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An Integral Representation of an Eigenfunction of the Laplacian on the Euclidean Space

M. HASHIZUME, A. KOWATA, K. MINEMURA and K. OKAMOTO

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§ 1. Introduction

The classical theory about Dirichlet problem shows that certain classes of harmonic functions on the unit disk are given by the Poisson integral (cf. [1]). However, as Helgason proved in [4], to obtain arbitrary harmonic functions one has to consider the Poisson integral of "hyperfunctions". He also proved that any eigenfunction of the laplacian (with respect to the Poincaré metric) can be given by the Poisson integral of hyperfunctions.

The present paper deals with the similar problem about eigenfunctions of the laplacian on the n-dimensional euclidean space.

Suggested by the work of Ehrenpreis [2], we define the map $\mathscr{P}_\lambda$ which is an analogue of the Poisson integral [see § 4].

In our case, contrary to the usual Poisson integral, it is not sufficient to consider the hyperfunctions to obtain arbitrary eigenfunctions of the laplacian, but one should consider a certain space $\tilde{\mathscr{B}}(S^{n-1})$ which contains the space of hyperfunctions on the $(n-1)$-dimensional unit sphere as a proper subspace. We shall prove in § 5 that our map $\mathscr{P}_\lambda$ gives an isomorphism of $\tilde{\mathscr{B}}(S^{n-1})$ onto the space of the eigenfunctions of the laplacian.

In this paper we deal with the case where $\lambda \neq 0$. We shall discuss the case where $\lambda=0$ in the forthcoming paper [5].

§ 2. Review of the repesentation theory of $SO(n)$

In this section we summarize briefly the representation theory of $\boldsymbol{SO}(n)$. $\boldsymbol{SO}(n)$ acts on $\boldsymbol{R}^n$ and if we denote by H the isotropy subgroup of $\boldsymbol{SO}(n)$ at ${}^t(1, 0, \cdots, 0)=e_1$ in $\boldsymbol{R}^n$, then H consists of all elements of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \qquad (h \in \boldsymbol{SO}(n-1)).$$

The orbit of e_1 of $\boldsymbol{SO}(n)$ is canonically isomorphic to S^{n-1}, the unit sphere in $\boldsymbol{R}^n$. So we obtain an isomorphism

$$S^{n-1} \ni \omega = g\cdot e_1 \longleftrightarrow gH \in \boldsymbol{SO}(n)/H$$

where $g \in \boldsymbol{SO}(n)$.

For each non-negative integer m, let $\mathscr{H}^{n,m}$ denote the space of all homogeneous harmonic polynomials on $\boldsymbol{R}^n$ of degree m. The canonical action of $\boldsymbol{SO}(n)$ defines an irreducible (unitary) representation of $\boldsymbol{SO}(n)$ on $\mathscr{H}^{n,m}$, which we denote by τ_m. It is known that the representation τ_m is of class one with respect to H. Conversely, every irreducible representation of $\boldsymbol{SO}(n)$ of class one with respect to H is obtained in this way. Let $d(m)$ be the degree of τ_m. We choose an orthonormal basis $\{\psi_1, \cdots, \psi_{d(m)}\}$ of $\mathscr{H}^{n,m}$, where ψ_1 is the unit fixed vector for H. For each m, $\tau_m(g)(g \in \boldsymbol{SO}(n))$ is represented by the matrix $(t_{ij}^m(g))_{1 \leq i,j \leq d(m)}$ where $t_{ij}^m(g) = (\tau_m(g)\psi_i, \psi_j)$. Since ψ_1 is an H-fixed vector we see that $t_{1j}^m(g)$ can be regarded as a function on $\boldsymbol{SO}(n)/H$ which is isomorphic to S^{n-1}. Put $\psi_j^m(\omega) = \sqrt{d(m)}\, t_{1j}^m(g)$ where $\omega = gH \in S^{n-1}$ (we identify S^{n-1} with $\boldsymbol{SO}(n)/H$), then it is well known that, if we denote by $\Delta_{S^{n-1}}$ the Laplace-Beltrami operator on S^{n-1}, each $\psi_j^m (1 \leqq j \leqq d(m))$ satisfies the differential equation

$$\Delta_{S^{n-1}} \psi_j^m = -m(m+n-2)\psi_j^m,$$

and $\{\psi_j^m;\ 1 \leqq j \leqq d(m),\ m \geqq 0 \text{ integer}\}$ form a complete orthonormal basis of $L^2(S^{n-1})$ which is the Hilbert space of all square-integrable functions on S^{n-1} with respect to the $\boldsymbol{SO}(n)$-invariant measure $d\omega$ on S^{n-1}.

