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A Study of $\mathscr{D}'_{L^2}$-Valued Distributions on a Semi-Axis in connection with the Cauchy Problem for a Pseudo-Differential System

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In a previous paper [10] one of the present authors has investigated the fine Cauchy problem for a system of linear partial differential operators and obtained the following result: Let $\vec{P}(t, x, D_x)$ be an $N \times N$ matrix of linear partial differential operators with coefficients $\in C^\infty(R_{n+1})$. The fine Cauchy problem consists in finding a solution $\vec{u}=(u_1, u_2, \cdots, u_N)$, $u_j \in \mathscr{D}'(R^+_{n+1})$ to the equation

$$D_t\vec{u} + \vec{P}(t, x, D_x)\vec{u} = \vec{f} \qquad \text{in } R^+_{n+1}$$

with initial condition

$$\lim_{t \downarrow 0} \vec{u}(t, x) = \vec{\alpha},$$

when $\vec{\alpha}=(\alpha_1, \alpha_2, \cdots, \alpha_N)$, $\alpha_j \in \mathscr{D}'(R_n)$ and $\vec{f}=(f_1, f_2, \cdots, f_N)$, $f_j \in \mathscr{D}'(R^+_{n+1})$ are arbitrarily given, where $\lim_{t \downarrow 0} \vec{u}$ denotes the distributional boundary value of $\vec{u}$. If there exists a solution $\vec{u}$ for the problem, then $\vec{f}$ must have the canonical extension $\vec{f}_\sim$ over $t=0$ and $\vec{v}=\vec{u}_\sim$ satisfies the equation

$$D_t\vec{v} + \vec{P}(t, x, D_x)\vec{v} = \vec{f} - i\delta \otimes \vec{\alpha}.$$

Conversely, if $\vec{v}=(v_1, v_2, \cdots, v_N)$, $v_j \in \mathscr{D}'_+(R_{n+1})$ is a solution of this equation, then the restriction $\vec{u}=\vec{v}\,|\,R^+_{n+1}$ is a solution for our original Cauchy problem and $\vec{u}_\sim=\vec{v}$. If we replace $\vec{P}(t, x, D_x)$ by $\vec{A}(t)$, an $N \times N$ matrix of pseudo-differential operators [cf. p. 384 for definition], we shall have a right reason to consider the spaces $\mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ and $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ instead of $\mathscr{D}'(R^+_{n+1})$ and $\mathscr{D}'(R_{n+1})$ respectively. As a result, it will be natural to introduce the boundary value and the canonical extension in a suitable sense.

The present paper is also designed to be the introductory part of our subsequent paper [12] which will appear in this journal.

In Section 1 we discuss the space $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ and the spaces related to it. These spaces are all reflexive, ultrabornological and Souslin. Section 2 is devoted to discussions concerning the $\mathscr{D}'_{L^2}$-boundary value and the $\mathscr{D}'_{L^2}$-canonical extension. Various alternatives of these notions will also be considered. In Section 3 we shall introduce the operator $\vec{A}(t)$ referred to above and investigate the properties thereof. In Section 4 some pseudo-commutativity relation for $\vec{A}(t)$ will be discussed. In particular, when applied to a singular integral operator in the sense of A.P. Caldelón, our result will refine Theorem 4 in [3]. The final section is concerned with the fine Cauchy problem for a pseudo-differential system.

1. The space $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$

Let $R_{n+1}=R\times R_n$ be an $(n+1)$-dimensional Euclidean space with generic point (t, x), $x=(x_1, \cdots, x_n)$ and $R^+_{n+1}=\{(t, x)\in R_{n+1}: t>0\}$. As usual, we write $|x|=(\sum_{j=1}^n x_j^2)^{1/2}$ and $<x, \xi>=\sum_{j=1}^n x_i\xi_i$, where $\xi=(\xi_1, \cdots, \xi_n)\in \Xi_n$, the dual Euclidean space of R_n. If p is an n-tuple $(p_1, \cdots, p_n)$ of non-negative integers, the sum $\sum_{j=1}^n p_j$ will be denoted by $|p|$ and with $D_x=(D_1, \cdots, D_n)$, $D_j=\frac{1}{i}\frac{\partial}{\partial x_j}$ and $D_t=\frac{1}{i}\frac{\partial}{\partial t}$, we put $D_x^p=D_1^{p_1}\cdots D_n^{p_n}$.

Let L be a locally convex Hausdorff space and L' be its dual. We shall denote by L'_σ, L'_b and L'_c, respectively, the weak dual, the strong dual and the dual space L' with the topology of uniform convergence on absolutely convex, compact subsets of L. For a locally convex Hausdorff space M, following L. Schwartz [16, *p.* 18], the ε-product $L\varepsilon M$ is defined as the linear space of bilinear forms on $L'_c\times M'_c$ hypocontinuous with respect to the equicontinuous subsets of L', M' and provided with the ε-topology, that is, the topology of uniform convergence on the products of an equicontinuous subset of L' and an equicontinuous subset of M'. If we let $\mathscr{L}_\varepsilon(L'_c;M)$ be the space of continuous linear maps of L'_c into M with the topology of uniform convergence on the equicontinuous subsets of L', it is shown [16, *p.* 34] that there exist the canonical isomorphisms between $L\varepsilon M$, $\mathscr{L}_\varepsilon(L'_c; M)$ and $\mathscr{L}_\varepsilon(M'_c; L)$. Hence we can identify $L\varepsilon M$ with $\mathscr{L}_\varepsilon(L'_c; M)$ or with $\mathscr{L}_\varepsilon(M'_c; L)$ in accordance with these canonical isomorphisms.

As to the tensor product $L\otimes M$, every $\sum_{j=1}^n x_j\otimes y_j\in L\otimes M$ defines a bilinear form on $L'\times M'$; $(x', y')\to\sum_{j=1}^m <x', x_j><y', y_j>$, which is certainly an element of $L\varepsilon M$. In view of the fact that the linear map of $L\otimes M$ into $L\varepsilon M$ thus defined is injective, $L\otimes M$ is regarded as a linear subspace of $L\varepsilon M$. Equipped with the ε-topology, the space $L\otimes M$ will be denoted by $L\otimes_\varepsilon M$ [16, p. 47]. The π-topology (resp. the ι-topology) on $L\otimes M$ is defined as the finest locally convex topology on this vector space for which the canonical bilinear map $(x, y)\to x\otimes y$ of $L\times M$ into $L\otimes M$ is continuous (resp. separately continuous). $L\otimes_\pi M$ (resp. $L\otimes_\iota M$) will stand for the space $L\otimes M$ with the π-topology (resp. the ι-topology). The notations $L\hat{\otimes}_\varepsilon M$, $L\hat{\otimes}_\pi M$ and $L\hat{\otimes}_\iota M$ are used to represent the completions of $L\otimes M$ with topologies ε, π and ι

respectively. In what follows we often write $L(M)$ instead of $L\varepsilon M$.

In our later discussions we need the following

LEMMA 1 (*cf.* [17, *p.* 103]). *Let L be a nuclear Fréchet space and M a reflexive Fréchet space, then $L\varepsilon M$ is a reflexive Fréchet space and furthermore we have* $(L\varepsilon M)'_b = L'_b\varepsilon M'_b$.

Now let $\mathscr{H}$ be a locally convex Hausdorff space contained in $\mathscr{D}'(R_{n+1})$. Following L. Schwartz [16, p. 7] we shall say that $\mathscr{H}$ is a space of distributions if the identical map of $\mathscr{H}$ into $\mathscr{D}'(R_{n+1})$ is continuous, and that $\mathscr{H}$ is normal if (i) it is a space of distributions, (ii) $\mathscr{H}$ contains $\mathscr{D}(R_{n+1})$ as a dense subset and (iii) the identical map of $\mathscr{D}(R_{n+1})$ into $\mathscr{H}$ is continuous. It is shown in [16, p. 10] that if $\mathscr{H}$ is a normal space of distributions, then so is $\mathscr{H}'_c$.

As is well known, $\mathscr{D}'_t$ and $(\mathscr{D}'_{L^2})_x$ are complete normal spaces of distributions enjoying the approximation properties by truncation and regularization. It follows from Proposition 3 and Corollary 1 [16, p. 9, p. 47] that $\mathscr{D}'_t\widehat{\otimes}_\varepsilon(\mathscr{D}'_{L^2})_x = \mathscr{D}'_t((\mathscr{D}'_{L^2})_x$. Since $\mathscr{D}'_t$ is nuclear, we have $\mathscr{D}'_t\widehat{\otimes}_\varepsilon(\mathscr{D}'_{L^2})_x = \mathscr{D}'_t\widehat{\otimes}_\pi(\mathscr{D}'_{L^2})_x$ and therefore $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x) = \mathscr{D}'_t\widehat{\otimes}_\pi(\mathscr{D}'_{L^2})_x$.

PROPOSITION 1. *$\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ is a normal space of distributions.*

PROOF. Since the identical map $(\mathscr{D}'_{L^2})_x \to \mathscr{D}'_x$ is a continuous injection, it follows from Proposition 1 in [16, p. 20] that $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x) \subset \mathscr{D}'_t(\mathscr{D}'_x)$. On the other hand, owing to the kernel theorem [16, p. 93], $\mathscr{D}'_{t,x}$ is identified with $\mathscr{D}'_t(\mathscr{D}'_x)$ algebraically and topologically. Consequentely $\mathscr{D}'_t(\mathscr{D}'_{L^2}) \subset \mathscr{D}'_{t,x}$. If we consider $\mathscr{D}_{t,x}$ as a subspace of $\mathscr{D}'_{t,x}$ it is clear that $\mathscr{D}_{t,x}$ is a dense subset of $\mathscr{D}'_t(\mathscr{D}'_{L^2})$, which completes the proof.

REMARK. For any element $u \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$, there exists a sequence $\{\phi_j\}$, $\phi_j \in \mathscr{D}(R_{n+1})$ such that ϕ_j converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to u as $j\to\infty$. More precisely, if we let $\{\rho_j\}$ and $\{\alpha_j\}$ be respectively any sequences of reguralizations and multiplications in $\mathscr{D}'_t$, and let $\{\rho'_j\}$ and $\{\alpha'_j\}$ be corresponding sequences in $(\mathscr{D}'_{L^2})_x$, we can then apply the Banach-Steinhaus theorem to conclude that the sequence $\alpha_j\alpha'_j(u*(\rho_j\rho'_j)) \in \mathscr{D}(R_{n+1})$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to u.

Let us denote by $\bar{\mathscr{D}}_t((\mathscr{D}_{L^2})_x)$ the strict inductive limit of the Fréchet spaces $\mathscr{D}_{K_j}((\mathscr{D}_{L^2})_x)(=\mathscr{D}_{K_j}\widehat{\otimes}_\pi(\mathscr{D}_{L^2})_x)$, $j=1, 2, \cdots$, where we have designated by $\mathscr{D}_{K_j}$ the space of infinitely differentiable functions in R_t which vanish outside $K_j=[-j, j]$. We see from Lemma 1 that $\mathscr{D}_{K_j}((\mathscr{D}_{L^2})_x)$ is a reflexive Fréchet space. Consequently $\bar{\mathscr{D}}_t((\mathscr{D}_{L^2})_x)$ is reflexive. $\bar{\mathscr{D}}_t((\mathscr{D}_{L^2})_x)$ consists of all infinitely differentiable functions f in R_{n+1} such that $\operatorname{supp} f \subset [a, b]\times R_n$ for some bounded interval $[a, b]$ and $\max_t\left(\int |D_t^k D_x^p f(t, x)|^2 dx\right)^{\frac{1}{2}} < \infty$ for any $k, p=(p_1,\cdots, p_n)$. It is to be noted that $\bar{\mathscr{D}}_t((\mathscr{D}_{L^2})_x) = \mathscr{D}_t\widehat{\otimes}_\iota(\mathscr{D}_{L^2})_x$. In fact, $\mathscr{D}_t\otimes(\mathscr{D}_{L^2})_x$ is clearly a dense subset of $\bar{\mathscr{D}}_t((\mathscr{D}_{L^2})_x)$. Let G be any locally convex Hausdorff space.

To any separately continuous bilinear map u of $\mathscr{D}_t \times (\mathscr{D}_{L^2})_x$ into G, there is uniquely associated a linear map v of $\mathscr{D}_t \otimes (\mathscr{D}_{L^2})_x$ into G such that $u = v \circ \phi$, ϕ being a canonical map of $\mathscr{D}_t \times (\mathscr{D}_{L^2})_x$ into $\mathscr{D}_t \otimes (\mathscr{D}_{L^2})_x$. Observing that $\mathscr{D}_{K_j}$ and $(\mathscr{D}_{L^2})_x$ are Fréchet spaces, we see that the restriction of v to $\mathscr{D}_{K_j} \otimes (\mathscr{D}_{L^2})_x$ becomes continuous under the π-topology and admits a unique continuous extension taking $\mathscr{D}_{K_j} \hat{\otimes}_\pi (\mathscr{D}_{L^2})_x = \mathscr{D}_{K_j}((\mathscr{D}_{L^2})_x)$ into $\hat{G}$, the completion of G, which shows that v admits a unique continuous extension which takes $\bar{\mathscr{D}}_t((\mathscr{D}_{L^2})_x)$ into $\hat{G}$. Thus $\mathscr{D}_t \otimes_\iota (\mathscr{D}_{L^2})_x$ is a dense subspace of $\bar{\mathscr{D}}_t((\mathscr{D}_{L^2})_x)$, whereupon $\bar{\mathscr{D}}_t((\mathscr{D}_{L^2})_x) = \mathscr{D}_t \hat{\otimes}_\iota (\mathscr{D}_{L^2})_x$. It is shown [17, p. 104] that $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ is the strong dual of $\mathscr{D}_t((\mathscr{D}_{L^2})_x)$. With these in mind, we can state the following

PROPOSITION 2. *$\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ is a reflexive space with strong dual $\bar{\mathscr{D}}_t((\mathscr{D}_{L^2})_x) = \mathscr{D}_t \hat{\otimes}_\iota (\mathscr{D}_{L^2})_x$.*

A locally convex Hausdorff space E is said to be ultrabornological or of type (β) if E is an inductive limit of Banach spaces B_ι, $\iota \in I$. It follows from this definition that an ultrabornological space is barreled and bornological, and that a quasicomplete bornological Hausdorff space is ultrabornological.

