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Differential Operator without Dense Range

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§ 1. Introduction

Let Ω be a domain in the n-dimensional complex space $\mathbf{C}^n$, and let $\boldsymbol{H}(\Omega)$ be the space of all holomorphic functions in Ω equipped with the compact convergence topology. In this note, we shall study the range of a differential operator $\mathrm{P}\left(z, \frac{\partial}{\partial z}\right)$ with variable coefficients. If Ω is convex and $\mathrm{P}\left(\frac{\partial}{\partial z}\right)$ is a differential operator with constant coefficients, it is well known that $\mathrm{P}\left(\frac{\partial}{\partial z}\right)$ maps $\boldsymbol{H}(\Omega)$ onto itself, in particular the range of $\mathrm{P}\left(\frac{\partial}{\partial z}\right)$, $\mathrm{P}\left(\frac{\partial}{\partial z}\right)\boldsymbol{H}(\Omega)$, is dense in $\boldsymbol{H}(\Omega)$. Now we shall be interested in the case where an operator has variable coefficients. For example, if the coefficients of an operator $\mathrm{P}\left(z, \frac{\partial}{\partial z}\right)$ have a common zero in Ω, every holomorphic function in $\mathrm{P}\left(z, \frac{\partial}{\partial z}\right)\boldsymbol{H}(\Omega)$ vanishes at the point, so that $\mathrm{P}\left(z, \frac{\partial}{\partial z}\right)\boldsymbol{H}(\Omega)$ cannot be dense in $\boldsymbol{H}(\Omega)$. The purpose of this note is to construct an operator with polynomial coefficients without dense range even if its coefficients have no common zero in some polydisc Ω. The essential idea is due to I. Wakabayashi [4], who proved that for some domain of holomorphy $\mathbf{D}$ in $\mathbf{C}^3$, the equation $\frac{\partial u}{\partial z}=f$ cannot be solved for some holomorphic function f in $\mathbf{D}$.

§ 2. Construction of a differential operator without dense range

We use Wermer's example for a domain of holomorphy (see Gunning-Rossi [1], p. 38). Let F be the holomorphic mapping of $\mathbf{C}^3$ into $\mathbf{C}^3$ defined by

$$\begin{aligned} F(z_1, z_2, z_3) &= (w_1, w_2, w_3) \\ &= (z_1,\ z_1z_2+z_3,\ z_1z_2^2-z_2+2z_2z_3). \end{aligned}$$

Then, for sufficiently small $b\left(0<b<\frac{1}{2}\right)$ F maps the polydisc Δ_b,

$$\Delta_b=\{(z_1, z_2, z_3);\ |z_1|<1+b,\ |z_2|<1+b,\ |z_3|<b\},$$

biholomorphically onto its image $\mathbf{D}=F(\Delta_b)$. Let π be the complex plane $\{(w_1, 1, 0)\}$. Then $\pi\cap\mathbf{D}$ is the annular domain $\left\{\frac{1}{1+b}<|w_1|<1+b\right\}$ in the π plane. Now, we recall the following

PROPOSITION 1. (Y. Tsuno [3], p. 148): *Let* $\mathrm{P}\left(\frac{d}{dz}\right)$ *be any differential operator with constant coefficients of order* $\geqq 1$, *and* Ω *a non-simply connected domain in* $\mathbf{C}$. *Then,* $\mathrm{P}\left(\frac{d}{dz}\right)\boldsymbol{H}(\Omega)$ *is not dense in* $\boldsymbol{H}(\Omega)$.

REMARK. Professor Hikosaburo Komatsu kindly informed the author that this proposition could also be proved using the index of an operator instead of analytic functionals. (see [5])

From this proposition and the fact that every holomorphic function $f(w_1)$ on $\pi\cap\mathbf{D}$ is a restriction of some holomorphic function $\tilde{f}(w_1, w_2, w_3)\in\boldsymbol{H}(\mathbf{D})$ to the plane $\pi\cap\mathbf{D}$ (Cartan's Theorem B), any operator $\mathrm{P}\left(\frac{\partial}{\partial w_1}\right)=\sum_{k=0}^{m}a_k\left(\frac{\partial}{\partial w_1}\right)^k$, ($m\geqq 1$, $a_m\neq 0$, a_k: constant) has not a dense range in $\boldsymbol{H}(\mathbf{D})$.

Since $\mathbf{D}$ is biholomorphic to Δ_b, we can pull back the operator $\mathrm{P}\left(\frac{\partial}{\partial w_1}\right)$ to that on $\boldsymbol{H}(\Delta_b)$. Now,

$$\frac{\partial}{\partial w_1}=\frac{\partial z_1}{\partial w_1}\frac{\partial}{\partial z_1}+\frac{\partial z_2}{\partial w_1}\frac{\partial}{\partial z_2}+\frac{\partial z_3}{\partial w_1}\frac{\partial}{\partial z_3}$$
$$=\frac{\partial}{\partial z_1}-\frac{z_2^2}{1-2z_3}\frac{\partial}{\partial z_3}+\frac{z_1z_2^2-z_2+2z_2z_3}{1-2z_3}\frac{\partial}{\partial z_3}.$$