Using the fact that S^{n-1} is compact, every function ψ in $C^\infty(S^{n-1})$ can be expanded in an absolutely and uniformly convergent Fourier series:

$$\psi(\omega) = \sum_{m \geqq 0} \sum_{j=1}^{d(m)} (\psi, \psi_j^m)\psi_j^m(\omega).$$

More briefly we write this

$$\psi = \sum_{m \geqq 0} \langle C_m, \Phi_m(\omega) \rangle,$$

here

$$C_m = {}^t((\psi, \psi_1^m), \cdots, (\psi, \psi_{d(m)}^m)) \in \boldsymbol{C}^{d(m)},$$

$$\Phi_m(\omega) = {}^t(\psi_1^m(\omega), \cdots, \psi_{d(m)}^m(\omega)),$$

$$\langle C_m, \Phi_m(\omega) \rangle = \sum_{j=1}^{d(m)} (\psi, \psi_j^m)\psi_j^m(\omega).$$

§ 3. Some results on eigenfunctions

In this section we shall prove three lemmas which we need in the following sections.

Lemma 1. *Let $J_\nu(z)$ be the Bessel function of order $\nu > 0$, then for any complex number z, we have*

(i) $$|J_\nu(z)| \leqq \frac{\left|\frac{z}{2}\right|^\nu}{\Gamma(\nu+1)} \exp\left(\left|\frac{z}{2}\right|^2\right)$$

(ii) $$|J_\nu(z)| \geqq \frac{1}{2} \frac{\left|\frac{z}{2}\right|^\nu}{\Gamma(\nu+1)} \quad \text{if} \quad \nu \geqq \frac{|z|^2}{4 \log \frac{3}{2}} - 1.$$

PROOF. From the power series expansion of the Bessel function $J_\nu(z)$, we have

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{n! \frac{\Gamma(\nu+n+1)}{\Gamma(\nu+1)}} \right\}.$$

Put $\theta = \sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n}}{n! \frac{\Gamma(\nu+n+1)}{\Gamma(\nu+1)}}$. Then it is easy to see

$$|\theta| \leqq \sum_{n=1}^{\infty} \frac{\left|\frac{z}{2}\right|^{2n}}{n!(\nu+1)^n} = \exp\left\{\frac{|z|^2}{4(\nu+1)}\right\}.$$

For $\nu > 0$ and arbitary $z \in \boldsymbol{C}$,

$$\exp\left\{\frac{|z|^2}{4(\nu+1)}\right\} - 1 < \exp\left(\frac{|z|^2}{4}\right) - 1$$

shows that

$$|J_\nu(z)| \leqq \left|\frac{z}{2}\right|^\nu \frac{1}{\Gamma(\nu+1)} (1 + |\theta|) < \frac{\left|\frac{z}{2}\right|^\nu \exp\left(\frac{|z|^2}{4}\right)}{\Gamma(\nu+1)}.$$

This proves (i) in the lemma.

Next we notice that

$$|J_\nu(z)| = \left|\frac{z}{2}\right|^\nu \frac{1}{\Gamma(\nu+1)} |1+\theta| \geqq \left|\frac{z}{2}\right|^\nu \frac{1}{\Gamma(\nu+1)} (1 - |\theta|).$$

Suppose that

$$|z|^2 \leqq 4 \log \frac{3}{2} (\nu+1).$$

Then

$$|\theta| \leqq \exp\left\{\frac{z^2}{4(\nu+1)}\right\} - 1 \leqq \frac{1}{2}$$

Hence, we obtain

$$|J_\nu(z)| \geqq \left|\frac{z}{2}\right|^\nu \frac{1}{\Gamma(\nu+1)}(1-|\theta|) \geqq \frac{\left|\frac{z}{2}\right|^\nu}{2\Gamma(\nu+1)}.$$

This completes the proof of the lemma.

