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On Function-theoretic Separative Conditions on Compactifications of Hyperbolic Riemann Surfaces

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Introduction

In the theory of compactifications of hyperbolic Riemann surfaces, there have been considered various conditions which require that ideal boundary points are separated in some function-theoretic sense. In order to extend Fatou's and Beurling's theorems to Riemann surfaces, Z. Kuramochi introduced notions of $H.B.$ separative and $H.D.$ separative metrics (cf. [10]). The present author [19] defined separative compactifications rather than separative metrics and simplified Kuramochi's definitions: A compactification R^* of a hyperbolic Riemann surface R is called $H.D.$ (resp. $H.B.$) separative if any two closed sets in R which are separated in R^* are also separated in the Royden compactification up to a set of capacity zero (resp. in the Wiener compactification up to a set of harmonic measure zero.) In [19], it was shown that $H.B.$ separative compactifications are nothing but resolutive ones, i.e., the quotient spaces of the Wiener compactification and that the quotient spaces of the Royden compactification are $H.D.$ separative but the converse is not true. Another notion of separativeness is the regularity introduced by F-Y. Maeda [12]: A resolutive compactification R^* of R is called regular if continuous functions on $\Delta=R^*-R$ whose Dirichlet solutions belong to HD separate points of Δ.

In this paper, we shall introduce another notions of separativeness. The first of them is of Kuramochi's type: $H.M.$ separativeness, which is defined in the same fashion as $H.D.$ separativeness using the harmonic measure on the Royden compactification instead of capacity (§4). The other notions will be defined in terms of curves (§6): A metrizable compactifiation R^* of R is said to satisfy condition (E) (resp. (G)) if almost every curve in R (resp. Green lines) tending to the ideal boundary Δ terminates at one point on Δ. Here, "almost every" is in the sense of extremal length (resp. Green measure). The main purpose of this paper is to investigate relations among these various separative conditions. In §1 and §2, we prepare basic definitions and results which are necessary for the subsequent theories. In §3, we focus our attention to singular points on the Kuramochi boundary and to

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poles of such points defined on the Royden boundary and on the Wiener boundary. Some of the results in this section will be used to supply examples in §4; while the other results (especially Theorem 3) concerning a characterization of singular points in terms of poles may be interesting in its own right.

Relations among *H.D.* separativeness, *H.M.* separativeness and regularity are studied in §4. The results in this section are summarized at the end of the section. Then, in §5, we consider the Martin compactification of a Riemann surface belonging to $O_{HD}-O_{HB}$. We shall show that such a compactification is neither *H.D.* separative nor *H.M.* separative nor regular. Furthermore, we shall remark that, on its boundary, a normal derivative in the sense of F-Y. Maeda [12] is not uniquely determined. Finally, in §6, we introduce conditions (E) and (G) and investigate relations among *H.D.* or *H.M.* separativeness and these conditions. Our results in this last section improve those given F-Y. by Maeda ([11; Theorem 2]) and M. Ohtsuka ([16; Theorem 1]).

Notation and terminology

Let R be a hyperbolic Riemann surface. For a subset A of R, we denote by ∂A and A^i the (relative) boundary and the interior of A respectively. We call a closed or open subset A of R *regular* if ∂A is non-empty and consists of at most a countable number of analytic arcs clustering nowhere in R. An *exhaustion* will mean an increasing sequence $\{R_n\}_{n=1}^{\infty}$ of relatively compact domains on R such that $\bigcup_{n=1}^{\infty} R_n=R$ and each ∂R_n consists of a finite number of closed analytic Jordan curves. We fix a closed disk K_0 in R once for all and let $R_0=R-K_0$.

We denote by $BC=BC(R)$ the space of all bounded continuous (real-valued) functions on R and by $C_0=C_0(R)$ the subspace of BC whose functions have compact supports in R. Let $HB=HB(R)$ be the space of all bounded harmonic functions on R and $HD=HD(R)$ be the space of all harmonic functions on R with finite Dirichlet integral (or finite Dirichlet norm). We denote $HBD=HD\cap BC$.

§1 Preliminaries

1.1 Wiener functions ([3])

For a finite continuous function f on R, we shall denote by $\bar{\mathscr{W}}_f$ (resp. $\underline{\mathscr{W}}_f$) the family of all superharmonic (resp. subharmonic) functions on R such that $s\geqq f$ on $R-K_s$ (resp. $s\leqq f$ on $R-K_s$) for some compact set K_s in R. If $\bar{\mathscr{W}}_f$ and $\underline{\mathscr{W}}_f$ are non-empty, then we set $\bar{h}_f(a)=\inf\{s(a);\ s\in\bar{\mathscr{W}}_f\}$ and $\underline{h}_f(a)=\sup\{s(a);\ s\in\underline{\mathscr{W}}_f\}$ $(a\in R)$. It is known that $\bar{h}_f$, $\underline{h}_f$ are harmonic and $\underline{h}_f\leqq\bar{h}_f$. If $\bar{h}_f=\underline{h}_f$, then f is said to be harmonizable. We write $h_f=\bar{h}_f=\underline{h}_f$ if f is harmonizable. A finite continuous function f on R is called a *Wiener function* if $|f|$ has a superharmonic majorant and f is harmonizable. If a Wiener function f satisfies $h_f=0$, then f is called a *Wiener potential.* We denote by W (resp. W_0) the family of all finite continuous Wiener functions (resp. Wiener potentials) on R and set $BCW=W\cap BC$ (resp. $BCW_0=W_0\cap BC$). It is known that both BCW and BCW_0 are vector lattices with respect to the maximum and minimum operations and also contain C_0

1.2 Dirichlet functions and Dirichlet principle

We follow C. Constantinescu and A. Cornea [3] for the definition and properties of Dirichlet functions. Let f be a Dirichlet function on R and F be a non-polar[1)] closed set in R. Then there exists a uniquely determined Dirichlet function f^F which minimizes the Dirichlet norm $||g||$ among Dirichlet functions g such that $g=f$ $q.p.$ (quasi überall)[1)] on F and which is equal to f on F and is harmonic in $R-F$.

Properties of f^F([3]):

(A, 1) $||f^F||\leqq||f||$ and $(g-f^F, f^F)$[2)]$=0$ for any Dirichlet function g such that $g=f$ $q.p.$ on F.

(A, 2) If $f\geqq 0$ on F, then $f^F\geqq 0$.

(A, 3) If $F_1\subset F_2$, then $f^{F_1}=(f^{F_1})^{F_2}=(f^{F_2})^{F_1}$.

(A, 4) $(a_1f_1+a_2f_2)^F=a_1f_1^F+a_2f_2^F(a_1, a_2$: constant).

(A, 5) If G is a component of $R-F$, then $f^F=f^{\partial F}=f^{\partial G}$ on G.

We denote by BCD (resp. BCD_0) the family of all bounded continuous Dirichlet functions (resp. Dirichlet potentials) on R. It is known that both BCD and BCD_0 are vector lattices with respect to the maximum and minimum operations. Furthermore, BCD is decomposed into the direct sum of two parts HBD and BCD_0 (Royden decomposition). It is known ([3]) that $BCD\subset BCW$ and $BCD_0\subset BCW_0$.

1.3 Compactifications

If R^* a compact Hausdorff space and if there is a homeomorphism of R into R^* such that the image of R is open and dense in R^*, then we may identify the image of R with R and call R^* a *compactification* of R. $\varDelta=R^*-R$ is called an ideal boundary of R. We shall say that a subfamily Q of BC separates points of $\varDelta$ if, for any two distinct points ξ_1 and ξ_2 of $\varDelta$, there exists a function f in Q such that $\overline{\lim}_{a\to\xi_1} f(a)<\underline{\lim}_{a\to\xi_2} f(a)$ or $\underline{\lim}_{a\to\xi_1} f(a)>\overline{\lim}_{a\to\xi_2} f(a)$. Given a compactification R^*, let $C(\varDelta)$ (resp. $C(R^*)$) be the space of all finite

1) See p. 30 in [3].

2) $(g-f^F, f^F)$ is the mixed Dirichlet integral of $g-f^F$ and f^F.

continuous functions on Δ (resp. R^*).

Let Q be a non-empty subfamily of BC. If a compactification R^* of R satisfies the following:

1) every $f \in Q$ can be continuously extended over R^*,

2) Q separates points of Δ,

then R^* is called a *Q-compactification* of R. It is known ([3]) that a Q-compactifiation always exists and is unique up to a homeomorphism. Thus it will be denoted by R_Q^* and its ideal boundary by Δ_Q.

Properties of Q-compactifications:

(a) Let R^* be a compactification. If $Q \subset BC \cap C(R^*)$ separates points of Δ, then $R^* = R_Q^*$.

(b) If R^* is metrizable, then there exists a countable subfamily Q of BC such that $R^* = R_Q^*$.

(c) Let Q be a vector sublattice of BC containing C_0 and constants. If A and B are closed subsets of R such that $\bar{A}^* \cap \bar{B}^* = \emptyset$ in R_Q^*, then there exists a function f in Q such that $f=0$ on A and $=1$ on B.

We refer to [3] for the definitions and properties of the Martin compactification R_M^*, the Kuramochi compactification R_N^*, the Royden compactifiation R_D^* and the Wiener compactification R_W^*. For a subset A of R, we shall denote by $\bar{A}^*$ (resp. $\bar{A}^M$, $\bar{A}^N$, $\bar{A}^D$, $\bar{A}^W$) the closure of A in R^* (resp. R_M^*, R_N^*, R_D^*, R_W^*).

Let R_1^* and R_2^* be two compactifications of R. If there is a continuous mapping π of R_2^* onto R_1^* whose restriction to R is the identity mapping and $\pi^{-1}(R) = R$, then we shall say that π is a *canonical mapping* of R_2^* onto R_1^* and that R_1^* is a quotient space of R_2^* It is known ([3]) that if $Q_1 \subset Q_2$, then $R_{Q_1}^*$ is a *quotient space* of $R_{Q_2}^*$. We note that R_M^*, R_N^* and R_D^* are quotient spaces of R_W^*. Furthermore R_N^* is a quotient space of R_D^*.

We shall frequently use the following fact: Let R^* be a compactification of R and A be a closed set in $\Delta = R^* - R$. For any neighborhood U of A in R^*, there exists a regular closed set F in R such that $\bar{F}^*$ is a neighborhood of A and $\bar{F}^* \subset U$.

1.4 Harmonic measures and harmonic boundaries

Let R^* be a compactification of R and let $\Delta = R^* - R$. Given a function f (extended real-valued) on Δ, we consider the following classes:

$$\bar{\mathscr{S}}_f = \bar{\mathscr{S}}_f^{R^*} = \left\{ \begin{array}{l} s;\ \text{superharmonic, bounded below on } R, \\ \varliminf_{a \to \xi} s(a) \geqq f(\xi) \qquad \text{for } \xi \in \Delta \end{array} \right\} \cup \{\infty\}$$

and

$$\underline{\mathscr{S}}_f = \underline{\mathscr{S}}_f^{R^*} = \{s;\ -s \in \bar{\mathscr{S}}_{-f}^{R^*}\}.$$

Let $\bar{H}_f(a)=\bar{H}^{R^*}_f(a)=\inf\{s(a);\ s\in\bar{\mathscr{S}}_f\}$ and $\underline{H}_f(a)=\underline{H}^{R^*}_f(a)=\sup\{s(a);\ s\in\underline{\mathscr{S}}_f\}$ $(a\in R)$. It is known (Perron-Brelot) that $\bar{H}_f$ (resp. $\underline{H}_f$) is either harmonic, $\equiv+\infty$ or $\equiv-\infty$. If $\bar{H}_f=\underline{H}_f$ and are harmonic, then we say that f is resolutive (with respect to R^*) and $H_f=\bar{H}_f=\underline{H}_f$ is called the Dirichlet solution of f (with respect to R^*). If any function in $C(\varDelta)$ is resolutive, then we say that R^* is resolutive. It is known ([3]) that R^*_M, R^*_N, R^*_D and R^*_W are resolutive. We denote by $\omega^Q=\omega^Q_a(a\in R)$ the harmonic measure on $\varDelta_Q(Q=M, N, D, W)$. Let G be a domain on R. Then $\bar{G}^Q(Q=M, N, D, W)$ is a resolutive compactification of G (cf. Hilfssatz 8.2 in [3]). We denote by $\omega^{Q,G}=\omega^{Q,G}(a)(a\in G)$ the harmonic measure on $\bar{G}^Q-G$.

Let R^* be a compactification of R. For a (Green) potential p on R, we set $\Gamma_p=\{b\in\varDelta;\ \underline{\lim}_{a\to b}p(a)=0\}$ and $\Gamma=\bigcap_p\Gamma_p$. Then Γ is a non-empty compact subset of $\varDelta$ and is called the *harmonic boundary* of R^*. We denote by Γ_W (resp. Γ_D) the harmonic boundary of R^*_W (resp. R^*_D).

Properties of harmonic boundaries (cf.[3]):

(i) The support of ω^Q is equal to $\Gamma_Q(Q=W, D)$.

(ii) If π is the canonical mapping of R^*_W onto R^*_D, then $\pi(\Gamma_W)=\Gamma_D$.

(iii) A Riemann surface R belongs to $O_{HB}-O_G$ (resp. $O_{HD}-O_G$) if and only if Γ_W (resp. Γ_D) consists of a single point.

1.5. Capacity in the sense of G. Choquet

Let X be a compact Hausdorff space and $\mathscr{K}$ be the family of all compact sets in X. A finite-valued function $\varPsi$ on $\mathscr{K}$ is said to be a *capacity* (on X) in the sense of G. Choquet [2] if it has the following properties:

(a) If $K_1\subset K_2$, then $\varPsi(K_1)\leqq\varPsi(K_2)$.

(b) $\varPsi(K_1\cup K_2)+\varPsi(K_1\cap K_2)\leqq\varPsi(K_1)+\varPsi(K_2)$.

(c) Given $K\in\mathscr{K}$ and $\varepsilon>0$, there is an open set G in X such that $K\subset K'\subset G$ $(K'\in\mathscr{K})$ implies $\varPsi(K')<\varPsi(K)+\varepsilon$.

By definition, any positive (Radon) measure on X is a capacity. For a set A in X, we define $\varPsi_i(A)=\sup\{\varPsi(K);\ K\in\mathscr{K}$ and $K\subset A\}$ and $\varPsi_e(A)=\inf\{\varPsi_i(G);\ G$ is open and $A\subset G\}$. A set A in X is said to be $(\varPsi-)$ capacitable if $\varPsi_e(A)=\varPsi_i(A)$. G. Choquet [2] proved that any analytic set is capacitable, and hence any Borel set is capacitable. By definition, we see that if A is capacitable, then $\varPsi_e(A)=\sup\{\varPsi(K);\ K\in\mathscr{K}$ and $K\subset A\}$.

§2 Harmonic measures and capacities

2.1 Reduced functions

We follow [3] (see p. 21) for the definition of the Dirichlet problem on an open set in R and use the same notation as there. Let G be a domain on

R. Let F be a relatively closed set in G and s be a non-negative superharmonic function on G. We introduce the following function:

$$s_F^G = \inf\{v;\ \text{superharmonic} \geqq 0 \text{ on } G,\ v \geqq s\ q.p. \text{ on } F\}.$$

Then s_F^G is superharmonic on G and $0 \leqq s_F^G \leqq s$. When $G=R$, we write $s_F^G = s_F$ for simplicity.

