group can be obtained as sub-representations from reducible ones.

The author should like to express his hearty thanks to Professor K. Okamoto for his kind advice and to Professor H. Ozeki for much encouragement.

2. Some ideals in the universal enveloping algebra

By the assumption mentioned in 1, we can find a non-zero vector $f_\lambda \in X^\lambda$ and a unitary character σ of K such that

$$kf_\lambda=\sigma(k)f_\lambda \qquad \text{for every} \quad k \in K.$$

The representation σ of K defines the infinitesimal representation σ_* of the Lie algebra $\mathfrak{k}$. For each $X \in \mathfrak{k}_0$ and $x \in G$, we have

$$\begin{aligned}(Xf_\lambda)(x)&=\left[\frac{d}{dt}f_\lambda(\exp\text{-}tX\cdot x)\right]_{t=0}\\ &=\left[\frac{d}{dt}\sigma(\exp tX)\cdot f_\lambda(x)\right]_{t=0}\\ &=\sigma_*(X)\cdot f_\lambda(x).\end{aligned}$$

Hence $\qquad Xf_\lambda=\sigma_*(X)f_\lambda \qquad$ for every $\quad X \in \mathfrak{k}_0$.

Thus we have $\qquad Xf_\lambda=\sigma_*(X)f_\lambda \qquad$ for every $\quad X \in \mathfrak{k}$.

Define the subspaces $\tilde{\mathfrak{k}}$ and $\hat{\mathfrak{k}}$ of $\mathfrak{U}$ by

$$\tilde{\mathfrak{k}}=\{X-\sigma_*(X);\ X\in\mathfrak{k}\},\qquad \hat{\mathfrak{k}}=\{X+\sigma_*(X);\ X\in\mathfrak{k}\}.$$

And we define a left-ideal $\mathfrak{L}_\lambda$ in $\mathfrak{U}$ by

$$\mathfrak{L}_\lambda=\{u\in\mathfrak{U};\ uf_\lambda=0\},$$

then $\tilde{\mathfrak{k}}$ is a subspace of $\mathfrak{L}_\lambda$ and the $\mathfrak{U}$-submodule $\mathfrak{U}\cdot f_\lambda$ of X^λ is isomorphic to the $\mathfrak{U}$-module $\mathfrak{U}/\mathfrak{L}_\lambda$.

Let $\mathfrak{U}(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$, which can be regarded as an associative subalgebra of $\mathfrak{U}$. For every $u\in\mathfrak{U}$, there exists a unique $p_u\in\mathfrak{U}(\mathfrak{a})$ such that $u-p_u\in\hat{\mathfrak{k}}\mathfrak{U}+\mathfrak{U}\mathfrak{n}$, where $\hat{\mathfrak{k}}\mathfrak{U}$ denotes the subspace of $\mathfrak{U}$ generated by $\{uv;\ u\in\hat{\mathfrak{k}},\ v\in\mathfrak{U}\}$ and the same for $\mathfrak{U}\mathfrak{n}$. Let $u\to u^t$ denote the linear anti-automorphism of the associative algebra $\mathfrak{U}$ determined by the following conditions:

(1) $X^t=-X$ if $X\in\mathfrak{g}$

(2) $(uv)^t=v^tu^t$ for $u,\ v\in\mathfrak{U}$.

And for each $u\in\mathfrak{U}$, let $p^u\in\mathfrak{U}(\mathfrak{a})$ be the polynomial function on $\mathfrak{a}'$ given by $p^u(\upsilon)=p_{u^t}(-\upsilon)$ where $\upsilon\in\mathfrak{a}'$.

Definition: Let Z be a $\mathfrak{U}$-module. A non-zero vector $\eta\in Z$ is called *a highest weight vector* if η is an eigenvector for every $X\in\mathfrak{a}+\mathfrak{n}$.

Note. Let Z be a $\mathfrak{U}$-module. Then a non-zero vector $\eta\in Z$ is a highest weight vector if and only if (i) η is an eigenvector for every $H\in\mathfrak{a}$ and (ii) $X\eta=0$ for every $X\in\mathfrak{n}$.

The above Note follows easily from $[\mathfrak{a},\mathfrak{n}]=\mathfrak{n}$. A highest weight vector $\eta\in Z$ determines an element υ of $\mathfrak{a}'$, called the weight of η, by $H\cdot\eta=\upsilon(H)\eta$, where $H\in\mathfrak{a}$.

For a $\mathfrak{U}$-module Z, let Z' denote its dual vector space. We define a $\mathfrak{U}$-module structure on Z' by $\langle z, uz'\rangle=\langle u^tz, z'\rangle$ for every $u\in\mathfrak{U}$, $z\in Z$ and $z'\in Z'$, where $\langle\,,\,\rangle$ denotes the canonical pairing of Z and Z'. The space Z' equipped with this $\mathfrak{U}$-module structure is called the $\mathfrak{U}$-module dual to Z.

Lemma 1. *Let Z be a $\mathfrak{U}$-module and Z' the $\mathfrak{U}$-module dual to Z. We assume that* (i) *there exists a non-zero vector $\psi\in Z$ such that $u\psi=0$ for every $u\in\tilde{\mathfrak{k}}$ and that* (ii) *there exists a highest weight vector $\eta\in Z'$ of the weight $-\mu\in\mathfrak{a}'$. Then we have*

$$\langle u\psi,\eta\rangle=p^u(\mu)\langle\psi,\eta\rangle\qquad \textit{for}\quad u\in\mathfrak{U}.$$

Proof. The vector space $\mathfrak{U}$ can be decomposed directly in the form $\mathfrak{U}=\mathfrak{U}(\mathfrak{a})\oplus(\mathfrak{U}\tilde{\mathfrak{k}}+\mathfrak{n}\mathfrak{U})$, and the corresponding projection $\mathfrak{U}\to\mathfrak{U}(\mathfrak{a})$ is given by

$u \to p^t_{u^t}$. Thus each $u \in \mathfrak{U}$ can be written in the form:

$$u = p^t_{u^t} + u_1 + u_2$$

where $u_1 \in \mathfrak{U}\tilde{\mathfrak{k}}$ and $u_2 \in \mathfrak{n}\mathfrak{U}$. By the conditions on ψ and η, we have

$$\begin{aligned} <u\psi, \eta> &= <\psi, p_{u^t}\eta + u_2^t\eta> = <\psi, p_{u^t}\eta> \\ &= <\psi, p_{u^t}(-\mu)\eta> \\ &= p^u(\mu)<\psi, \eta>. \end{aligned}$$

Q.E.D.

DEFINITION. A complex vector space Z is called a $(K, \mathfrak{U})$-module if it is both a K-module and a $\mathfrak{U}$-module and if $k(uz) = (ku)\cdot(kz)$ for every $k \in K$, $u \in \mathfrak{U}$ and $z \in Z$.

Note. X^λ is a $(K, \mathfrak{U})$-module.

Let Z be a $(K, \mathfrak{U})$-module and Z' its dual vector space. We define a K-action on Z' by $<z, kz'> = <k^{-1}z, z'>$ for $k \in K$, $z \in Z$ and $z' \in Z'$. Then the space Z' becomes a $(K, \mathfrak{U})$-module with this K-module structure and the dual $\mathfrak{U}$-module structure, which is called the $(K, \mathfrak{U})$-module dual to Z.

DEFINITION. Let Z be a $(K, \mathfrak{U})$-module and Z' the $(K, \mathfrak{U})$-module dual to Z. A non-zero vector $\eta \in Z'$ is called *a K-effective highest weight vector* if (i) η is a highest weight vector of the $\mathfrak{U}$-module Z' and if (ii) $K\cdot\eta$ is non-singularly paired to Z, i.e., if only the zero vector in Z is orthogonal to $K\cdot\eta$.

LEMMA 2. *Let Z be a $(K, \mathfrak{U})$-module and Z' the $(K, \mathfrak{U})$-module dual to Z. We assume that* (i) *there exists a non-zero vector $\psi \in Z$ such that $k\psi = \sigma(k)\psi$ for every $k \in K$ and that* (ii) *there exists a K-effective highest weight vector $\eta \in Z'$ of a weight $-\mu \in \mathfrak{a}'$. Then we have the following:*

(1) $<\psi, \eta> \neq 0$.

(2) *For each $u \in \mathfrak{U}$, the following conditions* (A) *and* (B) *are equivalent:*

(A) $u\psi = 0$.

(B) $p^v(\mu) = 0$ *for every element v in the K-submodule of $\mathfrak{U}$ generated by u.*

PROOF. (1) By the K-effectiveness of η, we have

$$<K\cdot\psi, \eta> = <\psi, K\cdot\eta> \neq \{0\}.$$

By the condition (i), we have

$$K\cdot\psi = \{c\psi;\ c \in \boldsymbol{C}^*\}.$$

Thus we have $<\psi, \eta> \neq 0$.

(2) First suppose (A). It is sufficient to show $p^{ku}(\mu)=0$ for every $k \in K$. By Lemma 1, we have

$$p^{ku}(\mu)<\psi, \eta> = <(ku)\psi, \eta> = <k(u \cdot k^{-1}\psi), \eta>$$
$$= \sigma(k^{-1})<k(u\psi), \eta> = 0.$$

Since $<\psi, \eta> \neq 0$, we have $p^{ku}(\mu)=0$.

Next suppose (B). Applying Lemma 1 to ku, we have

$$<(ku)\psi, \eta> = p^{ku}(\mu)<\psi, \eta> = 0.$$

On the otherhand,

$$<(ku)\psi, \eta> = <k(u \cdot k^{-1}\psi), \eta> = \sigma(k^{-1})<k(u\psi), \eta>$$
$$= \sigma(k^{-1})<u\psi, k^{-1}\eta>.$$

Hence $\qquad \sigma(k^{-1})<u\psi, k^{-1}\eta> = 0 \qquad$ for every $\quad k \in K$.

Since $\sigma(k^{-1}) \neq 0$, we have $<u\psi, K \cdot \eta> = 0$. Therefore, by the K-effectiveness of η, we have $u\psi = 0$. Q.E.D.

We shall apply the above lemmas to the $(K, \mathfrak{U})$-module X^{λ}. Define the element $\delta \in (X^{\lambda})'$ by $<f, \delta> = f(e)$ where $f \in X^{\lambda}$, and e is the identity element of G.

LEMMA 3. *δ is a K-effective highest weight vector of X^{λ} of the weight $-\mu$.*

PROOF. For every $H \in \mathfrak{a}_0$ and $f \in X^{\lambda}$, we have

$$<f, H \cdot \delta> = <-Hf, \delta> = (-Hf)(e)$$
$$= \left[\frac{d}{dt} f(\exp tH)\right]_{t=0} = \left[\frac{d}{dt} e^{-t\mu(H)} f(e)\right]_{t=0}$$
$$= -\mu(H) f(e) = <f, -\mu(H)\delta>.$$

Therefore $\qquad H\delta = -\mu(H)\delta \qquad$ for every $\quad H \in \mathfrak{a}_0$.

Hence $\qquad H\delta = -\mu(H)\delta \qquad$ for every $\quad H \in \mathfrak{a}$.