Fix a non zero complex number μ and let us consider the differential equation

$$\Delta f = \mu f, \quad f \in C^\infty(\boldsymbol{R}^n),$$

where

$$\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i{}^2}.$$

We fix one of the square roots of μ and denote it by λ. We denote by $C^\infty(\boldsymbol{R}^n)_\lambda$ the space of all functions f of $C^\infty(\boldsymbol{R}^n)$ which satisfy $\Delta f = \lambda^2 f$. Then we have the following

LEMMA 2. *For any $f \in C^\infty(\boldsymbol{R}^n)_\lambda$ and for each non-negative integer m, there exists a unique constant $C_m \in \boldsymbol{C}^{d(m)}$ such that*

$$f(x) = \sum_{m \geqq 0} r^{(2-n)/2} J_{m+(n-2)/2}(\lambda r) < C_m, \Phi_m(\omega) >$$

where $x = r\omega (r > 0, \omega \in S^{n-1})$ and $J_\nu(z)$ is the Bessel function of order ν. The right hand side converges absolutely and uniformly on every compact subset in $\boldsymbol{R}^n$.

PROOF. For each f in $C^\infty(\boldsymbol{R}^n)_\lambda$ and for each real number r, we put $f_r(\omega) = f(r\omega) (\omega \in S^{n-1})$. Then f_r has an absolutely and uniformly convergent Fourier expansion

$$f_r(\omega) = \sum_{m \geqq 0} \sum_{j=1}^{d(m)} (f_r, \psi_j^m) \psi_j^m(\omega)$$

(See §2.). If we put $b_j^m(r) = (f_r, \psi_j^m)$, then

$$f_r(\omega) = \sum_{m \geqq 0} \sum_{j=1}^{d(m)} b_j^m(r) \psi_j^m(\omega).$$

Using the assumption that f satisfies the equation $\Delta f = \lambda^2 f$, we have

$$\Delta b_j^m(r) \psi_j^m(\omega) = \lambda^2 b_j^m(r) \psi_j^m(\omega).$$

On the other hand we know that Δ is expressed in polar coordinates in the form

$$\Delta=-\left(\frac{\partial^2}{\partial r^2}+\frac{n-1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta_{S^{n-1}}\right).$$

If we recall that $\Delta_{S^{n-1}}\psi_j^m=-m(m+n-2)\psi_j^m$, we obtain the following equation for b_j^m:

$$\frac{d^2 b_j^m}{dr^2}+\frac{n-1}{r}\frac{d b_j^m}{dr}+\left(\lambda^2-\frac{m(m+n-2)}{r^2}\right)b_j^m=0.$$

A fundamental system of solutions of this differential equation is given as follows:

(1) $$r^{(2-n)/2}J_{m+(n-2)/2}(\lambda r),\ r^{(2-n)/2}N_{m+(n-2)/2}(\lambda r)$$

when $m+(n-2)/2$ is an integer,

(2) $$r^{(2-n)/2}J_{m+(n-2)/2}(\lambda r),\ r^{(2-n)/2}J_{-m-(n-2)/2}(\lambda r)$$

when $m+(n-2)/2$ is not an integer.

Here $J_\nu(z)$ is the Bessel function of order ν and $N_\nu(z)$ is the Neumann function of order ν. On the other hand the solution must be a restriction of C^∞-function on $\boldsymbol{R}^n$, therefore b_j^m is a constant multiple of $r^{(2-n)/2}J_{m+(-2+n)/2}(\lambda r)$. Consequently, there exists a unique constant $C_{m,j}\in\boldsymbol{C}(1\leqq j\leqq d(m))$ such that $b_j^m(r)=C_{m,j}r^{(2-n)/2}J_{m+(n-2)/2}(\lambda r)$. If we put $C_m={}^t(C_{m,1},\cdots,C_{m,d(m)})\in\boldsymbol{C}^{d(m)}$ and $\Phi_m(\omega)={}^t(\psi_1^m(\omega),\cdots,\psi_{d(m)}^m(\omega))$, then by the above formula we have

$$f_r(\omega)=\sum_{m\geqq 0}r^{(2-n)/2}J_{m+(n-2)/2}(\lambda r)\langle C_m,\Phi_m(\omega)\rangle.$$

Now the last statement of the lemma is clear from Lemma 1, (i).