$\mathscr{L}_c(E; F)$ is a Souslin space, that is, a continuous image of a Polish space, if E is a strict inductive limit of a sequence of separable Fréchet spaces and if F is a countable union of images, under continuous linear maps, of separable Fréchet spaces. The result was stated without proof by L. Schwartz [19, p. 602]. We shall make use of this fact which can be verified without much labor and show the following

PROPOSITION 3. *$\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ is an ultrabornological Souslin space.*

PROOF. The strong dual of an (LF)-space in the strict sense is ultrabornological if the latter is reflexive [6, p. 111]. It follows that $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ is ultrabornological.

That the space $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ is a Souslin space is a consequence of Schwartz's theorem referred to just before, since we can take $E = \mathscr{D}_t$ and $F = (\mathscr{D}'_{L^2})_x = \bigcup_{m=0}^{\infty} \mathscr{H}_{(-m)}$. Thus the proof is complete.

As a generalization of the preceding proposition we shall show the following Theorem 1, where F is a closed subset of R_t and $\mathscr{D}'_F$ denotes the subspace of $\mathscr{D}'_t$ which consists of all the one-dimensional distributions with support contained in F. $\mathscr{D}'_F$ is provided with the induced topology, so it is nuclear.

THEOREM 1. *$\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$ is a reflexive, ultrabornological Souslin space.*

PROOF. $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ being reflexive, we see that $\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$ is semireflexive as a closed subspace of $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$. Consequently if we can show that $\mathscr{D}'_F(('_{L^2})_x)$ is bornological, then we can conclude that it is reflexive and ultrabornological. That $\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$ is a Souslin space follows from the fact that $\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$ is a closed subspace of $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ which is known by Proposition 3 to be a Souslin space. Thus to complete the proof of our theorem it remains to show that $\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$ is bornological. To this end, we shall first consider a special case where F is a compact subset K of R_t. $\mathscr{D}'_K$ is the strong dual of a nuclear Fréchet space $\mathscr{E}(K)$ which is obtained by restriction to the set K of infinitely differentiable functions of t. It follows from Lemma 1 that $\mathscr{D}'_K((\mathscr{D}'_{L^2})_x)$ is the strong dual of a reflexive Fréchet space $\mathscr{E}(K)\hat{\otimes}_\pi(\mathscr{D}_{L^2})_x$, and it results that $\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$ is bornological. Now we shall turn to the general case by following the process due to K. Fujikata and K. Miyazaki [4, p. 23]. Let $\{\alpha_j\}$ be a partition of unity subordinate to the covering C_j, $j=1, 2, \cdots$, where $C_j=\left\{t\in R_t\colon j-1-\frac{1}{3}<|t|<j+\frac{1}{3}\right\}$, Putting

$$F_1=F\cap\{\bigcup_{j=1}^{\infty}\bar{C}_{2j-1}\},\qquad F_2=F\cap\{\bigcup_{j=1}^{\infty}\bar{C}_{2j}\},$$

$$\alpha=\sum_{j=1}^{\infty}\alpha_{2j-1},\qquad \beta=\sum_{j=1}^{\infty}\alpha_{2j},$$

$$Q'_j=\left\{t\in R_t\colon|t|<2j-1+\frac{1}{2}\right\},\qquad Q''_j=\left\{t\in R_t\colon|t|<2j+\frac{1}{2}\right\},$$

we obtain

(i) $F=F_1\cup F_2$,

(ii) $(\operatorname{supp}\alpha)\cap F_1$, $(\operatorname{supp}\beta)\cap F\subset F_2$ and $\alpha+\beta=1$,

(iii) $Q'_j\cap F_1$, $Q''_j\cap F_2$ are compact for each j.

Now we can write down: $\mathscr{D}'_{F_1}((\mathscr{D}'_{L^2})_x)=\prod_{j=1}^{\infty}\mathscr{D}'_{\bar{C}_{2j-1}}((\mathscr{D}'_{L^2})_x)$ and $\mathscr{D}'_{F_2}((\mathscr{D}'_{L^2})_x)=\prod_{j=1}^{\infty}\mathscr{D}'_{\bar{C}_{2j}}((\mathscr{D}'_{L^2})_x)$. Using the fact that the product space of a countable number of bornological spaces is bornological, we see that $\mathscr{D}'_{F_1}((\mathscr{D}'_{L^2})_x)$ and $\mathscr{D}'_{F_2}((\mathscr{D}'_{L^2})_x)$ are bornological. Consider the map $\theta\colon\mathscr{D}'_{F_1}((\mathscr{D}'_{L^2})_x)\times\mathscr{D}'_{F_2}((\mathscr{D}'_{L^2})_x)\ni(u_1, u_2)\to u_1+u_2\in\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$. Then θ is linear and continuous. For any given $u\in\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$, if we put $u_1=\alpha u$, $u_2=\beta u$, then $u_1\in\mathscr{D}'_{F_1}((\mathscr{D}'_{L^2})_x)$, $u_2\in\mathscr{D}'_{F_2}((\mathscr{D}'_{L^2})_x)$ and $u_1+u_2=u$, that is, θ is onto. Furthermore if u converges in $\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$ to 0, then u_1, u_2 converges respectively in $\mathscr{D}'_{F_1}((\mathscr{D}'_{L^2})_x)$, $\mathscr{D}'_{F_2}((\mathscr{D}'_{L^2})_x)$ to 0. Then we see that the map θ is epimorphic and therefore $\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$ is isomorphic to $(\mathscr{D}'_{F_1}((\mathscr{D}'_{L^2})_x)\times\mathscr{D}'_{F_2}((\mathscr{D}'_{L^2})_x))/\operatorname{Ker}\theta$. Consequently, $\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$ is bornological, which was to be proved.

If $F=[0, \infty)$, we shall use the notation $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ instead of $\mathscr{D}'_F((\mathscr{D}'_{L^2})_x)$. Similarly for $(\mathscr{D}'_t)_-((\mathscr{D}'_{L^2})_x)$. As an immediate consequence of Theorem 1, we have

COROLLARY 1. *$(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ is a reflexive, ultrabornological Souslin space.*

We note that the strong dual of $(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ is $\mathscr{D}(\bar{R}_t^+)\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x$. Here $\mathscr{D}(\bar{R}_t^+)$ is the set of infinitely differentiable functions in $\bar{R}_t^+$ which vanish outside a compact subset and it is a reflexive (LF)-space with the usual topology. We omit the proof since the method of proving Proposition 2 will be applied.

We shall denote by $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(\bar{R}_{n+1}^+)$ the space which is obtained by restriction to R_{n+1}^+ of all the distributions $\in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$. The space will be identified with the quotient space $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)/(\mathscr{D}'_t)_-((\mathscr{D}'_{L^2})_x)$ equipped with the quotient topology. We shall also denote by $\mathring{\mathscr{D}}(\bar{R}_t^+)$ the closed subspace of $\mathscr{D}(R_t)$ which consists of infinitely differentiable functions with support contained in $[0, \infty)$.

Finally we shall show

PROPOSITION 4. *$\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(\bar{R}_{n+1}^+)$ is a reflexive, ultrabornological Souslin space and $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(\bar{R}_{n+1}^+)$ is isomorphic to $\mathscr{D}'(\bar{R}_t^+)((\mathscr{D}'_{L^2})_x)=(\mathring{\mathscr{D}}(R_t^+)\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x)'_b$.*

PROOF. According to the reasoning just before Proposition 2, $\mathring{\mathscr{D}}(\bar{R}_t^+)\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x$ is reflexive and an (LF)-space in the strict sense. Here we can infer that $\mathscr{D}'(\bar{R}_t^+)((\mathscr{D}'_{L^2})_x)$ is the strong dual of $\mathring{\mathscr{D}}(\bar{R}_t^+)\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x$. It follows that $\mathscr{D}'(\bar{R}_t^+)((\mathscr{D}'_{L^2})_x)=(\mathring{\mathscr{D}}(\bar{R}_t^+)\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x)'$ is ultrabornological. Consider the identical map J: $\mathring{\mathscr{D}}(\bar{R}_t^+)\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x\to\mathscr{D}(R_t)\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x$ which is a monomorphism. The dual map ${}^tJ$: $\mathring{\mathscr{D}}'_t((\mathscr{D}'_{L^2})_x)\to\mathscr{D}'(\bar{R}_t^+)((\mathscr{D}'_{L^2})_x)$ is continuous and onto. Here $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ is a Souslin space and $\mathscr{D}'(\bar{R}_t^+)((\mathscr{D}'_{L^2})_x)$ is ultrabornological. The open mapping theorem [19, p. 604] then shows that ${}^tJ$ is an epimorphism, whereupon $\mathscr{D}'(\bar{R}_t^+)((\mathscr{D}'_{L^2})_x)$ is isomorphic to the quotient space $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)/\mathrm{Ker}\,{}^tJ=\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)/(\mathscr{D}'_t)_-((\mathscr{D}'_{L^2})_x)=\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(\bar{R}_{n+1}^+)$. Thus we can also see that $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)(R_{n+1}^+)$ is are reflexive, ultrabornological Souslin space. The proof is complete.

2. $\mathscr{D}'_{L^2}$-boundary values and $\mathscr{D}'_{L^2}$-canonical extensions

Given $\varphi\in\mathscr{D}(R_t^+)$, then φ_λ, $\lambda>0$, will be defined by letting $\varphi_\lambda(t)=\dfrac{1}{\lambda}\varphi\left(\dfrac{t}{\lambda}\right)$.

LEMMA 2. *Let E be a locally convex Hausdorff space and v a continuous linear map of $\mathscr{D}(R_t^+)$ into E. If we assume that $v(\phi)=v(\phi_\lambda)$ for every non-negative $\phi\in\mathscr{D}(R_t^+)$ with $\int_0^\infty\phi(t)\,dt=1$, then there exists a unique $e_0\in E$ such that $v(\phi)=\left(\int_0^\infty\phi(t)\,dt\right)e_0$ for every $\phi\in\mathscr{D}(R_t^+)$.*

PROOF. It is clear that $v(\phi)=v(\phi_\lambda)$ holds for every $\phi\in\mathscr{D}(R_t^+)$. Now let e' be any elemeht of E', and consider a linear form $\mathscr{D}(R_t^+)\ni\phi\to\langle e', v(\phi)\rangle$. Since it is continuous, there exists a unique distribution $T_{e'}\in\mathscr{D}'(R_t^+)$ such that $<T_{e'}, \phi> = <e', v(\phi)>$. It follows then from our assumption that $<T_{e'}, \phi> = <T_{e'}, \phi_\lambda>$, which implies that $T_{e'}(t) = T_{e'}(\lambda t)$ for every $\lambda>0$ and therefore $\frac{d}{dt} T_{e'}=0$, that is, $v\left(\frac{d\phi}{dt}\right)=0$ for any $\phi \in \mathscr{D}(R_t^+)$. Let ϕ_0 be a fixed non-negative element of $\mathscr{D}(R_t^+)$ such that $\int_0^\infty \phi_0(t)dt=1$. If we put $e_0=v(\phi_0)$, then, since any $\phi \in \mathscr{D}(R_t^+)$ can be written in the form $\phi=\left(\int_0^\infty \phi(t)\, dt\right)\phi_0+\frac{d}{dt}\chi$, $\chi \in \mathscr{D}(R_t^+)$, we obtain $v(\phi)=\left(\int_0^\infty \phi(t)\, dt\right)e_0$, as desired.

Now let us consider a distribution $u \in \mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x) \subset \mathscr{D}'(R_{n+1}^+)$ which is identified with a continuous linear map of $\mathscr{D}(R_t^+)$ into $(\mathscr{D}'_{L^2})_x$. Suppose $u(\varepsilon t, x)$ converges in $\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$ to a distribution v as $\varepsilon \downarrow 0$. Then Lemma 2 shows that v is independent of t and can be written in the form $Y_t \otimes \alpha$, where Y_t is the Heaviside function and $\alpha \in (\mathscr{D}'_{L^2})_x$. α is called the $\mathscr{D}'_{L^2}$-boundary value of u and denoted by $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u$. From this definition we also see that if $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u=\alpha$ and $\gamma \in \mathscr{E}(R_t)$, then $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} \gamma u=\gamma(0)\alpha$. By making use of this observation, we shall show that $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u=\alpha$ is equivalent to saying that $\phi_\varepsilon u$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to $\delta_t\otimes\alpha$ for any non-negative $\phi \in \mathscr{D}(R_t^+)$ with $\int_0^\infty \phi(t)dt=1$. Suppose that $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u=\alpha$. Then for any $\psi \in \mathscr{D}(R_t)$ we have $<\phi_\varepsilon(t)u(t, \cdot), \psi(t)> = <(\psi u)(\varepsilon t, \cdot), \phi(t)>$, and the product ψu has the $\mathscr{D}'_{L^2}$-boundary value $\psi(0)\alpha \in (\mathscr{D}'_{L^2})_x$. Thus $\lim_{\varepsilon\downarrow 0} <\phi_\varepsilon u, \psi> =\psi(0)\alpha= <\delta_t\otimes\alpha, \psi>$. Conversely if $\phi_\varepsilon u$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to $\delta_t\otimes\alpha$ and if $\Psi \in \mathscr{D}(R_t)$ is such that $\psi(t)=1$ in a 0-neighborhood, then $<\phi_\varepsilon u, \psi>$ converges in $(\mathscr{D}'_{L^2})_x$ to $<\delta_t\otimes\alpha, \psi>=\psi(0)\alpha=\alpha$. Since $<\phi_\varepsilon u, \psi>=<u, \phi_\varepsilon\psi>=<u, \phi_\varepsilon>$ for sufficiently small $\varepsilon>0$, it follows that $<u(\varepsilon t, \cdot), \phi>$ converges in $(\mathscr{D}'_{L^2})_x$ to α.