The same calculation as above as to $X \in \mathfrak{n}_0$ shows us that

$$X\delta = 0 \qquad \text{for every} \quad X \in \mathfrak{n}.$$

Thus δ is a highest weight vector of the weight $-\mu$. Suppose that $<f, K \cdot \delta> = \{0\}$ for some $f \in X^{\lambda}$. Then we have $f|K=0$ by the definition of δ. By the condition of $f \in X^{\lambda}$, we have $f=0$. Thus δ is K-effective. Q.E.D.

Let $\mathfrak{U}^K$ be the subalgebra of $\mathfrak{U}$ consisting of all K-fixed elements in $\mathfrak{U}$. Let dk denote the Haar measure on K normalized by $\int_K dk=1$. For each $u \in \mathfrak{U}$, define an element $u_0 \in \mathfrak{U}^K$ by $u_0=\int_K ku\,dk$. We put $\mathfrak{C}_\lambda=\{u-p^u(\mu);\ u \in \mathfrak{U}^K\}$ and define left ideals $\mathfrak{L}_\lambda^{\min}$ and $\mathfrak{L}_\lambda^{\max}$ in $\mathfrak{U}$ by the following:

$$\mathfrak{L}_\lambda^{\min}=\mathfrak{U}\tilde{\mathfrak{k}}+\mathfrak{U}\mathfrak{C}_\lambda$$

$$\mathfrak{L}_\lambda^{\max}=\{u \in \mathfrak{U};\ p^{(vu)_0}(\mu)=0 \text{ for every } v \in \mathfrak{U}\}.$$

PROPOSITION 4. 1) $\mathfrak{L}_\lambda^{\min}\subset\mathfrak{L}_\lambda\subset\mathfrak{L}_\lambda^{\max}$

2) *For a proper left ideal $\mathfrak{L}$ in $\mathfrak{U}$, $\mathfrak{L}_\lambda^{\min}\subset\mathfrak{L}$ implies $\mathfrak{L}\subset\mathfrak{L}_\lambda^{\max}$.*

PROOF. 1) For $u \in \mathfrak{U}^K$, and $k \in K$,

$$\begin{aligned}\langle [u-p^u(\mu)]f_\lambda,\ k^{-1}\delta\rangle &= \langle k([u-p^u(\mu)]f_\lambda),\ \delta\rangle\\ &=\sigma(k)\langle [u-p^u(\mu)]f_\lambda,\ \delta\rangle\\ &=0\end{aligned}$$

where we have used Lemma 1 and Lemma 3. Thus

$$\langle [u-p^u(\mu)]f_\lambda,\ K\cdot\delta\rangle=\{0\}.$$

Owing to the K-effectiveness of δ, we have $[u-p^u(\mu)]f_\lambda=0$, which implies that $\mathfrak{C}_\lambda\subset\mathfrak{L}_\lambda$. Thus $\mathfrak{L}_\lambda^{\min}\subset\mathfrak{L}$ is proved.

Next we shall prove that $\mathfrak{L}_\lambda\subset\mathfrak{L}_\lambda^{\max}$. For every $u \in \mathfrak{L}_\lambda$ and $v \in \mathfrak{U}$,

$$\begin{aligned}p^{(vu)_0}(\mu)\langle f_\lambda,\ \delta\rangle &= \langle (vu)_0 f_\lambda,\ \delta\rangle\\ &=\left\langle \int_K k(vu)\,dk\cdot f_\lambda,\ \delta\right\rangle\\ &=\left\langle \int_K [k(vu)]f_\lambda\,dk,\ \delta\right\rangle\\ &=\left\langle \int_K \sigma(k^{-1})k(vuf_\lambda)\,dk,\ \delta\right\rangle\\ &=0.\end{aligned}$$

Thus we have $p^{(vu)_0}(\mu)=0$ for every $u \in \mathfrak{L}_\lambda$ and $v \in \mathfrak{U}$, owing to Lemma 2 (1).

Hence $$u \in \mathfrak{L}_\lambda^{\max}.$$

Therefore $$\mathfrak{L}_\lambda\subset\mathfrak{L}_\lambda^{\max}.$$

2) Let us assume that the assertion does not hold, i.e., we assume that

there exists a proper left ideal $\mathfrak{L}$ in $\mathfrak{U}$ such that $\mathfrak{L}_\lambda^{\min}\subset\mathfrak{L}$ and $\mathfrak{L}\not\subset\mathfrak{L}_\lambda^{\max}$. Then we can find $u\in\mathfrak{L}$ and $v\in\mathfrak{U}$ suth that $p^{(vu)_0}(\mu)\neq 0$. Since $\mathfrak{L}$ is a left-ideal in $\mathfrak{U}$, the element vu belongs to $\mathfrak{L}$, and so we can find $u\in\mathfrak{L}$ such that $p^{u_0}(\mu)\neq 0$. First we shall prove that $K\cdot\mathfrak{L}\subset\mathfrak{L}$. For every $X\in\mathfrak{k}$ and $v\in\mathfrak{L}$, we have

$$\mathrm{ad}X\cdot v=Xv-vX\in\mathfrak{L}$$

since $\mathfrak{U}\tilde{\mathfrak{k}}\subset\mathfrak{L}$. Therefore $\exp X\cdot\mathfrak{L}\subset\mathfrak{L}$ for every $X\in\mathfrak{k}$. As the subgroup K of G is connected, it follows that $K\cdot\mathfrak{L}\subset\mathfrak{L}$.

This implies that $u_0\in\mathfrak{L}$, since $u_0=\int_K ku\,dk$ and $u\in\mathfrak{L}$.

On the other hand, by the definition of $\mathfrak{C}_\lambda$, we have

$$u_0-p^{u_0}(\mu)\in\mathfrak{C}_\lambda\subset\mathfrak{L}\,.$$

Thus we have $p^{u_0}(\mu)\in\mathfrak{L}$.
Now our assumption "$p^{u_0}(\mu)\neq 0$" implies that $\mathfrak{L}$ contains a non-zero scalar element, which contradicts the fact that $\mathfrak{L}$ is a "proper" left ideal of $\mathfrak{U}$.

Q.E.D.

PROPOSITION 5. *$\mathfrak{L}_\lambda^{\max}$ is a maximal left ideal in $\mathfrak{U}$.*

PROOF. This follows immediately from Proposition 4 (2). Q.E.D.

Let Z^λ denote the quotient $\mathfrak{U}$-module $\mathfrak{U}/\mathfrak{L}_\lambda^{\max}$. For every $k\in K$, $u\in\mathfrak{L}_\lambda^{\max}$ and $v\in\mathfrak{U}$, we have

$$\begin{aligned}p^{(v\cdot ku)_0}(\mu)&=p^{[k(k^{-1}v\cdot u)]_0}(\mu)\\&=p^{(k^{-1}v\cdot u)_0}(\mu)\\&=0\end{aligned}$$

by the definition of $\mathfrak{L}_\lambda^{\max}$. Therefore $ku\in\mathfrak{L}_\lambda^{\max}$. Thus we have proved that $\mathfrak{L}_\lambda^{\max}$ is K-invariant. And so the $\mathfrak{U}$-module Z^λ admits the canonical K-module structure, and Z^λ is a $(K, \mathfrak{U})$-module.

Proposition 4 and Proposition 5 imply that "$\mathfrak{U}f_\lambda$ is an irreducible $\mathfrak{U}$-module if and only if $\mathfrak{L}_\lambda=\mathfrak{L}_\lambda^{\max}$."

3. K-module structures on X^λ

In this section we shall replace the K-module structure on X^λ by a suitable one.

Let h be a $\boldsymbol{C}$-valued real analytic function on G such that (1) $h|K=f_\lambda|K$ and (2) $h(kan)=h(k)$ for every $k\in K$, $a\in A_+$, and $n\in N$. We note that the function h does not take the value 0 anywhere.

Let $\lambda_0=(1, \mu)$ be the pair of the trivial character 1 of M and element

$\mu\in\mathfrak{a}'$. Using λ_0, we define π^{λ_0}, $X^{(\lambda_0)}$, X^{λ_0} in the same way as we have defined ones as to λ. For each $f\in X^{\lambda_0}$, hf is contained in X^{λ}, and $f\to hf$ defines a linear isomorphism A of X^{λ_0} onto X^{λ}.

For every $k\in K$ and $f\in X$, we put $\tau(k)f=\sigma(k^{-1})kf$. "$\tau$" determines a "new" K-module structure on X^{λ} which we shall call the $\tau(K)$-module structure on X^{λ}. We note that the $\mathfrak{U}$-module X^{λ} with this $\tau(K)$-module structure is still a $(K, \mathfrak{U})$-module, which we shall sometimes refer to the $(\tau(K), \mathfrak{U})$-module.

LEMMA 6. *The mapping A is a K-isomorphism of X^{λ_0} onto the $\tau(K)$-module X^{λ}, i.e., the following diagram is commutative.*

$$\begin{array}{ccc} X^{\lambda_0} & \xrightarrow{K\text{-action}} & X^{\lambda_0} \\ \downarrow A & & \downarrow A \\ X^{\lambda} & \xrightarrow{\tau(K)\text{-action}} & X^{\lambda} \end{array}$$

PROOF. For every $k\in K$ and $f\in X^{\lambda_0}$, we have

$$\begin{aligned}\tau(k)Af&=\sigma(k^{-1})k(Af)=\sigma(k^{-1})k(hf)\\ &=\sigma(k^{-1})kh\cdot kf=h\cdot kf\\ &=A(kf).\end{aligned}$$

Q.E.D.

LEMMA 7. *The projection $\mathfrak{U}\to\mathfrak{U}\cdot f_{\lambda}$ is a K-homomorphism of $\mathfrak{U}$ onto the $\tau(K)$-submodule $\mathfrak{U}\cdot f_{\lambda}$ of X^{λ}, i.e., the following diagram is commutative.*

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{K\text{-action}} & \mathfrak{U} \\ \downarrow \text{projection} & & \downarrow \text{projection} \\ \mathfrak{U}\cdot f_{\lambda} & \xrightarrow{\tau(K)\text{-action}} & \mathfrak{U}\cdot f_{\lambda} \end{array}$$

PROOF. For every $k\in K$ and $u\in\mathfrak{U}$, we have

$$\begin{aligned}(ku)f_{\lambda}&=k(u\cdot k^{-1}f_{\lambda})=k(u\cdot\sigma(k^{-1})f_{\lambda})\\ &=\sigma(k^{-1})k(uf_{\lambda})=\tau(k)(uf_{\lambda}).\end{aligned}$$

Q.E.D.

Note. Lemma 7 implies that the $(\tau(K), \mathfrak{U})$-module $\mathfrak{U}\cdot f_{\lambda}$ is isomorphic to the $(K, \mathfrak{U})$-module $\mathfrak{U}/\mathfrak{L}_{\lambda}$.