LEMMA 3. *Put* $F_m(x)=\int_{S^{n-1}}e^{i\lambda\langle x,\omega\rangle}\Phi_m(\omega)d\omega$. *Then*

$$F_m(x)=i^m a_n\left(\frac{\lambda r}{2}\right)^{(2-n)/2}J_{m+(n-2)/2}(\lambda r)\Phi_m(\omega),$$

where $x=r\omega$, $(r>0,\ \omega\in S^{n-1})$ *and* $a_n=\dfrac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}}$.

PROOF. For arbitrary $g\in\boldsymbol{SO}(n)$, there exist $h,h'\in H$ such that $g=hu_\theta h'$ where

$$u_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1_{n-2} \end{pmatrix} \quad (0 \leqq \theta \leqq \pi).$$

Using this decomposition, for any $\psi \in C^\infty(\boldsymbol{SO}(n))$, we have

$$\int_{SO(n)} \psi(g)dg = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_H \int_H \int_0^\pi \psi(hu_\theta h') \sin^{n-2}\theta \, dh \; dh' d\theta.$$

where dg (resp. dh, dh') is the normalized Haar measure on $\boldsymbol{SO}(n)$ (resp. H). In view of §2, if we recall that $\omega \in S^{n-1}$ is written as $\omega = g\cdot e_1 (g = hu_\theta h' \in \boldsymbol{SO}(n))$ and H is the isotropy subgroup of $\boldsymbol{SO}(n)$ at e_1, we obtain by the definition of Φ_m that $\Phi_m(hu_\theta h'\cdot e_1) = \tau_m(h)\Phi_m(u_\theta e_1)$ and that $e^{i\lambda<re_1,\omega>} = e^{i\lambda<re_1, hu_\theta h'\cdot e_1>} = e^{i\lambda r\cos\theta}$.

Using the above integral formula, we obtain

$$F_m(r) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_H \int_H \int_0^\pi e^{i\lambda r\cos\theta} \tau_m(h) \Phi_m(u_\theta e_1)(\sin\theta)^{n-2} d\theta \, dh dh'.$$

If we remark that

$$\int_H \int_H t_{ij}^m(hgh')dhdh' = \delta_{i1}\delta_{j1} t_{11}^m(g)$$

for $g \in \boldsymbol{SO}(n)$, we can show that every component of $F_m(r)$ vanishes except the first one. It is not difficult to see that the first component of $\Phi_m(u_\theta e_1)$ is expressed in terms of the Gegenbauer polynomials. The lemma follows immediately from the equation in ([8], p. 71, line 1).

§4. Definitions of $\tilde{\mathscr{B}}(S^{n-1})$ and $\mathscr{P}_\lambda$

Let $\mathscr{A}(S^{n-1})$ be the space of all analytic functions on S^{n-1} and $\mathscr{B}(S^{n-1})$ the space of all hyperfunctions on S^{n-1}. Then, because of the compactness of S^{n-1}, each element of $\mathscr{B}(S^{n-1})$ is regarded as a continuous linear functional on $\mathscr{A}(S^{n-1})$. In this section we fix a non-zero complex number λ once for all and consider the analytic function $e^{i\lambda<x,\omega>}$ on the product space $\boldsymbol{R}^n \times S^{n-1}$. For any hyperfunction T on S^{n-1}, we define a function on $\boldsymbol{R}^n$ as follows. For any fixed $x \in \boldsymbol{R}^n$, we denote by $f(x)$ the value of T at the function $e^{i\lambda<x,\omega>}$. Here we regard $e^{i\lambda<x,\omega>}$ as an analytic function on S^{n-1}, x being fixed. Then we shall show below that $f \in C^\infty(\boldsymbol{R}^n)_\lambda$. Thus, putting $f = \mathscr{P}_\lambda T$, we obtain a linear map $\mathscr{P}_\lambda$ of $\mathscr{B}(S^{n-1})$ into $C^\infty(\boldsymbol{R}^n)_\lambda$. As is seen below, $\mathscr{P}_\lambda$ is not surjective, so that we extend the domain of the definition of $\mathscr{P}_\lambda$ as follows. For any non-negative integer m, let τ_m be the irreducible unitary representation of $\boldsymbol{SO}(n)$ of class one with respect to H on $\mathscr{H}^{n,m}$ which is defined in § 2. We denote by $d(m)$ the degree of τ_m. Let $\prod_{m=0}^{\infty} \boldsymbol{C}^{d(m)}$ be the product set of all the complex euclidean space $\boldsymbol{C}^{d(m)}$ of dimension $d(m)$. Then $\prod_{m=0}^{\infty} \boldsymbol{C}^{d(m)}$ has canonically the structure of a vector space. We write $\mathscr{F}$ for the vector subspace of $\prod_{m=0}^{\infty} \boldsymbol{C}^{d(m)}$ consisting of all $(C_m)_{m\geqq 0}$ (where $C_m \in \boldsymbol{C}^{d(m)}$) which satisfies $\sum_{m\geqq 0} \|C_m\| s^m < \infty$ for all s $(0<s<1)$. Now we consider the Fourier expansions of hyperfunctions on S^{n-1}. For any T in $\mathscr{B}(S^{n-1})$ one can show that there exists a unique element $(C_m)_{m\geqq 0}$ in $\mathscr{F}$ such that