Lemma 3. *Let s be a real number. If a sequence $\{u_j\}$, $u_j \in \mathscr{H}_{(s)}(R_n)$, is bounded in $\mathscr{H}_{(s)}(R_n)$ and converges in $(\mathscr{D}'_{L^2})_x$ to 0, then u_j converges in $\mathscr{H}_{(s-1)}(R_n)$ to 0.*

Proof. By our assumption there exists a constant C such that $\int |\hat{u}_j|^2(1+|\xi|^2)^s d\xi \leqq C$. Given $\varepsilon>0$, we can take N so large that

$$\int_{|\xi|>N} |\hat{u}_j|^2(1+|\xi|^2)^{s-1}d\xi \leqq \frac{1}{1+N^2}\int |\hat{u}_j|^2(1+|\xi|^2)^s d\xi \leqq \frac{C}{1+N^2}<\varepsilon,$$

where $\hat{u}_j$ is the Fourier transform of u_j. Let χ be the characteristic function of the set $\{\xi \in \Xi_n\colon |\xi| \leqq N\}$ and we put $\hat{v}_j=\chi(\xi)\hat{u}_j(1+|\xi|^2)^s$. For any integer l with $l+s\geqq 0$ we have

$$\int |\hat{v}_j|^2(1+|\xi|^2)^l d\xi=\int_{|\xi|\leqq N} |\hat{u}_j|^2(1+|\xi|^2)^{l+2s}d\xi$$

$$\leqq (1+N^2)^{l+s}\int |\hat{u}_j|^2(1+|\xi|^2)^s d\xi \leqq C(1+N^2)^{l+s},$$

which shows that the sequence $\{v_i\}$ is bounded in $(\mathscr{D}_{L^2})_x$. Since $\{u_j\}$ converges in $(\mathscr{D}'_{L^2})_x$ to 0 as $j\to\infty$, it follows that $\sup_k |(u_j, v_k)| = \sup_k |<u_j, \bar{v}_k>|$ converges to 0 as $j\to\infty$. Consequently the inequalities

$$\sup_k |(u_j, v_k)| \geqq |(u_j, v_j)| = \frac{1}{(2\pi)^n}\int_{|\xi|\leqq N} |\hat{u}_j|^2(1+|\xi|^2)^s d\xi$$

$$\geqq \frac{1}{(2\pi)^n}\int_{|\xi|\leqq N} |u_j|^2(1+|\xi|^2)^{s-1} d\xi$$

yield that $\int_{|\xi|\leqq N} |\hat{u}_j|^2(1+|\xi|^2)^{s-1}d\xi < 2\varepsilon$ for sufficiently large j, which completes the proof.

REMARK. Let f be a $\mathscr{D}'_{L^2}$-valued continuous function of t with support $\subset[0, a]$ such that $f(t)=o(t^k)$ in $(\mathscr{D}'_{L^2})_x$ as $t\downarrow 0$. Then there exists a non-negative integer m such that f is an $\mathscr{H}_{(-m)}$-valued continuous function of t and $\|f(t)\|_{(-m)}=o(t^k)$ as $t\downarrow 0$. In fact, the set $\left\{\frac{f(t)}{t^k}\right\}_{0<t<a}$ is bounded in $(\mathscr{D}'_{L^2})_x$ and therefore there exists a non-negative integer m such that $f(t)\in\mathscr{H}_{(-m+1)}$ and $\|f(t)\|_{(-m+1)}=O(t^k)$. By Lemma 3, $f(t)$ is an valued $\mathscr{H}_{(-m)}$-continuous function of t and $\lim_{t\downarrow 0}\frac{\|f(t)\|_{(-m)}}{t^k}=0$.

LEMMA 4. *Let E be a Fréchet space and F an inductive limit of Banach spaces F_j, $j=1, 2, \cdots$, with norm $\|\cdot\|_{(j)}$ and assume that every bounded subset of F belongs to some F_j and bounded there. Let $\{u_\gamma\}_{\gamma\in\Gamma}$ be a family of continuous linear maps u_γ of E into F and assume that $\{u_\gamma(x)\}_{\gamma\in\Gamma}$ is bounded in F for every $x\in E$. Then there exists an m_0 such that $u_\gamma(x)\in F_{m_0}$ for any $x\in E$ and the seminorm $x\to\sup_\gamma\|u_\gamma(x)\|_{(m_0)}$ is continuous.*

PROOF. Let us consider the set

$$\mathbf{F}_m=\{\{y_\gamma\}_{\gamma\in\Gamma}:\ y_\gamma\in F_m \text{ and } \{\|y_\gamma\|_{(m)}\}_{\gamma\in\Gamma} \text{ is bounded}\}.$$

If we put $\|\{y_\gamma\}\|=\sup_\gamma\|y_\gamma\|_{(m)}$ for $\{y_\gamma\}_{\gamma\in\Gamma}\in\mathbf{F}_m$, then $\mathbf{F}_m$ is a Banach space with norm $\|\cdot\|$. $G_m=\{(x, \{u_\gamma(x)\}_{\gamma\in\Gamma})\in E\times\mathbf{F}_m\}$ is a Fréchet space and closed in $E\times\mathbf{F}_m$. Consider the projection P_m of G_m into E. As a continuous image of a Fréchet space, the set $E_m=P_m(G_m)$ is of the 1st or of the 2nd category. On the other hand we have $E=\bigcup_m E_m$. In fact, let $x\in E$. Since $\{u_\gamma(x)\}_{\gamma\in\Gamma}$ is bounded, there exists an m such that $u_\gamma(x)\in F_m$ and $\{\|u_\gamma(x)\|_{(m)}\}_{\gamma\in\Gamma}$ is bounded, that is, $(x, \{u_\gamma, (x)\})\in G_m$ and therefore $x\in E_m$. Since E is a Fréchet space, it follows that $E=E_{m_0}$ for some m_0. Then the projection P_{m_0} has a continuous inverse $E\ni x\to(x, \{u_\gamma(x)\})\in G_{m_0}$. This means that $u_\gamma(x)\in F_{m_0}$ for any $x\in E$ and the norm $x\to\sup_\gamma\|u_\gamma(x)\|_{(m)}$ is continuous. Thus the proof is complete.

Let $u \in \mathscr{D}'(R_t^+)$ and $I=(a, b) \subset\subset (0, \infty)$. u is said to be of order $\leqq l$ on $\bar{I}$ if there exists a constant C such that $|<u, \phi>| \leqq C \sup_t |D_t^l \phi(t)|$ for any $\phi \in \mathscr{D}(R_t^+)$. Then, $\mathscr{D}_{\bar{I}}$ being dense in $\mathscr{D}_{\bar{I}}^l$, u will be uniquely extended to a continuous linear form on $\mathscr{D}_{\bar{I}}^l$.

Now we are prepared to apply S. Łojaciewicz's method [13, p.p. 17–18] in proving the following

THEOREM 2. *Let a be any positive number. Given $u \in \mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$, then $\mathscr{D}'_{L^2}\text{-}\lim_{t \downarrow 0} u = \alpha \in (\mathscr{D}'_{L^2})_x$ if and only if there exists a $(\mathscr{D}'_{L^2})_x$-valued continuous function $f(t)$, $t \in [0, a]$, such that for a non-negative integer k,*

$$u = Y_t \otimes \alpha + D_t^k f \qquad \text{in } (0, a) \times R_n$$

and

$$f(t) = o(t^k) \qquad \text{as } t \downarrow 0.$$

More precisely, f can be chosen an $\mathscr{H}_{(-m)}$-valued continuous function with $\|f(t)\|_{(-m)} = o(t^k)$ as $t \to 0$, for some non-negative integer m.

PROOF. Let u be written in the form as asserted in our theorem. Let $g(t) = \dfrac{f(t)}{t^k}$. Now, given $\phi \in \mathscr{D}(R_t^+)$, there can be found a $\psi \in \mathscr{D}(R_t^+)$ such that $\psi_\varepsilon = t^k D_t^k \phi_\varepsilon$. Since, then, $g(t) \to 0$ in $\mathscr{D}'_{L^2}$ as $t \downarrow 0$, we obtain for $\varepsilon \downarrow 0$

$$<D_t^k f, \phi_\varepsilon> = (-1)^k \int_0^\infty f(t) D_t^k \phi_\varepsilon \, dt = (-1)^k \int_0^\infty g(t) \psi_\varepsilon \, dt \to 0.$$

This means that $\mathscr{D}'_{L^2}\text{-}\lim_{t \downarrow 0} D_t^k f = 0$, so we have

$$\begin{aligned} \mathscr{D}'_{L^2}\text{-}\lim_{t \downarrow 0} u &= \mathscr{D}'_{L^2}\text{-}\lim_{t \downarrow 0} (Y_t \otimes \alpha + D_t^k f) \\ &= \mathscr{D}'_{L^2}\text{-}\lim_{t \downarrow 0} (Y_t \otimes \alpha) = \alpha. \end{aligned}$$

Suppose $\mathscr{D}'_{L^2}\text{-}\lim_{t \downarrow 0} u = \alpha$ holds. Without loss of generality, we may assume that $a=1$ and $\alpha=0$. Let us consider the intervals $I=(0, 1)$ and $I_\nu = (\theta^{\nu+2}, \theta^\nu)$, $\nu = 0, 1, \cdots$, where $\theta = \dfrac{1}{2}$, and we put $u_\nu(t, x) = u(\theta^\nu t, x)$. Now we can regard u_ν as a continuous map of $\mathscr{D}_{\bar{I}_0}$ into $(\mathscr{D}'_{L^2})_x$. Here $\mathscr{D}_{\bar{I}_0}$ is a Fréchet space and $(\mathscr{D}'_{L^2})_x = \bigcup_{m=0}^\infty \mathscr{H}_{(-m)}$. In view of Lemmas 3 and 4, we can take a non-negative integer m and a 0-neighborhood V of $\mathscr{D}_{\bar{I}_0}$ such that $\|u_\nu(\phi)\|_{(-m)} \leqq 1$ and $\lim_{\nu \to \infty} \|u_\nu(\phi)\|_{(-m)} = 0$ for any $\phi \in V$, where $V = \{\phi \in \mathscr{D}_{\bar{I}_0} : \sup_t |D_t^l \phi| \leqq 1\}$, l being a non-negative integer. $\mathscr{D}_{\bar{I}_0}^l$ is the closure of $\mathscr{D}_{\bar{I}_0}$ with respect to the norm $\sup_t |D_t^l \phi|$, so that u_ν can be uniquely extended to a continuous map of $\mathscr{D}^l_{\bar I_0}$ into $\mathscr{H}_{(-m)}$. By the same method as in [9, p. 399] we can find a function $G \in \mathscr{D}^l_{\bar I_0 \times \bar I_0}$ such that if we put $f_\nu(t) = u_\nu(g_t)$, where $g_t(s) = G(t, s)$, then $f_\nu(t)$ is an $\mathscr{H}_{(-m)}$-valued continuous function with support $\subset \bar I_0$ and

(1) $u_\nu = D_t^{2l+2} f_\nu \qquad \text{in } I_0.$

Since $\{g_t\}_{t \in \bar I_0}$ forms a compact subset of $\mathscr{D}^l_{\bar I_0}$, it follows from the Banach-Steinhaus theorem that the sequence of $\mathscr{H}_{(-m)}$-valued continuous functions $f_\nu(t)$ uniformly converges to 0 as $\nu \to \infty$, hence we can choose $\lambda_\nu > 0$ so that

(2) $\sup_t \|f_\nu(t)\|_{(-m)} \leqq \lambda_\nu \downarrow 0 \text{ as } \qquad \nu \to \infty.$

Since, for any $\psi \in \mathscr{D}_{\bar I_\nu}$, we can write

$$\begin{aligned} u(\psi) = \langle u(t, \cdot), \psi(t) \rangle_t &= \langle u(\theta^\nu t, \cdot), \theta^\nu \psi(\theta^\nu t) \rangle_t \\ &= \langle u_\nu(t, \cdot), \theta^\nu \psi(\theta^\nu t) \rangle_t \\ &= \langle D_t^{2l+2} f_\nu(t), \theta^\nu \psi(\theta^\nu t) \rangle_t \\ &= \langle D_t^{2l+2}(\theta^{\nu(2l+2)} f_\nu(\theta^{-\nu} t)), \psi(t) \rangle_t, \end{aligned}$$

so $F_\nu(t) = \theta^{\nu(2l+2)} f_\nu(\theta^{-\nu} t)$ will be an $\mathscr{H}_{(-m)}$-valued continuous function with support $\subset \bar I_\nu$ such that