4. Some preparation from Kostant-Rallis [5], [6] and its application

The space $\mathfrak{p}$ admits the canonical K-module structure. Let $\mathfrak{p}'$ denote the K-module dual to $\mathfrak{p}$, and let $S=S(\mathfrak{p})$ and $S'=S(\mathfrak{p}')$ denote the symmetric algebras over $\mathfrak{p}$ and $\mathfrak{p}'$ respectively. These algebras carry the K-module structures extended from ones on $\mathfrak{p}$ and $\mathfrak{p}'$. The algebra S' may be regarded as the polynomial ring on $\mathfrak{p}$, while the algebra S may be regarded as the ring of differential operators on S' with constant coefficients. Define subspaces J and J^+ of S by

$$J=\{x\in S;\ kx=x \text{ for every } k\in K\}$$

$$J_+=\{x \in J;\ \text{the constant part of } x \text{ is zero}\}$$
$$=J\cap\sum_{i=1}^{\infty}S^i$$

where S^i is the subspace of S consisting of all homogeneous elements of the degree i. And we define the subspace H' of S' by

$$H'=\{f \in S';\ xf=0 \text{ for every } x \in J_+\}.$$

The Killing form B of $\mathfrak{g}$ determines the K-module isomorphism of $\mathfrak{p}'$ onto $\mathfrak{p}$, which can be extended to the K-module isomorphism of S' onto S. The image of H' under this isomorphism we denote by H. The spaces H and J are K-submodules of S, and so one can make up the K-module $H\otimes J$. The mapping of $H\otimes J$ to S defined by $\sum_i f_i\otimes g_i\to\sum_i f_ig_i$ is a K-module homomorphism, where $f_i \in H$ and $g_i \in J$.

LEMMA 8. $S=H\otimes J$.

PROOF. This is a result of Kostant-Rallis [5]. But we deal with a slightly different situation, and so we need some remarks. First we put

$$\tilde{G}^c=\operatorname{Int}\mathfrak{g}=\text{the group of inner automorphisms of } \mathfrak{g}$$
$$\tilde{G}=\{g \in \tilde{G}^c;\ g(\mathfrak{g}_0)\subset\mathfrak{g}_0\}$$
$$\tilde{G}_0=\text{the identity component of } \tilde{G}$$
$$\tilde{K}_\theta^c=\{g \in \tilde{G}^c;\ g(\mathfrak{k})\subset\mathfrak{k} \text{ and } g(\mathfrak{p})\subset\mathfrak{p}\}$$
$$\tilde{K}_\theta=\tilde{K}_\theta^c\cap\tilde{G}$$
$$\tilde{K}=\text{the identity component of } \tilde{K}_\theta\,.$$

Then, as is well known, one has

$$\tilde{G}_0\cong\operatorname{Int}\mathfrak{g}_0=\text{the group of inner automorphisms of } \mathfrak{g}_0$$
$$\tilde{K}=\tilde{K}_\theta\cap\tilde{G}_0\,.$$

Further one can see easily that $G/Z\cong\tilde{G}_0$ and $K/Z\cong\tilde{K}$, where Z denotes the center of G.

Since the group G acts on $\mathfrak{g}$ as the adjoint representation, we have

$$J=\{x \in S;\ kx=x \text{ for every } k \in \tilde{K}\}.$$

Let $\tilde{J}$ denote the space of all $\tilde{K}_\theta^c$-fixed vectors in S, i.e.,

$$\tilde{J}=\{x \in S;\ kx=x \text{ for every } k \in \tilde{K}_\theta^c\}$$

and we define $\tilde{H}$ in the same way as H by using $\tilde{J}$. Then the theorem of Kostant-Rallis [5] shows us that

$$S=\tilde{H}\otimes\tilde{J}.$$

Some considerations added in the process of proving the Kostant-Rallis' theorem lead us to $J=\tilde{J}$.(*)

Therefore $$H=\tilde{H}$$

Thus we have $$S=H\otimes J$$ Q.E.D.

Let $S(\mathfrak{g})$ denote the symmetric algebra of $\mathfrak{g}$. Then, as is well known, there exists a unique linear isomorphism β of $S(\mathfrak{g})$ onto $\mathfrak{U}$ such that (1) $\beta(X^k)=(\beta(X))^k$ for every $X\in\mathfrak{g}$ and (2) (with the obvious identification) β is the identity map on $\mathfrak{g}$. This mapping β is called the symmetrization and has the following property: if $X_1, \cdots, X_n$ are elements of $\mathfrak{g}$, then

$$\beta(X_1\cdots X_n)=\frac{1}{n!}\sum_{\sigma\in S_n}X_{\sigma(1)}\cdots X_{\sigma(n)}$$

where S_n denotes the permutation group of n-numbers $\{1, \cdots, n\}$.

We set $J^*=\beta(J)$ and $H^*=\beta(H)$. The spaces J^* and H^* are K-submodules of $\mathfrak{U}$, since β is a K-homomorphism.

LEMMA 9. $\mathfrak{U}=\mathfrak{U}\mathfrak{k}+H^*J^*$.

Proof. For each non-negative integer n, let $\mathfrak{U}^n$ be the set of all elements in $\mathfrak{U}$ whose degrees are equal to or smaller than n. We shall prove by induction on n that $\mathfrak{U}^n$ is contained in $\mathfrak{U}\mathfrak{k}+H^*J^*$. This is obvious if $n=0$, since subspaces H^* and J^* of $\mathfrak{U}$ contain non-zero scalar elements of $\mathfrak{U}$. Now let us assume that $\mathfrak{U}^{n-1}\subset\mathfrak{U}\mathfrak{k}+H^*J^*$.

Let $u\in\mathfrak{U}^n$ be an element of the form $u=\beta(X_1\cdots X_n)$, where $X_1, \cdots, X_n$ are elements of $\mathfrak{g}$. As β is a degree-preserving linear isomorphism of $S(\mathfrak{g})$ onto $\mathfrak{U}$, the fact $\mathfrak{U}^n\subset\mathfrak{U}\mathfrak{k}+H^*J^*$ follows if we prove that the above u belongs to the space $\mathfrak{U}\mathfrak{k}+H^*J^*$.

The element u is written in the form

$$u=\frac{1}{n!}\sum_{\sigma\in S_n}X_{\sigma(1)}\cdots X_{\sigma(n)}.$$

We shall prove $u\in\mathfrak{U}\mathfrak{k}+H^*J^*$ in case that (1) some of X_i's $(1\leq i\leq n)$ are contained in $\mathfrak{k}$ or that (2) $\{X_1, \cdots, X_n\}$ is a subset of $\mathfrak{p}$. Although (1) and (2) don't cover all the possible cases, these 2-cases are sufficient as one can see easily.

(1) We assume $X_p\in\mathfrak{k}$, and rewrite each term $X_{\sigma(1)}\cdots X_{\sigma(n)}$ in the expan-

(*) This remark is due to Mr. Sakane.

sion of u in the following way:

$$X_{\sigma(1)}\cdots X_{\sigma(n)}\equiv X_{\sigma(1)}X_{\sigma(2)}\cdots\hat{X}_p\cdots X_{\sigma(n)}X_p(\bmod\,\mathfrak{U}^{n-1})$$
$$\equiv X_{\sigma(1)}X_{\sigma(2)}\cdots\hat{X}_p\cdots X_{\sigma(n)}(X_p-\sigma_*(X_p))(\bmod\,\mathfrak{U}^{n-1}).$$

Since $X_{\sigma(1)}X_{\sigma(2)}\cdots\hat{X}_p\cdots X_{\sigma(n)}(X_p-\sigma_*(X_p))\in\mathfrak{U}\tilde{\mathfrak{k}}$ and $\mathfrak{U}^{n-1}\subset\mathfrak{U}\tilde{\mathfrak{k}}+H^*J^*$ by the inductive assumption, it follows that

$$X_{\sigma(1)}\cdots X_{\sigma(n)}\in\mathfrak{U}\tilde{\mathfrak{k}}+H^*J^*,$$

therefore $u\in\mathfrak{U}\tilde{\mathfrak{k}}+H^*J^*$.

(2) The element $X_1\cdots X_n\in S(\mathfrak{p})=H\otimes J$ can be decomposed into the form:

$$X_1\cdots X_n=\sum_{i=1}^{m}u_iv_i$$

we have

$$u=\sum_{i=1}^{m}\beta(u_iv_i)$$
$$\equiv\sum_{i=1}^{m}\beta(u_i)\beta(v_i)\qquad(\bmod\,\mathfrak{U}^{n-1}).$$

Since $\sum_{i=1}^{m}\beta(u_i)\beta(v_i)\in H^*J^*$ and $\mathfrak{U}^{n-1}\subset\mathfrak{U}\tilde{\mathfrak{k}}+H^*J^*$ by the inductive assumption, the element u is in $\mathfrak{U}\tilde{\mathfrak{k}}+H^*J^*$. Q.E.D.

PROPOSITION 10. *The projection $u\to u\cdot f_\lambda$ induces a K-homomorphism of H^* onto the $\tau(K)$-module $\mathfrak{U}\cdot f_\lambda$, that is,*

$$\mathfrak{U}\cdot f_\lambda=H^*\cdot f_\lambda\,.$$

PROOF. By Lemma 9, it is sufficient to prove that (1) $uf_\lambda=0$ if $u\in\tilde{\mathfrak{k}}$, (2) $J^*f_\lambda=\boldsymbol{C}\cdot f_\lambda$ and (3) $\boldsymbol{C}\subset H^*$.

Assertions (1) and (3) are obvious by the definitions of $\tilde{\mathfrak{k}}$ and H^*, so we shall prove the assertion (2). First we note that the subspace V of X^λ consisting of all elements $f\in X^\lambda$ such that $kf=\sigma(k)f$ for every $k\in K$ is one-dimensional, since each $f\in V$ is entirely determined by $f(e)$. Therefore

$$V=\{cf_\lambda;\ c\in\boldsymbol{C}\}.$$

Let u be an arbitrary element of J^*. Then for every $k\in K$,

$$k(uf_\lambda)=(ku)(kf_\lambda)=\sigma(k)\cdot uf_\lambda$$

since $u\in\mathfrak{U}^K$ and by the definition of f_λ.

Hence $$uf_\lambda \in V.$$

Therefore $$uf_\lambda = cf_\lambda \quad \text{for some} \quad c \in \boldsymbol{C}.$$

Thus the assertion (2) holds since $\boldsymbol{C} \subset J^*$. Q.E.D.

5. Irreducibility theorem

Let Γ be the set of all equivalence classes of irreducible K-modules which admit non-zero M-fixed vectors, and, for each $\gamma \in \Gamma$, let V_γ be a K-module in the class γ. If $\gamma \in \Gamma$, then γ' will denote the class of K-modules contragredient to V_γ. Let $l(\gamma) = \dim V_\gamma^M$ = the dimension of the subspace of V_γ consisting of all M-fixed vectors in V_γ. Then the multiplicity of γ in the K-module X^{λ_0} is equal to $l(\gamma)$ by Frobenius' reciprocity theorem. The results in Kostant-Rallis [6] and the same remark as in the proof of Lemma 8 lead us to the assertion that the K-module X^{λ_0} is isomorphic to H^*. Hence the $\tau(K)$-module X^λ is also isomorphic to the K-module H^*, and the multiplicities of γ in H^* and in the $\tau(K)$-module X^λ are both equal to $l(\gamma)$. Let H_γ^* denote the isotypic component of H^* of type γ, that is, H_γ^* is the sum of all K-submodules of H^* which are isomorphic to V_γ, and let X_γ^λ denote that of X^λ of type γ. Then H^* and X^λ decompose directly in the following form:

$$H^* = \bigoplus_{\gamma \in \Gamma} H_\gamma^*$$

$$X^\lambda = \bigoplus_{\gamma \in \Gamma} X_\gamma^\lambda$$

where both H_γ^* and X_γ^λ are isomorphic to a direct sum of $l(\gamma)$-copies of the K-module V_γ.