$$T(\psi)=\sum_{m\geqq 0}\int_{S^{n-1}} <C_m, \Phi_m(\omega)> d\omega \qquad \text{for all } \psi \in A(S^{n-1})$$

where the right hand side is absolutely convergent, (see [3]).

Next we introduce the vector subspace $\tilde{\mathscr{F}}$ of all elements $(C_m)_{m\geqq 0}$ in $\prod_{m=0}^{\infty} \boldsymbol{C}^{d(m)}$ satisfying $\sum_{m\geqq 0} \frac{\|C_m\|}{\Gamma\left(m+\frac{n}{2}\right)} s^m<\infty$ for all $s>0$.

Then, as is easily seen, every element $(C_m)_{m\geqq 0} \in \mathscr{F}$ satisfies

$$\sum_{m\geqq 0} \frac{|p(m)|\,\|C_m\|}{\Gamma\left(m+\frac{n}{2}\right)} s^m < \infty$$

for any polynomial p and all $s>0$.

The formula of Cauchy-Hadamard about the radius of convergence implies that $\tilde{\mathscr{F}}$ contains $\mathscr{F}$ as a proper subspace. Let $\tilde{\mathscr{A}}(S^{n-1})$ be the set of all ψ in $\mathscr{A}(S^{n-1})$ such that the series

$$\sum_{m\geqq 0}\int_{S^{n-1}} <C_m, \Phi_m(\omega)> \psi(\omega) d\omega$$

is convergent absolutely for any $(C_m)_{m\geqq 0} \in \tilde{\mathscr{F}}$. We remark that every element of the orthonormal basis $\{\psi_j^m \mid 1\leqq i\leqq d(m),\ m\geqq 0,\}$ lies in $\tilde{\mathscr{A}}(S^{n-1})$. For any $(C_m)_{m\geqq 0} \in \tilde{\mathscr{F}}$, we define a linear mapping $T[(C_m)_{m\geqq 0}]$ from $\tilde{\mathscr{A}}(S^{n-1})$ into $\boldsymbol{C}$ by

$$T[(C_m)_{m\geqq 0}]\psi=\sum_{m\geqq 0}\int_{S^{n-1}} <C_m, \Phi_m(\omega)> \psi(\omega) d\omega$$

for any ψ in $\tilde{\mathscr{A}}(S^{n-1})$. Moreover we denote by $\tilde{\mathscr{B}}(S^{n-1})$ the set of all $T[(C_m)_{m\geqq 0}]$ where $(C_m)_{m\geqq 0} \in \tilde{\mathscr{F}}$.

PROPOSITION 1. *The mapping*

$$\tilde{\mathscr{F}} \ni (C_m)_{m\geqq 0} \longmapsto T[(C_m)_{m\geqq 0}] \in \tilde{\mathscr{B}}(S^{n-1})$$

is an onto-isomorphism.

