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Nonimbedding Theorems of Lie Algebras

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§ 1.

Let N be a Lie or associative algebra, and $\mathfrak{D}$ a set of derivations of N which contains all the inner derivations. Two sequences of subspaces $\{N\mathfrak{D}^i\}$ and $\{N\mathfrak{D}_i\}$ are defined inductively as follows ([2, 3]):

$$N\mathfrak{D}^i = N\mathfrak{D}^{i-1}\mathfrak{D}$$

$$N\mathfrak{D}_i = \{x \in N;\ x\mathfrak{D} \subseteq N\mathfrak{D}_{i-1}\},$$

where $N\mathfrak{D}^0 = N$ and $N\mathfrak{D}_0 = 0$. N is called $\mathfrak{D}$-*nilpotent of class* n, when $N\mathfrak{D}^{n-1} \neq 0$ and $N\mathfrak{D}^n = 0$.

Several authors have investigated the nonimbedding of nilpotent algebras. Namely, Chao [1] showed that a non-abelian Lie algebra A such that its center is 1-dimensional or dim $A/[A, A]=2$ cannot be any $N\mathfrak{J}^i$, where $\mathfrak{J}$ is the algebra of all inner derivations of a nilpotent Lie algebra N. Ravisankar [2] improved this result as follows: Such an algebra A cannot be any $N\mathfrak{D}^i$ of a $\mathfrak{D}$-nilpotent algebra N. Moreover, Tôgô and Maruo [3] proved the following theorems among other things:

Let N be a $\mathfrak{D}$-nilpotent algebra, and A a non-abelian subalgebra of N.

(a) If dim $A/[A, A]=2$, then it is impossible that

$$N\mathfrak{D}^i \supseteq A \supseteq N\mathfrak{D}^{i+1} \qquad (i \geq 1).$$

(b) If A is mapped into $[A, A]$ by every derivation of A, then it is impossible that

$$N\mathfrak{D}^i \supseteq A \supseteq N\mathfrak{D}^{i+1} \qquad (i \geq 1) \quad \text{if } A \text{ is Lie,}$$

$$A = N\mathfrak{D}^1 \text{ or } N\mathfrak{D}^i \supseteq A \supseteq N\mathfrak{D}^{i+1} \qquad (i \geq 2) \quad \text{if } A \text{ is associative.}$$

The purpose of the present paper is to improve these two results about nonimbedding of algebras. Hereafter, we suppose that N is a Lie or associative algebra, $\mathfrak{J}(N)$ is the Lie algebra of all inner derivations of N, $\mathfrak{D}$ is a subset of the derivation algebra of N which contains $\mathfrak{J}(N)$, and N is $\mathfrak{D}$-nilpotent of class n.

We shall need the following result stated in [3].

LEMMA. *The sequences $\{N\mathfrak{D}^i\}$ and $\{N\mathfrak{D}_i\}$ are monotone decreasing and* *increasing respectively, and*

$$[N\mathfrak{D}^i, N\mathfrak{D}^j] \subseteq N\mathfrak{D}^{i+j+1}$$

$$[N\mathfrak{D}^i, N\mathfrak{D}_j] \subseteq N\mathfrak{D}_{j-i-1} .$$

§ 2.

THEOREM 1. *Let A be a non-abelian subalgebra of N. If* $\dim A = \dim [A, A] + 2$, *then it is impossible that*

$$N\mathfrak{D}^i \supseteq A \supseteq N\mathfrak{D}^{3i} \qquad (i \geq 1)$$

except for the case where $[A, [A, A]] = 0$ *and* $3i \geq n-1$.

PROOF. Let us suppose that $N\mathfrak{D}^i \supseteq A \supseteq N\mathfrak{D}^{3i}$ for $i \geq 1$.

At first we assume that $[A, [A, A]] = 0$. Then it is obvious that A is 3-dimensional. If $3i \leq n-2$, then $A \supseteq N\mathfrak{D}^{3i} \supseteq N\mathfrak{D}^{n-2}$. On the other hand we have

$$[N\mathfrak{D}^{n-2}, A] \subseteq [N\mathfrak{D}^{n-2}, N\mathfrak{D}^1] \subseteq N\mathfrak{D}^n = 0 ,$$

and therefore $N\mathfrak{D}^{n-2}$ is contained in the center of A, which is 1-dimensional. However, this is contrary to the fact that $N\mathfrak{D}^{n-2} \supset N\mathfrak{D}^{n-1} \neq 0$. Hence it must be $3i \geq n-1$.