Properties of s_F (cf. [3]):

(B, 1) $s_F = H_s^{R-F}$ on $R-F$ and $s_F = s$ on F except at irregular boundary points of $R-F$.

(B, 2) If $F_1 \subset F_2$ and $s_1 \leqq s_2$ $q.p.$ on F_1, then $(s_1)_{F_1} \leqq (s_2)_{F_2}$.

(B, 3) If $F_1 \subset F_2$, then $s_{F_1} = (s_{F_1})_{F_2} = (s_{F_2})_{F_1}$.

(B, 4) $(a_1 s_1 + a_2 s_2)_F = a_1 (s_1)_F + a_2 (s_2)_F$ (a_1, a_2: constants $\geqq 0$).

(B, 5) $s_{F_1 \cup F_2} + s_{F_1 \cup F_2} \leqq s_{F_1} + s_{F_2}$.

PROPOSITION 1. *Let G be a regular domain on R. Let $\{F_n\}_{n=1}$ be a sequence of regular closed subsets of R such that $F_n \supset F_{n+1}$ $(n=1, 2, \cdots)$ and $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Let u* (resp. u_0) *be the limit function of $\{1_{F_n}\}_{n=1}^{\infty}$*(resp. $\{1_{F_n \cap G}^G\}_{n=1}^{\infty}$). *Then we have*

(a) $u - u_0 = u_{R-G}$ *on* G.

(b) *Assume* $\overline{R-G}^W \cap \bar{F}_1^W = \emptyset$. *Then* $u=0$ *if and only if* $u_0 = 0$.

PROOF (a) If we set $g_n = 0$ on ∂G and $=1$ on $\partial F_n \cap G$, then $1_{F_n \cap G}^G = H_{g_n}^{G-F_n}$ on $G-F_n$. Since $(1_{F_n})_{(R-G) \cup F_n} = 1_{F_n}$ by (B, 3) it follows from (B, 1) that

$$1_{F_n} - 1_{F_n \cap G}^G = H_{1_{Fn} - g_n}^{G-F_n}$$

Since $\varliminf_{a \to b} H_{1_{Fn}}^G(a) \geqq 1_{F_n}(b) - g_n(b)$ for $b \in \partial G \cup (\partial F_n \cap G)$, we obtain that

$$1_{F_n} - 1_{F_n \cap G}^G \leqq H_{1_{Fn}}^G \qquad \text{on } G - F_n.$$

By letting $n \to \infty$, we have $u - u_0 \leqq H_u^G$ on G. On the other hand, since $u - u_0$ is a non-negative superharmonic function on G and

$$\varliminf_{a \to b} (u(a) - u_0(a)) = u(b) \qquad \text{for } b \in \partial G,$$

we have $u - u_0 \geqq H_u^G$. Thus $u - u_0 = H_u^G$ on G. Since $H_u^G = u_{R-G}$ on G by (B, 1), we have (a).

(b) Since $u_0 \leqq u$, $u=0$ implies $u_0 = 0$. Conversely, suppose $u_0 = 0$. Then, by (a), we have $u = u_{R-G}$ on R. On the other hand, it follows from Proposition 1 in [19] that $u_{F_1} = u$ on R. Thus $u = (u_{R-G})_{F_1} \leqq (1_{R-G})_{F_1} \leqq \min(1_{R-G}, 1_{F_1})$. Since $\overline{R-G}^W \cap \bar{F}_1^W = \emptyset$, it follows from Lemma 4 in [19] that $(1_{R-G})_{F_1} = 0$ on Γ_W. Hence $(1_{R-G})_{F_1}$ is a potential. Therefore we have $u=0$.

PROPOSITION 2. *Let* $\{F_n\}_{n=1}^{\infty}$ *be a sequence of regular closed subsets of* R_0 *such that*

(a) $\overline{R-F_n^{i^W}}\cap\bar{F}_{n+1}^W=\emptyset$ (*resp.* $\overline{R-F_n^{i^D}}\cap\bar{F}_{n+1}^D=\emptyset$) $(n=1, 2, \cdots)$,

(b) $\bigcap_{n+1}^{\infty} F_n=\emptyset$,

(c) $1_{F_n}\to 0$ *as* $n\to\infty$.

Then we can find a sequence $\{\phi_n\}_{n=1}^{\infty}$ *of functions in BCW* (*resp. BCD*) *such that*

(α) $0\leqq\phi_n\leqq 1$ *on R*, $\phi_n=0$ *on* $(R-F_{2n-1}^i)\cup F_{2n+1}$ *and* $=1$ *on* ∂F_{2n},

(β) ϕ_n *is harmonic in* $F_{2n-1}^i-F_{2n+1}-\partial F_{2n}$.

Furthermore, if we set $f_n=\sum_{k=1}^{n}\phi_k$, *then* f_n *is a function in BCW* (*resp. BCD*) *and converges to a function f in BCW as* $n\to\infty$.

PROOF. First we consider the case of Wiener functions. Let n be fixed. By (a), we can find g_n in BCW such that $0\leqq g_n\leqq 1$, $g_n=0$ on $R-F_n^i$ and $=1$ on F_{n+1}. We set

$$g_n'=\begin{cases} g_n & \text{on } R-(F_n^i-F_{n+1}) \\ H_{g_n}^{F_n^i-F_{n+1}} & \text{on } F_n^i-F_{n+1}. \end{cases}$$

By Hilfssatz 6.5 in [3], we see that g_n' is a function in BCW. If we set $\phi_n=\min(g_{2n-1}', 1-g_{2n}')$, then we see that ϕ_n satisfies (α) and (β). We set $f_n=\sum_{k=1}^{n}\phi_k$. Then f_n tends to a bounded continuous function f on R. Since $f_n\leqq f\leqq f_n+1_{F_{2n+1}}$ on R $(n=1, 2, \ldots)$, we have $0\leqq\bar{h}_f-\underline{h}_f\leqq 1_{F_{2n+1}}$ on $R\,(n=1, 2, \ldots)$. By letting $n\to\infty$, we obtain that $\bar{h}_f=\underline{h}_f$. Since $|f|$ is bounded, f is a function in BCW.

Secondly we consider the case of Dirichlet functions. Since we can choose g_n in BCD in this case, we obtain ϕ_n in BCD satisfying (α) and (β) in the same way as above by considering $g_n'=g_n^{R-(F_n^i-F_{n+1})}$. The rest of the proof is the same as above.

COROLLARY. *In the above proposition, if each* f_n *is a function in* BCW_0, *then so is f*.

PROOF. Since $h_{f_n}\leqq h_f\leqq h_{f_n}+1_{F_{2n+1}}$ and $h_{f_n}=0$ $(n=1, 2, \cdots)$, by letting $n\to\infty$, we obtain that $h_f=0$.

2.2 Harmonic measures on the ideal boundary

Let R^* be a resolutive compactification and ω be the harmonic measure on Δ. For a closed subset A of Δ, we consider the following class:

$$\mathscr{S}_{A,R^*}=\left\{\begin{array}{l} s;\ \text{superharmonic} \geqq 0 \text{ on } R,\ s\geqq 1 \text{ on } U\cap R \text{ for} \\ \text{some neighborhood } U \text{ of } A \text{ in } R^* \end{array}\right\}.$$

Then the function $1_A(a)=\inf\{s(a);\ s\in\mathscr{S}_{A,R^*}\}$ $(a\in R)$ is harmonic on R and $0\leqq 1_A\leqq 1$.

LEMMA 1. *Let A be a closed subset of Δ and let χ_A be the characteristic function of A. Then $1_A=\bar{H}_{\chi_A}=\omega(A)$.*

PROOF. By an elementary discussion, we can show that $1_A=\bar{H}_{\chi_A}$. On the other hand, it follows from Hilfssatz 8.3 in [3] that $\bar{H}_{\chi_A}=\omega(A)$.

LEMMA 2. *Let A be a closed subset of Δ and $\{U_n\}_{n=1}^{\infty}$ be a sequence of neighborhoods of A in R^*. Then there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of regular closed sets in R such that*

(a) *The closure $\bar{F}_n^*$ of each F_n is a neighborhood of A in R^*,*

(b) *$U_n\cap R\supset F_n$ $(n=1, 2, \dots)$ and $\bigcap_{n=1}^{\infty}F_n=\emptyset$,*

(c) *$\overline{R-F_n^i}^*\cap\bar{F}^*_{n+1}=\emptyset$ $(n=1, 2, \dots)$,*

(d) *$1_{F_n}\to\omega(A)$ as $n\to\infty$.*

PROOF. Let a_0 be a fixed point in R. Then we can find a sequence $\{s_n\}_{n=1}^{\infty}$ in $\mathscr{S}_{A,R^*}$ such that $s_n(a_0)\to\omega_{a_0}(A)$ as $n\to\infty$. By assumption, $s_n\geqq 1$ on $V_n\cap R$ for some neighborhood V_n of A. Hence we may assume that $U_n\supset V_n$, $V_n\supset V_{n+1}$ $(n=1, 2,\dots)$ and $\bigcap_{n=1}^{\infty}(V_n\cap R)=\emptyset$. Then there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of regular closed sets in R such that $V_n\cap R\supset F_n$, $\bar{F}_n^*$ is a neighborhood of A, $\overline{R-F_n^i}^*\cap\bar{F}^*_{n+1}=\emptyset$. This sequence satisfies (a), (b) and (c). Since it is a decreasing sequence and $\bigcap_{n=1}^{\infty}F_n=\emptyset$, 1_{F_n} tends to a harmonic function u on R as $n\to\infty$. Since $s_n\geqq 1_{F_n}\geqq 1_A$, by letting $n\to\infty$, we have $1_A(a_0)\geqq u(a_0)\geqq 1_A(a_0)$. Since $u\geqq 1_A$, it follows from the maximum principle that $u=1_A$. By Lemma 1, we obtain (d). This completes the proof.

As for a resolutive compactification R^* of R, we have

LEMMA 3. *Let G be a domain on R. Then $\bar{G}^*$ is a resolutive compactification of G. For a closed subset B of Δ, we denote by $u(a)$ $(a\in G)$ the harmonic measure of $B\cap\bar{G}^*$ with respect to G. Theu we have*

(a) *$\omega(B)-u=(\omega(B))_{R-G}=H^G_{\omega(B)}$ on G.*

(b) *Assume $\overline{R-G}^*\cap B=\emptyset$. Then $\omega(B)=0$ if and only if $u=0$.*

PROOF. (a) First setting $R=G$, $A=B\cap\bar{G}^*$, $U_n=\bar{G}^*$ $(n=1, 2,\dots)$ in Lemma 2, we obtain a sequence $\{\delta_n\}_{n=1}^{\infty}$ of regular closed sets in G such that $\bigcap_{n=1}^{\infty}\delta_n=\emptyset$, $\overline{G-\delta_n^i}^*\cap\bar{\delta}^*_{n+1}=\emptyset$ $(n=1, 2,\dots)$ and $1^G_{\delta_n}\to u$ as $n\to\infty$. Since each $\bar{\delta}_n^*$ is a neighborhood of $B\cap\bar{G}^*$ in G^*, there is a neighborhood V_n of B in R^* such that $V_n\cap\bar{G}^*\subset\bar{\delta}_n^*$ $(n=1, 2,\dots)$. Secondly setting $R=R$, $A=B$, $U_n=V_n$ $(n=1, 2,\dots)$ in Lemma 2, we have a sequence $\{F_n\}_{n=1}^{\infty}$ of regular closed sets in R such that $\bigcap_{n=1}^{\infty}F_n=\emptyset$, $\overline{R-F_n^i}^*\cap\bar{F}^*_{n+1}=\emptyset$ $(n=1, 2,\dots)$ and $1_{F_n}\to\omega(B)$ as $n\to\infty$. Since

$F_n \cap G \subset U_n \cap G \subset \delta_n$ and $\overline{F_n \cap G^*}$ is a neighborhood of $B \cap \bar{G}^*$ $(n=1, 2, \cdots)$, we see that $1^G_{F_n \cap G} \to u$ as $n \to \infty$. It follows from (a) in Proposition 1 that

$$\omega(B) - u = (\omega(B))_{R-G} = H^G_{\omega(B)} \qquad \text{on } G.$$

(b) Since $\overline{R-G^*} \cap B = \emptyset$, we can take $\{F_n\}_{n=1}^{\infty}$ in (a) in such way that $\overline{R-G^*} \cap \bar{F}_n^* = \emptyset$. Then $\overline{R-G^W} \cap \bar{F}_n^W = \emptyset$. Thus it follows from (b) in Proposition 1 that $\omega(B) = \lim_{n\to\infty} 1_{F_n} = 0$ if and only if $u = \lim_{n\to\infty} 1^G_{F_n} = 0$.

Corollary (cf. [19; Lemma 6]). *For a closed subset B of $\Delta_Q (Q=D, W)$, $\omega^Q(B)=0$ if and only if $\omega^{Q, R_0}(B)=0$.*

2.3 Full-superharmonic functions[3)]

Let s be a non-negative full-superharmonic function on R_0 and F be a closed set in R. We refer to [3] for the definition of full-superharmonic functions and the (full-) reduced function $s_{\bar{F}}$.

Properties of $s_{\bar{F}}$ ([3]):

(C, 1) If s is a Dirichlet function on R, $s=0$ on K_0 and s is a non-negative full-superharmonic function on R_0, then

$$s_{\bar{F}} = s^{K_0 \cup F} \qquad \text{on } R_0 - F.$$

(C, 2) If $F_1 \subset F_2$ and $s_1 \leqq s_2$ *q.p.* on F_1, then $(s_1)_{\bar{F}} \leqq (s_2)_{\bar{F}_2}$.
(C, 3) If $F_1 \subset F_2$, then $s_{\bar{F}_1} = (s_{\bar{F}_1})_{\bar{F}_2} = s_{\bar{F}_2})_{\bar{F}_1}$.
(C, 4) $(a_1 s_1 + a_2 s_2)_{\bar{F}} = a_1 (s_1)_{\bar{F}} + a_2 (s_2)_{\bar{F}} (a_1, a_2 \geqq 0)$.
(C, 5) $s_{\widetilde{F_1 \cup F_2}} + s_{\widetilde{F_1 \cup F_2}} \leqq s_{\bar{F}_1} + s_{\bar{F}_2}$.
(C, 6) If $s_n \uparrow s$ as $n \uparrow \infty$, then $(s_n)_{\bar{F}} \uparrow s_{\bar{F}}$ as $n \uparrow \infty$.

Lemma 4 (cf. [19]). *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of regular closed subsets of R_0 such that $F_n \supset F_{n+1}$ $(n=1, 2, \cdots)$ and $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Then $1_{\bar{F}_n}$ converges locally uniformly on R_0 and in Dirichlet norm as $n \to \infty$. Furthermore, setting $u = \lim_{n\to\infty} 1_{\bar{F}_n}$, we have*

(α) *If F is a regular closed subset of R_0 such that $F \supset F_{n_0}$ for some n_0, then $u_{\bar{F}} = u$ on R_0.*

(β) *If u is positive, then $\sup_{F_n} u = 1$ for each n.*

(γ) $\|1_{\bar{F}_n}\|^2 = \int_{\partial K_0} \frac{\partial}{\partial \nu}(1_{\bar{F}_n}) ds$ *and* $\|u\|^2 = \int_{\partial K_0} \frac{\partial u}{\partial \nu} ds$.