Therefore

$$\left\{\sum_{k=0}^{m}a_k\left(\frac{\partial}{\partial z_1}-\frac{z_2^2}{1-2z_3}\frac{\partial}{\partial z_2}+\frac{z_1z_2^2-z_2+2z_2z_3}{1-2z_3}\frac{\partial}{\partial z_3}\right)^k\right\}\boldsymbol{H}(\Delta_b)$$

is not dense in $\boldsymbol{H}(\Delta_b)$. (Note that $1-2z_3\neq 0$ in Δ_b.) We define the operator $\mathrm{Q}\left(z, \frac{\partial}{\partial z}\right)$ as follows:

$$\mathrm{Q}\left(z, \frac{\partial}{\partial z}\right)=(1-2z_3)^{2m-1}\sum_{k=0}^{m}a_k\left(\frac{\partial}{\partial z_1}-\frac{z_2^2}{1-2z_3}\frac{\partial}{\partial z_2}+\frac{z_1z_2^2-z_2+2z_2z_3}{1-2z_3}\frac{\partial}{\partial z_3}\right)^k,$$

where $m\geqq 1$, $a_m\neq 0$ and a_k is a constant number. Then $\mathrm{Q}\left(z, \frac{\partial}{\partial z}\right)$ has polynomial coefficients which have no common zero in Δ_b and $\mathrm{Q}\left(z, \frac{\partial}{\partial z}\right)\boldsymbol{H}(\Delta_b)$ is not dense in $\boldsymbol{H}(\Delta_b)$.

Thus we have the following

PROPOSITION 2. *Let* $\mathrm{Q}\left(z, \frac{\partial}{\partial z}\right)$ *be the differential operator defined as above. Then* $\mathrm{Q}\left(z, \frac{\partial}{\partial z}\right)$ *has not a dense range in* $\boldsymbol{H}(\Delta_b)$ *even if its polynomial coefficients have no common zero in* Δ_b.

§ 3. Some comments

Now, we have the following Cauchy-Kowalewski theorem due to J. Leray.

THEOREM (J. Leray [2], p. 399). *Let* $\mathrm{P}\left(z, \frac{\partial}{\partial z}\right)$ *be a differential operator of order* m *whose coefficients are holomorphic on* $\{z \mid |z_j| \leqq R, j=1, \cdots, n\}$, *and* $\mathrm{P}_m\left(z, \frac{\partial}{\partial z}\right)$ *be its principal part.*

Suppose that $\mathrm{P}_m(0, \mathrm{N}) \neq 0$ *where* $\mathrm{N}=(1, 0, \cdots, 0)$. *Then the unique holomorphic solution* $u(z)$ *of the Cauchy Problem*

$$\begin{cases} \mathrm{P}\left(z, \dfrac{\partial}{\partial z}\right)u(z)=v(z) \\ \left(\dfrac{\partial}{\partial z_1}\right)^k u(z)\Big|_{z_1=0}=w_k(z_2, \cdots, z_n), \qquad k=0, 1, \cdots, m-1, \end{cases}$$

where $v(z)$ *is holomorphic on* $\{|z_j| \leqq R, j=1, \cdots, n\}$ *and* $w_k(z)$ *is holomorphic on* $\{|z_j| \leqq r, j=2, \cdots, n\}$, *exists in* $\left\{z \,\middle|\, \|z\| < \frac{1}{12nm} q \inf(qR, r)\right\}$, *where*

$$\|z\|^2=\sum_{j=1}^{n}|z_j|^2, \qquad q=q(R)=\frac{|\mathrm{P}_m(0, \mathrm{N})|}{\sup\limits_{\substack{|z_j|=R \\ |p_j|=1}} |\mathrm{P}_m(z, p)|}.$$

We shall apply this theorem to find a sufficient condition for an operator to have a dense range. For a given operator $\mathrm{P}\left(z, \frac{\partial}{\partial z}\right)$ of order m with entirely holomorphic coefficients, we define the quantities $q(R)$ and l as follows; $q(R)$ is the same as in the theorem, and $l=\sup\limits_{R>0} q(R)^2 \cdot R$. Let Δ be the unit polydisc, i.e. $\Delta=\{z \mid |z_j|<1, j=1,\cdots,n\}$. Then we obtain

PROPOSITION 3. *Asusme that* $l>12n^{3/2}m$, *then* $\mathrm{P}\left(z, \frac{\partial}{\partial z}\right)\boldsymbol{H}(\Delta)$ *is dense in* $\boldsymbol{H}(\Delta)$.

PROOF. Let $v(z)$ be any polynomial and apply the Cauchy-Kowalewski Theorem with $w_k(z)=0$, $k=0, 1, \cdots, m-1$. Then there exists a solution $u(z)$

of $P\left(z, \frac{\partial}{\partial z}\right)u(z)=v(z)$, where $u(z)$ is holomorphic in $\left\{z \mid \|z\|<\frac{1}{12nm}\cdot q(R)^2\cdot R\right\}$ for any $R>0$, that is, $u(z)$ is holomorphic in $\left\{\|z\|<\frac{l}{12nm}\right\}$. If $l>12n^{3/2}m$, Δ is contained in the ball $\left\{\|z\|<\frac{l}{12nm}\right\}$. Therefore for any polynomial $v(z)$, we can find $u(z)\in \boldsymbol{H}(\Delta)$ such that $P\left(z, \frac{\partial}{\partial z}\right)u(z)=v(z)$. This proves the proposition.

REMARK. From Cauchy's inequality, it is easy to see that $l=\infty$ if and only if P_m has constant coefficients and $P_m(0, N)\neq 0$. And if P_m has constant coefficients, $P\left(z, \frac{\partial}{\partial z}\right)$ maps $\boldsymbol{H}(\mathbf{C}^n)$ onto itself by the Cauchy-Kowalewski Theorem.

References

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