Our main purpose is to study the $\mathfrak{U}$-module structure of X^λ with the help of the $\tau(K)$-module structure of X^λ. Since the $(\tau(K), \mathfrak{U})$-submodule $\mathfrak{U}\cdot f_\lambda$ of X^λ is isomorphic to the $(K, \mathfrak{U})$-module $\mathfrak{U}/\mathfrak{L}_\lambda$, the multiplicity of γ in $\mathfrak{U}/\mathfrak{L}_\lambda$ is equal to or smaller than $l(\gamma)$ for each $\gamma \in \Gamma$. Moreover as the canonical projection $\mathfrak{U}/\mathfrak{L}_\lambda \to Z^\lambda = \mathfrak{U}/\mathfrak{L}_\lambda^{\max}$ is a surjective K-homomorphism, the multiplicity of γ in $\mathfrak{U}/\mathfrak{L}_\lambda^{\max}$ is not larger than that of $\mathfrak{U}/\mathfrak{L}_\lambda$. Therefore we have the following inequality:

$$\text{the multiplicity of } \gamma \text{ in } Z^\lambda \leqq l(\gamma) \text{ for each } \gamma \in \Gamma.$$

$$\mathfrak{U}/\mathfrak{L}_\lambda \cong \mathfrak{U}\cdot f_\lambda \hookrightarrow X^\lambda$$

$\downarrow$ canonical projection (surjective $(K, \mathfrak{U})$-module homomorphism)

$$Z_\lambda = \mathfrak{U}/\mathfrak{L}_\lambda^{\max}$$

Now consider the following diagram, where notations "$\cong$" and "$\hookrightarrow$" denote a $(K, \mathfrak{U})$-module isomorphism and a $(\tau(K), \mathfrak{U})$-module inclusion respectively. If the multiplicity of γ in Z^λ is equal to $l(\gamma)$ for every $\gamma \in \Gamma$, then every mapping in the above diagram must be a K-module isomorphism, which means in turn a $\mathfrak{U}$-module isomorphism since each of them is a $(K, \mathfrak{U})$-homomorphism. Hence the $\mathfrak{U}$-module X^λ is isomorphic to the irreducible $\mathfrak{U}$-module Z^λ, and so X^λ is an irreducible $\mathfrak{U}$-module.

Conversely, if the $\mathfrak{U}$-module X^λ is irreducible, every mapping in the above becomes a $\mathfrak{U}$-isomorphism, which is, at the same time, a K-module isomorphism. Therefore the multiplicity of γ in Z^λ is equal to $l(\gamma)$ for every $\gamma \in \Gamma$. Thus we have proved the following theorem:

THEOREM 11. *The $\mathfrak{U}$-module X^λ is irreducible if and only if the multiplicity of γ in Z^λ is equal to $l(\gamma)$ for every $\gamma \in \Gamma$.*

In the following we shall give formulae for the multiplicities of γ in $\mathfrak{U}\cdot f_\lambda$ and Z^λ, which are a key to determine the $\mathfrak{U}$-module structure of X^λ.

For $\gamma \in \Gamma$, we put

$$E_\gamma = \mathrm{Hom}_K(V_\gamma, H_\gamma^*)$$

$$= \text{the space of all } K\text{-homomorphisms of } V_\gamma \text{ into } H_\lambda^*$$

$$E_{\gamma'} = \mathrm{Hom}_K(V_{\gamma'}, H_{\gamma'}^*)$$

where γ' denotes the class of K-modules contragredient to V_γ. The K-module $V_{\gamma'}$ may be identified with the dual K-module (K, V'_γ) whose vector space is the dual space of V_γ and the K-action on which is contragredient to that on V_γ. Let $d(\gamma)$ denote the dimension of V_γ, which is equal to that of V'_γ. Let $\{v_j\}_{1\leqq j\leqq d(\gamma)}$ be a basis of V_γ, and $\{v'_j\}_{1\leqq j\leqq d(\gamma)}$ its dual basis. Define the bilinear mapping of $E_{\gamma'}\times E_\gamma$ to $\mathfrak{U}$ by

$$Z(\phi', \phi) = \sum_{j=1}^{d(\gamma)} \phi'(v'_j)\phi(v_j)$$

where $\phi' \in E_{\gamma'}$ and $\phi \in E_\gamma$.

LEMMA 12. (1) *The mapping Z does not depend on the choice of a basis $\{v_j\}_{1\leqq j\leqq d(\gamma)}$.*

(2) *Image of Z is contained in $\mathfrak{U}^K$.*

PROOF. The first assertion (1) is an easy exercise of the linear algebra. As the K-module V'_γ is contragredient to V_γ, $\{kv'_j\}_{1\leqq j\leqq d(\gamma)}$ forms the dual basis of $\{kv_j\}_{1\leqq j\leqq d(\gamma)}$ for each $k \in K$. And so, using (1), we have

$$\sum_{i=1}^{d(\gamma)} \phi'(kv'_j)\phi(kv_j) = \sum_{j=1}^{d(\gamma)} \phi'(v'_j)\phi(v_j)\,.$$

Therefore

$$
\begin{aligned}
kZ(\phi', \phi) &= \sum_{j=1}^{d(\gamma)} k\phi'(v_j') \cdot k\phi(v_j) \\
&= \sum_{j=1}^{d(\gamma)} \phi'(kv_j')\phi(kv_j) \\
&= \sum_{j=1}^{d(\gamma)} \phi'(v_j')\phi(v_j) \\
&= Z(\phi', \phi) .
\end{aligned}
$$

Q.E.D.

LEMMA 13. *Let $\gamma \in \Gamma$ and $\phi \in E_\gamma$ be fixed. Then $\phi(V_\gamma) \subset \mathfrak{L}_\lambda^{\max}$ holds if and only if $p^{Z(\phi', \phi)}(\mu) = 0$ for every $\phi' \in E_{\gamma'}$.*

PROOF. First we assume that $\phi(V_\gamma) \subset \mathfrak{L}_\lambda^{\max}$. Since $\mathfrak{L}_\lambda^{\max}$ is a left-ideal in $\mathfrak{U}$, $\phi(V_\gamma) \subset \mathfrak{L}_\lambda^{\max}$ implies that $Z(\phi', \phi) \subset \mathfrak{L}_\lambda^{\max}$ for every $\phi' \in E_{\gamma'}$. Hence by the definition of $\mathfrak{L}_\lambda^{\max}$ and by Lemma 12 (2), it follows that $p^{Z(\phi', \phi)}(\mu) = 0$ for every $\phi' \in E_{\gamma'}$.

Next we assume that $\phi(V_\gamma) \not\subset \mathfrak{L}_\lambda^{\max}$. Then any non-zero element $u \in \phi(V_\gamma)$ is not contained in $\mathfrak{L}_\gamma^{\max}$ since $\phi(V_\gamma)$ is K-irreducible and $\mathfrak{L}_\lambda^{\max}$ is K-stable. By definition of $\mathfrak{L}_\lambda^{\max}$, there exists $v \in \mathfrak{U}$ which satisfies $p^{(vu)_0}(\mu) \neq 0$. In the following, we use some notations which will be introduced in 6. Since the projection $H^* \rightarrow \mathfrak{U}/\mathfrak{L}_{-\bar{\lambda}+2\rho}^{\max}$ is surjective and the space H^* is stable under s-operation, $\mathfrak{U}$ can be decomposed in the form of $\mathfrak{U} = H^* + (\mathfrak{L}_{-\bar{\lambda}+2\rho}^{\max})^s$. Using Lemma 19 in 7, we obtain

$$
p^{(v'u)_0}(\mu) = \overline{p^{(v'u)_0^s}(-\bar{\mu}+2\rho)} = \overline{p^{(u^s v'^s)_0}(-\bar{\mu}+2\rho)} = 0,
$$

for every $v' \in (\mathfrak{L}_{-\bar{\lambda}+2\rho}^{\max})^s$. So we can assume that the above $v \in \mathfrak{U}$ belongs to the space H^*. The γ'-component $v_{\gamma'}$ of v can be written as follows:

$$
v_{\gamma'} = d(\gamma') \int_K \overline{\chi_{\gamma'}(k)} \cdot kv \, dk = d(\gamma) \int_K \overline{\chi_\gamma(k^{-1})} \cdot kv \, dk,
$$

where χ_γ(resp. $\chi_{\gamma'}$) denotes the character of γ(resp. γ'). Then we have

$$
\begin{aligned}
p^{(v_{\gamma'} u)_0}(\mu) &= p^{d(\gamma) \int_K \overline{\chi_\gamma(k^{-1})} \cdot [kv \cdot u]_0 dk}(\mu) \\
&= p^{[v \cdot d(\gamma) \int_K \overline{\chi_\gamma(k^{-1})} k^{-1} u dk]_0}(\mu) \\
&= p^{[vu]_0}(\mu) \neq 0.
\end{aligned}
$$

This implies that $p^{[H_{\gamma'}^* \cdot \phi(V_\gamma)]_0}(\mu) \neq \{0\}$, and one can find $\phi' \in E_{\gamma'}$ which satisfies $p^{[\phi'(V_{\gamma'})\phi(V_\gamma)]_0}(\mu) \neq 0$. Now we define an endomorphism f of $V_{\gamma'}$ as follows:

$$
\langle f(x), y \rangle = p^{[\phi'(x)\phi(y)]_0}(\mu) \qquad \text{for every} \quad x \in V_{\gamma'} \text{ and } y \in V_\gamma.
$$

It is easy to prove that f is a K-homomorphism and the image of f is not

$\{0\}$. Since $V_{\gamma'}$ is K-irreducible, there exists $\alpha \in \boldsymbol{C}-\{0\}$ satisfying $f(x)=\alpha x$ for every $x \in V_{\gamma'}$. Thus we obtain

$$p^{Z(\phi',\phi)}(\mu)=\sum_{i=1}^{d(\gamma)} p^{[\phi'(v_i')\phi(v_i)]_0}(\mu)=\sum_{i=1}^{d(\gamma)} \alpha < v_i', v_i >$$

$$=\alpha\cdot d(\gamma)\neq 0. \qquad \text{Q.E.D.}$$

Both dimensions of E_γ and $E_{\gamma'}$ are equal to $l(\gamma)$. Let $\{\phi_j\}_{1\leqq j\leqq l(\gamma)}$ and $\{\phi_j'\}_{1\leqq j\leqq l(\gamma)}$ be bases of E_γ and $E_{\gamma'}$ respectively. We set $P_{ij}^\gamma=p^{\phi_j(v_i)}$ for $1\leqq i\leqq d(\gamma)$, $1\leqq j\leqq l(\gamma)$ and $R_{ij}^\gamma=p^{Z(\phi_i',\phi_j)}$ for $1\leqq i\leqq l(\gamma)$, $1\leqq j\leqq l(\gamma)$. We define matrices $P^\gamma(\lambda)$ and $R^\gamma(\lambda)$ by $P^\gamma(\lambda)=(P_{ij}^\gamma(\mu))$ and $R^\gamma(\lambda)=(R_{ij}^\gamma(\mu))$. We note that, for a different choice of bases of E_γ and $E_{\gamma'}$, the matrices $P^\gamma(\lambda)$ and $R^\gamma(\lambda)$ are replaced by $P^\gamma(\lambda)B$ and $AR^\gamma(\lambda)B$ where A and B are invertible $l(\gamma)\times l(\gamma)$-matrices with coefficients in $\boldsymbol{C}$. Thus ranks of these matrices are determined independent of the choice of bases of E_γ and $E_{\gamma'}$.