Lemma 5. *Let s be a non-uegative full-superharmonic function on R_0 and F be a closed subset of R_0. If G is a component of $R_0 - F$, then $s_{\bar{F}} = s_{\widetilde{\partial F}} = s_{\widetilde{\partial G}}$ on G.*

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3) This is called $\overline{\text{super}}$harmonic by Z. Kuramochi [6] and "positive vollsuperharmonisch" in [3].

PROOF. Let D be a relatively compact open disk in R such that $K_0 \subset D$ and $(D \cup \partial D) \cap F = \emptyset$. For each integer $n > 0$, we set $s_n = \min(s_{\widetilde{R_0 - D}}, n)$. Since s_n is bounded and the total mass of the measure associated with s_n is finite, it follows from Satz 17.3 in [3] that s_n is a Dirichlet function. Hence it follows from (A, 5) that $(s_n)_{\tilde{F}} = (s_n)_{\widetilde{\partial F}} = (s_n)_{\widetilde{\partial G}}$ on G. Since $s_{\widetilde{R_0 - D}} = s$ on $R_0 - (D \cup \partial D)$, by letting $n \to \infty$, we complete the proof by (C, 6).

2.4 Relative full-reduced functions

Let G be a regular open subset of R. Let F be a non-polar closed subsets of G such that $\overline{R-G}^D \cap \bar{F}^D = \emptyset$. Then there exists a function f in BCD such that $f=0$ on $R-G$ and $=1$ on F. Since $f^{(R-G)\cup F}$ does not depend on the choice of such an f, we shall denote it by $1_{\tilde{F}}^G$. If F is a regular closed set, then $1_{\tilde{F}}^G$ is continuous. We note that if F is a regular closed subset of R_0, then $1_{\tilde{F}}^{R_0} = 1_{\tilde{F}}$ on R_0. Let $\{F_n\}_{n=1}^\infty$ be a decreasing sequence of regular closed subsets of G such that $\bigcap_{n=1}^\infty F_n = \emptyset$. Suppose $\overline{R-G}^D \cap \bar{F}_1^D = \emptyset$. Then $1_{\tilde{F}_n}^G$ is defined for each n. By an argument similar to the proof of Lemma 4 (see [19; Proposition 2]), we can show that $1_{\tilde{F}_n}^G$ tends to a function, say u, on G locally uniformly and in Dirichlet norm as $n \to \infty$. Furthermore u is harmonic in G.

The following Lemma is known ([6], [10]).

LEMMA 6. *Let u be the function defined above. Suppose $u \not\equiv 0$ and $C_t = \{z \in G;\ u(z) = t\}(0 < t < 1)$. Then*

$$\int_{C_t} \frac{\partial u}{\partial \nu} ds = \|u\|^2 \qquad \textit{for almost all } t,\ 0 < t < 1.$$

LEMMA 7 ([9; Theorem 5]). *Let G be a regular open subset of R_0. Let $\{F_n\}_{n=1}^\infty$ be a decreasing sequence of regular closed subsets of G such that $\bigcap_{n=1}^\infty F_n = \emptyset$. Suppose $\overline{R-G}^D \cap \bar{F}_1^D = \emptyset$. Then $\lim_{n\to\infty} 1_{\tilde{F}_n} = 0$ if and only if $\lim_{n\to\infty} 1_{\tilde{F}_n}^G = 0$.*

PROOF. By (A, 2), (A, 3) and (A, 4), we see that

$$1_{\tilde{F}_n}^G \leqq 1_{\tilde{F}_n} \qquad \text{on } R\ (n=1, 2, \dots).$$

On the other hand, it follows from the Dirichlet principle (A, 1) that

$$\|1_{\tilde{F}_n}\| \leqq \|1_{\tilde{F}_n}^G\| \qquad (n=1, 2, \dots).$$

These two inequalities imply our assertion.

2.5 Full-reduced functions on the ideal boundary

Let R^* be a compactification of R. Let u be a non-negative full-superharmonic function on R_0. For a closed subsets A of Δ, we consider the following class:

$$\mathscr{S}^u_{A,R^*}=\left\{\begin{array}{l} s;\ \text{full-superharmonic} \geqq 0 \text{ on } R_0,\ s\geqq u \text{ on } U\cap R_0 \\ \quad \text{for some neighborhood } U \text{ of } A \text{ in } R^*. \end{array}\right\}$$

Then the function

$$u_{\tilde{A}}(a)=\inf\{s(a);\ s\in\mathscr{S}^u_{A,R^*}\}(a\in R_0)$$

is harmonic, full-superharmonic on R_0 and $0\leqq u_{\tilde{A}}\leqq u$. We denote $1_{\tilde{A}}$ by $\tilde{\omega}(A)=\tilde{\omega}_a(A)$.

REMARK: For the Kuramochi compactification, the above function $u_{\tilde{A}}$ does not necessarily equal the one defined in [3] (p. 197). However, for $u=1$, we can prove that they are identical.

By a discussion similar to that in the proof of Lemma 2, we can prove

LEMMA 8 (cf. [19; Lemma 6]). *Let u and A be as above. Let $\{U_n\}_{n=1}^{\infty}$ be any sequence of neighborhoods of A in R^*. Then there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of regular closed subsets of R_0 such that*

(a) *$\bar{F}_n^*$ is a neighbohood of A,*

(b) *$U_n\cap R_0\supset F_n$ $(n=1, 2,\cdots)$ and $\bigcap_{n=1}^{\infty}F_n=\emptyset$,*

(c) *$\overline{R-F_n^{i*}}\cap\bar{F}_{n+1}^*=\emptyset$ $(n=1, 2,\cdots)$,*

(d) *$u_{\tilde{F}_n}$ decreases to $u_{\tilde{A}}$ as $n\to\infty$.*

LEMMA 9. *Let u be a Dirichlet function on R such that u is a non-negative full-superharmonic function on R_0. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of regular closed sets in R which satisfies* (a)-(d) *in Lemma* 8. *Then $u_{\tilde{A}}$ is a Dirichlet function and we have*

(i) *$\|u_{\tilde{F}_n}-u_{\tilde{A}}\|\to 0$ as $n\to\infty$ and $\|u_{\tilde{F}_n}\|$ decreases to $\|u_{\tilde{A}}\|$ as $n\to\infty$. In particular, $\|1_{\tilde{F}_n}-\tilde{\omega}(A)\|\to 0$ and $\|1_{\tilde{F}_n}\|$ decreases to $\|\tilde{\omega}(A)\|$* as $n\to\infty$.

(ii) *If F is a regular closed subset of R_0 such that $\bar{F}^*$ is a neighborhood of A in R^*, then $(u_{\tilde{F}}-u_{\tilde{A}}, u_{\tilde{A}})=0$ and $\|u_{\tilde{A}}\|\leqq\|u_{\tilde{F}}\|$.*

PROOF. (i) By (C, 1) and (A, 1), we see that

$$(u_{\tilde{F}_n}-u_{\tilde{F}_m}, u_{\tilde{F}_m})=0 \qquad \text{if } m>n.$$

It follows that $\|u_{\tilde{F}_n}\|$ is decreasing and $\{u_{\tilde{F}_n}\}_{n=1}^{\infty}$ is a Cauchy sequence in Dirichlet norm. Since $u_{\tilde{F}_n}$ tends to $u_{\tilde{A}}$ on R_0 as $n\to\infty$, we see that $u_{\tilde{A}}$ is a Dirichlet function and $\|u_{\tilde{F}_n}-u_{\tilde{A}}\|\to 0$ as $n\to\infty$. It also follows that $\|u_{\tilde{F}_n}\|$ decreases to $\|u_{\tilde{A}}\|$ as $n\to\infty$.

(ii) We may assume that $F\supset F_1$. Then we have

$$(u_{\tilde{F}}-u_{\tilde{F}_n}, u_{\tilde{F}_n})=0$$

for each n. By letting $n\rightarrow\infty$, we obtain that $(u_{\tilde{F}}-u_{\tilde{A}}, u_{\tilde{A}})=0$. Hence $\|u_{\tilde{A}}\| \leqq \|u_{\tilde{F}}\|$.

By the aid of (C, 2)-(C, 5) and Lemma 8, we can show the following:

(D, 1) If $A_1\subset A_2$ and $u_1\leqq u_2$, then $(u_1)_{\tilde{A}_1}\leqq(u_2)_{\tilde{A}_2}$.

(D, 2) If $A_1\subset A_2$, then $u_{\tilde{A}_1}=(u_{\tilde{A}_1})_{\tilde{A}_2}=(u_{\tilde{A}_2})_{\tilde{A}_1}$.

(D, 3) $(a_1u_1+a_2u_2)_{\tilde{A}}=a_1(u_1)_{\tilde{A}}+a_2(u_2)_{\tilde{A}}$ (a_1, a_2; constant $\geqq 0$).

(D, 4) $u_{\widetilde{A_1\cup A_2}}+u_{\widetilde{A_1\cap A_2}}\leqq u_{\tilde{A}_1}+u_{\tilde{A}_2}$.

Since R_D^* is a quotient space of R_W^* and a full-superharmonic function is superharmonic, we have the following

LEMMA 10. $\omega^{W,R_0}(\bar{F}_1^W\cap\bar{F}_2^W\cap\Delta_W)\leqq\omega^{D,R_0}(\bar{F}_1^D\cap\bar{F}_2^D\cap\Delta_D)$

$$\leqq\tilde{\omega}(\bar{F}_1^D\cap\bar{F}_2^D\cap\Delta_D)$$

for any regular closed sets F_1 and F_2 in R.

We can easily see

LEMMA 11. *Let R^* be an arbitrary compactification of R and let $\{F_n\}_{n=1}^\infty$ be a sequence of regular closed subsets of R_0 such that $F_n\supset F_{n+1}$ $(n=1, 2,\cdots)$ and $\bigcap_{n=1}^\infty F_n=\emptyset$. We set $A=\bigcap_{n=1}^\infty \bar{F}_n^*$. If u is a non-negative full-superharmonic function on R_0, then $u_{\tilde{A}}\geqq\lim_{n\rightarrow\infty}u_{\tilde{F}_n}$.*

2.6 Full-reduced functions on the Royden boundary

LEMMA 12. *Let u be a bounded continuous, non-negative, full-superharmonic function on R_0. If u is a Dirichlet function on R_0 and F is a regular closed subset of R_0, then*

$$u_{\tilde{F}}\geqq u_{\widetilde{\bar{F}^D\cap\Delta_D}}.$$

PROOF. Since u and $u_{\tilde{F}}$ are bounded continuous Dirichlet functions on R_0, $v=u-u_{\tilde{F}}$ can be continuously extended over $R_0\cup\Delta_D$. We denote by v^* the continuous extension of v. For each $\varepsilon>0$, we set $U_\varepsilon=\{\in R_0\cup\Delta_D; v^*(z)<\varepsilon\}$. Since $v^*=0$ on $\bar{F}^D$, U_ε is an open neighborhood of $\bar{F}^D\cap\Delta_D$ and $u_{\tilde{F}}+\varepsilon>u$ on $U_\varepsilon\cap R_0$. Hence $u_{\tilde{F}}+\varepsilon\geqq u_{\widetilde{\bar{F}^D\cap\Delta_D}}$. Since ε is arbitrary, we have $u_{\tilde{F}}\geqq u_{\widetilde{\bar{F}^D\cap\Delta_D}}$.

By the above lemma and Lemma 11, we obtain

COROLLARY 1. *Let $\{F_n\}_{n=1}^\infty$ be a sequence of regular closed subsets of R_0 such that $F_n\supset F_{n+1}$ $(n=1, 2,\cdots)$ and $\bigcap_{n=1}^\infty F_n=\emptyset$ and let $A=\bigcap_{n=1}^\infty\bar{F}_n^D$. Then $u_{\tilde{F}_n}$ converges to $u_{\tilde{A}}$ locally uniformly and in Dirichlet norm as $n\rightarrow\infty$.*

COROLLARY 2. *Let $\{R_n\}_{n=1}^\infty$be an exhaustion of R and let F be a regular closed subset of R_0. Then $u_{\widetilde{F-R_n}}$ converges to $u_{\widetilde{\bar{F}^D\cap\Delta_D}}$ locally uniformly and in*

Dirichlet norm as $n\to\infty$. *In particular,* $1_{\widetilde{F-R_n}}$ *converges to* $\tilde{\omega}(\bar{F}^D\cap\Delta_D)$ *locally uniformly and in Dirichlet norm as* $n\to\infty$.

2.7 Capacity on the Royden boundary ([19])

Let A be a closed subset of Δ_D. Then, by (i) in Lemma 9, we see that $\|\tilde{\omega}(A)\|<\infty$. We define

$$C(A)=\frac{1}{2\pi}\int_{\partial K_0}\frac{\partial\tilde{\omega}(A)}{\partial\nu}ds$$

and call $C(A)$ the capacity of A (with respect to K_0). By (γ) of Lemma 4, we can show that $C(A)=(1/2\pi)\|\tilde{\omega}(A)\|^2$. It follows from (D, 1), (D, 4) and Lemma 9 that $A\to C(A)$ is a capacity in the sense of G. Choquet [2].

We can show that if π is the canonical mapping of R_D^* onto R_N^*, then $C(\pi^{-1}(A))=\tilde{C}(A)$ for any closed set A in Δ_N, where $\tilde{C}$ is the Kuramochi capacity (see [3]).

Proposition 3. $C(\Lambda_D)=0$ *where* $\Lambda_D=\Delta_D-\Gamma_D$.

Proof. Since Λ_D is an open set, it is sufficient to show that an arbitrary compact subset K of Λ_D is of capacity zero. By Hilfssatz 9.1 in [3], we see that there exists a finite continuous Green potential p with finite energy such that $\lim_{a\to K}p(a)=\infty$. Since p is a continuous Dirichlet function, so is $p_0=p-p_{K_0}$. For any $\varepsilon>0$, there exists a regular closed subset F of R_0 such that $\bar{F}^D$ is a neighborhood of K and $p_0\geqq 1/\varepsilon$ on F. Since $\min(\varepsilon p_0, 1)=0$ on K_0 and $=1$ on F, it follows from (C, 1) and (A, 1) that

$$\|1_{\tilde{F}}\|\leqq\|\min(\varepsilon p_0, 1)\|.$$

Hence, by (ii) of Lemma 9, we have

$$\|\tilde{\omega}(K)\|\leqq\|1_{\tilde{F}}\|\leqq\|\min(\varepsilon p_0, 1)\|\leqq\varepsilon\|p_0\|.$$

Since ε is arbitrary, we have $\tilde{\omega}(K)=0$. Hence $C(K)=0$. This completes the proof.

Corollary. *If* ξ *is a point in* Δ_D *with* $C(\{\xi\})>0$, *then it is contained in* Γ_D.