PROOF. By definition it is easy to see that the mapping is linear and surjective. So we have only to prove that it is injective. Let $(C_m)_{m\geqq 0}$ be an element of $\tilde{\mathscr{F}}$ such that

$$T[(C_m)_{m\geqq 0}]=0.$$

Then we have

$$\sum_{m\geqq 0} \int_{S^{n-1}} \langle C_m, \Phi_m(\omega)\rangle \psi(\omega)d\omega=0 \qquad \text{for all } \psi \in \tilde{\mathscr{A}}(S^{n-1}).$$

As we remarked before, $\tilde{\mathscr{A}}(S^{n-1})$ contains a complete orthonormal system. It follows at once that $C_m=0$ for all $m\geqq 0$. This completes the proof.

Since $\mathscr{F}$ is contained in $\tilde{\mathscr{F}}$ as a proper subspace, $\mathscr{B}(S^{n-1})$ is a proper subspace of $\tilde{\mathscr{B}}(S^{n-1})$. The following proposition assures that the domain of the definition of $\mathscr{P}_\lambda$ can be extended to $\tilde{\mathscr{B}}(S^{n-1})$.

PROPOSITION 2. *For any* $T[(C_m)_{m\geqq 0}]$ *in* $\tilde{\mathscr{B}}(S^{n-1})$,

$$f(x)=\sum_{m\geqq 0} \int_{S^{n-1}} e^{i\lambda\langle x,\omega\rangle} \langle C_m, \Phi_m(\omega)\rangle d\omega$$

is absolutely and uniformly convergent on every compact subset in $\boldsymbol{R}^n$, *and* f *defines an element of* $C^\infty(\boldsymbol{R}^n)_\lambda$.

PROOF. We fix $r_0>0$. For any x in $\boldsymbol{R}^n$ such that $\|x\|<r_0$, putting $x=r\omega$, we have

$$\sum_{m\geqq 0} \left| \int_{S^{n-1}} e^{i\lambda\langle x,\xi\rangle} \langle C_m, \Phi_m(\xi)\rangle d\xi \right|$$

$$\leqq |a_n| \sum_{m\geqq 0} |r^{(2-n)/2} J_{m+(n-2)/2}(\lambda r) \langle C_m, \Phi_m(\omega)\rangle |$$

$$\leqq |a_n| \sum_{m\geqq 0} |r^{(2-n)/2} J_{m+(n-2)/2}(\lambda r)|\, d(m)\|C_m\|$$

$$\leqq |a_n| \sum_{m\geqq 0} r^{(2-n)/2} \frac{\left|\frac{\lambda r}{2}\right|^{m+(n-2)/2}}{\Gamma\left(m+\frac{n}{2}\right)} \exp\left(\frac{|\lambda r|^2}{4}\right) d(m)\|C_m\|$$

$$= |a_n| \left(\frac{|\lambda|}{2}\right)^{(n-2)/2} \exp\left(\frac{|\lambda r|^2}{4}\right) \sum_{m\geqq 0} \frac{d(m)\|C_m\|}{\Gamma\left(m+\frac{n}{2}\right)} \left(\frac{r}{2}\right)^m$$

$$\leqq |a_n| \left(\frac{|\lambda|}{2}\right)^{(n-2)/2} \exp\left(\frac{|\lambda r_0|^2}{4}\right) \sum_{m\geqq 0} \frac{d(m)\|C_m\|}{\Gamma\left(m+\frac{n}{2}\right)} \left(\frac{r_0}{2}\right)^m.$$

Since $(C_m)_{m\geqq 0} \in \tilde{\mathscr{F}}$ and $d(m)$ is a polnomial in m, the above series is convergent. This shows the absolute and uniform convergence of f. Next if we notice

$$\Delta\left(\int_{S^{n-1}} e^{i\lambda<x,\omega>} < C_m, \Phi_m(\omega) > d\omega\right)$$

$$=\lambda^2\left(\int_{S^{n-1}} e^{i\lambda<x,\omega>} < C_m, \Phi_m(\omega) > d\omega\right)$$

It follows immediately that f lies in $C^\infty(\boldsymbol{R}^n)_\lambda$ by the uniform convergence of f. This completes the proof of Proposition 2.