(3) $u = D_t^{2l+2} F_\nu(t) \qquad \text{in } I_\nu,$

(4) $\sup_t \|F_\nu(t)\|_{(-m)} \leqq \lambda_\nu \theta^{\nu(2l+2)}.$

If we put $q_\nu(t) = F_{\nu+1}(t) - F_\nu(t)$, $t \in \bar I_{\nu+1} \cap \bar I_\nu$, then, since $D_t^{2l+2} q_\nu = 0$ in $I_{\nu+1} \cap I_\nu$, so there is a polynomial $\tilde q_\nu$ such that $\tilde q_\nu(t) = q_\nu(t)$ for $t \in \bar I_{\nu+1} \cap \bar I_\nu$, where q_ν is determined by taking $t_0 = \theta^{\nu+2} < t_1 < \cdots < t_{2l+1} = \theta^{\nu+1}$ and by putting $\tilde q_\nu(t) = \sum_{j=0}^{2l+1} q_\nu(t_j) \times \prod_{j \neq k} \frac{t - t_k}{t_j - t_k}$. By a simple estimation we obtain

(5) $D_t^{2l+2} \tilde q_\nu = 0,$

(6) $\|\tilde q_\nu(t)\|_{(-m)} \leqq K \lambda_\nu \theta^\nu (\theta^{\nu(2l+1)} + t^{2l+1}) \qquad \text{for } t \in [\theta^{\nu+2}, 1],$

where K is a constant independent of ν. Now let us define continuous functions $\tilde F_\nu(t)$ on $[\theta^{\nu+2}, 1]$ by putting $\tilde F_0 = F_0$ and

$$\tilde F_\nu = \begin{cases} F_\nu & \text{on } \bar I_\nu \\ \tilde F_{\nu-1} + \tilde q_{\nu-1} & \text{on } [\theta^{\nu+1}, 1] \end{cases}$$

for $\nu = 1, 2, \cdots$. Note that the restriction of F_ν to $[\theta^{\nu+1}, \theta^\nu]$ is equal to $\tilde F_{\nu-1} + \tilde q_{\nu-1}$. For any $\nu \geqq \nu_0$, ν_0 being any given positive integer, we have for $t \in [\theta^{\nu_0+2}, 1]$

$$\|\tilde{F}_{\nu+k}(t)-\tilde{F}_\nu(t)\|_{(-m)}=\|\tilde{q}_\nu(t)+\cdots+\tilde{q}_{\nu+k-1}(t)\|_{(-m)}$$

$$\leqq K\sum_{j=\nu}^{\nu+k-1}\lambda_j\theta^j(\theta^{j(2l+1)}+t^{2l+1})\leqq 4K\lambda_\nu\theta^\nu.$$

This shows that $\{\tilde{F}_\nu\}$ uniformly converges on $[\theta^{\nu_0+1}, 1]$. Let $f(t)=\lim_{\nu\to\infty}\tilde{F}_\nu(t)$, $t\in(0, 1]$. f is an $\mathscr{H}_{(-m)}$-valued continuos functions on $(0, 1]$ and

$$(7)\quad f(t)=\tilde{F}_\nu(t)+\sum_{j=\nu}^{\infty}\tilde{q}_j(t), \qquad t\in[\theta^{\nu+2}, 1],$$

whence $D_t^{2l+2}f=u$ in I since $D_t^{2l+2}\tilde{F}_\nu=u$ in $(\theta^{\nu+2}, 1)$ and $D_t^{2l+2}\tilde{q}_j=0$. Owing to the estimates (4), (5), we have for $t\in I_\nu$

$$(8)\quad \|\tilde{F}_\nu(t)\|_{(-m)}=\|F_\nu(t)\|_{(-m)}\leqq\lambda_\nu\theta^{\nu(2l+2)}$$
$$\leqq\lambda_\nu\theta^{-4(l+1)}t^{2l+2},$$

$$(9)\quad \|\tilde{q}_\nu(t)\|_{(-m)}\leqq K\lambda_\nu\theta^\nu(\theta^{\nu(2l+1)}+t^{2l+1})$$
$$\leqq 2K\lambda_\nu\theta^{\nu(2l+2)}$$
$$\leqq 2K\lambda_\nu\theta^{-4(l+1)}t^{2l+2},$$

$$(10)\quad \|\tilde{q}_{\nu+1}(t)\|_{(-m)}\leqq K\lambda_{\nu+1}\theta^{\nu+1}(\theta^{(\nu+1)(2l+1)}+t^{2l+1})$$
$$\leqq K\lambda_\nu\theta^\nu(\theta^{\nu(2l+1)}+t^{2l+1})\theta$$
$$\leqq\theta(2K\lambda_\nu\theta^{-4(l+1)})t^{2l+2}.$$

From these, together with (7), we obtain that $f(t)=o(t^{2l+2})$ as $t\downarrow 0$. Thus the proof is complete.

Let $\phi\in\mathscr{D}(R_t^+)$ be such that $\phi\geqq 0$ and $\int_0^\infty\phi(t)\,dt=1$. Let $\rho=Y*\phi$ and put $\rho_{(\varepsilon)}(t)=\rho\left(\frac{t}{\varepsilon}\right)$ for any $\varepsilon>0$. Consider a $u\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. Then $\rho_{(\varepsilon)}u$ will always be understood an element of $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$. If $\rho_{(\varepsilon)}u$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to v_ϕ as $\varepsilon\downarrow 0$, then v_ϕ does not depend on the choice of ϕ. In fact, this follows from Lemma 2, together with the equations $v_\phi=v_{\phi_\lambda}$, $\lambda>0$, which can be easily verified. The limit element v will be referred to as the $\mathscr{D}'_{L^2}$-canonical extension of u over $t=0$ and denoted by $u_\sim$. It is to be noticed that $(u_\sim|R_{n+1}^+)_\sim=u_\sim$. The same will be the case for $u\in\mathscr{D}'(R_t^-)((\mathscr{D}'_{L^2})_x)$. Then its canonical extension over $t=0$ will be denoted by $u^\sim$.

Proposition 5. *Let $u\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. If $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}u=\alpha$, then u has the $\mathscr{D}'_{L^2}$-canonical extension $u_\sim$.*

Proof. Owing to Theorem 2 we have a local representation of u:

$$u=Y_t\otimes\alpha+D_t^k f \qquad \text{in } (0, a)\times R_n,$$

where f is an $\mathscr{H}_{(-m)}$-valued continuous function with the properties described there. Then we have for $t<a$

$$\rho_{(\varepsilon)}u=\rho_{(\varepsilon)}\otimes\alpha+\rho_{(\varepsilon)}D_t^k f$$
$$=\rho_{(\varepsilon)}\otimes\alpha+D_t^k(\rho_{(\varepsilon)}f(t))+\sum_{j=1}^{k}(-1)^j\binom{k}{j}D_t^{k-j}((D_t^j\rho_{(\varepsilon)})f)),$$

whence, observing that $\rho_{(\varepsilon)}f\to f$ and $(D_t^j\rho_{(\varepsilon)})f\to 0$ in $\mathscr{D}'(-\infty, a)((\mathscr{D}'_{L^2})_x)$ as $\varepsilon\downarrow 0$, we can establish the conclusion of our proposition.

We shall say that $u\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ is $\mathscr{D}'_{L^2}$-canonical if $(u\,|\,R^+_{n+1})_\sim=u$ holds. In what follows, we shall write u instead of $(u\,|\,R^+_{n+1})_\sim$. Then we can show the following

PROPOSITION 6. *Let $u\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ and put $v=Y*u$. u is $\mathscr{D}'_{L^2}$-canonical if and only if v has the $\mathscr{D}'_{L^2}$-boundary value* 0 *and is $\mathscr{D}'_{L^2}$-canonical,*

PROOF. Suppose that u is $\mathscr{D}'_{L^2}$-canonical. We shall first show that $\mathscr{D}'_{L^2}$-$\lim_{t\downarrow 0} v=0$. Let ϕ be an arbitrary element of $\mathscr{D}(R_t^+)$ such that $\phi(t)\geqq 0$ and $\int\phi(t)dt=1$ and γ an element of $\mathscr{D}(R_t)$ such that $\gamma(t)=1$ in a 0-neighborhood of R_t. Then, observing that $<(1-\gamma)u, \check{Y}*\phi_\varepsilon>=0$ for $\varepsilon>0$ small enough, we obtain

$$<Y*u, \phi_\varepsilon>=<\gamma u, \check{Y}*\phi_\varepsilon>+<(1-\gamma)u, \check{Y}*\phi_\varepsilon>$$
$$=<u, \gamma(1*\phi_\varepsilon)>-<\gamma u, Y*\phi_\varepsilon>$$
$$=<u, \gamma>-<\rho_{(\varepsilon)}u, \gamma>,$$

which implies that $\lim_{\varepsilon\downarrow 0}<Y*u, \phi_\varepsilon>=0$, that is, $\mathscr{D}'_{L^2}$-$\lim_{t\downarrow 0} v=0$ as desired. That v is $\mathscr{D}'_{L^2}$-canonical can be seen as follows. Owing to Proposition 5, $(Y*u)_\sim$ exists. Let $\alpha_0, \alpha_1, \cdots, \alpha_k\in(\mathscr{D}'_{L^2})_x$ be such that

$$(Y*u)_\sim-Y*u=\delta\otimes\alpha_0+D_t\delta\otimes\alpha_1+\cdots+D_t^k\delta\otimes\alpha_k.$$

Differentiating both sides of the equation and noting that $D_t(\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}(Y*u))=-iu$, we have

$$D_t\delta\otimes\alpha_0+\cdots+D_t^{k+1}\delta\otimes\alpha_k=0,$$

whence $\alpha_0=\cdots=\alpha_k=0$, that is, $Y*u$ is $\mathscr{D}'_{L^2}$-canonical.

The converse is trivial from the equations

$$\rho_{(\varepsilon)}u=i\rho_{(\varepsilon)}D_t(Y*u)=iD_t(\rho_{(\varepsilon)}(Y*u))-\phi_\varepsilon(Y*u),$$

since, then, $\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}u=iD_t(Y*u)=u$. Thus the proof is complete.

REMARK. In a previous paper [10], it is really shown that, given the space $\mathscr{H}_{(\sigma,s)}(R^+_{n+1})$ [7, p. 51], where σ and s are fixed, then (1) the $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u$ exists for every $u \in \mathscr{H}_{(\sigma,s)}(\bar{R}^+_{n+1})$ if and only if $\sigma > \frac{1}{2}$, (2) the $\mathscr{D}'_{L^2}$-canonical extension $u_\sim$ exists for every $u \in \mathscr{H}_{(\sigma,s)}(\bar{R}^+_{n+1})$ if and only if $\sigma > -\frac{1}{2}$, (3) $u_\sim \in \mathscr{H}_{(\sigma,s)}(R_{n+1})$ for every $u \in \mathscr{H}_{(\sigma,s)}(\bar{R}^+_{n+1})$ if and only if $|\sigma| < \frac{1}{2}$.

Let $u \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$. If, for $\varepsilon \downarrow 0$, $u(\varepsilon t, x)$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to a limit independent of t, we can write $\lim_{\varepsilon\downarrow 0} u(\varepsilon t, x) = 1_t \otimes \alpha$ with $\alpha \in (\mathscr{D}_{L^2})_x$. When this is the case, we shall call α the section of u for $t=0$ and denote it by $u(0, \cdot)$ [13, p. 15]. We shall also say that u has no mass on the hyperplane $t=0$, if $\varepsilon u(\varepsilon t, x)$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to 0 as $\varepsilon \to 0$ [13, p. 23]. It is clear that if u has the section for $t=0$, then u and $D_t u$ have no mass on $t=0$. Now we can show the following Theorem 3 which is an analogue to Theorem 2. However, the proof will be omitted since it can be carried out in a similar way as shown there.

THEOREM 3. *Let a be any positive number. Given $u \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$, then $u(0, \cdot) = \alpha \in (\mathscr{D}'_{L^2})_x$ if and only if there exists a $(\mathscr{D}'_{L^2})_x$-valued continuous function $f(t)$, $t \in [-a, a]$, such that for a non-negative integer k,*

$$u = 1_t \otimes \alpha + D_t^k f \qquad \text{in } (-a, a) \times R_n,$$

and

$$f(t) = o(|t|^k) \qquad \text{as } t \to 0.$$

More precisely, f can be chosen an $\mathscr{H}_{(-m)}$-valued continuous function with $\|f(t)\|_{(-m)} = o(|t|^k)$ as $t \downarrow 0$, for some non-negative integer m.

PROPOSITION 7. *Let $u \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$. Then u is $\mathscr{D}'_{L^2}$-canonical if and only if u has no mass on $t=0$.*

PROOF. Suppose u is $\mathscr{D}'_{L^2}$-canonical. Then by Proposition 6, $(Y*u)(\varepsilon t, x)$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to 0, whence $D_t\{(Y*u)(\varepsilon t, x)\} = -i\varepsilon u(\varepsilon t, x) \to 0$ in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$. Thus u has no mass on $t=0$.