§3 Singular points on the Kuramochi boundary

3.1 Singular points and thin sets

For $b\in\Delta_N$, let $\tilde{g}_b$ be the Kuramochi kernel (with respect to R_0) ([3]). Let $\tilde{C}$ be the Kuramochi capacity on $R_0\cup\Delta_N$. We denote by Δ_1 the set of all minimal points in Δ_N. Let b be a point in Δ_N. If $\tilde{C}(\{b\})>0$, then b is called *singular*. Furthermore if $\omega^N(\{b\})>0$, then b is called *strictly singular*. We denote by Δ_S (resp. Δ_{SS}) the set of all singular (resp. strictly singular) points[4]. Then $\Delta_{SS} \subset \Delta_S \subset \Delta_1$. A point b in Δ_1 belongs to Δ_S if and only if $\tilde{g}_b$ is bounded. It is known that if R belongs to $O_{HD}-O_G$, then Δ_{SS} consists of only one point. Z. Kuramochi [8] constructed a Riemann surface with $\Delta_S - \Delta_{SS} \neq \emptyset$.

The following lemma is known (cf. [3; Folgesatz 17.22]).

LEMMA 13. *Let b be a point in $R_0 \cup \Delta_N$ and $F_\alpha = \{z \in R_0;\ \tilde{g}_b(z) \geqq \alpha\}$ $(0<\alpha<\sup \tilde{g}_b)$. Then we have*

(a) $(\tilde{g}_b)_{\tilde{F}_\alpha} = \min(\tilde{g}_b, \alpha) = \tilde{g}_b$ *on* $R_0 - F_\alpha$,

(b) $\|\tilde{g}_b\|^2_{R_0-F_\alpha} = \|\min(\tilde{g}_b, \alpha)\|^2 = 2\pi\alpha$,

(c) *If b is a point in Δ_S, then $\tilde{g}_b = (\sup \tilde{g}_b)\tilde{\omega}(\{b\})$ and $\|\tilde{g}_b\| < +\infty$.*

A closed set F in R is said to be *thin* at $b \in \Delta_1$ if $(\tilde{g}_b)_{\tilde{F}} \neq \tilde{g}_b$.

Properties of thinness (cf. [3]):

(E, 1) If $F_1 \subset F_2$ and F_2 is thin at b, then so is F_1.

(E, 2) If both F_1 and F_2 are thin at b, then so is $F_1 \cup F_2$.

(E, 3) If $b \notin \bar{F}^N$, then F is thin at b.

(E, 4) If $b \in \Delta_S$, then F is thin at b if and only if $(\tilde{\omega}(\{b\}))_{\tilde{F}} \lneqq \tilde{\omega}(\{b\})$.

The following proposition is essentially due to Z. Kuramochi ([17; Theorem 8]).

PROPOSITION 4. *Let b be a point in Δ_S. Let F_1, F_2 be regular closed subsets of R_0 such that $\bar{F}_1^D \cap \bar{F}_2^D = \emptyset$. If F_1 is not thin at b, then F_2 is thin at b.*

PROOF. Let $\{V_n\}_{n=1}^\infty$ be a sequence of regular closed subsets of R_0 such that $\bar{V}_n^N$ is a neighborhood of b, $V_n \supset V_{n+1}$ $(n=1, 2, \cdots)$ and $\bigcap_{n=1}^\infty \bar{V}_n^N = \{b\}$. We set $u = \tilde{\omega}(\{b\}) = \lim_{n\to\infty} 1_{\tilde{V}_n}$. Let $f_n = 1_{\tilde{F}}^G$ where $G = R_0 - F_2$ and $F = V_n \cup F_1$ $(n=1, 2, \cdots)$ and let $v = \lim_{n\to\infty} f_n$. Since $b \notin \overline{F_1 - V_n^i}^N$, $F_1 - V_n^i$ is thin at b by (E, 3). Since $F_1 = (F_1 \cap V_n) \cup (F_1 - V_n^i)$ and F_1 is not thin at b, $F_1 \cap V_n$ is not thin at b. Hence $u_{\widetilde{F_1 \cap V_n}} = u$ $(n=1, 2, \cdots)$ by (E, 4) Since $u_{\widetilde{F_1 \cap V_n}} \leqq 1_{\widetilde{F_1 \cap V_n}} \leqq 1_{\tilde{V}_\alpha} \to u$ as $n \to \infty$, we obtain that $\lim_{n\to\infty} 1_{\widetilde{F_1 \cap V_n}} = u$. Since $\|1_{\widetilde{F_1 \cap V_n}}\| \leqq \|f_n\|$, $\|1_{\widetilde{F_1 \cap V_n}}\| \to \|u\|$ and $\|f_n\| \to \|v\|$ (cf. Lemma 4), we have $0 < \|u\| \leqq \|v\|$. Hence $v \neq 0$. We set $C_t = \{z \in R_0 - F_2, v(z) = t\}$ $(0<t<1)$. It follows from Lemma 6 that there exists a subset E of (0, 1) everywhere dense in (0, 1) such that

$$\int_{C_t} \frac{\partial v}{\partial \nu} ds = \|v\|^2_{R_0-F_2} \qquad \text{for } t \in E.$$

By Lemma 3 in [10], we see that

4) Δ point in $\Delta_S - \Delta_{SS}$ is called a singular point of first kind and a point in Δ_{SS} is called a singular point of second kind by Z. Kuramochi [7].

$$\int_{C_t} u_{\tilde{F}_2} \frac{\partial v}{\partial \nu} ds$$

is a constant for all $t \in E$. Let t_1 be an arbitrary number in E. Since $0 < u_{\tilde{F}_2} \leqq u < 1$ on $R_0 - F_2$, we can find $\delta > 0$ such that

$$\int_{C_{t1}} u_{\tilde{F}_2} \frac{\partial v}{\partial \nu} ds \leqq \|v\|^2_{R_0 - F_2} - \delta.$$

Hence we have

$$\|v\|^2_{R_0 - F_2} - \delta \geqq \int_{C_{t1}} u_{\tilde{F}_2} \frac{\partial v}{\partial \nu} ds = \lim_{t \to 1, t \in E} \int_{C_t} u_{\tilde{F}_2} \frac{\partial v}{\partial \nu} ds.$$

Let t_2 be any number in E such that $t_2 > t_1$. Since $u \geqq v$, we obtain that

$$\int_{C_{t2}} u \frac{\partial v}{\partial \nu} ds \geqq \int_{C_{t2}} v \frac{\partial v}{\partial \nu} ds = t_2 \int_{C_{t2}} \frac{\partial v}{\partial \nu} ds$$

$$= t_2 \|v\|^2_{R_0 - F_2}.$$

Thus we have

$$\lim_{t \to 1, t \in E} \int_{C_t} u \frac{\partial v}{\partial \nu} ds \geqq \|v\|^2_{R_0 - F_2} \geqq \delta + \lim_{t \to 1, t \in E} \int_{C_t} u_{\tilde{F}_2} \frac{\partial v}{\partial \nu} ds.$$

This shows that $u \neq u_{\tilde{F}_2}$. Hence F_2 is thin at b by (E, 4).

3.2 Poles of Kuramochi boundary point

Let R^* be a compactification of R. Let b be a point in $\Delta_1 (\subset \Delta_N)$. If $(\tilde{g}_b)_{\widetilde{\{\xi\}}} = \tilde{g}_b$ for $\xi \in \Delta$, we say that ξ is a (full-) pole of b on Δ. We denote by $\Phi(b)$ (resp. $\Phi_W(b)$)the set of all poles of b on Δ_D (resp. Δ_W). By definition, we see that the set of all poles of b on Δ is closed, and hence both $\Phi(b)$ and $\Phi_W(b)$ are closed. The following lemma shows that both $\Phi(b)$ and $\Phi_W(b)$ are non-empty.

Lemma 14. *Let R^* be a compactification of R and b be a point in Δ_1. Then we have*

(a) *If a closed set F in R is not thin at b, then there exists at least one pole of b on Δ which is contained in $\bar{F}^* \cap \Delta$.*

(b) *If $(\tilde{g}_b)_{\tilde{A}} = \tilde{g}_b$ for a closed subset A of Δ, then there exists at least one pole of b on Δ which is contained in A.*

Proof. Suppose every $\xi \in \bar{F}^* \cap \Delta$ is not a pole of b. Since $(\tilde{g}_b)_{\widetilde{\{\xi\}}} \neq \tilde{g}_b$, we can find a regular closed set F_ξ in R such that $\bar{F}^*_\xi$ is a neighborhood of ξ and F_ξ is thin at b. Since $\bar{F}^* \cap \Delta$ is compact, there exists a finite family $\{F_{\xi_k}\}_{k=1}^n$ of regular closed sets in R such that $\bigcup_{k=1}^n \bar{F}^*_{\xi_k}$ is a neighborhood of $\bar{F}^* \cap \Delta$. Since $\bigcup_{k=1}^n \bar{F}^*_{\xi_k} = \overline{\bigcup_{k=1}^n F^*_{\xi_k}}$, we can find a relatively compact open set D

in R such that

$$F-D\subset\bigcup_{k=1}^{n}F_{\xi_k}.$$

Since $\bigcup_{k=1}^{n}F_{\xi_k}$ is thin at b by (E, 2), $F-D$ is thin at b by (E, 1). Hence we see that F is thin at b. This a contradiction. Thus we obtain (a). Similarly we can show (b).

LEMMA 15. *Let* π (resp. π_W) *be the canonical mapping of* R_D^* (resp. R_W^*) *onto* R_N^*. *If b is a point in* Δ_1, *then* $\Phi(b)\subset\pi^{-1}(b)$ *and* $\Phi_W(b)\subset\pi_W^{-1}(b)$.

PROOF. Let b_0 be a point in Δ_N. If $b_0\neq b$, then by (E, 3) $(\tilde{g}_b)_{\widetilde{\{b_0\}}}\neq\tilde{g}_b$. By continuity of π, $(\tilde{g}_b)_{\widetilde{\pi^{-1}(b_0)}}=(\tilde{g}_b)_{\widetilde{\{b_0\}}}$. Hence, for $\xi\in\pi^{-1}(b_0)$, $(\tilde{g}_b)_{\widetilde{\{\xi\}}}\leqq(\tilde{g}_b)_{\widetilde{\{b_0\}}}\neq\tilde{g}_b$, so that $(\tilde{g}_b)_{\widetilde{\{\xi\}}}\neq\tilde{g}_b$. Thus, $\pi^{-1}(b_0)\cap\Phi(b)=\emptyset$, and hence $\Phi(b)\subset\pi^{-1}(b)$. Similarly we have $\Phi_W(b)\subset\pi_W^{-1}(b)$.

3.3 Poles on the Royden boundary

PROPOSITION 5. *Let b be a point in* Δ_S *and F be a regular closed set in R. Then F is thin at b if and only if* $\bar{F}^D\cap\Phi(b)=\emptyset$.

PROOF. The "if" part follows from (a) in Lemma 14. To prove "only if" part, suppose $\bar{F}^D\cap\Phi(b)\neq\emptyset$. Since $\tilde{g}_b$ is a Dirichlet function by (c) in Lemma 13, $(\tilde{g}_b)_{\widetilde{\bar{F}^D\cap\Delta_D}}\leqq(\tilde{g}_b)_{\bar{F}}$ by Lemma 12. For any $\xi\in\bar{F}^D\cap\Phi(b)$, $\tilde{g}_b=(\tilde{g}_b)_{\widetilde{\{\xi\}}}\leqq(\tilde{g}_b)_{\widetilde{\bar{F}^D\cap\Delta_D}}\leqq(\tilde{g}_b)_{\bar{F}}\leqq\tilde{g}_b$. Therefore, $(\tilde{g}_b)_{\bar{F}}=\tilde{g}_b$, i.e., F is not thin at b. Thus the "only if" part is proved.

COROLLARY 1. *Let* $b\in\Delta_S$ *and let* $\mathscr{G}_b=\{G\subset R;\ R-G$ *is a regular closed set in R and is thin at* $b\}$. *Then* $\{\bar{G}^D;\ G\in\mathscr{G}_b\}$ *is a fundamental system of neighborhoods of* $\Phi(b)$ *in* R_D^*.

COROLLARY 2. *If b is a point in* Δ_S, *then* $\Phi(b)$ *consists of only one point.*

PROOF. Suppose $\Phi(b)$ contains two distinct points ξ_1 and ξ_2. Then we can find two regular closed sets F_1 and F_2 in R such that $\bar{F}_k^D$ is a neighborhood of ξ_k in $R_D^*(k=1, 2)$ and $\bar{F}_1^D\cap\bar{F}_2^D=\emptyset$. It follows from the Proposition that neither F_1 nor F_2 is thin at b. This is a contradiction by Proposition 4.

THEOREM 1. *For* $b\in\Delta_1$, $\tilde{C}(\{b\})=C(\Phi(b))$ *and* $\omega^N(\{b\})=\omega^D(\Phi(b))$.

PROOF. Let π be the canonical mapping of R_D^* onto R_N^*. By virtue of Lemma 15, we have $\tilde{C}(\{b\})=C(\pi^{-1}(b))\geqq C(\Phi(b))$ and $\omega^N(\{b\})=\omega^D(\pi^{-1}(b))\geqq\omega^D(\Phi(b))$. Since $\tilde{C}(\{b\})=0$ and $\omega^N(\{b\})=0$ for $b\in\Delta_1-\Delta_S$, it is sufficient to prove the theorem for $b\in\Delta_S$. Again by Lemma 15, it is enough to show that $C(\pi^{-1}(b)-\Phi(b))=0$ and $\omega^D(\pi^{-1}(b)-\Phi(b))=0$. Let K be an arbitrary compact subet of $\pi^{-1}(b)-\Phi(b)$. Since K and $\Phi(b)$ are compact and $K\cap\Phi(b)=\emptyset$, we can find two regular closed sets F_1 and F_2 in R such that $\bar{F}_1^D\cap\bar{F}_2^D=\emptyset$,

$\bar{F}_1^D$ is a neighborhood of K and $\bar{F}_2^D$ is a neighborhood of $\Phi(b)$. Since $\Phi(b)\cap\bar{F}_2^D\neq\emptyset$, F_2 is not thin at b by Proposition 5. Hence it follows from Proposition 4 that F_1 is thin at b. Let $\{V_n\}_{n=1}^{\infty}$ be a sequence of regular closed subsets of R_0 such that $\bar{V}_n^N$ is a neighborhood of b, $V_n\supset V_{n+1}$ $(n=1, 2,\cdots)$ and $\bigcap_{n=1}^{\infty}\bar{V}_n^N=\{b\}$. Since $K\subset\pi^{-1}(b)$, $\bar{V}_n^D$ is a neighborhood of K in R_D^*. Hence $U_n=\bar{V}_n^D\cap\bar{F}_1^D$ is a neighborhood of K in R_D^*. Therefore, applying Lemma 8 with $u=1$, $A=K$ and the above U_n, we obtain a sequence $\{\delta_n\}_{n=1}^{\infty}$ of regular closed sets in R such that $\bar{\delta}_n^D$ is a neighborhood of K, $\delta_n\subset F_1\cap V_n$, $\bigcap_{n=1}^{\infty}\delta_n=\emptyset$, $\overline{R-\delta_n^{iD}}\cap\bar{\delta}_{n+1}^D=\emptyset$ and $1_{\tilde{\delta}_n}$ decreases to $1_{\tilde{K}}=\tilde{\omega}(K)$. Since F_1 is thin at b, δ_n is also thin at b by (E, 1). Suppose $\tilde{\omega}(K)>0$. Since the measure associated with $1_{\tilde{\delta}_n}$ is supported by $\bar{\delta}_n^N$, the measure associated with $\tilde{\omega}(K)$ is supported by $\bigcap_{n=1}^{\infty}\bar{\delta}_n^N=\{b\}$. Hence we see that $\tilde{\omega}(K)=c_0\tilde{g}_b$ for some $c_0>0$. It follows from (c) in Lemma 13 that $\tilde{\omega}(K)=c\tilde{\omega}(\{b\})$ for some $c>0$. Since $\sup\tilde{\omega}(K)=\sup\tilde{\omega}(\{b\})=1$ by (β) in Lemma 4, we have $c=1$. Hence $\tilde{\omega}(K)=\tilde{\omega}(\{b\})$. Since $(\tilde{\omega}(K))_{\tilde{\delta}_n}=\tilde{\omega}(K)$ by (α) in Lemma 4, $(\tilde{\omega}(\{b\}))_{\tilde{\delta}_n}=\tilde{\omega}(\{b\})$. This shows that δ_n is not thin at b by (E, 4). This is a contradiction. Thus $\tilde{\omega}(K)=0$ and $C(K)=0$. It follows that $C(\pi^{-1}(b)-\Phi(b))=0$. Since $0\leqq\omega^{D,R_0}(K)\leqq\tilde{\omega}(K)=0$, $\omega^{D,R_0}(K)=0$ for the above K. Hence, by the Corollary to Lemma 3, $\omega^D(K)=0$, and hence $\omega^D(\pi^{-1}(b)-\Phi(b))=0$.