So we assume that $[A, [A, A]] \neq 0$. Then $N\mathfrak{D}^{3i+2} \supseteq [A, [A, A]] \neq 0$ implies that $3i+2 \leq n-1$. In the case where $A = N\mathfrak{D}^{3i}$,

$$\dim N\mathfrak{D}^{3i+3} \leq \dim N\mathfrak{D}^{3i} - 3 = \dim A - 3 = \dim [A, [A, A]] \leq \dim N\mathfrak{D}^{9i+2}.$$

As $3i+3 \leq n$ and $[A, [A, A]] \neq 0$, it must be $3i+3 \geq 9i+2$, so we get $6i \leq 1$, which is impossible. So we may assume that $N\mathfrak{D}^i \supseteq A \supset N\mathfrak{D}^{3i}$. Then we have

$$A \supset N\mathfrak{D}^{3i} \supset N\mathfrak{D}^{3i+1} \supset N\mathfrak{D}^{3i+2} \supseteq [A, [A, A]] .$$

Therefore $\dim A/[A, [A, A]] = 3$ implies that $\dim A/N\mathfrak{D}^{3i} = 1$ and $[A, [A, A]] = N\mathfrak{D}^{3i+2}$. When we put $A = \{a\} + N\mathfrak{D}^{3i}$,

$$[A, A] = [a, N\mathfrak{D}^{3i}] + [N\mathfrak{D}^{3i}, N\mathfrak{D}^{3i}] \subseteq N\mathfrak{D}^{4i+1} \subseteq N\mathfrak{D}^{3i+2} = [A, [A, A]] ,$$

which is a contradiction. Hence the theorem is proved.

The following two examples show that there exist algebras N and A such that

$$N\mathfrak{D}^1 \supseteq A \supseteq N\mathfrak{D}^3, \quad \dim A/[A, A] = 2 \quad \text{and} \quad [A, [A, A]] = 0,$$

both for Lie and associative cases.

Example 1. Let N and A be the Lie algebras as follows.

$$N=\{a_1, a_2, a_3, b_1, b_2, c_1, c_2, d\}, \qquad A=\{b_1, b_2, d\},$$

$$[a_1, a_2]=b_1, \qquad [a_2, a_3]=b_2, \qquad [a_2, b_1]=c_1, \qquad [a_2, b_2]=c_2,$$

$$[a_1, c_2]=[a_3, c_1]=[b_1, b_2]=d\,.$$

In addition, we suppose that a product which is not in the table is 0, and $\mathfrak{D}=\mathfrak{J}(N)$. Then we have

$$N\mathfrak{D}^1=\{b_1, b_2, c_1, c_2, d\}\supset A\supset N\mathfrak{D}^3=\{d\}=[A, A]\,.$$

Example 2.[1] We take associative subalgebras N and A of the matrices algebra of order 5 as follows:

$$N=\{e_{21}, e_{32}, e_{43}, e_{54}, e_{31}, e_{42}, e_{53}, e_{41}, e_{52}, e_{51}\}$$

$$A=\{e_{31}, e_{53}, e_{51}\},$$

and we take $\mathfrak{D}=\mathfrak{J}(N)$. Then $\dim A/[A, A]=2$ and

$$N\mathfrak{D}^1=\{e_{31}, e_{42}, e_{53}, e_{41}, e_{52}, e_{51}\}\supset A\supset N\mathfrak{D}^3=\{e_{51}\}.$$

THEOREM 2. *Let A be a non-abelian subalgebra of N. If* $\dim A/[A, A]$ $=m$, *then it is impossible that*

$$N\mathfrak{D}^i\supseteq A\supseteq N\mathfrak{D}^{i+1} \qquad (i\geq m-1)\,.$$