Conversely, suppose u has no mass on $t=0$. Let $\phi_1 \in \mathscr{D}(R^+_t)$, $\phi_2 \in \mathscr{D}(R^-_t)$ be such that $\phi_1(t) \geqq 0$, $\phi_2(t) \geqq 0$, $\int \phi_1(t)dt = \int \phi_2(t)dt = 1$. If we put $\rho_1 = Y*\phi_1$, $\rho_2 = Y*\phi_2$, then $\chi = \rho_1 - \rho_2 \in \mathscr{D}(R_t)$. Now $\chi_{(\varepsilon)} u$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to 0 as $\varepsilon \downarrow 0$, and $(1-\rho_{2(\varepsilon)})u = 0$. Since we can write $\rho_{1(\varepsilon)} u = u + \chi_{(\varepsilon)} u - (1-\rho_{2(\varepsilon)})u$, it follows that $\rho_{1(\varepsilon)} u$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to u, which completes the proof.

In an entirely similar way we can show the following

PROPOSITION 8. *Let $u \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ have no mass on $t=0$. If $u_1 = u \mid R^+_{n+1}$*

has the $\mathscr{D}'_{L^2}$-canonical extension $u_{1\sim}$, then $u_2=u\,|\,R^-_{n+1}$ has the $\mathscr{D}'_{L^2}$-canonical extension $u_{\tilde{2}}$, and we can write $u=u_{1\sim}+u_{\tilde{2}}$.

When u has no mass on $t=0$, we shall obtain

PROPOSITION 9. *Let $u\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$. If u has no mass on $t=0$ and $\mathscr{D}'_{L^2}$-$\lim_{t\downarrow 0} u_1=\mathscr{D}'_{L^2}$-$\lim_{t\uparrow 0} u_2=\alpha$, where $u_1=u\,|\,R^+_{n+1}$ and $u_2=u\,|\,R^-_{n+1}$, then u has the section α for $t=0$.*

PROOF. For any $\alpha>0$ there exist integers $k,m\geqq 0$ and $\mathscr{H}_{(-m)}$-valued continuous functions $f_1(t)$ and $f_2(t)$ defined, respectively, on $[0,a]$ and on $[-a,0]$, for which

$$u_1=Y\otimes\alpha+D_t^kf_1,\ u_2=(1-Y)\otimes\alpha+D_t^kf_2 \qquad \text{in } (-a,a)\times R_n,$$

where $\|f_1\|_{(-m)}, \|f_2\|_{(-m)}=o(|t|^k)$ as $t\downarrow 0$ and we define $f_1(t)=0$ for $t<0$ and $f_2(t)=0$ for $t>0$. Whence we have

$$u_{1\sim}+u_{\tilde{2}}=1_t\otimes\alpha+D_t^k(f_1+f_2),$$

which means that $u_{1\sim}+u_{\tilde{2}}$ has the section α for $t=0$. Since $u-u_{1\sim}-u_{\tilde{2}}$ has no mass on $t=0$ and, in addition, its support lies on $t=0$, we must have that $u=u_{1\sim}+u_{\tilde{2}}$.

Let $u\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. We shall say that u has a weak $\mathscr{D}'_{L^2}$-boundary value α and we write w-$\mathscr{D}'_{L^2}$-$\lim_{t\downarrow 0} u=\alpha$ if $<u,\phi_\varepsilon>$ converges weakly in $(\mathscr{D}'_{L^2})_x$ to α as $\varepsilon\downarrow 0$, where ϕ is chosen an arbitrary non-negative function $\in\mathscr{D}(R_t^+)$ with $\int_0^\infty\phi(t)\,dt=1$.

PROPOSITION 10. *Let $u\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. Then w-$\mathscr{D}'_{L^2}$-$\lim_{t\downarrow 0} u$ exists if and only if $\lim_{t\downarrow 0} u$ exists and the set $\{u(\varepsilon t,x)\}_{0<\varepsilon\leqq 1}$ is bounded in $\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$.*

PROOF. The "only if" part is trivial. The "if" part can be verified as follows: $u_\varepsilon=u(\varepsilon t,x)$ is considered as a continuous map of $\mathscr{D}(R_t^+)$ into $(\mathscr{D}'_{L^2})_x$. We can apply the Banach-Steinhaus theorem to conclude that $<u_\varepsilon,\phi>$ weakly converges in $(\mathscr{D}'_{L^2})_x$.

Along the same line as in the proof of Theorem 2 we can prove the following

THEOREM 2′. *Let a be any positive number. Given $u\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$, then w-$\mathscr{D}'_{L^2}$-$\lim_{t\downarrow 0} u=\alpha\in\mathscr{D}'_{L^2}$ if and only if for some non-negative integer m there exists an $\mathscr{H}_{(-m)}$-valued continuous function $f(t)$, $t\in[0,a]$, such that for a non-negative integer k*

$$u=Y\otimes\alpha+D_t^kf \qquad \textit{in } (0,a)\times R_n$$

and

$$<f(t), \psi> = o(t^k) \qquad as\ t \downarrow 0$$

for any $\psi \in (\mathscr{D}_{L^2})_x$.

Let $\phi \in \mathscr{D}(R_t^+)$ be taken in such a way that $\phi \geqq 0$ and $\int_0^\infty \phi(t)dt = 1$, and let $\rho_{(\varepsilon)}$ be defined as before. Let $u \in \mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. We shall say that u has a weak $\mathscr{D}'_{L^2}$-canonical extension if, for any $\psi \in \mathscr{D}(R_t)$, $<\rho_{(\varepsilon)} u, \psi>$ converges weakly in $(\mathscr{D}'_{L^2})_x$. When this is the case, there exists a unique $v \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ such that $\lim_{\varepsilon \downarrow 0} <\rho_{(\varepsilon)} u, \psi> = <v, \psi>$. Here v is called the weak $\mathscr{D}'_{L^2}$-canonical extension and denoted by $u_\sim$.

Proposition 11. *Let $u \in \mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. Then u has the weak $\mathscr{D}'_{L^2}$-canonical extension $u_\sim$ if and only if $\rho_{(\varepsilon)} u$ converges in $\mathscr{D}'(R_{n+1})$ and the set $\{\rho_{(\varepsilon)} u\}_{0<\varepsilon\leqq 1}$ is bounded in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ for any ϕ.*

If the limit in defining the notions such that the $\mathscr{D}'_{L^2}$-canonical, the section and the like is understood in the weak sense, then we can show the corresponding analogues to Theorem 3 and Propositions 5, 6, 7, 8 and 9.

A sequence $\{\phi_k\}$, $\phi_k \in \mathscr{D}(R_t)$, will be referred to as a δ-sequence if $\phi_k \geqq 0$, $\int \phi_k dt = 1$ and supp ϕ_k converges to $\{0\}$ as $k \to \infty$. Let $u \in \mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. If $<u, \phi_k>$ converges in $(\mathscr{D}'_{L^2})_x$ for every δ-sequence $\{\phi_k\}$, where $\phi_k \in \mathscr{D}(R_t^+)$, then the limit is called the strict $\mathscr{D}'_{L^2}$-boundary value of u. The strict $\mathscr{D}'_{L^2}$-canonical extension of u over $t=0$ will be defined in an obvious way. Similarly for the section of u for $t=0$ in the strict sense if $u \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$. With the aid of these concepts, we shall be able to give some refinement of the results already obtained in this section. For instance, the following proposition is a refinement of Theorem 2.

Proposition 12. *Let $u \in \mathscr{D}(R_t^+)((\mathscr{D}'_{L^2})_x)$. u has a strict $\mathscr{D}'_{L^2}$-boundary value $\alpha \in (\mathscr{D}'_{L^2})_x$ if and only if for some non-negative integer m and $a>0$, there exists an $\mathscr{H}_{(-m)}$-valued bounded measurable function $w(t)$ in $t \in [0, a]$ such that*

$$u = w \qquad in\ \mathscr{D}'((0, a) \times R_n)$$

and

$$\lim_{t \downarrow 0} \|w(t) - \alpha\|_{(-m)} = 0.$$

This can be shown by making use of Lemma 3. But the proof is omitted.

3. Operator of order r which maps $(\mathscr{D}'_{L^2})_x$ into itself

Let r be an arbitrary real number and let OP_r be the set of linear maps of $(\mathscr{D}'_{L^2})_x$ into itself which are at the same time continuous operators of $\mathscr{H}_{(s+r)}(R_n)$ into $\mathscr{H}_{(s)}(R_n)$ for any real s. OP_r is a locally convex Hausdorff space, where the topology is defined by the operator norms $\|\cdot\|_{(s+r\to s)}$ of the spaces $\mathscr{L}(\mathscr{H}_{(s+r)}, \mathscr{H}_{(s)})$. Let l be a non-negative integer or ∞. We denote by $\mathfrak{C}^l_{(r)}$ the set of OP_r-valued C^l functions of $t\in R_t$. We shall note that any OP_r-valued C^l function $A(t)$ defined on $[0, \infty)$ can be extended to a function $\in \mathfrak{C}^l_{(r)}$. It is trivial if $l<\infty$. Let $l=\infty$. In [20] R.T. Seeley considered the sequences $\{a_k\}$, $\{b_k\}$ of real numbers such that (i) $b_k<0$, (ii) $\sum_{k=0}^{\infty}|a_k|\,|b_k|^n<\infty$ for $n=0, 1,\cdots$, (iii) $\sum_{k=0}^{\infty}a_k b_k^n=1$ for $n=0, 1,\cdots$ and (iv) $b_k\to-\infty$ as $k\to\infty$. Let ϕ be a C^∞ function on R_t with $\phi(t)=1$ for $0\leqq t\leqq 1$, $\phi(t)=0$ for $t>2$. We define $A(t)=\sum_{k=0}^{\infty}a_k\phi(b_k t)A(b_k t)$ for $t<0$. It is easy to verify that $A(t)$ is a C^∞ function on $(-\infty, 0)$. We can write

$$\sum_{k=0}^{\infty}a_k\phi(b_k t)A(b_k t)-A(0)=\sum_{k=0}^{\infty}a_k(\phi(b_k t)A(b_k t)-A(0)).$$

Then there exists for any given $\varepsilon>0$ an integer $N>0$ such that

$$\sum_{k=N}^{\infty}\|a_k(\phi(b_k t)A(b_k t)-A(0))\|_{(s+r\to s)}\leqq 2\max_{0\leqq t\leqq 2}\|A(t)\|_{(s+r\to s)}\sum_{k=N}^{\infty}|a_k|<\varepsilon,$$

whence it follows that $\lim_{t\uparrow 0}A(t)=A(0)$. Similarly, with the aid of (ii) and (iii), we can also show that $\lim_{t\uparrow 0}A^{(j)}(t)=A^{(j)}(0)$, $j=1, 2,\cdots$.

Let $A^*(t)$ be denoted for each t the adjoint with respect to the scalar product $(\phi, \psi)=<\phi, \bar{\psi}>$ between $\mathscr{H}_{(s)}(R_n)$ and $\mathscr{H}_{(-s)}(R_n)$. Then $A(t)\in\mathfrak{C}^l_{(r)}$ implies $A^*(t)\in\mathfrak{C}^l_{(r)}$.

In the rest of this section $A(t)$ will be understood to belong to $\mathfrak{C}^\infty_{(r)}$. Let $\phi\in\mathscr{D}(R_{n+1})$. For each $t\in R_t$, $A(t)\phi(t, \cdot)\in(\mathscr{D}_{L^2})_x$ and $A(t)\phi(t, \cdot)$ is a $(\mathscr{D}_{L^2})_x$-valued C^∞ function of t, whence $A(t)\phi(t, \cdot)$, when considered as a function of t and x, is an infinitely differentiable function which, in what follows, will often be denoted by $A(t)\phi(t, x)$. Now we shall define $A(t)u$ for every $u\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. Let $\{\phi_j\}$, $\phi_j\in\mathscr{D}(R_{n+1})$, be a sequence such that ϕ_j converges in $\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$ to u. $A(t)\phi_j(t, \cdot)\in\mathscr{D}'(R_n^+)((\mathscr{D}'_{L^2})_x)$ for each j. Let B be any bounded subset of $(\mathscr{D}_{L^2})_x$. Then, for any $\psi_1\in\mathscr{D}(R_t)$ and $\psi_2\in B$, we have

$$(A(t)\phi_j(t, x), \psi_1\otimes\psi_2)=(\phi_j(t, x), \psi_1 A^*(t)(\psi_2)),$$

where the set $\{\psi_1 A^*(t)(\psi_2): \psi_2\in B\}$ is equicontinuous in $\mathscr{D}(R_t^+)\hat{\otimes}_\iota(\mathscr{D}_{L^2})_x$.

Thus the sequence $A(t)\phi_j(t,\cdot)$ will converge in $\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$ to an element of $\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. The limit is defined as $A(t)u(t,\cdot)\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. If $u\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$, then $A(t)u$ will also be defined in an obvious fashion. In any way, owing to the Banach-Steinhaus theorem, the map $u\to A(t)u$ will be continuous.

PROPOSITION 13. *Let $u\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$. If u has a $\mathscr{D}'_{L^2}$- boundary value α, then $A(t)u$ also has a $\mathscr{D}'_{L^2}$-boundary value, which is equal to $A(0)\alpha$.*

PROOF. Our assumption implies that $\phi_\varepsilon u$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to $\delta\otimes\alpha$ as $\varepsilon\downarrow 0$, and therefore $\phi_\varepsilon A(t)u=A(t)\phi_\varepsilon u$ converges in $\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ to $A(t)(\delta\otimes\alpha)=\delta\otimes A(0)\alpha$, completing the proof.

REMARK. By the same method as above, we can prove the analogues for the canonical extension, the section for $t=0$ and the like.