THEOREM 14. *For each $\gamma \in \Gamma$, we have*
(1) *the multiplicity of γ in $\mathfrak{U}\cdot f_\lambda=$ rank of $P^\gamma(\gamma)$*
(2) *the multiplicity of γ in $Z^\lambda=$ rank of $R^\gamma(\lambda)$.*

PROOF. (1) We put

$$s=\text{the multiplicity of } \gamma \text{ in } \mathfrak{U}\cdot f_\lambda\,.$$

As the rank of the matrix $P^\gamma(\lambda)$ is independent of the choice of a basis of E_γ, we select the basis $\{\phi_1, \cdots, \phi_{l(\gamma)}\}$ of E_γ such that K-submodules $\phi_1(V_\gamma), \cdots, \phi_s(V_\gamma)$ are mapped isomorphically into $\mathfrak{U}\cdot f_\lambda$ and $\phi_{s+1}(V_\gamma), \cdots, \phi_{l(\gamma)}(V_\gamma)$ fall into the null-space under the projection $H^*\to\mathfrak{U}\cdot f_\lambda$. With such a selection of $\{\phi_j\}_{1\leqq j\leqq l(\gamma)}$, one has

$$\phi_j(v_i)f_\lambda=0 \qquad \text{if} \quad 1\leqq i\leqq d(\gamma) \text{ and } s+1\leqq j\leqq l(\gamma).$$

Using Lemma 1 and Lemma 2, one has

$$p^{\phi_j(v_i)}(\mu)=0 \qquad \text{if} \quad 1\leqq i\leqq d(\gamma) \text{ and } s+1\leqq j\leqq l(\gamma).$$

Hence the rank of $P^\gamma(\lambda)\leqq s$.

To prove the equality, we shall prove that vectors $\{x_1, \cdots, x_s\}$ are linearly independent, where x_j's are defined by

$$x_j=\begin{pmatrix} p^{\phi_j(v_1)}(\mu) \\ \vdots \\ p^{\phi_j(v_{d(\gamma)})}(\mu) \end{pmatrix} \qquad \text{for} \quad 1\leqq j\leqq s.$$

Let $a_1, \cdots, a_s$ be such scalars that $\sum_{j=1}^s a_j x_j=0$. If we set $\phi=\sum_{i=1}^s a_i\phi_i \in E_\gamma$, then we have $p^{\phi(v_i)}(\mu)=0$ for every i. Using Lemma 2 (2), we have

$$\phi(v)f_\lambda=0 \qquad \text{for every} \quad v \in V_\gamma\,.$$

Hence
$$\sum_{j=1}^{s} a_j\phi_j(v)f_\lambda=0 \qquad \text{for every} \quad v \in V_\gamma\,.$$

On the other hand, $\{\phi_j(v_i)f_\lambda;\ 1\leqq i\leqq d(\gamma),\ 1\leqq j\leqq s\}$ form a basis of $\mathfrak{U}\cdot f_\lambda$ by the choice of $\{\phi_j\}$. Therefore we have $a_1, \cdots, a_s=0$.

(2) Let t be the multiplicity of γ in Z^λ. As the rank of the matrix $R^\gamma(\lambda)$ is independent of the choice of a basis of E_γ, we select the basis $\{\phi_1, \cdots, \phi_{l(\gamma)}\}$ of E_γ such that K-modules $\phi_1(V_\gamma), \cdots, \phi_t(V_\gamma)$ are mapped isomorphically into Z^λ and $\phi_{t+1}(V_\gamma), \cdots, \phi_{l(\gamma)}(V_\gamma)$ fall into the null-space under the projection $H^*\to Z^\lambda$. With such a selection of $\{\phi_j\}$, one has $\phi_j(V_\gamma)\subset\mathfrak{L}_\lambda^{\max}$ if $t+1\leqq j\leqq l(\gamma)$. By Lemma 13, one has $R_{ij}^\gamma(\mu)=0$ for $1\leqq i\leqq l(\gamma)$ and $t+1\leqq j\leqq l(\gamma)$. Hence

$$\text{the rank of} \quad R^\gamma(\lambda)\leqq t.$$

To prove the equality, we shall prove that vectors $\{y_1, \cdots, y_t\}$ are linearly independent, where y_j's are defined by

$$y_j=\begin{pmatrix} R_{1j}^\gamma(\mu) \\ \vdots \\ R_{l(\gamma)j}^\gamma(\mu)\end{pmatrix}=\begin{pmatrix} p^{Z(\phi_1',\ \phi_j)}(\mu) \\ \vdots \\ p^{Z(\phi_{l(\gamma)}',\phi_j)}(\mu)\end{pmatrix}$$

for $1\leqq j\leqq t$. Let $a_1, \cdots, a_t$ be such scalars that $\sum_{j=1}^{t} a_j y_j=0$. If we set $\phi=\sum_{j=1}^{t} a_j\phi_j \in E_\gamma$, then $p^{Z(\phi_i',\phi)}(\mu)=0$ for every i. Hence $\phi(V_\gamma)\subset\mathfrak{L}_\lambda^{\max}$, by using Lemma 13. This means that the image of $\phi(V_\gamma)$ is equal to zero under the projection $H^*\to Z^\lambda$. On the other hand, each K-module $\phi_j(V_\gamma)$ is K-isomorphic to its image in Z^λ for $1\leqq j\leqq t$. Therefore we have $a_1, \cdots, a_t=0$.

Q.E.D.

Combining Theorem 11 and Theorem 14, we have the following:

COROLLARY 15. 1) *The* $\mathfrak{U}$*-module* X^λ *is irreducible if and only if the matrix* $R^\gamma(\lambda)$ *is regular for every* $\gamma \in \Gamma$.

2) *The* $\mathfrak{U}$*-module* $\mathfrak{U}\cdot f_\lambda$ *is irreducible if and only if the rank of* $P^\gamma(\lambda)$ *is equal to that of* $R^\gamma(\lambda)$ *for every* $\gamma \in \Gamma$.

6. A formula for the matrix $R^\gamma(\lambda)$

The matrix $R^\gamma(\lambda)$ depends on the choice of bases of E_γ and $E_{\gamma'}$, and the different choice of their bases changes $R^\gamma(\lambda)$ into a matrix of the form $AR^\gamma(\lambda)B$, where A and B are non-singular $l(\gamma)\times l(\gamma)$-matrices with coefficients in $\boldsymbol{C}$.

Let $\{,\}$ be a K-invariant positive definite Hermitian inner product on V_γ, which is $\boldsymbol{C}$-linear on the first component. we fix an orthonormal basis $\{v_1, \cdots, v_{d(\gamma)}\}$ of V_γ with respect to $\{,\}$, and let $\{v_1', \cdots, v_{d(\gamma)}'\}$ be its dual basis of V_γ'. Then we can define the conjugate-linear $\boldsymbol{R}$-isomorphism T_γ of V_γ' onto V_γ by

$$T_\gamma\Big(\sum_{i=1}^{d(\gamma)} a_i v_i'\Big)=\sum_{i=1}^{d(\gamma)} \bar{a}_i v_i$$

where $a_1, \cdots, a_{d(\gamma)} \in \boldsymbol{C}$. We can see easily that the mapping T_γ has the following property:

Lemma 16. 1) *$v'(v)=\{v, T_\gamma v'\}$ for every $v \in V_\gamma$ and $v' \in V_{\gamma'}$*

2) *The mapping T_γ commutes with K-actions on V_γ and $V_{\gamma'}$, i.e., the following diagram is commutative:*

$$\begin{array}{ccc} V_{\gamma'} & \xrightarrow{\textit{K-action}} & V_{\gamma'} \\ \downarrow T_\gamma & & \downarrow T_\gamma \\ V_\gamma & \xrightarrow{\textit{K-action}} & V_\gamma \end{array}$$

We define a $\boldsymbol{R}$-linear isomorphism $u \to u^s$ of $\mathfrak{U}$ by the conditions:

(1) $X^s=-X$ if $X \in \mathfrak{g}_0$

(2) $(uv)^s=v^s u^s$ for $u, v \in \mathfrak{U}$

(3) $(\alpha u)^s=\bar{\alpha} u^s$ for $\alpha \in \boldsymbol{C}$ and $u \in \mathfrak{U}$.

Then $u \to u^s$ is a conjugate-linear anti-automorphism of the universal enveloping algebra $\mathfrak{U}$. As one can see easily, this mapping has the property:

$$(ku)^s=ku^s \qquad \text{for every} \quad k \in K \text{ and } u \in \mathfrak{U}$$

$$(H^*)^s=H^*.$$

Proposition 17. *For every $\gamma \in \Gamma$, we have*

$$(H_\gamma^*)^s=H_{\gamma'}^* .$$

Proof. Let ϕ be an arbitrary element of E_γ, and we consider the following commutative diagram:

$$\begin{array}{ccccccc} V_\gamma' & \xrightarrow{T_\gamma} & V_\gamma & \xrightarrow{\phi} & H_\gamma^* & \xrightarrow{s} & (H_\gamma^*)^s \\ \downarrow K\text{-action} & & \downarrow K\text{-action} & & \downarrow K\text{-action} & & \downarrow K\text{-action} \\ V_\gamma' & \xrightarrow{T_\gamma} & V_\gamma & \xrightarrow{\phi} & H_\gamma^* & \xrightarrow{s} & (H_\gamma^*)^s \end{array}$$

We set $\phi^s=s\circ\phi\circ T_\gamma$. Then ϕ^s is a $\boldsymbol{C}$-linear K-homomorphism of V_γ' into $(H_\gamma^*)^s$, i.e., $\phi^s \in \mathrm{Hom}_K(V_\gamma', (H_\gamma^*)^s)$. And so the K-module $(H_\gamma^*)^s$ contains K-submodules which are isomorphic to V'_γ.

Now let $\{\phi_1, \cdots, \phi_{l(\gamma)}\}$ be a basis of E_γ. Then we have $l(\gamma)$-numbers of K-homomorphisms $\phi_1^s, \cdots, \phi_{l(\gamma)}^s$ of V'_γ into H^*, which are linearly independent. As the images of $\phi_1^s, \cdots, \phi_{l(\gamma)}^s$ span the K-submodule $(H_\gamma^*)^s$ of H^*, we have

$$(H_\gamma^*)^s \subset H_{\gamma'}^* .$$

As we have already explained in the previous section, $H_{\gamma'}$ is a sum of $l(\gamma)$-copies of K-submodules of H^* of type γ, and so equality holds, i.e., $(H_\gamma^*)^s = H_{\gamma'}^*$. Q.E.D.