PROPOSITION 6. *Let π be the canonical mapping of R_D^* onto R_N^* and set* $\Delta_S^D=\{\xi\in\Delta_D;\ C(\{\xi\})>0\}$ $(\subset\Gamma_D)$. *Then*

(a) *Φ induces a one-to-one mapping of Δ_S onto Δ_S^D.*

(b) *π restricted on Δ_S^D is a one-to-one mapping of Δ_S^D onto Δ_S.*

Furthermore, $\pi\circ\Phi$ is the identity on Δ_S and $\Phi\circ\pi$ is the identity on Δ_S^D.

PROOF. By Corollary 2 to Proposition 5, we see that Φ induces a mapping of Δ_S into Δ_D. By Theorem 1, $\Phi(b)\in\Delta_S^D$ for any $b\in\Delta_S$. Let $\xi\in\Delta_S^D$. Since $1_{\widetilde{\pi(\xi)}}\geqq 1_{\widetilde{\{\xi\}}}>0$, $\pi(\xi)\in\Delta_S$. On the other hand, $\Phi(\pi(\xi))\in\Delta_S^D\cap(\pi^{-1}(\pi(\xi))$ by the above and Lemma 15. As shown in the proof of Theorem 1, $C(\pi^{-1}(\pi(\xi))-\Phi(\pi(\xi))=0$. Since $C(\{\xi\})>0$, it follows that $\xi=\Phi(\pi(\xi))$. Therefore, Φ is an onto mapping, $\Phi\circ\pi$ is an identity on Δ_S^D, and hence π is one-to-one.

Next, let $b\in\Delta_S$. Again by Lemma 15, $\pi(\Phi(b))\subset\pi(\pi^{-1}(b))=\{b\}$. Thus, $\pi(\Phi(b))=b$, so that π is an onto mapping, $\pi\circ\Phi$ is an identity on Δ_S and Φ is one-to-one on Δ_S.

3.4 Characterizations of singular points by poles

THEOREM 2. *For each $b\in\Delta_1$, either $\Phi(b)$ consists of only one point or contains an uncountable number of points accordingly as b is singular or not.*

PROOF. If b is singular, then, by Corollary 2 to Proposition 5, $\Phi(b)$ consists of only one point. Next suppose b is a point in $\Delta_1-\Delta_S$. Let $\{F_n\}_{n=1}^\infty$ be a sequence of regular closed subsets of R_0 such that $\bar F_n^N$ is a neighborhood of b, $\overline{R-F_n^{iN}}\cap\bar F_{n+1}^N=\emptyset$ $(n=1, 2,\dots)$ and $\bigcap_{n=1}^\infty\bar F_n^N=\{b\}$. Then $1_{\bar F_n}\to 0$ as $n\to\infty$. For each m, n $(m>n)$, let $f_{n,m}=1_F^G$ where $F=F_m$ and $G=F_n^i$. By Lemma 7, $1_{\bar F_n}\to 0$ implies $f_{1,n}\to 0$ as $n\to\infty$. It follows that $\|f_{1,n}\|\to 0$ as $n\to\infty$. Thus, there is n_1 such that $\|f_{1,n_1}\|<1/2$. By induction, we can find a subsequence $\{F_{n_k}\}_{k=1}^\infty$ of $\{F_n\}_{n=1}^\infty$ such that $\|f_{n_k,n_{k+1}}\|<1/2^{k+1}$ $(k=1, 2,\dots)$. Thus we may assume that $\|f_{n,n+1}\|<1/2^{n+1}$ $(n=1, 2,\dots)$ from the beginning.

We set

$$\phi_n=0 \text{ on } (R-F_{2n-1}^i)\cup F_{2n+1}, =f_{2n-1,2n} \text{ on } F_{2n-1}^i-F_{2n},$$

$$=1-f_{2n,2n+1} \text{ on } F_{2n}^i-F_{2n+1} \text{ and} =1 \text{ on } \partial F_{2n}$$

$(n=1, 2,\dots)$. Then ϕ_n is a function in BCD and $\|\phi_n\|<1/2^n$. Then, it is easy to see that $f=\sum_{n=1}^\infty\phi_n$ belongs to BCD. For each $\alpha(0<\alpha<1)$, we set

$$\Omega_{\alpha,n}=\{z\in F_{2n-1};\ f(z)\geqq\alpha\}\cup F_{2n}$$

and

$$C_\alpha=\{z\in R;\ f(z)=\alpha\}.$$

Then $\Omega_{\alpha,n}$ and C_α are regular closed and $\partial\Omega_{\alpha,n}\subset C_\alpha$. Since $(\tilde g_b)_{\tilde\Omega_{\alpha,n}}=\tilde g_b$ on R_0, $(\tilde g_b)_{\widetilde{\partial\Omega}_{\alpha,n}}=\tilde g_b$ on $R_0-\Omega_{\alpha,n}$ by Lemma 5. This shows that $(\tilde g_b)_{\tilde C_\alpha}=\tilde g_b$ on R_0 for each α. We set $A_\alpha=\bar C_\alpha^D\cap\Delta_D$. For an arbitrary $\alpha(0<\alpha<1)$, let $\{R_n\}_{n=1}^\infty$ be an exhaustion of R such that $C_\alpha-R_n$ is regular closed in R. Since $C_\alpha=(C_\alpha-R_n)\cup(C_\alpha\cap(R_n\cup\partial R_n))$ and $C_\alpha\cap(R_n\cup\partial R_n)$ is thin at b by (E, 3), we have $(\tilde g_b)_{\widetilde{C_\alpha-R_n}}=(\tilde g_b)_{\tilde C_\alpha}=\tilde g_b$ on R_0 for each n. Hence it follows from Lemma 11 that

$$\tilde g_b\geqq(\tilde g_b)_{\tilde A_\alpha}\geqq\lim_{n\to\infty}(\tilde g_b)_{\widetilde{C_\alpha-R_n}}=\tilde g_b \qquad \text{on } R_0.$$

Hence $(\tilde g_b)_{\tilde A_\alpha}=\tilde g_b$ on R_0 for each α. By (b) in Lemma 14, there exists at least one pole $z(\alpha)$ of b on A_α for each α. If $\alpha\neq\alpha'$, then $A_\alpha\cap A_{\alpha'}=\emptyset$ since $f\in BCD$. Hence $\Phi(b)$ is uncountable. This completes the proof.

As for $\Phi_W(b)$, we have

THEOREM 3. *For a point b in $\Delta_1-\Delta_{SS}$, $\Phi_W(b)$ contains an uncountable number of points. For a point b in Δ_{SS}, $\Phi_W(b)$ does not necessarily consist of a single point.*

PROOF. First let b be a point in $\Delta_1-\Delta_{SS}$. Let $\{F_n\}_{n=1}^\infty$ be a sequence of regular closed sets in R such that $\bar F_n^N$ is a neighborhood of b in R_N^*, $\overline{R-F_n^{iN}}\cap\bar F_{n+1}^N=\emptyset$ $(n=1, 2,\dots)$ and $\bigcap_{n=1}^\infty\bar F_n^N=\{b\}$. Then, by assumption, we see that

$1_{F_n}\to 0$ as $n\to\infty$. Since $\overline{R-\bar{F}_n^{iW}}\cap\bar{F}_{n+1}^W=\emptyset$ $(n=1, 2,\cdots)$ and $\bigcap_{n=1}^{\infty} F_n=\emptyset$, we can apply Proposition 2 to this case and obtain a function f in BCW such that $f=1$ on ∂F_{2n} and $f=0$ on $\partial F_{2n-1}(n=1, 2, \cdots)$. For each $(0<\alpha<1)$, we set

$$C_\alpha=\{z \in R; f(z)=\alpha\} \text{ and } A_\alpha=\bar{C}_\alpha^W\cap\Delta_W.$$

By a discussion similar to the proof of Theorem 2, we obtain that $(\tilde{g}_b)_{\tilde{A}_\alpha}=\tilde{g}_b$ on R_0 for each α and that $\Phi_W(b)$ is uncountable.

Secondly suppose R belongs to $O_{HD}-O_{HB}$. Then Δ_{SS} consists of a single point b. Furthermore Γ_D consists of a single point $\Phi(b)$ and Γ_W contains at least two distinct points ξ_1 and ξ_2. Then we can find two regular closed sets F_1 and F_2 in R such that $\bar{F}_k^W$ is a neighborhood of ξ_k $(k=1, 2)$ and $\bar{F}_1^W\cap\bar{F}_2^W=\emptyset$. Since both ξ_1 and ξ_2 are mapped to $\Phi(b)$ by the canonical mapping of R_W^* onto R_D^*, we see that $\Phi(b)\in\bar{F}_k^D$ $(k=1, 2)$. Hence $(\tilde{g}_b)_{\tilde{F}_k}=\tilde{g}_b$ on R_0 by Proposition 5, and hence $(\tilde{g}_b)_{\widetilde{\bar{F}_k^W\cap\Delta_W}}=\tilde{g}_b$ on R_0 $(k=1, 2)$ by Lemma 11. This shows that both ξ_1 and ξ_2 belong to $\Phi_W(b)$. This completes the proof.

3.5 A property of Riemann surfaces belonging to $O_{HD}-O_{HB}$

PROPOSITION 7 (cf. [3; Satz 9.10]). *If ξ is a point in Δ_D with $C(\{\xi\})>0$, then there exists a fundamental system of open connected neighborhoods of ξ in R_D^*.*

PROOF. Let π be the canonical mapping of R_D^* onto R_N^*. By Proposition 6, we have $\Phi(\pi(\xi))=\xi$. By Theorem 1, we see that $\tilde{\omega}(\{\xi\})=1_{\{\tilde{\xi}\}}=1_{\widetilde{\pi(\xi)}}=(\sup \tilde{g}_{\pi(\xi)})^{-1}\tilde{g}_{\pi(\xi)}$. We set $u=\tilde{\omega}(\{\xi\})$. Let U be an arbitrary neighborhood of ξ in R_D^* such that $U\cap R$ is a regular open set in R and $K_0\cap((U\cap R)\cup\partial(U\cap R))=\emptyset$. Then $F=R-U\cap R$ is a regular closed set in R. Since $\xi\notin\bar{F}^D$, it follows from Proposition 5 that F is thin at $\pi(\xi)$, and hence $u_{\tilde{F}}\neq u$. Hence there exists a connected component G of $R_0-F=U\cap R$ such that $u_{\tilde{F}}<u$ on G. Since $u_{\widetilde{R-G}}=u_{\widetilde{\partial G}}\leq u_{\tilde{F}}$ on G by Lemma 5, we see that $u_{\widetilde{R-G}}\neq u$. Hence $\xi\notin\overline{R-G^D}$ by Proposition 5. Thus $\xi\in\bar{G}^D$. Since $\bar{G}^D-\overline{\partial G}^D$ is open in R_D^* (cf., [3; Satz 9.9]) and $\xi\notin R_D^*-U(\subset\overline{\partial G}^D)$, $\bar{G}^D-\overline{\partial G}^D$ is an open connected neighdorhood of ξ in R_D^*. This completes the proof.

COROLLARY 1. *Let π be the canonical mapping of R_W^* onto R_D^*. If ξ is a point in Δ_D with $C(\{\xi\})>0$, then $\pi^{-1}(\xi)$ is connected.*

PROOF. Let $\{U_\alpha\}_{\alpha\in A}$ be a fundamental system of open connected neighborhoods of ξ in R_D^* where A is an index set. Since $\overline{U_\alpha\cap R}^W$ is connected and $\{\overline{U_\alpha\cap R}^W; \alpha\in A\}$ is a lower directed family, we see that $\pi^{-1}(\xi)=\bigcap_{\alpha\in A}\overline{U_\alpha\cap R}^W$ is connected.

COROLLARY 2 ([3; Satz 9. 10]). *If ξ is a point in Δ_D with $\omega^D(\{\xi\})>0$,* *then there exists a fundamental system of open connected neighborhoods of* ξ *in* R_D^*.

THEOREM 4. *If R belongs to* $O_{HD}-O_{HB}$, *then there exists a bounded continous Green potential p on R such that* $\omega^D(\bar{A}_\alpha^D \cap \Delta_D)>0$ *for some* $\alpha>0$ *where* $A_\alpha=\{z \in R; p(z) \geqq \alpha\}$.

PROOF. If $R \in O_{HD}-O_{HB}$, then Γ_D consists of a single point ξ with $\omega^D(\{\xi\})>0$. Let π be the cannonical mapping of R_W^* onto R_D^*. Since $\pi(\Gamma_D)=\Gamma_W$, we have $\Gamma_W \subset \pi^{-1}(\Gamma_D)=\pi^{-1}(\xi)$. Since Γ_W contains at least two distinct points and is totally disconnected (cf. [3; Satz 9.6]), the connectedness of $\pi^{-1}(\xi)$ (the above Corollary 1) implies the existence of a point $z \in \Lambda_W=\Delta_W-\Gamma_W$ such that $\pi(z)=\xi$. lt follows from Hilfssatz 8.4 in [3] that there exists a bounded continuous Green potential p on R such that $\lim_{a\to z} p(a)>0$. Let α be a real numder such that $0<\alpha<\lim_{a\to z} p(a)$. Since $\pi(z)=\xi$, we see that $\xi \in \bar{A}_\alpha^D \cap \Delta_D$. This completes the proof.

COROLLARY. *If R belongs to* $O_{HD}-O_{HB}$, *then there exists a bounded continuous Green potential p on R such that* $C(\bar{A}_\alpha^D \cap \Delta_D)>0$ *for some* $\alpha>0$.

§4 Function-theoretic separative conditions

In this section, for a given compactification R^* of R and a closed subset A of Δ, we set $\mathscr{V}(A)=\{F; F$ is regular closed in R and $\bar{F}^*$ is a neighborhood of A in $R^*\}$.