By $\tilde{\mathscr{H}}_{(\sigma,s)}$ we mean the set of all $u\in\mathscr{D}'(R_{n+1})$ with the property that $\phi u\in\mathscr{H}_{(\sigma,s)}(R_{n+1})$ for any $\phi\in C_0^\infty(R_t)$. Here the topology is given as a local space [7, p. 42]. Then we have

PROPOSITION 14. *$A(t)$ is a continuous linear map of $\tilde{\mathscr{H}}_{(\sigma,s+r)}$ into $\tilde{\mathscr{H}}_{(\sigma,s)}$ for any real σ, s.*

PROOF. Let $\phi\in\mathscr{D}(R_t)$ be given. It suffices to show that there exists a constant C such that

$$||\phi(t)A(t)u||_{(\sigma,s)}\leqq C||\phi(t)u||_{(\sigma,s+r)}$$

for every $u\in\tilde{\mathscr{H}}_{(\sigma,s)}$, whence if we put $A_1(t)=\phi(t)A(t)$, we have only to show that

$$||A_1(t)u||_{(\sigma,s)}\leqq C||u||_{(\sigma,s+r)}$$

for any $u\in\mathscr{D}(R_{n+1})$, C being a constant.

Let $\sigma=0$. Then we have

$$||A_1(t)u||^2_{(\sigma,s)}=\int||A_1(t)u(t,\cdot)||^2_{(s)}dt$$

$$\leqq\sup_t||A_1(t)||^2_{(s+r\to s)}\int_{-\infty}^{\infty}||u(t,\cdot)||^2_{(s+r)}dt.$$

Let $\sigma=m$, a positive integer. It is well known that, for every s, the norm $||u||_{(m,s)}$ is equivalent to the norm

$$\left(\int||u(t,\cdot)||^2_{(s+m)}dt+\cdots+\int||D_t^m u(t,\cdot)||^2_{(s)}dt\right)^{1/2}.$$

Since $D_t^j(A_1(t)u)=\sum_{k=0}^{j}\binom{j}{k}(D_t^kA_1(t))D_t^{j-k}u$ and

$$||(D_t^k A_1(t))D_t^{j-k}u||^2_{(m+s-j)} \leqq \sup_t ||D_t^k A_1(t)||^2_{(m+s+r-j\to m+s-j)})||D_t^{j-k}u||_{(m+s+r-j)},$$

we see that $||A_1(t)u||_{(m,s)} \leqq C_2||u||_{(m,s+r)}$ with a constant C_2. $\{\mathscr{H}_{(\sigma,s+r)}\}_{0\leqq\sigma\leqq m}$ forms a Hilbert scale and $A_1(t)$ is continuous of $\mathscr{H}_{(0,s+r)}$ into $\mathscr{H}_{(0,s)}$ and of $\mathscr{H}_{(m,s+r)}$ into $\mathscr{H}_{(m,s)}$. In virtue of the interpolation theorem we can conclude that $A_1(t)$ is continuous of $\mathscr{H}_{(\sigma,s+r)}$ into $\mathscr{H}_{(\sigma,s)}$ for $0\leqq\sigma\leqq m$, where m can be chosen arbitrarily large. Similarly, $A_1^*(t)$ is continuous of $\mathscr{H}_{(\sigma,s+r)}$ into $\mathscr{H}_{(\sigma,s)}$ for $\sigma\geqq 0$, then its adjoint $A_1(t)=A_1^{**}(t)$ is continuous of $\mathscr{H}_{(-\sigma,-s)}$ into $\mathscr{H}_{(-\sigma,-s-r)}$. Thus the proof is complete.

4. Pseudo-commutativity for Calderón's singular integral operators

For any real $\beta\geqq 0$, $B_\beta(R_n)$ will stand for the class of bounded functions f on R_n such that the distributional derivatives $D^\alpha f$, $0\leqq|\alpha|\leqq[\beta]$, coincide with bounded functions and such that $D^\alpha f$, $|\alpha|=[\beta]$, satisfy a uniform Hölder condition of order $\beta-[\beta]$. The norm $||f||_\beta$ of a function f in $B_\beta(R_n)$ will be by definition the least upper bound for the absolute value of its derivatives of order $\leqq[\beta]$ and the Hölder constants of the derivatives of order $[\beta]$.

Let us consider a function $h(x,\xi)$, $x\in R_n$, $\xi\in \Xi_n$, with the following properties: for any fixed $x\in R_n$, $h(x,\xi)$ is homogeneous of degree 0 in ξ, $\in C^\infty(\Xi_n\setminus\{0\})$ and for each ξ, $|\xi|=1$, $h(x,\xi)$ and its derivatives with respect to coordinates of ξ of orders not exceeding $2n$ are functions of x belonging to $B_\beta(R_n)$, with bounded norms. The least upper bound of these norms is called the norm of h and denoted by $||h||_\beta$, that is,

$$||h||_\beta=\max_{0\leqq|\alpha|\leqq 2n}\left\{\sup_{|\xi|=1}\left\|\left(\frac{\partial}{\partial\xi}\right)^\alpha h(x,\xi)\right\|_\beta\right\}.$$

Let $a_0(x)$ be the mean value of $h(x,\xi)$ on $|\xi|=1$ and $k(x,z)$ is the inverse Fourier transform of $h(x,\xi)-a_0(x)$ with respect to ξ. An operator $f\to Kf$ of the form

$$Kf=a_0(x)f(x)+\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}k(x,x-y)f(y)dy$$

is said to be a B_β singular integral operator. We will call h the symbol of K and write $h=\sigma(K)$. We define the norm $||K||_\beta$ by $||K||_\beta=||h||_\beta$ where $h(x,\xi)=a_0(x)+\hat{k}(x,\xi)$,

In the case where $n\geqq 2$, let $\{Y_{lm}\}$, $m=0,1,\cdots$, $l=1,2,\cdots,d(m)$, be a complete orthogonal system of spherical harmonics of degree m, where $d(m)=g(m)-g(m-2)$, $g(m)=\binom{m+n-1}{n-1}$ and we set $g(-1)=g(-2)=0$. Then we can expand the B_β singular integral operator K in the series

$$(Kf)(x)=a_0(x)f(x)+\sum_{m=1}^{\infty}\sum_{l=1}^{d(m)} a_{lm}(x)(G_{lm}f)(x),$$

where G_{lm} are the Giraud operators

$$(G_{lm}f)(x)=\lim_{\varepsilon\downarrow 0}\int_{|x-y|>\varepsilon}|x-y|^{-n}Y_{lm}(x-y)f(y)dy,$$

and we have the estimates $\|a_0(x)\|_\beta\leqq C$, $\|a_{lm}\|_\beta\leqq Cm^{-(3/2)n}\|K\|_\beta$, $\|G_{lm}f\|_{(s)}\leqq Cm^{(n-2)/2}\gamma_m\|f\|_{(s)}$ with $\gamma_m=-i^m(2\sqrt{\pi})^{-n}\Gamma(m)\left(\Gamma\left(\frac{m+n}{2}\right)\right)^{-1}$ and $d(m)\leqq Cm^{n-2}$ ([3], [15]).

For $n=1$ we have the expression

$$(Kf)(x)=a_0(x)f(x)+a_1(x)\lim_{\varepsilon\downarrow 0}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy,$$

where $a_0, a_1 \in B_\beta$.

Let Λ and S be operators with symbols $|\xi|$ and $(1+|\xi|^2)^{1/2}$ respectively. Then, for any B_∞ singular integral operator K, the product KS^γ is an operator belonging to the class OP_γ. In this section we shall study the order of the operator $S^\gamma K-KS^\gamma$ to give a refinement of Calderón's result [3, p. 72].

Now, the operator S^α can be written in the form

$$S^\alpha(\chi)=G_{-\alpha}*\chi, \qquad \chi\in(\mathscr{D}'_{L^2})_x,$$

where

$$G_\alpha(x)=\begin{cases} C_\alpha\,\mathrm{P.f.}[|x|^{(\alpha-n)/2}K_{(n-\alpha)/2}(|x|)] & \text{for } \alpha\neq 0, -2, -4,\cdots,\\ (1-\Delta)^k & \text{for } \alpha=-2k,\ k=0, 1, 2,\cdots,\end{cases}$$

where $C_\alpha=\left\{2^{(n+\alpha-2)/2}\pi^{n/2}\Gamma\left(\frac{\alpha}{2}\right)\right\}^{-1}$ and the modified Bessel function of the kind $K_{\frac{n-\alpha}{2}}(|x|)$, which is analytic except for the origin [1, p. 415; 18, p. 47]. third G_α belongs to the space $\mathscr{D}'_{L^2}$ and $\alpha\to G_\alpha$ is analytic [18, p. 47]. If $\alpha<0$ then $|x|^\beta G_\alpha(x)\in L^1(R_n)$ for any β with $|\alpha|<\beta$.

We shall first show the following proposition, where we have used the notation $[b]$ to denote the multiplication $\chi\to b\chi$.

PROPOSITION 15. *Let $b\in B_\beta(R_n)$, $\beta>1$. Then, for any γ such that $-\beta+1<\gamma<\beta$, we have with a constant $C(\beta,\gamma)$ such that*

$$\|(S^\gamma[b]S^{1-\gamma}-[b]S)\chi\|_{(0)}\leqq C(\beta,\gamma)\|b\|_\beta\|\chi\|_{(0)}, \qquad \chi\in C_0^\infty(R_n).$$

PROOF. (a) We first assume that $\gamma\geqq 1$. Put $A_\gamma=S^\gamma[b]-[b]S^\gamma$. If $\gamma=2k$, k a positive integer, then we have for any $\chi\in C_0^\infty(R_n)$

$$A_\gamma\chi=A_{2k}\chi=(1-\Delta)^k(b\chi)-b(1-\Delta)^k\chi$$

$$=\sum_{\substack{|p|+|q|\leqq 2k\\ q<2k}} C_{pq}D_x^p b D_x^q x,\ C_{pq} \text{ being constants,}$$

whence we obtain with a constant C_1

$$||A_{2k}x||_{(0)}\leqq C_1||b||_\beta||x||_{(2k-1)}=C_1||b||_\beta||x||_{(\gamma-1)},$$

which, by continuity, remains valid for any $x\in(\mathscr{D}_{L^2})_x$. From this it follows that

$$||A_\gamma S^{1-\gamma}x||_{(0)}\leqq C_1||b||_\beta||S^{1-\gamma}x||_{(\gamma-1)}=C_1||b||_\beta||x||_{(0)}.$$

If γ is not an even positive integer, then we can write

$$A_\gamma(x)=\int G_{-\gamma}(x-y)(b(y)-b(x))dy,$$

where

$$b(y)-b(x)=\sum_{1\leqq|p|\leqq[\beta]-1}\frac{i^{|p|}}{p!}(D^p b)(x)(y-x)^p+B_1(x,\ y)+B_2(x,\ y),$$

$$B_1=\sum_{|q|=[\beta]}\frac{i^{|q|}[\beta]}{p!}(y-x)^q\int_0^1(1-t)^{[\beta]-1}((D^q b)(x+t(y-x))-(D^q b)(x))dt,$$

$$B_2=\sum_{|q|=[\beta]}\frac{i^{|q|}[\beta]}{q!}(y-x)^q(D^q b)(x)\int_0^1(1-t)^{[\beta]-1}dt.$$

In view of the inequalities

$$\begin{aligned}|((-ix)^p G_{-\gamma}(x))^\wedge| &= |(iD_\xi)^p(1+|\xi|^2)^{\gamma/2}|\\ &\leqq C_2(1+|\xi|^2)^{(\gamma-|p|)/2}\leqq C_2(1+|\xi|^2)^{(\gamma-1)/2},\end{aligned}$$

we obtain

$$\left\|(D^p b)(x)\int(y-x)^p G_{-\gamma}(x-y)x(y)dy\right\|_{(0)}\leqq C_2||b||_\beta||x||_{(\gamma-1)}. \tag{11}$$

In a similar way we have with a constant C_3

$$\left\|\int G_{-\gamma}(x-y)B_2(x,\ y)x(y)dy\right\|_{(0)}\leqq C_3||b||_\beta||x||_{(\gamma-1)}. \tag{12}$$

By assumption $1\leqq\gamma<\beta$. Hence $|x|^\beta G_{-\gamma}(x)\in L^1(R_n)$. Then we have with constants C_4, C_5

$$\begin{aligned}&\left\|\int G_{-\gamma}(x-y)B_1(x,\ y)x(y)dy\right\|_{(0)} \\ &\leqq C_4||b||_\beta\left\|\int|y-x|^\beta|G_{-\gamma}(x-y)|\,|x(y)|dy\right\|_{(0)}\end{aligned} \tag{13}$$

$$\leqq C_5\|b\|_\beta\|\chi\|_{(0)}\leqq C_5\|b\|_\beta\|\chi\|_{(\gamma-1)}.$$

From these estimates (11), (12) and (13) we have with a constant $C=C(\beta,\gamma)$

$$\|A_\gamma S^{1-\gamma}\chi\|_{(0)}\leqq C\|b\|_\beta\|S^{1-\gamma}\chi\|_{(\gamma-1)}=C\|b\|_\beta\|\chi\|_{(0)}.$$

(b) Next, let $\gamma\leqq 0$. Then $1\leqq 1-\gamma<\beta$. From (a) we see that $S^{1-\gamma}[b]S^\gamma-[b]S$ is a continuous map of L^2 into itself. Thus its dual operator $S^\gamma[\bar{b}]S^{1-\gamma}-S[\bar{b}]$ is also continuous with the same norm. With the aid of the inequality $\|(S[b]-[b]S)\chi\|_{(0)}\leqq C(\beta,1)\|b\|_\beta\|\chi\|_{(0)}$, we obtain

$$\|(S^\gamma[b]S^{1-\gamma}-[b]S)\chi\|_{(0)}\leqq(C(\beta,1-\gamma)+C(\beta,1))\|b\|_\beta\|\chi\|_{(0)}.$$

(c) Finally, consider the case where $0<\gamma<1$. Let k be a positive integer such that $1+\frac{2\gamma}{k}<\beta$ and put $\varepsilon=\frac{\gamma}{k}$. From (a) and (b) it follows that $S^{1+\varepsilon}[b]S^{-\varepsilon}-S^{1+2\varepsilon}[b]S^{-2\varepsilon}$ and $S^{-\varepsilon}[b]S^{1+\varepsilon}-[b]S$ are the continuous maps of L^2 into itself, whence it follows that the latter is a continuous map of $\mathscr{H}_{(1+2\varepsilon)}$ into itself. In virtue of the interpolation theorem it is immediate that $S^{-\gamma}[b]S^{1+\gamma}-[b]S$ is continuous of $\mathscr{H}_{(\delta)}$ into itself for δ with $0\leqq\delta\leqq 1+2\varepsilon$. Thus, if we let $\delta=j\varepsilon, j=1,2,\cdots,k$, it results that $S^{(j-1)\varepsilon}[b]S^{1-(j-1)\varepsilon}-S^{j\varepsilon}[b]S^{1-j\varepsilon}$ is a continuous map of L^2 into itself with norm $\leqq C_j(\beta,\gamma)\|b\|_\beta$, which, combined with the equation: $S^\gamma[b]S^{1-\gamma}-[b]S=-\sum_{j=1}^{k}(S^{(j-1)\varepsilon}[b]S^{1-(j-1)\varepsilon}-S^{j\varepsilon}[b]S^{1-j\varepsilon})$, yields that

$$\|(S^\gamma[b]S^{1-\gamma}-[b]S)\chi\|_{(0)}\leqq C(\beta,\gamma)\|b\|_\beta\|\chi\|_{(0)}.$$

This ends the proof.