Note. Let $\{v_1, \cdots, v_{d(\gamma)}\}$ be an orthonormal basis of V_γ with respect to $\{,\}$, and let $\{v'_1, \cdots, v'_{d(\gamma)}\}$ be its dual basis of V'_γ. Then for every $\phi \in E_\gamma$, we have

$$\phi^s(v'_j) = \phi(v_j)^s$$

since $\phi^s(v'_j) = (s \circ \phi \circ T_\gamma)(v'_j) = (s \circ \phi)(v_j) = \phi(v_j)^s$.

If $\{\phi_1, \cdots, \phi_{l(\gamma)}\}$ is a basis of E_γ, then $\{\phi_1^s, \cdots, \phi_{l(\gamma)}^s\}$ is a basis of $E_{\gamma'}$. Henceforward we shall make it a rule to use these bases of E_γ and $E_{\gamma'}$ when we construct the matrix $R^\gamma(\lambda)$, and further make it a rule to use orthonormal basis of V_γ with respect to $\{,\}$ when we construct the matrix $P^\gamma(\lambda)$. With respect to these bases, the matrix $R^\gamma(\lambda)$ is given by

$$R^\gamma(\lambda) = (p^{Z(\phi_i^s, \phi_j)}(\mu)).$$

Let us define $-\bar{\mu}$ by the following conditions:

(1) $(-\bar{\mu})(H) = -\overline{\mu(H)}$ for every $H \in \mathfrak{a}_0$

(2) $-\bar{\mu}$ is a $\boldsymbol{C}$-valued $\boldsymbol{C}$-linear function on $\mathfrak{a}$, i.e., $-\bar{\mu} \in \mathfrak{a}'$.

The pair $-\bar{\lambda}+2\rho = (\varepsilon, -\bar{\mu}+2\rho)$ of a character ε of M and an element $-\bar{\mu}+2\rho \in \mathfrak{a}'$ defines a character of B, and by using $-\bar{\lambda}+2\rho$, we define $\pi^{-\bar{\lambda}+2\rho}$, $\pi_*^{-\bar{\lambda}+2\rho}$, $X^{(-\bar{\lambda}+2\rho)}$, $X^{-\bar{\lambda}+2\rho}$, in the same way as we have defined ones about λ. We can find an element $f \in X^{-\bar{\lambda}+2\rho}$ such that $f|K = f_\lambda|K$, which we denote by $f_{-\bar{\lambda}+2\rho}$. Obviously one has

$$\pi^{-\bar{\lambda}+2\rho}(k) f_{-\bar{\lambda}+2\rho} = \sigma(k) f_{-\bar{\lambda}+2\rho} \qquad \text{for every} \quad k \in K .$$

We can choose f_λ and $f_{-\bar{\lambda}+2\rho}$ such that $f_\lambda(e) = f_{-\bar{\lambda}+2\rho}(e) = 1$, and from now on, we fix f_λ and $f_{-\bar{\lambda}+2\rho}$ as such.

For each $f \in X^\lambda$ and $g \in X^{-\bar{\lambda}+2\rho}$, we put

$$(f, g) = \int_K f(k)\overline{g(k)}\, dk .$$

Then it is well known that $(,)$ is a non-singular pairing of the $\mathfrak{U}$-modules X^λ and $X^{-\bar{\lambda}+2\rho}$, i.e.,

$$(\pi_*^\lambda(u) f, g) = (f, \pi_*^{-\bar{\lambda}+2\rho}(u^s) g)$$

for every $u \in \mathfrak{U}$, $f \in X^{\lambda}$ and $g \in X^{-\bar{\lambda}+2\rho}$.

Note. $(f_{\lambda}, f_{-\bar{\lambda}+2\rho})=1$.

The relation between matrices $R^{\gamma}(\lambda)$ and $P^{\gamma}(\lambda)$ is established by the following theorem.

THEOREM 18. $R^{\gamma}(\lambda)=P^{\gamma}(-\bar{\lambda}+2\rho)^{*}P^{\gamma}(\lambda)$
*for every $\gamma \in \Gamma$, where * denotes the Hermitian conjugate of the matrix.*

PROOF. Let $\{v_j\}_{1\leq j\leq d(\gamma)}$ be an orthonormal basis of V_{γ} with respect to $\{,\}$. The K-action on V_{γ} defines a matrix representation T of K by

$$(kv_1, \cdots, kv_{d(\gamma)})=(v_1, \cdots, v_{d(\gamma)})\cdot T(k)$$

where $T(k)$ is a unitary matrix for every $k \in K$, i.e.,

$$T(k) \in U(d(\gamma))$$

since $\{,\}$ is K-invariant. The well-known orthogonality relation tells us that

$$\int_K T_{ij}(kk')\,T_{mn}(k'^{-1})\,dk'=\frac{\delta_{jm}}{d(\gamma)}T_{in}(k).$$

By Lemma 1 and Lemma 2, we have

$$\begin{aligned}
R^{\gamma}_{ij}(\lambda)&=p^{Z(\phi_i^s,\phi_j)}(\mu)(f_{\lambda}, f_{-\bar{\lambda}+2\rho})\\
&=(\pi_*^{\lambda}(Z(\phi_i^s, \phi_j))f_{\lambda}, f_{-\bar{\lambda}+2\rho})\\
&=(\pi_*^{\lambda}(\sum_{t=1}^{d(\gamma)}\phi_i(v_t)^s\phi_j(v_t))f_{\lambda}, f_{-\bar{\lambda}+2\rho})\\
&=\sum_{t=1}^{d(\gamma)}(\pi_*^{\lambda}(\phi_j(v_t))f_{\lambda}, \pi_*^{-\bar{\lambda}+2\rho}(\phi_j(v_t))f_{-\bar{\lambda}+2\rho})\\
&=\sum_{t=1}^{d(\gamma)}\int_K[\pi_*^{\lambda}(\phi_j(v_t))f_{\lambda}](k^{-1})\times\overline{[\pi_*^{-\bar{\lambda}+2\rho}(\phi_i(v_t))f_{-\bar{\lambda}+2\rho}](k^{-1})}\,dk\\
&=\sum_{t=1}^{d(\gamma)}\int_K[\pi_*^{\lambda}(k\phi_j(v_t))f_{\lambda}](e)\times\overline{[\pi_*^{-\bar{\lambda}+2\rho}(k\phi_i(v_t))f_{-\bar{\lambda}+2\rho}](e)}\,dk\\
&=\sum_{t,m,n=1}^{d(\gamma)}\int_K T_{mt}(k)[\pi_*^{\lambda}(\phi_j(v_m))f_{\lambda}](e)\times\overline{T_{nt}(k)[\pi_*^{-\bar{\lambda}+2\rho}(\phi_i(v_n))f_{-\bar{\lambda}+2\rho}](e)}\,dk\\
&=\sum_{t,m,n=1}^{d(\gamma)}\int_K T_{mt}(k)\,T_{tn}(k^{-1})\,dk\cdot p^{\phi_j(v_m)}(\mu)\cdot\overline{p^{\phi_i(v_n)}(-\bar{\mu}+2\rho)}\,dk\\
&=\sum_{t,m,n=1}^{d(\gamma)}\frac{\delta_{nm}}{d(\gamma)}T_{tt}(e)\cdot P^{\gamma}_{mj}(\lambda)\cdot\overline{P^{\gamma}_{ni}(-\bar{\lambda}+2\rho)}
\end{aligned}$$

$$=\sum_{m=1}^{d(\lambda)} P^{\gamma}_{mj}(\lambda)\cdot\overline{P^{\gamma}_{mi}(-\bar{\lambda}+2\rho)}$$

$$=[P^{\gamma}(-\bar{\lambda}+2\rho)^{*}\cdot P^{\gamma}(\lambda)]_{ij}\,. \qquad \text{Q.E.D.}$$

7. Hermitian structures

In this section we shall consider under which conditions the $(K, \mathfrak{U})$-module Z^{λ} admits non-degenerate or positive-definite invariant Hermitian structures.

Definition. A non-degenerate Hermitian form $< , >$ on Z^{λ} is called *invariant* if (1) $<uz, w>=<z, u^{s}w>$ for every $u \in \mathfrak{U}$ and $z, w \in Z^{\lambda}$ and if (2) $<kz, kw>=<z, w>$ for every $k \in K$ and $z, w \in Z^{\lambda}$.

Lemma 19. *For every* $u \in \mathfrak{U}^{K}$, *we have*

(1) $$p^{u}(\mu)=\overline{p^{u^{s}}(-\bar{\mu}+2\rho)}$$

(2) $$p_{u}(-\mu)=\overline{p_{u^{s}}(\bar{\mu}-2\rho)}$$

Proof. (1) Let (,) be the $\mathfrak{U}$-invariant pairing of X^{λ} and $X^{-\bar{\lambda}+2\rho}$ as is defined in the previous section. Then for every $u \in \mathfrak{U}^{K}$, we have

$$\begin{aligned} p^{u}(\mu) &= p^{u}(\mu)(f_{\lambda}, f_{-\bar{\lambda}+2\rho}) \\ &= (\pi_{*}^{\lambda}(u)f_{\lambda}, f_{-\bar{\lambda}+2\rho}) \\ &= (f_{\lambda}, \pi_{*}^{-\bar{\lambda}+2\rho}(u^{s})f_{-\bar{\lambda}+2\rho}) \\ &= \overline{p^{u^{s}}(-\bar{\mu}+2\rho)}(f_{\lambda}, f_{-\bar{\lambda}+2\rho}) \\ &= \overline{p^{u^{s}}(-\bar{\mu}+2\rho)}. \end{aligned}$$

Thus we have proved (1). The assertion (2) follows immediately from (1) and $p^{u}(\mu)=p_{u^{t}}(-\mu)$. Q.E.D.

Proposition 20. *The following conditions are equivalent:*

(1) Z^{λ} *admits a non-degenerate invariant Hermitian structures*

(2) $p^{u}(\mu)=\overline{p^{u^{s}}(\mu)}$ *for every* $u \in \mathfrak{U}^{K}$

(3) $p_{u}(-\mu)=p_{u}(\bar{\mu}-2\rho)$ *for every* $u \in \mathfrak{U}^{K}$.

Proof. Let ψ be the image of $1 \in \mathfrak{U}$ under the projection $\mathfrak{U}\rightarrow Z^{\lambda}=\mathfrak{U}/\mathfrak{L}_{\lambda}^{\max}$.