4.1 General notion of separative compactification

Let $R\cup\Gamma$ be a compactification of R and Ψ be a capacity on Γ in the sense of Choquet. For a subset E of R, we denote by E^a the closure of E in $R\cup\Gamma$.

DEFINITION 1. Let R^* be a compactification of R. Then R^* is said to be *Ψ-separative* if $\Psi(F_1^a \cap F_2^a)=0$ for any regular closed sets F_1 and F_2 in R such that $\bar{F}_1^* \cap \bar{F}_2^*=\emptyset$ in R^*.

The following lemma follows immediately from the definition.

LEMMA 16. *Let* R_1^* *and* R_2^* *be two compactifications of R. If* R_2^* *is a quotient space of* R_1^* *and* R_1^* *is Ψ-separative, then* R_2^* *is also Ψ-separative.*

PROPOSITION 8. *Let* R^* *be a compactification of R. Suppose, for any two distinct points* ξ_1 *and* ξ_2 *in* Δ, *there exist* $A \in \mathscr{V}(\{\xi_1\})$ *and* $B \in \mathscr{V}(\{\xi_2\})$ *such that*

(a) $\bar{A}^* \cap \bar{B}^*=\emptyset$ *in* R^*,

(b) $\Psi(A^a \cap B^a)=0$.

Then R^* is Ψ-separative.

Proof. Let F_1 and F_2 be two regular closed sets in R such that $\bar{F}_1^* \cap \bar{F}_2^* = \emptyset$. We shall show that $\Psi(F_1^a \cap F_2^a)=0$. Let $\alpha_k=\bar{F}_k^* \cap \Delta$ $(k=1, 2)$. We may assume that $\alpha_k \neq \emptyset$ $(k=1, 2)$. For any $\xi \in \alpha_1$ and $\eta \in \alpha_2$, we can find $A_{\xi,\eta} \in \mathscr{V}(\{\xi\})$ and $B_{\xi,\eta} \in \mathscr{V}(\{\eta\})$ which satisfy (a) and (b). First, fix ξ. Since α_2 is compact, there exists a finite number of points $\{\eta_k\}_{k=1}^n$ in α_2 such that $B_\xi = \bigcup_{k=1}^n B_{\xi,\eta_k} \in \mathscr{V}(\alpha_2)$. We may assume that $A_\xi = \bigcap_{k=1}^n A_{\xi,\eta_k}$ belongs to $\mathscr{V}(\{\xi\})$. Since $A_\xi^a \cap B_\xi^a \subset \bigcup_{k=1}^n (A_{\xi,\eta_k}^a \cap B_{\xi,\eta_k}^a)$, we see that $\Psi(A_\xi^a \cap B_\xi^a) \leqq \sum_{k=1}^n \Psi(A_{\xi,\eta_k}^a \cap B_{\xi,\eta_k}^a \cap \Gamma)=0$. Next, varying η, we can similarly show that there exist $U \in \mathscr{V}(\alpha_1)$ and $V \in \mathscr{V}(\alpha_2)$ such that $\bar{U}^* \cap \bar{V}^* = \emptyset$ in R^* and $\Psi(U^a \cap V^a)=0$. Since $F_1^a \cap \Gamma \subset U^a \cap \Gamma$ and $F_2^a \cap \Gamma \subset V^a \cap \Gamma$, we odtain that $\Psi(F_1^a \cap F_2^a)=0$. Hence R^* is Ψ-separative.

Corollary. *Let f be any non-constant function in BC. Suppose there is a dense subset E of* $[\inf f, \sup f]$ *such that $\Psi(\{f \leqq r_1\}^a \cap \{f \geqq r_2\}^a)=0$ for any $r_1, r_2 \in E$ with $r_1 < r_2$. Then $R_{\{f\}}^*$ is Ψ-separative.*

Proof. Let ξ_1 and ξ_2 be two distinct points of $\Delta = R_{\{f\}}^* - R$. We may assume that $t_1 = \lim_{z \to \xi_1} f(z) < \lim_{z \to \xi_2} f(z) = t_2$. Let r_1 and r_2 be numbers in E such that $t_1 < r_1 < r_2 < t_2$. Then we can find $A \in \mathscr{V}(\{\xi_1\})$ and $B \in \mathscr{V}(\{\xi_2\})$ such that $f < r_1$ on A and $f > r_2$ on B. We see that

$$A^a \cap B^a \subset \{f \leqq r_1\}^a \cap \{f \geqq r_2\}^a.$$

Thus

$$\Psi(A^a \cap B^a) \leqq \Psi(\{f \leqq r_1\}^a \cap \{f \geqq r_2\}^a)=0.$$

Hence it follows from the proposition that $R_{\{f\}}^*$ is Ψ-separative.

Theorem 5. *There is always a maximum Ψ-separative compactification of R up to a homeomorphism, i.e., there exists a Ψ-separative compactification R_Ψ^* of R such that any other Ψ-separative compactification of R is a quotient space of R_Ψ^*.*

Proof. We set $Q_0 = \{f \in BC;\ R_{\{f\}}^* \text{ is } \Psi\text{-separative}\}$. Let R^* be any Ψ-separative compactification of R. If we set $Q = BC \cap C(R^*)$, then $R^* = R_Q^*$. Let f be any function in Q. Then it follows from Lemma 16 that $R_{\{f\}}^*$ is Ψ-seprarative. Hence f belongs to Q_0 and $Q \subset Q_0$. This shows that R_Q^* is a quotient space of $R_{Q_0}^*$. Now, we shall show that $R_{Q_0}^*$ itself is Ψ-separative. If Q_0 consists of only constant functions, then $R_{Q_0}^*$ is the one-point compactification, and is trivially Ψ-separative. Suppose Q_0 contains non-constant functions and let ξ_1 and ξ_2 be two distinct points in Δ_{Q_0}. Then there exists a function f in Q_0 such that $\lim_{z\to\xi_1} f(z) < \lim_{z\to\xi_2} f(z)$. Choose α, β such that $\lim_{z\to\xi_1} f(z) < \alpha < \beta < \lim_{z\to\xi} f(z)$. Then we can find $A \in \mathscr{V}(\{\xi_1\})$ and $B \in \mathscr{V}(\{\xi_2\})$ such that $f<\alpha$ on A and $f>\beta$ on B. Then $\bar{A}^* \cap \bar{B}^* = \emptyset$ in $R^*_{\{f\}}$. Since $R^*_{\{f\}}$ is Ψ-separative, we have $\Psi(A^a \cap B^a)=0$. Hence it follows from Proposition 8 that $R^*_{Q_0}$ is Ψ-separative.

4.2 *H.D.* separativeness, *H.M.* separativeness and regularity

Let R^* be a resolutive compactification of R. We introduce the following class:

$$C_D(\varDelta) = \{f \in C(\varDelta);\ H_f^{R^*} \in HD\}.$$

DEFINITION 2. A resolutive compactification R^* of R is said to be *regular* if $C_D(\varDelta)$ is dense in $C(\varDelta)$ with respect to the uniform convergence topology.

DEFINITION 3. A compactification R^* of R is said to be *H.D. separative* if $C(\bar{F}_1^D \cap \bar{F}_2^D)=0$ for any regular closed sets F_1 and F_2 in R such that $\bar{F}_1^* \cap \bar{F}_2^* = \emptyset$ in R^*.

DEFINITION 4. A compactification R^* of R is said to be *H.M. separative* if $\omega^D(\bar{F}_1^D \cap \bar{F}_2^D)=0$ for any regular closed sets F_1 and F_2 in R such that $\bar{F}_1^* \cap \bar{F}_2^* = \emptyset$ in R^*.

REMARK: (i) Definition 2 is due to F–Y. Maeda [12].

(ii) Definition 3 is equivalent to the original one defined by Z. Kuramochi [10] in case R^* is metrizable (see Theorem 2 in [19]).

(iii) *H.D.* separativeness is the Ψ-separativeness with $\Gamma = \varDelta_D$ and $\Psi = C$.

(iv) *H.M.* separativeness is the Ψ-separativeness with $\Gamma = \varDelta_D$ and $\Psi = \omega_{a_0}^D (a_0 \in R)$.

(v) Resolutivity is the Ψ-separativeness with $\Gamma = \varDelta_W$ and $\Psi = \omega_{a_0}^W (a_0 \in R)$ (see Corollary 2 to Theorem 1 in [19]).

PROPOSITION 9. *A compactification R^* of R is regular if and only if there exists a non-empty subfamily Q of the vector sum $HBD + BCW_0$ such that $R^* = R_Q^*$.*

PROOF. Suppose R^* is regular. For $f \in C(R^*)$ we denote its restrictions to $\varDelta$ and R by $f_\varDelta$ and f_R respectively. We set $C_D(R^*) = \{f \in C(R^*);\ H_{f_\varDelta}^{R^*} \in HD\}$ and $Q = \{f_R;\ f \in C_D(R^*)\}$. Since $C_D(\varDelta) = \{f_\varDelta;\ f \in C_D(R^*)\}$ is dense in $C(\varDelta)$, Q separates points of $\varDelta$. Hence $R^* = R_Q^*$. Let f be any function in Q. By Hilfssatz 8.2 in [3], we see that $f - H_f^{R^*}$ is contained in BCW_0. Thus $Q \subset HBD + BCW_0$. Conversely suppose, for a given R^*, there exists a non-empty subfamily Q of $HBD + BCW_0$ such that $R^* = R_Q^*$. It is easy to see that $C_D(\varDelta_Q)$ is a vector sublattice of $C(\varDelta_Q)$ with respect to the maximum and minimum operations and contains constants. Let b_1 and b_2 be two distinct points of Δ_Q. Then we can find a function f in Q such that $\lim_{a\to b_2} f(a) \neq \lim_{a\to b_1} f(a)$. Let $\psi(b)=\lim_{a\to b} f(a)$ for $b \in \Delta_Q$. Then $\psi(b_1)\neq\psi(b_2)$. Since $H_\psi^{R^*}=h_f \in HBD$, $\psi \in C_D(\Delta_Q)$. Thus $C_D(\Delta_Q)$ separates points of Δ_Q. Hence $C_D(\Delta_Q)$ is dense in $C(\Delta_Q)$ with respect to the uniform convergence topology by the Stone-Weierstrass theorem. Therefore R_Q^* is regular.

We introduce the following notation on types of compactifications:

(D) $R^*=R_Q^*$ for some $Q\subset BCD$.
(HD) R^* is $H.D.$ separative.
(HM) R^* is $H.M.$ separative.
(R) R^* is regular.
(W) R^* is resolutive.

Now we have the following two theorems.

THEOREM 6. $(D)\Rightarrow(R)\Rightarrow(W)$.

PROOF. Since $BCD=HBD+BCD_0\subset HBD+BCW_0$, Proposition 9 implies that $(D)\Rightarrow(R)$. The implication $(R)\Rightarrow(W)$ is a part of the definition of regularity.

THEOREM 7. $(D)\Rightarrow(HD)\Rightarrow(HM)\Rightarrow(W)$.

PROOF. The implication $(D)\Rightarrow(HD)$ is obvious by the definition of $H.D.$ separativeness (cf. Lemma 16). The last two implications follows from Lemma 10.

4.3 Exmaples

EXMAPLE 1. We set $R=\{|z|<1\}$. Let $\omega_a(a \in R)$ be the harmonic measure of the arc $\{e^{i\theta};\ |\theta|<\pi/2\}$ with respect to R. We set $Q=\{\omega_a\}$ and consider R_Q^*. Then we have

(a) R_Q^* is $H.D.$ separative.
(b) R_Q^* is not regular.

PROOF. (a) It is known ([19]) that R_Q^* is $H.D.$ separative.

(b) We set $\xi_0=\{\xi \in \Delta_Q;\ \omega_\xi=0\}$ and $\xi_1=\{\xi \in \Delta_Q;\ \omega_\xi=1\}$. For each $\varepsilon(0<\varepsilon<\pi/2)$, we denote by $u_\varepsilon(a)$ (resp. $v_\varepsilon(a)$) the harmonic measure of the arc $\{e^{i\theta};\ |\theta+\pi/2|<\varepsilon\}$ (resp. $\{e^{i\theta};\ |\theta-\pi/2|<\varepsilon\}$) with respect to R. Given $f \in C(\Delta_Q)$, let $M=\max_{\xi\in\Delta_Q}|f(\xi)|$. Then we can easily show that

$$3M(u_\varepsilon+v_\varepsilon)+f(\xi_0)+(f(\xi_1)-f(\xi_0))\omega_a \in \bar{\mathscr{S}}_f$$

and

$$-3M(u_\varepsilon+v_\varepsilon)+f(\xi_0)+(f(\xi_1)-f(\xi_0))\omega_a \in \underline{\mathscr{S}}_f.$$

Since $u_\varepsilon \to 0$ and $v_\varepsilon \to 0$ as $\varepsilon \to 0$, it follows that

$$H_f(a) = f(\xi_0) + (f(\xi_1) - f(\xi_0))\omega_a \qquad (a \in R).$$

Since ω_a is not a function in BCD, we see that $C_D(\Delta_Q) = \{f \in C(\Delta_Q); f(\xi_1) = f(\xi_0)\}$. Thus, $f_0(\xi) = \omega_\xi$ cannot belong to the closure of $C_D(\Delta_Q)$. Hence R_Q^* is not regular

EXAMPLE 2. Let R be a Riemann surface in $O_{HD} - O_{HB}$. Then it follows from Theorem 4 that there exists a bounded continuous Green potential p on R such that $\omega^D(\bar{A}_\alpha^D \cap \Delta_D) > 0$ for some $\alpha > 0$, where $A_\alpha = \{z \in R; p(z) \geqq \alpha\}$. We set $Q = \{p\}$ and consider R_Q^*. Then we have

(a) R_Q^* is regular.

(b) R_Q^* is not *H.M.* separative.

PROOF. (a) Since $p \in BCW_0$, we see that R_Q^* is regular by Proposition 9.

(b) We shall use the same notation as in the proof of Theorem 4. For each α_1 and α_2 $(0 < \alpha_1 < \alpha_2 < \overline{\lim}_{a \to \xi} p(a))$, we set $A = \{z \in R; p(z) \leqq \alpha_1\}$ and $B = \{z \in R; p(z) \geqq \alpha_2\}$. Since $\underline{\lim}_{a \to \xi} p(a) = 0$ by the definition of Γ_D, we see that $\xi \in \bar{A}^D \cap \bar{B}^D$, so that $\omega^D(\bar{A}^D \cap \bar{B}^D) > 0$. Obviously, $\bar{A}^* \cap \bar{B}^* = \emptyset$ in R_Q^*. Hence R_Q^* is not *H.M.* separative.

EXAMPLE 3. Let R be a Riemann surface with $\Delta_S - \Delta_{SS} \neq \emptyset$. Let b be a point in $\Delta_S - \Delta_{SS}$ and ξ be the unique pole of b on Δ_D. Then it follows from Theorem 1 that $C(\{\xi\}) > 0$ and $\omega^D(\{\xi\}) = 0$. Let $\{F_n\}_{n=1}^\infty$ be a sequence of regular colosed sets in R such that $\bar{F}_n^N$ is a neighdorhood of b in R_N^*, $\overline{R - F_n^i}^N \cap \bar{F}_{n+1}^N = \emptyset (n = 1, 2, \cdots)$ and $\bigcap_{n=1}^\infty \bar{F}_n^N = \{b\}$. Since $\omega^N(\{b\}) = 0$, $1_{F_n} \to 0$ as $n \to \infty$. Since $\overline{R - F_n^i}^D \cap \bar{F}_{n+1}^D = \emptyset$ $(n = 1, 2, \cdots)$, we obtain functions f_n is BCD and f in BCW as in Proposition 2. We set $Q = \{f\}$ and consider R_Q^*. Then we have

(a) R_Q^* is *H.M.* separative.