Corollary 2. *Let $b\in B_\beta(R_n)$, $\beta>1$. Then, for any γ, s such that $-\beta+1<\gamma+s<\beta$ and $-\beta+1<s<\beta$, we have with a constant $C(\beta,\gamma,s)$*

$$\|(S^\gamma[b]-[b]S^\gamma)\chi\|_{(s)}\leqq C(\beta,\gamma,s)\|b\|_\beta\|\chi\|_{(\gamma+s-1)},\qquad \chi\in C_0^\infty(R_n).$$

Proof. Putting $\chi_1=S^{\gamma+s-1}\chi$, we have $\|\chi_1\|_{(0)}=\|\chi\|_{(\gamma+s-1)}$ and

$$\begin{aligned}&\|(S^\gamma[b]-[b]S^\gamma)\chi\|_{(s)}\\&=(S^{s+\gamma}[b]S^{1-\gamma-s}-[b]S)\chi_1-(S^s[b]S^{1-s}-[b]S)\chi_1\|_{(0)}\\&\leqq C(\beta,\gamma+s)\|b\|_\beta\|\chi_1\|_{(0)}+C(\beta,s)\|b\|_\beta\|\chi_1\|_{(0)}\\&=C(\beta,\gamma,s)\|b\|_\beta\|\chi\|_{(\gamma+s-1)},\end{aligned}$$

where $C(\beta,\gamma,x)=C(\beta,\gamma+s)+C(\beta,s)$, which completes the proof.

Theorem 4. *Let $\beta>1$ and K be a B_β singular integral operator in the sense of Calderón. Then, for any γ, s such that $-\beta+1<\gamma+s<\beta$ and $-\beta+1$*

$<s<\beta$, *we have with a constant* $C(\beta, \gamma, s)$

$$||(S^{\gamma}K-KS^{\gamma})\chi||_{(s)}\leqq C(\beta, \gamma, s)||K||_{\beta}||\chi||_{(\gamma+s-1)}, \qquad \chi \in C_0^{\infty}(R_n),$$

PROOF. Let $n\geqq 2$. For any $\chi \in C_0^{\infty}(R_n)$ we have the expansion $K\chi= a_0\chi+\sum_{m=1}^{\infty}\sum_{l=1}^{d(m)}a_{lm}G_{lm}\chi$ in $\mathscr{H}_{(s+\gamma)}(R_n)$. Since S^{γ} is a continuous map of $\mathscr{H}_{(s+\gamma)}(R_n)$ into $\mathscr{H}_{(s)}(R_n)$, the series

$$S^{\gamma}K\chi=S^{\gamma}a_0\chi+\sum_{m=1}^{\infty}\sum_{l=1}^{d(m)}S^{\gamma}a_{lm}G_{lm}\chi$$

is convergent in $\mathscr{H}_{(s)}(R_n)$. On the other hand, the series

$$KS^{\gamma}\chi=a_0S^{\gamma}\chi+\sum_{m=1}^{\infty}\sum_{l=1}^{d(m)}a_{lm}G_{lm}S^{\gamma}\chi$$

is convergent in $\mathscr{H}_{(s)}(R_n)$.

With the aid of Corollary 2 we have

$$\begin{aligned}||S^{\gamma}a_{lm}G_{lm}-a_{lm}G_{lm}S^{\gamma}\chi||_{(s)}&=||(S^{\gamma}a_{lm}-a_{lm}S^{\gamma})G_{lm}\chi||_{(s)}\\&\leqq C(\beta, \gamma, s)||a_{lm}||_{\beta}||G_{lm}\chi||_{(s+\gamma-1)}\\&\leqq C_1(\beta, \gamma, s)m^{-(3/2)n}||K||_{\beta}m^{(n-2)/2}||\chi||_{(s+\gamma-1)}\\&=C_1(\beta, \gamma, s)m^{-n-1}||K||_{\beta}||\chi||_{(s+\gamma-1)}.\end{aligned}$$

Since $d(m)\leqq Cm^{n-2}$, C being a constant, we have

$$\begin{aligned}||KS^{\gamma}\chi-S^{\gamma}K\chi||_{(s)}&\leqq C_2(\beta, \gamma, s)||K||_{\beta}||\chi||_{(s+\gamma-1)}(1+\sum_{m=1}^{\infty}m^{-3})\\&=C_3(\beta, \gamma, s)||K||_{\beta}||\chi||_{(s+\gamma-1)},\end{aligned}$$

where C_1, C_2 and C_3 are constants independent of χ and K.

In the case where $n=1$, we have the expression $K\chi=a_0(x)\chi(x)+a_1(x)\lim_{\varepsilon\downarrow 0}\int_{|x-y|>\varepsilon}\frac{\chi(y)}{x-y}dy$. Since the Hilbert transform is a continuous map of $\mathscr{H}_{(s)}(R_n)$ into itself for any s, we obtain the estimate

$$||(S^{\gamma}K-KS^{\gamma})\chi||_{(s)}=C(\beta, \gamma, s)||K||_{\beta}||\chi||_{(\gamma+s-1)}, \qquad \chi \in C_0^{\infty}(R_n).$$

Thus the proof is complete.

5. Fine Cauchy problem for a system of pseudo-differential operators

This final section will be devoted to some general investigations about the fine Cauchy problem for a system of pseudo-differential operators. As for differential operators, by one of the present authors [9], the problem was formulated and investigated from a distribution-theoretic view-point, where the notions such as distributional boundary value and canonical extension over $t=0$ were proved to be fundamental. Our present aim is to generalize the results obtained there to a system of pseudo-differential operators.

For given $\vec{f}=(f_1,\cdots,f_l)$ with $f_j \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ and $\vec{\alpha}=(\vec{\alpha}_0,\cdots,\vec{\alpha}_{m-1})$, $\vec{\alpha}_j=(\alpha_{j1},\cdots,\alpha_{jl})$ with $\alpha_{jk} \in (\mathscr{D}'_{L^2})_x$ we shall consider the Cauchy problem for a system of pseudo-differential operators in the unknown vector distribution $\vec{u}=(u_1,\cdots,u_l)$ with $u_j \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$:

$$(14)\quad \begin{cases} P\vec{u}=D_t^m\vec{u}+\sum_{j=1}^{m}\vec{A}_j(t)D_t^{m-j}\vec{u}=\vec{f} & \text{in } R_{n+1}, \\ (\vec{u}(0,\cdot),(D_t\vec{u})(0,\cdot),\cdots,(D_t^{m-1}\vec{u})(0,\cdot))=\vec{\alpha}, \end{cases}$$

where $\vec{A}_i(t)$ are $l\times l$ matrices of operators $A_{i,jk}(t)\in\mathfrak{E}^\infty_{(\tau)}$ and $\vec{u}(0,\cdot)=(u_1(0,\cdot),\cdots,u_l(0,\cdot))$, $u_j(0,\cdot)$ being the section of u_j for $t=0$.

Substituting $u_{i,k}=D_t^{k-1}u_i$, $i=1,2,\cdots,l$, $k=1,2,\cdots,m-1$, we obtain the system:

$$\begin{cases} D_t u_{j,1}-u_{j,2}=0, \\ \quad\vdots \\ D_t u_{j,m-1}-u_{j,m}=0, \\ D_t u_{j,m}+\sum_{i=1}^{m}\sum_{k=1}^{l}A_{i,jk}(t)u_{k,m-i+1}=f_j, \qquad j=1,2,\cdots,l, \end{cases}$$

with the initial conditions

$$(u_{j,1}(0,\cdot),\cdots,u_{j,m}(0,\cdot))=(\alpha_{j,0},\cdots,\alpha_{j,m-1}),\qquad j=1,2,\cdots,l,$$

which is a special case of the Cauchy problem for a pseudo-differential system written in matrix notation

$$(15)\quad \begin{cases} D_t\vec{u}+\vec{A}(t)\vec{u}=\vec{f} & \text{in } R_{n+1}, \\ \vec{u}(0,\cdot)=\vec{\alpha}, \end{cases}$$

where $\vec{u}=(u_1,\cdots,u_N)$, $\vec{f}=(f_1,\cdots,f_N)$, $\vec{\alpha}=(\alpha_1,\cdots,\alpha_N)$, $N=lm$, and u_j, $f_j \in \mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ and $\alpha_j \in (\mathscr{D}'_{L^2})_x$. We shall write $\vec{u}\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ and we shall say that $\vec{u}$ has the section for $t=0$ if this is a case for each component u_j. The terms $\mathscr{D}'_{L^2}$-canonical, $\mathscr{D}'_{L^2}$-canonical extension, $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0} u$ and the like shoud be understood in a similar way.

Put $Y_l=\frac{1}{(l-1)!}t_+^{l-1}$, l being a non-negative integer, where we set $Y_0=\delta_t$. Note that Y_1 is the Heaviside function Y. Let $\vec{u}\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$. Then so does $Y*\vec{u}$ and we have

(16) $$Y_k*(\vec{A}(t)\vec{u})=\sum_{j=0}^{k}\binom{k}{j}(-i)^j Y_j*(D_t^j\vec{A}(t)(Y_k*\vec{u})).$$

THEOREM 5. *For given* $\vec{f}=(f_1,\cdots,f_N)\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ *and* $\vec{\alpha}=(\vec{\alpha}_1,\cdots,\alpha_N)\in(\mathscr{D}'_{L^2})_x$, *suppose that there exists a solution* $\vec{u}=(u_1,\cdots,u_N)\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ *for the Cauchy problem* (15), *then* $\vec{f}$ *has no mass on* $t=0$ *and the restrictions* $\vec{f}_1=\vec{f}\,|\,R^+_{n+1}$, $\vec{f}_2=\vec{f}\,|\,R^-_{n+1}$ *have the* $\mathscr{D}'_{L^2}$*-canonical extensions* $\vec{f}_{1^\sim}$, $\vec{f}_{\tilde{2}}$ *and* $\vec{f}=\vec{f}_{1^\sim}+\vec{f}_{\tilde{2}}$. *The* $\mathscr{D}'_{L^2}$*-canonical extensions* $\vec{u}_{1^\sim}$, $\vec{u}_{\tilde{2}}$ *of* $\vec{u}_1=\vec{u}\,|\,R^+_{n+1}$, $\vec{u}_2=\vec{u}\,|\,R^-_{n+1}$ *are solutions of equations*:

(17) $$D_t(\vec{u}_{1^\sim})+\vec{A}(t)\vec{u}_{1^\sim}=\vec{f}_{1^\sim}-i\delta\otimes\vec{\alpha},$$

(18) $$D_t(\vec{u}_{\tilde{2}})+\vec{A}(t)\vec{u}_{\tilde{2}}=\vec{f}_{\tilde{2}}+i\delta\otimes\vec{\alpha}.$$

Conversely, if $\vec{v}_1\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ *and* $\vec{v}_2\in(\mathscr{D}'_t)_-((\mathscr{D}'_{L^2})_x)$ *are solutions of* (17), (18) *respectively, then* $\vec{u}=\vec{v}_1+\vec{v}_2\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ *is a solution for the Cauchy problem* (15).