[Proof of (1)$\Rightarrow$(2)] First we shall prove $<\psi, \psi>\neq 0$. For every $u \in \mathfrak{U}$, we have

$$<u_{0}\psi, \psi>=<\int_{K} ku\,dk\cdot\psi, \psi>=\int_{K}<(ku)\psi, \psi>dk$$

$$=\int_K <k(u\psi),\ \psi> dk$$

$$=\int_K <u\psi,\ k^{-1}\psi> dk$$

$$=\int_K <u\psi,\ \psi> dk = <u\psi,\ \psi>.$$

On the other hand, $u_0\psi = p^{u_0}(\mu)\psi$ since $u_0 - p^{u_0}(\mu) \in \mathfrak{C}_\lambda \subset \mathfrak{L}_\lambda^{\max}$. Thus we have

$$<u\psi,\ \psi> = p^{u_0}(\mu) <\psi,\ \psi> \qquad \text{for every} \quad u \in \mathfrak{U}\,.$$

Now assume $<\psi, \psi> = 0$. Then we have $<u\psi, \psi> = 0$ for every $u \in \mathfrak{U}$, which contradicts the non-degeneracy of Hermitian form $<\ ,\ >$. Thus $<\psi, \psi> \neq 0$. For every $u \in \mathfrak{U}^K$, we have

$$p^u(\mu) <\psi,\ \psi> = <u\psi,\ \psi> = <\psi,\ u^s\psi>$$
$$= \overline{p^{u^s}(\mu)} <\psi,\ \psi>.$$

Therefore $p^u(\mu) = \overline{p^{u^s}(\mu)}$ for every $u \in \mathfrak{U}^K$.

[Proof of (2)$\Rightarrow$(1)] First we shall show that $p^{(v^s u)_0}(\mu)$ is determined only by $u\psi$ and $v\psi$.

(i) "If $u\psi = u'\psi$, then $p^{(v^s u)_0}(\mu) = p^{(v^s u')_0}(\mu)$ for every $v \in \mathfrak{U}$." In fact, the condition $u\psi = u'\psi$ means that $u - u' \in \mathfrak{L}_\lambda^{\max}$. By the definition of $\mathfrak{L}_\lambda^{\max}$, we have

$$p^{[v^s(u-u')]_0}(\mu) = 0 \qquad \text{for every} \quad v \in \mathfrak{U}\,.$$

Therefore $p^{(v^s u)_0}(\mu) = p^{(v^s u')_0}(\mu)$ for every $v \in \mathfrak{U}$.

(ii) "$p^{(v^s u)_0}(\mu) = \overline{p^{(u^s v)_0}(\mu)}$ for every $u, v \in \mathfrak{U}$." In fact, by the condition (2), it follows that

$$\overline{p^{(u^s v)_0}(\mu)} = p^{(u^s v)_0^s}(\mu)$$
$$= p^{(v^s u)_0}(\mu).$$

(iii) "If $v\psi = v'\psi$, then $p^{(v'^s u)_0}(\mu) = p^{(v^s u)_0}(\mu)$ for every $u \in \mathfrak{U}$." By using (i) and (ii) we have

$$p^{(v^s u)_0}(\mu) = \overline{p^{(u^s v)_0}(\mu)} = \overline{p^{(u^s v')_0}(\mu)}$$
$$= p^{(v'^s u)_0}(\mu).$$

Assertions (i) and (iii) tell us that $p^{(v^s u)_0}(\mu)$ is determined just by $u\psi$ and $v\psi$, and so we can define $<u\psi, v\psi> \in \boldsymbol{C}$ by

$$<u\psi,\ v\psi> = p^{(v^s u)_0}(\mu)\,.$$

The assertion (ii) shows that $<\ ,\ >$ is a Hermitian form on Z^{λ}.

(iv) "$<\ ,\ >$ is invariant."

For each $u, v, w \in \mathfrak{U}$ and $k \in K$, we have

$$<wu\psi, v\psi> = p^{(v^s wu)_0}(\mu) = p^{((w^s v)^s u)_0}(\mu)$$
$$= <u\psi, w^s v\psi>$$
$$<ku\psi, kv\psi> = p^{[(kv^s)(ku)]_0}(\mu) = p^{[kv^s \cdot ku]_0}(\mu)$$
$$= p^{(v^s u)_0}(\mu)$$
$$= <u\psi, v\psi>.$$

(v) "$<\ ,\ >$ is non-degenerate."

Let $u \in \mathfrak{U}$ be an element such that $<u\psi, v\psi>=0$ for every $v \in \mathfrak{U}$. Then by the definition of $<\ ,\ >$, we have

$$p^{(v^s u)_0}(\mu)=0 \qquad \text{for every} \quad v \in \mathfrak{U}\,.$$

Hence $u \in \mathfrak{L}_{\lambda}^{\max}$. Thus we have $u\psi=0$, which proves the assertion (v).

[Proof of (2)$\Leftrightarrow$(3)] By using Lemma 19, the condition (2) is equivalent to $p^u(\mu)=p^u(-\bar{\mu}+2\rho)$ for every $u \in \mathfrak{U}$, which is again equivalent to $p_u(-\mu)=p_u(\bar{\mu}-2\rho)$. Q.E.D.

Note. When there exists a non-degenerate invariant Hermitian structure $<\ ,\ >$ on Z^{λ}, the isotypic components Z_{γ}^{λ} and Z_{δ}^{λ} are orthogonal to each other with respect to $<\ ,\ >$ if $\gamma \neq \delta$. In fact, χ_{γ} denotes the character of γ, then we have for every $z \in Z_{\gamma}^{\lambda}$ and $w \in Z_{\delta}^{\lambda}$,

$$<z, w> = <d(\gamma)\int_K \overline{\chi_{\gamma}(k)} kz\,dk, w>$$
$$= d(\gamma)<z, \int_K \chi_{\gamma}(k)k^{-1}w\,dk>$$
$$= d(\gamma)<z, \int_K \overline{\chi_{\gamma}(k)} kw\,dk>$$
$$= 0\,.$$

Note. As one can see easily from the proof of Proposition 20, a non-degenerate invariant Hermitian structure on Z^{λ} is, if it exists, uniquely determined up to scalar multiples.

REMARK 1. Let W denote the Weyl group of the Riemannian symmetric pair (G, K). If ε is the trivial character of M and $f_{\lambda}|K=1$, the polynomial function p_u on $\mathfrak{a}$ has the following property: for $\nu, \nu' \in \mathfrak{a}'$, $\nu+\rho$ and $\nu'+\rho$ are

W-conjugate if and only if $p_u(\nu)=p_u(\nu')$ for every $u \in \mathfrak{U}^K$. (cf. Helgason[2]). Therefore, one can state that "when ε is the trivial character of M and $f_\lambda | K = 1$, the $(K, \mathfrak{U})$-module Z^λ admits a non-degenerate invariant Hermitian structure if and only if $\mu-\rho$ and $-(\bar{\mu}-\rho)$ are W-conjugate." This theorem is given explicitly in Kostant [4].

REMARK 2. The author has the conjecture that conditions in Proposition 20 are equivalent to the condition that "$\mu-\rho$ and $-(\bar{\mu}-\rho)$ are W-conjugate". In case of real 2×2 unimodular group, this conjecture can be shown true.

THEOREM 21. *Assume that λ satisfies conditions of Proposition 20. Then*
(1) *$R^\gamma(\lambda)$ is a Hermitian matrix for every $\gamma \in \Gamma$.*
(2) *The Hermitian structure on Z_λ is positive definite if and only if the matrix $R^\gamma(\lambda)$ is positive semi-definite for every $\gamma \in \Gamma$.*

PROOF. (1) We select bases of V_γ, $V_{\gamma'}$, E_γ and $E_{\gamma'}$ as in the previous section. Then we have,

$$\begin{aligned} Z(\phi_j^s, \phi_i)^s &= \Big[\sum_{k=1}^{d(\gamma)} \phi_j^s(v_k')\phi_i(v_k)\Big]^s \\ &= \Big[\sum_{k=1}^{d(\gamma)} \phi_j(v_k)^s \phi_i(v_k)\Big]^s \\ &= \sum_{k=1}^{d(\gamma)} \phi_i(v_k)^s \phi_j(v_k) \\ &= \sum_{k=1}^{d(\gamma)} \phi_i^s(v_k')\phi_j(v_k) \\ &= Z(\phi_i^s, \phi_j) \end{aligned}$$

Therefore, by Proposition 20, we have

$$\begin{aligned} \overline{R_{ji}^\gamma(\lambda)} &= \overline{p^{Z(\phi_j^s,\phi_i)}(\mu)} = \overline{p^{Z(\phi_i^s,\phi_j)^s}(\mu)} \\ &= p^{Z(\phi_i^s,\phi_j)}(\mu) = R_{ij}^\gamma(\lambda) \end{aligned}$$

(2) Suppose that the Hermitian structure on Z_λ is positive definite. It is enough to prove that the inequality $\sum_{i,j=1}^{l(\gamma)} \bar{z}_i R_{ij}^\gamma(\lambda) z_j \geqq 0$ holds for every $(z_1, \cdots, z_{l(\gamma)}) \in \boldsymbol{C}^{l(\gamma)}$. We put $u_k = \sum_{j=1}^{l(\gamma)} z_j \phi_j(v_k)$ for each $1 \leqq k \leqq d(\gamma)$. Then we have

$$\begin{aligned} \sum_{i,j=1}^{l(\gamma)} \bar{z}_i R_{ij}^\gamma(\lambda) z_j &= \sum_{i,j=1}^{l(\gamma)} \bar{z}_i p^{Z(\phi_i^s,\phi_j)}(\mu) z_j \\ &= \sum_{k=1}^{d(\gamma)} \sum_{i,j=1}^{l(\gamma)} \bar{z}_i p^{[\phi_i(v_k)^s \phi_j(v_k)]_0}(\mu) z_j \end{aligned}$$

$$=\sum_{k=1}^{d(\gamma)} p^{(u_k^* u_k)_0}(\mu)$$

$$=\sum_{k=1}^{d(\gamma)} <u_k\psi,\ u_k\psi> \geqq 0$$

where $\psi \in Z^\lambda$ is the image of $1 \in U$ under the projection $\mathfrak{U} \to Z^\lambda$.

For each $\gamma \in \Gamma$ the Hermitian form $<\ ,\ >$ may be regarded as a K-invarinat non-degenerate Hermitian structure on Z^λ_γ by Note to Proposition 20. Since K is compact, we can find K-submodules Z_1 and Z_2 of Z^λ_γ such that (i) $Z^\lambda_\gamma = Z_1 \oplus Z_2$ (direct sum), (ii) $<\ ,\ >$ is negative definite on Z_1 and positive definite on Z_2 and (iii) $Z_1 \perp Z_2$ with respect to $<\ ,\ >$. We choose a basis $\{\phi_1, \cdots, \phi_{l(\gamma)}\}$ of E_γ such that (a) $\phi_i(V_\gamma)\psi \subset Z_1$ if $1 \leqq i \leqq s$, (b) $\phi_i(V_\gamma)\psi \subset Z_2$ if $s+1 \leqq i \leqq t$ and (c) $\phi_i(V_\gamma)\psi = \{0\}$ if $t+1 \leqq i \leqq l(\gamma)$. Let $\{v_1, \cdots, v_{d(\gamma)}\}$ be an orthonormal basis of V_γ with respect to $\{\ ,\ \}$. As the matrix $R^\gamma(\lambda)$ is positive semi-definite, we have in particular

$$R^\gamma_{ii}(\gamma) \geqq 0 \qquad \text{for} \quad 1 \leqq i \leqq s\,.$$

Hence

$$\sum_{k=1}^{d(\gamma)} <\phi_i(v_k)\psi,\ \phi_i(v_k)\psi> \geqq 0 \qquad \text{for} \quad 1 \leqq i \leqq s\,.$$

This inequality combined with the fact that $<\ ,\ >$ is negative definite on Z_1 implies $Z_1 = \{0\}$.