(d) R_Q^* is not *H.D.* separative.

PROOF. (a) Let r_1, r_2 be real numbers such that $0 < r_1 < r_2 < 1$. We set $A = \{z \in R; f(z) \leqq r_1\}$ and $B = \{z \in R; f(z) \geqq r_2\}$. Since $f = f_n$ on $R - F_{2n+1}^i$, $A - F_{2n+1}^i \subset \{z \in R; f_n(z) \leqq r_1\}$ and $B - F_{2n+1}^i \subset \{z \in ; f_n(z) \geqq r_2\}$. Since f_n is a function in BCD, we see that $\overline{A - F_{2n+1}^i}^D \cap \overline{B - F_{2n+1}^i}^D = \emptyset$. Thus $\bar{A}^D \cap \bar{B}^D \subset (\overline{A - F_{2n+1}^i}^D \cap \bar{F}_{2n+1}^D) \cap (\overline{B - F_{2n+1}^i}^D \cup \bar{F}_{2n+1}^D) = \bar{F}_{2n+1}^D$. This shows that $F_n \in \mathscr{V}(\bar{A}^D \cap \bar{B}^D)$ for each n. Hence $\omega^D(\bar{A}^D \cap \bar{B}^D) \leqq 1_{F_n}$ for each n. By letting $n \to \infty$, we obtain that $\omega^D(\bar{A}^D \cap \bar{B}^D) = 0$. Therefore R_Q^* is *H.M.* separative by the Corollary to Proposition 8.

(d) We set $A = \{f \leqq 1/3\}$ and $B = \{f \geqq 2/3\}$. For each α $(0 < \alpha < 1)$, let $C_\alpha = \{z \in R; f(z) = \alpha\}$ and $A_\alpha = \bar{C}_\alpha^D \cap \Delta_D$. By a discussion similar to that in the proof of Theorem 2, we have $(\tilde{g}_b)_{\tilde{A}_\alpha} = \tilde{g}_b$ on R_0 for each α. Since $\Phi(b)$ consists of only one point ξ, we see that ξ belongs to A_α for each α by Lemma

14, (b). Since $C_\alpha \subset A$ for $\alpha \leqq 1/3$ and $C_\alpha \subset B$ for $\alpha \geqq 2/3$, we see that $\xi \in \bar{A}^D \cap \bar{B}^D$. Hence $0 < C(\{\xi\}) \leqq C(\bar{A}^D \cap \bar{B}^D)$. Therefore R_Q^* is not $H.D.$ separative.

Combining Theorem 6 and Theorem 7 with the above three examples, we have the following relations:

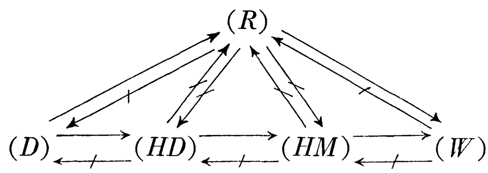

Diagram 1.

In the above diagram, $(A) \rightarrow (B)$ (reap. $(A) \nrightarrow (B)$) means that (A) implies (B) (resp. (A) does not imply (B)).

§5 Martin compactifications of Riemann surfaces belonging to $O_{HD} - O_{HB}$

In this section, let $\omega = \omega^W$ ad $\mu = \omega^M$ (the harmonic measures on Δ_W and Δ_M respectively). Let Δ_1 be the set of all minimal Martin boundary points of Δ_M in this section. It is known (cf. [3]) that Δ_1 is a Borel set and $\mu(\Delta_M - \Delta_1) = 0$.

5.1 Properties of Martin compactification of $R \in O_{HD} - O_{HB}$

The following lemma is due to J.L. Doob [4] (cf. [3]).

LEMMA 17. (a) *Let f be a resolutive function on Δ_M. Then the fine limit*[5] *of H_f exists and epuals f μ-almost everywhere on Δ_1.*

(b) *Let u be a bounded harmonic function on R. Then the fine limit f of u exists μ-almost everywhere on Δ_1 and u equals H_{f*} on R, where f^* is any extension of f over Δ_M.*

PROPOSITION 10. *A hyperbolic Riemann surface R does not belong to O_{HB} if and only if there exist two mutually disjoint compact subsets A_1 and A_2 of Δ_M such that $\mu(A_1) > 0$ and $\mu(A_2) > 0$.*

PROOF. Suppose R does not belong to O_{HB}. Let u be a non-constant bounded harmonic function on R. Then, by (b) in Lemma 17, we see that $u(a) = \int_{\Delta_1} \hat{u}\, d\mu_a$, where $\hat{u}$ is the fine limit of u on Δ_1. Hence we can find two mutually disjoint compact subsets A_1 and A_2 of Δ_M such that $\mu(A_1) > 0$ and $\mu(A_2) > 0$. Conversely suppose there exist two mutually disjoint compact

5) See [4] and [13].

subsets A_1 and A_2 of Δ_M such that $\mu(A_1)>0$ and $\mu(A_2)>0$. Since the greatest harmonic minorant of $\mu_a(A_1)$ and $\mu_a(A_2)$ is equal to $\mu_a(A_1\cap A_2)=0$, either $\mu_a(A_1)$ or $\mu_a(A_2)$ is a non-constant bounded harmonic function on R. Hence R does not belong to O_{HB}.

THEOREM 8. *The Martin compactifications of Riemann surfaces which belong to* $O_{HD}-O_{HB}$ *are not regular.*

PROOF. Let R be a Riemann surface which belongs to $O_{HD}-O_{HB}$. By Proposition 10, we can find two mutually disjoint compact subsets A_1 and A_2 of Δ_M such that $\mu(A_1)>0$ and $\mu(A_2)>0$. Suppose R_M^* is regular. Then we can find $f\in C_D(\Delta_M)$ such that $f\geqq 1$ on A_1 and $f\leqq 0$ on A_2. Since $R\in O_{HD}$, $H_f=$ constant. Hence. by (*a*) in Lemma 17, $f=$ constant μ-almost everywhere on Δ_1, which is a contradiction.

THEOREM 9. *The Martin compactifications of Riemann surfaces which belong to* $O_{HD}-O_{HB}$ *are not H.M. separative, and hence not H.D. separative.*

PROOF. Let R be a Riemann surface belonging to $O_{HD}-O_{HB}$. By Proposition 10, there exist two mutually disjoint compact subsets A_1 and A_2 of Δ_M such that $\mu(A_1)>0$ and $\mu(A_2)>0$. Then there exist two regular closed sets F_1 and F_2 in R such that $\bar{F}_k^M$ is a neighborhood of A_k in $R_M^*(k=1, 2)$ and $\bar{F}_1^M\cap\bar{F}_2^M=\emptyset$. We set $\alpha_k=\bar{F}_k^W\cap\Delta_W(k=1, 2)$. Since $\omega(\pi^{-1}(A))=\mu(A)$ for each compact subset A of Δ_M and $\alpha_k\supset\pi^{-1}(A_k)$ $(k=1, 2)$, we obtain that $0<\mu(A_k)=\omega(\pi^{-1}(A_k))\leqq\omega(\alpha_k)$ $(k=1, 2)$, where π is the canonical mapping of R_W^* onto R_M^*. Since the support of ω is equal to the harmonic boundary Γ_W of R_W^*, we see that $\alpha_k\cap\Gamma_W\neq\emptyset$ $(k=1, 2)$. On the other hand, it is known that Γ_D consists of a single point b. Since R_D^* is a quotient space of R_W^* and $\alpha_k\cap\Gamma_W\neq\emptyset(k=1, 2)$, it follows from Satz 8.6 in [3] that $b\in\bar{F}_1^D\cap\bar{F}_2^D$. Hence $\omega^D(\bar{F}_1^D\cap\bar{F}_2^D)\geqq\omega^D(\{b\})>0$. Therefore R_M^* is not *H.M.* separative, and hence is not *H.D.* separative.

5.2 Normal derivative on the Martin boundary

Let R^* be a resolutive compactification of R and $\lambda_a(a\in R)$ be the harmonic measure on Δ. We fix $a_0\in R$ once for all and let $\lambda\equiv\lambda_{a_0}$, $\mu\equiv\mu_{a_0}$. We set $R_D(\Delta)=\{f;$ resolutive on Δ and $H_f^{R^*}\in HD\}$.

DEFINTION 5 ([12]). Let u be a function in *HD*. We say that u has a *normal derivative* ψ on Δ (relative to a_0), or ψ is a normal derivative of u on Δ (relative to a_0), if $\psi f\in L^1(\Delta)$ and

$$(u, H_f^{R^*})=-\int\psi f d\lambda \qquad \text{for any } f\in R_D(\Delta).$$

F-M. Maeda ([12; Theorem 2]) proved that if the compactification R^* is regular, then the normal derivative of a function u in HD, if it exists, is uniquely determined λ-almost everywhere. We shall show that this result is not valid without regularity.

THEOREM 10. *For the Martin compactification of a Riemann surface which belong to $O_{HD}-O_{HB}$, a normal derivative of a function u in HD, if it exists, is not necessarily uniquely determined μ-almost everywhere.*

PROOF. Let A_1 and A_2 be as in the proof of Theorem 8. We set $\psi_1=\chi_{A_2}-(\mu(A_2)/\mu(A_1))\chi_{A_1}$ and $\psi_2=0$, where χ_A is the characteristic function of a subset A of Δ_M. Let f be any function in $R_D(\Delta_M)$. Since $R\in O_{HD}$, H_f is reduced to a constant. Hence it follows from (a) in Lemma 17 that f equals a constant μ-almost everywhere on Δ_M. Thus for any constant u, $(u, H_f)=0=-\int\psi_1 f d\mu=-\int\psi_2 f d\mu$. This shows that both ψ_1 and ψ_2 are normal derivatives of u on Δ_M. However ψ_1 is not equal to ψ_2 on a set of positive μ-measure.

§6 Extremal length and Green lines

In this section we assume that all compactifications are metrizable.

6.1 Family of curves and extremal length

In the following we consider only locally rectifiable curves and call them curves for simplicity. Let c be a curve on R. Then there exists a parameterization $z=z(t)$ $(0<t<1)$ of c such that $z=z(t)$ is non-constant on any subinterval of $(0, 1)$. We always consider such a parameterization of c and call it a parameterization of c for simplicity. We shall say that a curve c on R meets a subset A of R infinitely many times if there are a parameterization $z=z(t)$ $(0<t<1)$ of c and a sequence $\{t_n\}_{n=1}^{\infty}$ of real numbers such that $0<t_n<t_{n+1}$ $(n=1, 2,\cdots)$, $\lim_{n\to\infty} t_n=1$ and $z(t_n)\in A (n=1, 2,\cdots)$.

We shall say that a curve c on R starts at a point in R and tends to the ideal boundary of R if there is a parameterization $z=z(t)$ $(0<t<1)$ of c satisfying the following:

(i) $\bigcap_{\varepsilon>0}\overline{\{z(t);\, 0<t<\varepsilon\}}$ is a single point in R.

(ii) $\bigcap_{\varepsilon>0}\overline{\{z(t);\, 1-\varepsilon<t<1\}}$ is empty.

Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of regular closed sets in R such that $F_n\supset F_{n+1}$ $(n=1, 2,\cdots)$ and $\bigcap_{n=1}^{\infty} F_n=\emptyset$. Let c be a curve on R which starts at a point in R and tends to the ideal boundary of R. We shall say that c tends to the ideal boundary of R along $\{F_n\}_{n=1}^{\infty}$ if there is a parameterization $z=z(t)$ $(0<t<1)$ satisfying (i) and (ii) and a sequence $\{t_n\}_{n=1}^{\infty}$ of real numbers such that $0<t_n<t_{n+1}$, $\lim_{n\to\infty} t_n=1$ and $z(t)\in F_n$ for $t\geqq t_n$.

The extremal length (or module) of a family C of curves on R is defined as follows (cf. [17]). A non-negative Borel measurable linear density $\rho(z)|dz|$ is called admissible in association with C if $\int_c \rho(z)|dz| \geqq 1$ for each $c \in C$, and the *module* $M(C)$ of C is defined by $\inf\iint \rho^2 dxdy$, where inf is taken over all admissible $\rho(z)|dz|$ and $z=x+iy$ is a local parameter. The *extremal length* $\lambda(C)$ of C is defined by $1/M(C)$. We say that *almost every* curve on R has a property if the module of the family of exceptional curves vanishes.

Properties of modules:

(a) If $C_1 \subset C_2$, then $M(C_1) \leqq M(C_2)$.

(b) $M(\bigcup_{n=1}^{\infty} C_n) \leqq \sum_{n=1}^{\infty} M(C_n)$.

LEMMA 18. *Let F_1 and F_2 be regular colsed sets in R and C be the family of all curves on R each of which meets both F_1 and F_2 infinitely many times. If $\bar{F}_1^D \cap \bar{F}_2^D = \emptyset$, then $M(C)=0$.*

PROOF. We can find a function f in BCD such that $f=0$ on F_1 and $=1$ on F_2. We set $u=f^{F_1 \cup F_2}$. Then it can be seen that $\varepsilon|\operatorname{grad} u(z)||dz|$ is admissible in association with C for any $\varepsilon>0$. Thus we have

$$M(C) \leqq \varepsilon^2 \iint |\operatorname{grad} u|^2 dxdy = \varepsilon^2 \|u\|^2.$$

Since $\|u\| \leqq \|f\| < \infty$ and ε is arbitrary, we obtain that $M(C)=0$.

The following lemma is due to A. Pfluger [18].