PROOF. On the contrary, we suppose that there exists such a subalgebra A. Obviously we may assume that $A\supset N\mathfrak{D}^{i+1}$. Then,

$$N\mathfrak{D}^i\supseteq A\supset N\mathfrak{D}^{i+1}\supset N\mathfrak{D}^{i+2}\supset\cdots\supset N\mathfrak{D}^{2i+1}\supseteq[A, A]\,. \tag{1}$$

$$m=\dim A/[A, A]\geq\dim A/N\mathfrak{D}^{2i+1}\geq i+1\geq m$$

implies that $i=m-1$, $[A, A]=N\mathfrak{D}^{2i+1}$ and

$$\dim A/N\mathfrak{D}^{i+1}=\dim N\mathfrak{D}^{i+1}/N\mathfrak{D}^{i+2}=\cdots=\dim N\mathfrak{D}^{2i}/N\mathfrak{D}^{2i+1}=1\,.$$

Then we may express A as $A=\{a\}+N\mathfrak{D}^m$. Hence we have

$$N\mathfrak{D}^{2m-1}=[A, A]\subseteq[N\mathfrak{D}^{m-1}, N\mathfrak{D}^m]\subseteq N\mathfrak{D}^{2m}\,,$$

which implies that $N\mathfrak{D}^{2m-1}=[A, A]=N\mathfrak{D}^{2m}=0$. This contradicts our assumption. Hence the theorem is proved.

THEOREM 3. *Let A be a non-abelian subalgebra of N. If* $\dim A/[A, A]=$ m, *then it is impossible that*

[1] This example is due to the communication from Prof. Tôgô.

$$A \subseteq N\mathfrak{D}^{m-1} \quad \textit{and} \quad N\mathfrak{D}_i \supseteq A \supseteq N\mathfrak{D}_{i-1} \qquad (i \geq 1).$$

PROOF. We can prove this theorem as same as Theorem 2. We have only to replace (1) by the following

$$N\mathfrak{D}_i \supseteq A \supset N\mathfrak{D}_{i-1} \supset \cdots \supset N\mathfrak{D}_{i-m+1} \supset N\mathfrak{D}_{i-m} \supseteq [N\mathfrak{D}^{m-1}, N\mathfrak{D}_i] \supseteq [A, A].$$

THEOREM 4. *Let A be a non-abelian subalgebra of N, and we denote by $\mathfrak{D}(A)$ the derivation algebra of A. If*

$$A\mathfrak{D}(A) \subseteq [A, A],$$

then it is impossible that

$$N\mathfrak{D}^i \supseteq A \supseteq N\mathfrak{D}^{i+1} \qquad (i \geq 1).$$

This theorem was proved by Tôgô and Maruo [3] except for the case where $i=1$ and N is an associative algebra, but it remains valid also in this exceptive case as follows. The first half of the proof is same as [3].

PROOF. We suppose that there exists such a subalgebra A. Since A is a $\mathfrak{D}$-ideal as easily seen,

$$N\mathfrak{D}^{i+2} = N\mathfrak{D}^{i+1}\mathfrak{D} \subseteq A\mathfrak{D} \subseteq A\mathfrak{D}(A) \subseteq [A, A] \subseteq N\mathfrak{D}^{2i+1}.$$

Hence we have $i+2 \geq 2i+1$, which implies $i=1$ and $[A, A] = N\mathfrak{D}^3$. Then we get

$$N\mathfrak{D}^4 = [A, A]\mathfrak{D} \subseteq [A\mathfrak{D}, A] \subseteq [[A, A], A] \subseteq N\mathfrak{D}^5,$$

from which it follows $[A, [A, A]] = N\mathfrak{D}^4 = [A, A]\mathfrak{D} = 0$, that is $[A, A] \subseteq N\mathfrak{D}_1$. Then since $A\mathfrak{D} \subseteq [A, A] \subseteq N\mathfrak{D}_1$, we get $A \subseteq N\mathfrak{D}_2$. So from the preceding lemma it follows that

$$[A, A] \subseteq [N\mathfrak{D}^1, N\mathfrak{D}_2] \subseteq N\mathfrak{D}_0 = 0,$$

which is a contradiction. Hence we have the assertion.

References

[1] C.-Y. Chao, A nonimbedding theorem of nilpotent Lie algebras, Pacific J. Math., **22** (1967), 231-234.

[2] T.S. Ravisankar, Characteristically nilpotent algebras, Canad. J. Math., **23** (1971), 222-235.

[3] S. Tôgô and O. Maruo, Nonimbedding theorems of algebras, Hiroshima Math. J., **1** (1971), 5-16.

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