Therefore $Z^\lambda_\gamma = Z_2$ i.e., $<\ ,\ >$ is positive definite on Z^λ_γ.

Thus we have proved the sufficiency of the statement (2). Q.E.D.

8. An example

As an application of our theorems, we can construct all irreducible unitary representations of $SL(2, \boldsymbol{R})$ in a unified way. In case of $SL(2, \boldsymbol{R})$, conjecture in Remark 2 is true, and one knows necessary and sufficient conditions in order that the $\mathfrak{U}$-module X^λ or $\mathfrak{U}\cdot f_\lambda$ is irreducible, and those in order that the $(K, \mathfrak{U})$-module Z^λ admits a non-degenerate or positive definite invariant Hermitian structure.

We choose $\mathfrak{k}_0$, $\mathfrak{p}_0$ and so on as follows:

$$\mathfrak{k}_0 = \left\{\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix};\ x \in \boldsymbol{R}\right\}, \qquad \mathfrak{k} = \left\{\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix};\ x \in \boldsymbol{C}\right\}$$

$$\mathfrak{p}_0 = \left\{\begin{pmatrix} a & b \\ b & -a \end{pmatrix};\ a, b \in \boldsymbol{R}\right\}, \quad \mathfrak{p} = \left\{\begin{pmatrix} a & b \\ b & -a \end{pmatrix};\ a, b \in \boldsymbol{C}\right\}$$

$$\mathfrak{a}_0 = \left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix};\ a \in \boldsymbol{R}\right\}, \qquad \mathfrak{a} = \left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix};\ a \in \boldsymbol{C}\right\}$$

$$K = SO(2, \boldsymbol{R}) = \left\{k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix};\ \theta \in \boldsymbol{R}\right\}$$

$$M = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$$

$$A_+ = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix};\ a \in \boldsymbol{R}^+\right\}$$

The non-zero root system Δ is given by $\Delta = \{\alpha, -\alpha\}$ where α is defined by $\alpha(H) = 2a$ for every $H = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \in \mathfrak{a}$. We introduce the order in Δ such that α becomes a positive root. Then $\mathfrak{n}$, $\mathfrak{n}_0$ and N are given by

$$\mathfrak{n}_0 = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix};\ b \in \boldsymbol{R}\right\} \qquad \mathfrak{n} = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix};\ b \in \boldsymbol{C}\right\}$$

$$N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix};\ b \in \boldsymbol{R}\right\}$$

For each $n \in \boldsymbol{Z}$, let V_{2n} be the K-module such that (1) the space V_{2n} is isomorphic to $\boldsymbol{C}$, and (2) the K-action on V_{2n} is defined by $k_\theta v = e^{2in\theta} v$ for $k_\theta \in K$ and $v \in V_{2n}$. Let γ_{2n} denote the equivalence class of all irreducible K-modules which are isomorphic to V_{2n}, then the set Γ is given by

$$\Gamma = \{\gamma_{2n};\ n \in \boldsymbol{Z}\}$$

We put $\quad X_+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $X_- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$

Then by a theorem in Kostant-Rallis [8] and by an easy calculation, one can see that

$$H^*_{\gamma_{2n}} = \boldsymbol{C} \cdot X_+^n$$

$$H^*_{\gamma_{-2n}} = \boldsymbol{C} \cdot X_-^n$$

where n is a positive integer.

Each function of X^λ is entirely determined by its value on K, and so henceforward we shall regard each element $f \in X^\lambda$ not as a function on G but as a function on K.

Since $\dim_C \mathfrak{a}' = 1$, every $\mu \in \mathfrak{a}'$ is a scalar multiple of α, whose scalar we denote by μ.

If we choose function f_λ such that $f_\lambda(k_\theta) = e^{i\nu\theta}$ where $\nu \in \boldsymbol{Z}$, then we have

$$\sigma(k_\theta) = e^{-i\nu\theta}$$

$$\sigma_*(X) = -i\nu\theta \qquad \text{where} \quad X = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \in \mathfrak{k}.$$

And matrices $P^{\gamma_{2n}}(\lambda)$ and $R^{\gamma_{2n}}(\lambda)$ are given explicitly in the following form:

$$P^{\gamma_{2n}}(\lambda)=\left(\mu-\frac{\nu}{2}\right)\left(\mu-\frac{\nu}{2}+1\right)\cdots\left(\mu-\frac{\nu}{2}+n-1\right)$$

$$R^{\gamma_{2n}}(\lambda)=(-1)^n\left(\mu-\frac{\nu}{2}+n-1\right)\left(\mu-\frac{\nu}{2}+n-2\right)\cdots\left(\mu-\frac{\nu}{2}+1\right)\left(\mu-\frac{\nu}{2}\right)$$

$$\times\left(\mu+\frac{\nu}{2}-1\right)\left(\mu+\frac{\nu}{2}-2\right)\cdots\left(\mu+\frac{\nu}{2}-n\right)$$

$$P^{\gamma_{-2n}}(\lambda)=\left(\mu+\frac{\nu}{2}\right)\left(\mu+\frac{\nu}{2}+1\right)\cdots\left(\mu+\frac{\nu}{2}+n-1\right)$$

$$R^{\gamma_{-2n}}(\lambda)=(-1)^n\left(\mu+\frac{\nu}{2}+n-1\right)\left(\mu+\frac{\nu}{2}+n-2\right)\cdots\left(\mu+\frac{\nu}{2}+1\right)\left(\mu+\frac{\nu}{2}\right)$$

$$\times\left(\mu-\frac{\nu}{2}-1\right)\left(\mu-\frac{\nu}{2}-2\right)\cdots\left(\mu-\frac{\nu}{2}-n\right).$$

By means of these matrices one obtains the following conclusions:

[Case 1] $\lambda=(1, \mu)$ where 1 denotes the trivial character of M.

We define the function f_λ by $f_\lambda|K=1$. In this case matrices $P^{\gamma_{2n}}(\lambda)$ and $R^{\gamma_{2n}}(\lambda)$ are given by

$$P^{\gamma_{2n}}(\lambda)=\mu(\mu+1)\cdots(\mu+|n|-1)$$

$$R^{\gamma_{2n}}(\lambda)=(-1)^n(\mu+|n|-1)(\mu+|n|-2)\cdots(\mu+1)\mu$$

$$\times(\mu-1)(\mu-2)\cdots(\mu-|n|)$$

Then one can see:

(1) X^λ is irreducible if and only if μ is not integer.

(2) Among the above λ, X^λ admits an invariant positive definite Hermitian structure if and only if either (i) $\mu-\frac{1}{2}$ is pure imaginary or (ii) μ is real and $0<\mu<1$.

The representations (i) are called the representations of the principal series of class 1, while (ii) are called the representations of the supplementary series.

(3) When μ is an integer, X^λ is not irreducible and Z^λ admits a non-degenerate invariant Hermitian structure.

In order to make a further investigation in the case (3), we replace f_λ by the function $f_\lambda(k_\theta)=e^{i\nu\theta}$ where ν is an even integer. Then by using matrices $P^\gamma(\lambda)$ and $R^\gamma(\lambda)$, one finds out that

(I) Suppose μ is a positive integer, then

$$X^{\lambda}_{\geqq 2\mu}=\left\{e^{in\theta};\ \begin{matrix} n \text{ is an even integer} \\ n\geqq 2\mu \end{matrix}\right\}_C$$

and

$$X^{\lambda}_{\leqq -2\mu}=\left\{e^{in\theta};\ \begin{matrix} n \text{ is an even integer} \\ n\leqq -2\mu \end{matrix}\right\}_C$$

are irreducible $\mathfrak{U}$-submodules of X^{λ} and there exists a positive definite invariant Hermitian structure on each of them. And there are no proper $\mathfrak{U}$-submodules of X^{λ} other than them.

(II) If μ is a non-positive integer, X^{λ} has three proper $\mathfrak{U}$-invariant subspaces $X^{\lambda}_{\geqq 2\mu}$, $X^{\lambda}_{\leqq -2\mu}$, and $X^{\lambda}_{\geqq 2\mu}\cap X^{\lambda}_{\leqq -2\mu}$, among them only the $\mathfrak{U}$-module $X^{\lambda}_{\geqq 2\mu}\cap X^{\lambda}_{\leqq -2\mu}$ is irreducible, and $X^{\lambda}_{\geqq 2\mu}\cap X^{\lambda}_{\leqq -2\mu}$ admits a positive definite invariant Hermitian structure only when $\mu=0$ (in this case, $X^{\lambda}_{\geqq 2\mu}\cap X^{\lambda}_{\leqq -2\mu}=\boldsymbol{C}$).

The representations of (I) are representations of a branch of the discrete series.

[Case 2] $\lambda=(-1, \mu)$ where -1 denotes the alternating character of M.

Let us define the function f_{λ} by $f_{\lambda}(k_{\theta})=e^{i\nu\theta}$ where ν is an odd integer. Using matrices $P^{\gamma}(\lambda)$ and $R^{\gamma}(\lambda)$, one can see:

(1) X^{λ} is irreducible if and only if μ is not a half-integer.

(2) Among the above λ's, X^{λ} admits a positive definite invariant Hermitian structure if and only if $\mu-\frac{1}{2}$ is a pure-imaginary number. These unitary representations are representations of another branch of the principal series.

(3) When μ is a half-integer, X^{λ} contains proper $\mathfrak{U}$-invariant subspaces. And one can see:

(I) If μ is a positive half-integer, X^{λ} has only two $\mathfrak{U}$-invariant subspaces $X^{\lambda}_{\geqq 2\mu}$ and $X^{\lambda}_{\leqq -2\mu}$ defined by

$$X^{\lambda}_{\geqq 2\mu}=\left\{e^{in\theta};\ \begin{matrix} n \text{ is an odd integer} \\ n\geqq 2\mu \end{matrix}\right\}_C$$

and

$$X^{\lambda}_{\leqq -2\mu}=\left\{e^{in\theta};\ \begin{matrix} n \text{ is an odd integer} \\ n\leqq -2\mu \end{matrix}\right\}_C,$$

and they are irreducible $\mathfrak{U}$-submodules of X^{λ}, which admit positive definite invariant Hermitian structures. The unitary representations thus obtained are representations of another branch of the discrete series.

(II) If μ is a negative half-integer, X^{λ} has three proper $\mathfrak{U}$-invariant subspaces $X^{\lambda}_{\geqq 2\mu}$, $X^{\lambda}_{\leqq -2\mu}$ and $X^{\lambda}_{\geqq 2\mu}\cap X^{\lambda}_{\leqq -2\mu}$. Among them only the $\mathfrak{U}$-submodule $X^{\lambda}_{\geqq 2\mu}\cap X^{\lambda}_{\leqq -2\mu}$ is irreducible, and none of them admit positive definite invariant Hermitian structures.

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