LEMMA 19. *Let K be a closed set on $|z|=1$. Then the extremal length of the family of all curves in $1/2<|z|<1$ which connect the points of K to the points of $|z|=1/2$ is infinite if and only if the logarithmic capacity of K is zero.*

PROPOSITION 11. *Let K_0 be a closed disk in R. Let F_1 and F_2 be regular closed subsets of $R_0 = R - K_0$ such that $F_1 \cap F_2 = \emptyset$. Let C be a family of curves on R starting at points of K_0 and tending to the ideal boundary of R. If each member c in C meets both F_1 and F_2 infinitely many times, then $M(C) \leqq 2\pi C(\bar{F}_1^D \cap \bar{F}_2^D)$.*

PROOF. Applying Lemma 8 with $u=1$, $A=\bar{F}_1^D \cap \bar{F}_2^D$ and $U_n = R_D^* - K_0$, we obtain a sequence $\{\delta_n\}_{n=1}^{\infty}$ of regular closed sets in R_0 such that each $\bar{\delta}_n^D$ is a neighborhood of $\bar{F}_1^D \cap \bar{F}_2^D$ in R_D^*, $\bigcap_{n=1}^{\infty} \delta_n = \emptyset$, $\overline{R-\delta_n^{iD}} \cap \bar{\delta}_{n+1}^D = \emptyset$ and $1_{\bar{\delta}_n}$ decreases to $\tilde{\omega}(\bar{F}_1^D \cap \bar{F}_2^D)$. For each n, we set $C_n = \{c \in C;\, c \cap \delta_n = \emptyset\}$ and $C_0 = \bigcup_{n=1}^{\infty} C_n$. Since $\overline{F_1 - \delta_n^{iD}} \cap \overline{F_2 - \delta_n^{iD}} = \emptyset$, it follows from Lemma 18 that $M(C_n)=0$. Hence $M(C_0) \leqq \sum_{n=1}^{\infty} M(C_n) = 0$. Since each member of $C - C_0$ meets all δ_n, $|\operatorname{grad} 1_{\bar{\delta}_n}|$

$|dz|$ is admissible in association with $C-C_0$. Hence we have $M(C-C_0) \leqq \|1_{\tilde{\delta}_n}\|^2$ $(n=1, 2, \cdots)$ Thus we obtain.

$$M(C) \leqq M(C-C_0)+M(C_0)=M(C-C_0) \leqq \|1_{\tilde{\delta}_n}\|^2 \ (n=1, 2, \cdots).$$

By letting $n \to \infty$, we obtain that

$$M(C) \leqq \|\tilde{\omega}(\bar{F}_1^D \cap \bar{F}_2^D)\|^2 = 2\pi C(\bar{F}_1^D \cap \bar{F}_2^D).$$

COROLLARY. *If* $C(\bar{F}_1^D \cap \bar{F}_2^D)=0$, *then* $M(C)=0$.

6.2 Green lines and Dirichlet problems

For the following notation and definitions, we refer to M. Brelot-G. Choquet [1]. We denote by $g_a(z)=g(a, z)$ the Green function of R with pole at $a \in R$. Let a_0 be a fixed point in R and let $g_0(z)=g_{a_0}(z)$. We consider Green lines in R determined by g_0. Then the set L of all Green lines admits the Green measure g. By definition, g is a complete measure. A Green line l for which $\inf_{a \in l} g_0(a)=0$ is called a *regular Green line*. Any regular Green line tends to the ideal boundary of R as $g_0 \to 0$. The set of all regular Green lines will be denoted by L_r. It is known (cf. [1]) that L_r is a G_δ-set in L and $g(L-L_r)=0$. We shall say that *almost every* $l \in L_r$ has a property if the Green measure of the family of exceptional Green lines vanishes.

Given a real-valued function f on R and $l \in L_r$, let $\overline{\lim_l} f$ (resp. $\underline{\lim_l} f$) denote the upper limit $\overline{\lim}_{a \in l, g_0(a) \to 0} f(a)$ (resp. the lower limit $\underline{\lim}_{a \in l, g_0(a) \to 0} f(a)$). If $\overline{\lim_l} f = \underline{\lim_l} f$, then we say that f has a limit along l. Let ψ be an extended real-valued function on L_r. We define

$$\bar{\mathscr{F}}_\psi = \left\{ \begin{array}{l} s;\ \text{superharmonic, bounded below on } R, \\ \underline{\lim_l}\, s \geqq \psi(l) \ \text{ for almost every } \ l \in L_r \end{array} \right\} \cup \{\infty\}$$

and

$$\underline{\mathscr{F}}_\psi = \{-s;\ s \in \bar{\mathscr{F}}_{-\psi}\}.$$

Let $\bar{\mathscr{G}}_\psi(a)=\inf\{s(a);\ s \in \bar{\mathscr{F}}_\psi\}$ and $\underline{\mathscr{G}}_\psi(a)=\sup\{s(a);\ s \in \underline{\mathscr{F}}_\psi\}$ $(a \in R)$. Then it is known ([1]) that $\bar{\mathscr{G}}_\psi$(resp. $\underline{\mathscr{G}}_\psi$) is either harmonic, $\equiv +\infty$ or $\equiv -\infty$. If $\bar{\mathscr{G}}_\psi = \underline{\mathscr{G}}_\psi$ and are harmonic, then we write $\mathscr{G}_\psi = \bar{\mathscr{G}}_\psi = \underline{\mathscr{G}}_\psi$. It is known ([1]) that

$$\underline{\mathscr{G}}_\psi(a_0) \leqq \underline{\int} \psi\, dg \leqq \overline{\int} \psi\, dg \leqq \bar{\mathscr{G}}_\psi(a_0).$$

LEMMA 20. *Let f be a function in BC such that it has a limit $\psi(l)$ along almost every $l \in L_r$. Then we have*

(a) *If f is a function in BCW, then $\mathscr{G}_\psi = h_f$ and*

$$\int \psi\, dg = \mathscr{G}_\psi(a_0) = h_f(a_0).$$

(b) *If f is a function in BCW_0, then $\psi(l)=0$ for almost every $l \in L_r$.*

PROOF. (a) Since $\underline{\mathscr{W}}_f \subset \mathscr{F}_\psi$ and $\overline{\mathscr{W}}_f \subset \bar{\mathscr{F}}_\psi$, we obtain that $\underline{h}_f \leqq \underline{\mathscr{G}}_\psi \leqq \overline{\mathscr{G}}_\psi \leqq \bar{h}_f$. Hence we have ($a$). Then, ($b$) is obvious.

Let t_0 be a real number such that $K_0 = \{z;\ g_0(z) \geqq t_0\}$ is compact in R and $|\text{grad}\ g_0| \neq 0$ on $K_0 - \{a_0\}$. We shall call such a compact set K_0 a *Green disk* with center at a_0. For a subset Λ of L_r, we denote by $\Lambda(K_0)$ the family of curves consisting of the restrictions of $l \in \Lambda$ to $R - K_0$.

The following lemma is due to M. Ohtsuka [16].

LEMMA 21. *Let Λ be a subset of L_r. Then $g(\Lambda)=0$ if and only if $M(\Lambda(K_0)) = 0$.*

Let K_0 be a compact Green disk with center at a_0 and let $R_0 = R - K_0$. Although the following proposition follows from a result by M. Nakai ([14; Proposition 4.1]), we shall give an alternative proof.

PROPOSITION 12. *Let F_1 and F_2 be regular closed subsets of R_0 such that $F_1 \cap F_2 = \emptyset$. Let Λ be a subfamily of L_r whose member meets both F_1 and F_2 infinitely many times. Then we have*

$$\bar{g}(\Lambda) \leqq \omega_{a_0}^D(\bar{F}_1^D \cap \bar{F}_2^D),$$

where $\bar{g}$ means the outer measure induced by g.

PROOF. Let $\{\delta_n\}_{n=1}^\infty$ be as in the proof of Proposition 11. We set $\Lambda_n = \{l \in \Lambda;\ l \cap \delta_n = \emptyset\}$, $\Lambda_0 = \bigcup_{n=1}^\infty \Lambda_n$, $\tilde{\Lambda}_n = \{l \in \Lambda - \Lambda_0;\ l$ meets $R_0 - \delta_n^i$ infinitely many times$\}$ and $\tilde{\Lambda}_0 = \bigcup_{n=1}^\infty \tilde{\Lambda}_n$. As in the proof of Proposition 11, Lemma 18 implies $M(\Lambda_0(K)) = 0$. Since $\overline{R_0 - \delta_n^i}^D \cap \bar{\delta}_{n+1}^D = \emptyset$ and $\tilde{\Lambda}_n \cap \Lambda_{n+1} = \emptyset$, we have $M(\tilde{\Lambda}_n(K_0)) = 0$, $n = 1, 2, \cdots$, again by Lemma 18. Hence $M(\tilde{\Lambda}_0(K_0)) = 0$, and hence $M(\Lambda_0(K_0) \cup \tilde{\Lambda}_0(K_0)) = 0$. Since every $l \in \Lambda - (\Lambda_0 \cup \tilde{\Lambda}_0)$ tends to the ideal boundary of R along $\{\delta_n\}_{n=1}^\infty$, $1_{\delta_n} \in \bar{\mathscr{F}}_{\chi_{\Lambda - (\Lambda_0 \cup \tilde{\Lambda}_0)}}$ ($n = 1, 2, \cdots$). Thus we have

$$0 \leqq \bar{g}(\Lambda - (\tilde{\Lambda}_0 \cup \Lambda_0)) \leqq \overline{\mathscr{G}}_{\chi_{\Lambda - (\Lambda_0 \cup \tilde{\Lambda}_0)}}(a_0) \leqq 1_{\delta_n}(a_0)$$

($n = 1, 2, \cdots$). By letting $n \to \infty$, we obtain that

$$\bar{g}(\Lambda) \leqq \bar{g}(\Lambda - (\tilde{\Lambda}_0 \cup \Lambda_0)) + \bar{g}(\Lambda_0 \cup \tilde{\Lambda}_0) \leqq \omega_{a_0}^D(\bar{F}_1^D \cap \bar{F}_2^D).$$

COROLLARY. *If $\omega^D(\bar{F}_1^D \cap \bar{F}_2^D) = 0$, then $g(\Lambda) = 0$.*

6.3 Separative conditions (E) and (G).

DEFINITION 6. We shall say that a resolutive compactification R^* of R satisfies *condition* (E) if almost every curve on R which starts at a point in R and tends to the ideal boundary of R, has exactly one limit point in Δ.

Definition 7. We shall say that a resolutive compactification R^* of R satisfies *condition* (G) if, for every a_0, almost every Green line tends to one point in Δ.

REMARK: (i) By Lemma 21, condition (E) implies condition (G).
(ii) The condition (G) is said to be Green-compatible in [15].

The following results are known

LEMMA 22 ([11; Theorem 1] and [16; Theorem 1]).

(a) *If Q is a countable subfamily of BC such that each $f \in Q$ has a limit almost along every $l \in L_r$, then R^*_Q satisfies condition (G).*

(b) *If Q is a countable subfamily of BC such that each $f \in Q$ has a limit along almost every curve which starts at a point in R and tends to the ideal boundary of R, then R^*_Q satisfies condition (E).*

THEOREM 11. *The H.D. separativeness implies condition (E).*

PROOF. Since R^* is assumed to be metrizable, we can find a countable subfamily Q of BC such that $R^*=R^*_Q$. Let K_0 be a closed disk in R. We denote by C the family of all curves on R which starts at a point in K_0 and tends to the ideal boundary of R. Since R is covered by a countable family of closed disks, by (b) in Lemma 22, it is sufficient to prove that each $f \in Q$ has a limit along almost every curve in C. Let f be any non-constant function in Q. We may assume that $\inf f=0$ and $\sup f=1$. Let r and r' be two rational numbers such that $0<r<r'<1$. We set $C_{r,r'}=\{c \in C;\ c$ meets both $\{f\leqq r\}$ and $\{f\geqq r'\}$ infinitely many times$\}$. Since $\overline{\{f\leqq r\}}^* \cap \overline{\{f\geqq r'\}}^*=\emptyset$ in R^*_Q, it follows from the Corollary to Proposition 11 and *H.D.* separativeness that $M(C_{r,r'})=0$. Hence $M(\bigcup_{r,r'} C_{r,r'}) \leqq \sum_{r,r'} M(C_{r,r'})=0$. Since f has a limit along every curve $c \in C-\bigcup_{r,r'} C_{r,r'}$, we see that f has a limit along almost every curve in C.

By virtue of Theorem 3 in [19], this theorem implies the following results by M. Ohtsuka [16; Theorem 1 and Theorem 2]:

COROLLARY 1. *If Q is a countable subfamily of BCD and Q separates points of Δ, then almost every curve on R which starts at a point in R and tends to the ideal boundary, converges to a point in Δ.*

COROLLARY 2. *Every function in BCD has a limit along almost every curve which has the property in Corollary 1.*

Using the Corollary to Proposition 12 and (a) in Lemma 22, we can prove the following theorem by the same method as Theorem 11:

THEOREM 12. *The H.M. separativeness implies condition (G).*

COROLLARY 1 ([11]). *Almost every Green line converges to a point of the Kuramochi boundary.*

COROLLARY 2 ([5] and [14]). *Every function in BCD has a limit along almost every Green line.*

REMARK: We do not use the result by M. Godefroid ([5]) to obtain the above Corollary 1 (cf. [11]).

THEOREM 13. (a) *The H.M. separativeness does not imply condition (E).*
(b) *Regularity does not imply condition (E).*

PROOF. (a) Let $R=\{|z|<1\}$. Let K be a closed set on $|z|=1$ such that the logarithmic capacity of K is positive and the harmonic measure of K with respect to R is zero. Since K is compact in $D=\{|z|<2\}$, there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of regular compact sets in D such that $\partial K_n \cap \{|z|=1\}$ consists of a finite number of points, $K_n^i \supset K_{n+1}$ $(n=1, 2, \cdots)$ and $\bigcap_{n=1}^{\infty} K_n = K$. We set $F_n = K_n \cap R$ for each n. Then $1_{F_n} \to 0$ as $n \to \infty$ by the assumption on K. Since $\overline{R-\bar{F}_n^{iD}} \cap \bar{F}_{n+1}^D = \emptyset$ $(n=1, 2, \cdots)$, we obtain functions f_n in BCD and f in BCW as in Proposition 2. We set $Q=\{f\}$ and consider R_Q^*. By the same method as the proof of Example 3, (a), we see that R_Q^* is *H.M.* separative. Next we shall prove that R_Q^* does not satisfy condition (E). Let $K_0 = \{|z| \leqq 1/2\}$ and C be the family of all curves in $R-K_0$ which connect the points of ∂K_0 to the points of K. Then, by Lemma 19, we have $\lambda(C) < \infty$. Let c be any curve in C. Then c meets all ∂F_n. Since $f(z)=1$ for $z \in \partial F_{2k}$ and $=0$ for $z \in \partial F_{2k-1}$ $(k=1, 2, \cdots)$, c does not converge to a point of $\varDelta_Q$. Therefore, R_Q^* does not satisfy condition (E).

(b) Since $\overline{R-\bar{F}_n^{iW}} \cap \bar{F}_{n+1}^W = \emptyset$ $(1, 2, \cdots)$, we now take functions f_n and f in BCW constructed in the proof of Proposition 2 for this $\{F_n\}_{n=1}^{\infty}$. We set $Q=\{f\}$ and consider R_Q^*. By a discussion similar to the above, we see that R_Q^* does not satisfy condision (E). We shall prove that R_Q^* is regular. Since $\partial K_n \cap \{|z|=1\}$ consists of a finite number of points, by the definition of f_n we see that $\lim_{z\to\xi} f_n(z)=0$ if $\xi \in \{|z|=1\} - \bigcup_{k=1}^{\infty} \partial K_{2k} - K$. It follows that $\lim_{l} f_n = 0$ for almost all $l \in L_r$. Hence $\mathscr{G}_{f_n}=0$ for each n, and hence $h_{f_n}=0$ by Lemma 20 for each n. Thus, by the Corollary to Proposition 2, we see that $f \in BCW_0$. Therefore, R_Q^* is regular by Proposition 9.

COROLLARY. *Condition (G) does not imply condition (E).*

Finally, summarizing the above results, we have the following implication diagram for metrizable compactifications: Diagram 2.

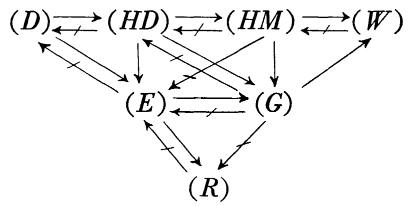

Diagram 2.

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