Finally we define the map $\mathscr{P}_\lambda$ of $\tilde{\mathscr{B}}(S^{n-1})$ into $C^\infty(\boldsymbol{R}^n)_\lambda$ as follows. For any $T = T[(C_m)_{m\geqq 0}]$ in $\tilde{\mathscr{B}}(S^{n-1})$, we define $\mathscr{P}_\lambda T$ to be the element f of $C^\infty(\boldsymbol{R}^n)_\lambda$ which is given in Proposition 2.

§ 5. The surjectivity of $\mathscr{P}_\lambda$.

Let us consider the differential equation $\Delta f = \lambda^2 f$. Then the following theorem says that every solution can be represented by an analogue of the "Poisson integral" of a unique element of $\tilde{\mathscr{B}}(S^{n-1})$.

THEOREM. *The map $\mathscr{P}_\lambda$ is an isomorphism of $\tilde{\mathscr{B}}(S^{n-1})$ onto $C^\infty(\boldsymbol{R}^n)_\lambda$.*

PROOF. Lemma 2 in § 3 and Proposition 2 in § 4 show that $\mathscr{P}_\lambda$ is injective.

Let f be an arbitrary element of $C^\infty(\boldsymbol{R}^n)_\lambda$. By Lemma 2 in § 3, there exists $(C'_m)_{m\geqq 0}$ such that

$$f_r(\omega) = \sum_{m\geqq 0} r^{(2-n)/2} J_{m+(n-2)/2}(\lambda r) < C'_m, \Phi_m(\omega) >.$$

We put

$$C_m = \frac{1}{i^m a_n}\left(\frac{\lambda}{2}\right)^{(n-2)/2} \cdot C'_m.$$

First, we show that $(C_m)_{m\geqq 0} \in \tilde{\mathscr{F}}$. From the absolute convergence, we have

$$\infty > (\|f_r\|_{L^2(S^{n-1})})^2$$

$$= \sum_{m\geqq 0} |r^{(2-n)/2} J_{m+(n-2)/2}(\lambda r)|^2 \|C'_m\|^2$$

$$= |a_n|^2 \left|\frac{\lambda}{2}\right|^{2-n} |r|^{2-n} \sum_{m\geqq 0} |J_{m+(n-2)/2}(\lambda r)|^2 \|C_m\|^2.$$

Using Lemma 1, (ii),

$$\infty > |a_n|^2 \left|\frac{\lambda}{2}\right|^{2-n} |r|^{2-n} \frac{1}{4} \sum_{m\geqq 0} \left[\frac{\left|\frac{\lambda r}{2}\right|^{m+(n-2)/2}}{\Gamma\left(m+\frac{n}{2}\right)} \|C_m\|\right]^2$$

$$= \frac{1}{4} |a_n|^2 \sum_{m\geqq 0} \left[\frac{\left|\frac{\lambda r}{2}\right|^{m} \|C_m\|}{\Gamma\left(m+\frac{n}{2}\right)}\right]^2.$$

From the Cauchy-Hadamard test,

$$\varlimsup_{m\to\infty} \left[\frac{\left|\frac{\lambda r}{2}\right|^{m} \|C_m\|}{\Gamma\left(m+\frac{n}{2}\right)}\right]^{\frac{2}{m}} \leqq 1.$$

So,

$$\varlimsup_{m\to\infty} \left[\frac{\left|\frac{\lambda r}{2}\right|^{m} \|C_m\|}{\Gamma\left(m+\frac{n}{2}\right)}\right]^{\frac{1}{m}} \leqq 1.$$

This implies that

$$\sum_{m\geqq 0} \frac{\left|\frac{\lambda r}{2}\right|^{m} \|C_m\|}{\Gamma\left(m+\frac{n}{2}\right)} < +\infty \qquad \text{for all } r.$$

Since $\lambda \neq 0$, we have that $(C_m)_{m\geqq 0}$ lies in $\mathscr{F}$.

Next, we put $T = T[(C_m)_{m\geqq 0}]$, then $T \in \mathscr{B}(S^{n-1})$. Then, using Lemma 3, it is easy to obtain $\mathscr{P}_\lambda T = f$. This completes the proof of the theorem.

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Department of Mathematics,
Faculty of Science,
Hiroshima University