PROOF. Let $\vec{u}\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ be a solution for the Cauchy problem (15). Since $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}\vec{u}_1=\mathscr{D}'_{L^2}\text{-}\lim_{t\uparrow 0}\vec{u}_2=\vec{\alpha}$, for any $\phi\in\mathscr{D}(R^+_t)$ such that $\phi(t)\geqq 0$, $\int\phi(t)dt=1$, $\lim_{\varepsilon\downarrow 0}\phi_\varepsilon\vec{u}=\delta\otimes\vec{\alpha}$, $\lim_{\varepsilon\downarrow 0}\check{\phi}_\varepsilon\vec{u}=-\delta\otimes\vec{\alpha}$ and, owing to Proposition 5, $\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon)}\vec{u}_1=\vec{u}_{1^\sim}$ and $\lim_{\varepsilon\downarrow 0}\check{\rho}_{(\varepsilon)}\vec{u}_2=\vec{u}_{\tilde{2}}$ exist. From the equations:

$$\rho_{(\varepsilon)}\vec{f}=D_t(\rho_{(\varepsilon)}\vec{u})+i\phi_\varepsilon\vec{u}+\vec{A}(t)\rho_{(\varepsilon)}\vec{u},$$

$$\check{\rho}_{(\varepsilon)}f=D_t(\check{\rho}_{(\varepsilon)}\vec{u})-i\phi_\varepsilon\vec{u}+\vec{A}(t)\check{\rho}_{(\varepsilon)}\vec{u},$$

we obtain

$$\vec{f}_{1^\sim}=D_t(\vec{u}_{1^\sim})+i\delta\otimes\vec{\alpha}+\vec{A}(t)(\vec{u}_{1^\sim}),$$

$$\vec{f}_{\tilde{2}}=D_t(\vec{u}_{\tilde{2}})-i\delta\otimes\vec{\alpha}+\vec{A}(t)(\vec{u}_{\tilde{2}})$$

and therefore $\vec{f}_{1^\sim}+\vec{f}_{\tilde{2}}=D_t(\vec{u}_{1^\sim}+\vec{u}_{\tilde{2}})+A(t)(\vec{u}_{1^\sim}+\vec{u}_{\tilde{2}})$. Since $\vec{u}$ has the section for $t=0$, $\vec{u}$ has no mass on $t=0$ and $\vec{u}=\vec{u}_{1^\sim}+\vec{u}_{\tilde{2}}$ and therefore $\vec{f}=\vec{f}_{1^\sim}+\vec{f}_{\tilde{2}}$ and $\vec{f}$ has no mass on $t=0$.

Conversely, let $\vec{v}_1$, $\vec{v}_2$ be solutions of (17), (18). Then for the interval $(0, 1)$ there exist non-negative integers k, m and a $\mathscr{D}'_{L^2}$-valued continuous function $\vec{g}(t)$ of t with support $\subset[0, 1]$ such that $\vec{v}_1=D_t^k\vec{g}(t)$ in $(0, 1)\times R_n$. Then, by the equation (16), we have

$$\frac{1}{i}Y_{k-1}*\vec{v}_1=-\sum_{j=1}^{k}\binom{k}{j}(-i)^j Y_j*(D_t^j\vec{A}(t)(Y_k\vec{v}*_1))+Y_k*\vec{f}_{1^\sim}+iY_k\otimes\vec{\alpha}.$$

By Proposition 6, $Y_k*\vec{f}_{1^\sim}$ is $\mathscr{D}'_{L^2}$-canonical and $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(Y_k*\vec{f}_{1^\sim})=0$ for $k\geqq 1$. Evidently $Y_k\otimes\alpha$ is $\mathscr{D}'_{L^2}$-canonical for $k\geqq 1$ and $\lim_{t\downarrow 0}(Y_k\otimes\vec{\alpha})=0$ for $k\geqq 2$ and $\lim_{t\downarrow 0}(Y\otimes\vec{\alpha})=\vec{\alpha}$. From the above equation we see that $Y_{k-1}*\vec{v}_1$ is also $\mathscr{D}'_{L^2}$-

canonical. Repeating this procedure we conclude that $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(\vec{v}_1 | R^+_{n+1}) = \vec{\alpha}$. Since $\vec{f}$ has no mass on $t=0$, so does $\vec{u}=\vec{v}_1+\vec{v}_2$ and therefore $\vec{u}$ has the section $\vec{\alpha}$ for $t=0$ and $D_t\vec{u}+\vec{A}(t)\vec{u}=\vec{f}$.

As an immediate consequence of the preceding theorem we have an analogue to Theorem 1 in [9, p. 18]:

Cororary 3. *For any given* $\vec{f} \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ *and* $\vec{\alpha} \in (\mathscr{D}'_{L^2})_x$, *if there exists a solution* $\vec{u} \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ *of the Cauchy problem*:

$$(19)\qquad \begin{cases} D_t\vec{u}+\vec{A}(t)\vec{u}=\vec{f} & \text{in } R^+_{n+1}, \\ \mathscr{D}'_{L^2}\text{-}\lim\limits_{t\downarrow 0} \vec{u}=\vec{\alpha}, \end{cases}$$

then $\vec{f}$ *has the* $\mathscr{D}'_{L^2}$*-canonical extension* $\vec{f}_\sim$ *and* $\vec{u}_\sim$ *is a solution of the equation*:

$$(20)\qquad D_t(\vec{u}_\sim)+\vec{A}(t)\vec{u}_\sim=\vec{f}_\sim - i\delta\otimes\vec{\alpha}.$$

Conversely, if $\vec{v} \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ *is a solution of* (20), *then* $\vec{u}=\vec{v}\,|\,R^+_{n+1} \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$*is a solution for the Cauchy problem* (19) *and* $\vec{u}_\sim=\vec{v}$.

Remark. For given $\vec{f}=(f_1,\cdots,f_l) \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ and $\vec{a}=(\vec{\alpha}_0,\cdots,\vec{\alpha}_{m-1}) \in (\mathscr{D}'_{L^2})_x$, if there exists a solution $\vec{u}=(u_1,\cdots,u_l) \in \mathscr{D}'(R^+_t)((\mathscr{D}'_{L^2})_x)$ of the Cauchy problem:

$$(21)\qquad \begin{cases} P\vec{u}=\vec{f} & \text{in } R^+_{n+1}, \\ \mathscr{D}'_{L^2}\text{-}\lim\limits_{t\downarrow 0} \vec{u}=\vec{\alpha}, \end{cases}$$

then f has the $\mathscr{D}'_{L^2}$-canonical extension $\vec{f}_\sim$ and $\vec{u}_\sim$ is a solution of the equation:

$$(22)\qquad P(\vec{u}_\sim)=\vec{f}_\sim+\sum_{k=0}^{m-1} D^k_t\delta\otimes\vec{r}_k(0),$$

where $\vec{r}_k(t)=-i\sum_{j=k+1}^{m}\sum_{l=1}^{j-k}(-1)^{j-l-k}\binom{j-l}{k}D_t^{j-l-k}\vec{A}_{m-j}(t)\vec{\alpha}_{l-1}$ and $\vec{A}_0$ is the unit matrix [11, p. 82]. We note that $\vec{r}_{m-k-1}(t)$ may be rewritten in the form

$$\vec{r}_{m-k-1}(t)=-i\vec{\alpha}_k+\sum_{j=0}^{k-1}\vec{B}_j(t)\vec{\alpha}_j,$$

where $\vec{B}_j(t)$ is a linear combination of derivative of $\vec{A}_j$ of order up to $k-1$.

Conversely, suppose $\vec{v} \in (\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$ is a solution of the equation (22): $P\vec{v}=\vec{f}_\sim+\sum_{k=0}^{m-1}D^k_t\delta\otimes\vec{r}_k(0)$. Then, by substitutions: $\vec{v}_1=\vec{v}$, $\vec{v}_2=D_t\vec{v}_1+i\delta\otimes\vec{\alpha}_0,\cdots$, $\vec{v}_m=D_t\vec{v}_{m-1}+i\delta\otimes\vec{\alpha}_{m-2}$, we get the equation written in the form:

$$\begin{cases} D_t\vec{v}_1=\vec{v}_2-i\delta\otimes\vec{\alpha}_0, \\ \quad\vdots \\ D_t\vec{v}_{m-1}=\vec{v}_m-i\delta\otimes\vec{\alpha}_{m-2}, \\ D_t\vec{v}_m=-\sum_{j=1}^{m}\vec{A}_j(t)D_t^{m-j}\vec{v}-i\delta\otimes\vec{\alpha}_{m-1}+\vec{f}. \end{cases}$$

Applying Corollary 3, we see that the restriction $\vec{u}=(u_1,\cdots, u_m)=(v_1,\cdots, v_m)\,|\,R_{n+1}^+$ is a solution for the Cauchy problem (21).

In the same way as in the proof of Theorem 5, we shall show the following

PROPOSITION 16. *Let $\vec{f}\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$ have the $\mathscr{D}'_{L^2}$-canonical extension $\vec{f}_\sim$. If $\vec{u}\in\mathscr{D}'_t((\mathscr{D}'_{L^2})_x)$ is a solution of*

$$D_t\vec{u}+\vec{A}(t)\vec{u}=\vec{f} \qquad \text{in } R_{n+1}^+,$$

then $\vec{u}\,|\,R_{n+1}^+$ has the $\mathscr{D}'_{L^2}$-boundary value.

PROOF. We can write $\vec{u}=D_t^k\vec{g}(t)$ in $(0,1)\times R_n$ with an $\mathscr{D}'_{L^2}$-valued continuous function $\vec{g}(t)$ of t with support $\subset[0,1]$. If we put $\vec{v}=D_t^k\vec{g}(t)\in(\mathscr{D}'_t)_+((\mathscr{D}'_{L^2})_x)$, then there exist $\vec{r}_0,\cdots,\vec{r}_l\in(\mathscr{D}'_{L^2})_x$ such that

$$D_t\vec{v}+\vec{A}(t)\vec{v}=\vec{f}_\sim+\delta_t\otimes\vec{r}_0+\cdots+D_t^l\delta_t\otimes\vec{r}_l \qquad \text{in } (-1,1)\times R_n.$$

Let k' be the smallest positive integer such that $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(Y_{k'}*\vec{v})$ exists. Then, applying the equation (16) with k replaced by k', we have

$$\frac{1}{i}Y_{k'-1}*\vec{v}=-\sum_{j=0}^{k'}\binom{k'}{j}(-i)^jY_j*(D_t^j\vec{A}(t)(Y_{k'}*\vec{v}))+$$

$$+Y_{k'}*\vec{f}_\sim+\frac{1}{i}Y_{k'}\otimes\vec{r}_0+\cdots+\frac{1}{i}Y_{k'-l}\otimes\vec{r}_l.$$

Since the right hand of the equation has the $\mathscr{D}'_{L^2}$-boundary value, so $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(Y_{k'-1}*\vec{v})$ must exist. Thus $k'=1$, which means the existence of $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow 0}(\vec{u}\,|\,R_{n+1}^+)$.

PROPOSITION 17. *Let $\vec{u}\in\mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$ be a solution of the equation:*

$$D_t\vec{u}+\vec{A}(t)\vec{u}=\vec{f} \qquad \text{in } R_{n+1}^+.$$

Then the following conditions are equivalent.

(a) *$\vec{u}$ is a $\mathscr{D}'_{L^2}$-valued continuous function of $t\in(t_1, t_2)$, $0<t_1<t_2\leqq\infty$.*

(b) *For any $\vec{g}$ such that $\vec{f}=D_t\vec{g}$, $\vec{g}$ is a $\mathscr{D}'_{L^2}$-valued continuous function of $t\in(t_1, t_2)$.*

PROOF. (a)$\Rightarrow$(b). Since $\vec{A}(t)\vec{u}$ is a $\mathscr{D}'_{L^2}$-valued continuous function of t, if we put $\vec{v}(t, \cdot)=\int_{t_1}^{t} \vec{A}(t')\vec{u}(t', \cdot)dt$, then $\vec{v}$ is a $\mathscr{D}'_{L^2}$-valued continuous function and $D_t(\vec{u}+\vec{v})=\vec{f}$ and therefore $\vec{g}=\vec{u}+\vec{v}$ is a $\mathscr{D}'_{L^2}$-valued continuous function of t.

(b)$\Rightarrow$(a). Let t_0 be any point such that $t_0 \in (t_1, t_2)$. Then the restriction $\vec{f}\,|\,(t_0, t_1)\times R_n$ has the $\mathscr{D}'_{L^2}$-canonical extension $\vec{f}_{\sim t_0}$ over $t=t_0$ and, owing to Proposition 16, $\mathscr{D}'_{L^2}\text{-}\lim_{t\downarrow t_0}\vec{u}=\vec{\alpha}_{t_0}$ exists. Thus we have

$$D_t(\vec{u}_{\sim t_0})+\vec{A}(t)(\vec{u}_{\sim t_0})=\vec{f}_{\sim t_0}+\delta_{t_0}\otimes\vec{\alpha}_{t_0}.$$

Let k' be the smallest positive integer such that $Y_{k'}*\vec{u}_{\sim t_0}$ is a $\mathscr{D}'_{L^2}$-valued continuous function of t in a right neighborhood of t_0. Applying the equation (16) with k replaced by k', we can show $k'=1$ in the same way as in the proof of Proposition 16. Since t_0 is arbitrary, we can conclude that $\vec{u}$ is a $\mathscr{D}'_{L^2}$-valued continuous functtion of t in (t_1, t_2). The proof is concluded.

As an immediate consequence we have the following

COROLLARY 4. *Let $\vec{u} \in \mathscr{D}'(R_t^+)((\mathscr{D}'_{L^2})_x)$ be a solution of the equation*:

$$D_t\vec{u}+\vec{A}(t)\vec{u}=\vec{f} \qquad \text{in } R_{n+1}^+.$$

Then the following conditions are equivalent:

(a) *$\vec{u}$ is a $\mathscr{D}'_{L^2}$-valued continuously differentiable function of $t \in (t_1, t_2)$, $0<t_1<t_2\leqq\infty$.*

(b) *$\vec{f}$ is a $\mathscr{D}'_{L^2}$-valued continuous function of $t \in (t_1, t_2)$.*

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