HIROSHIMA MATH. J.
2 (1972), 299–311

On the m-Accretiveness of Nonlinear Operators in Banach Spaces

Nobuyuki KENMOCHI

(Received September 19, 1972)

Introduction

In the theory of nonlinear contraction semigroups, the notion of accretive operators was introduced as a generalization of the notion of the infinitesimal generators, and studied by many authors (see e.g., [1], [2], [3], [4], [6], [8], [10], [11]).

In the present paper we study a multivalued accretive operator A from a Banach space X into itself. It is called m-accretive if the range of $I+A$ is the whole of X. The studies on the m-accretiveness of nonlinear operators were made by T. Kato [6], R.H. Martin, Jr. [9], G.F. Webb [12], the author [7] and others. The purpose of this paper is to give a necessary and sufficient condition for m-accretiveness; under certain conditions, an accretive operator A from X into X is m-accretive if and only if it is demiclosed and the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt}+Au(t) \ni z \\ u(0)=x \end{cases}$$

has a solution (in a certain sense) on $[0, \infty)$ for each $x \in D(A)$ and $z \in X$ (THEOREM 1). It was announced by F.E. Browder [2] that if the dual space of X is uniformly convex, then a densely defined singlevalued accretive operator A is m-accretive if and only if $-(A+z)$ is the weak infinitesimal generator of a nonlinear contraction semigroup on X for each $z \in X$. This was proved by M.G. Crandall and A. Pazy [4] in case X is a Hilbert space. In this paper we shall prove Browder's announcement in a more general form, namely, when A is multivalued.

§1. Definitions and notation

Throughout this paper let X be a real Banach space and X^* be its dual space. The natural pairing between $x \in X$ and $x^* \in X^*$ is denoted by $<x, x^*>$. The norms in X and X^* are denoted by $\|\cdot\|$ and the identity mapping in X by

I. For a subset E of X we denote by $\bar{E}$ and $co(E)$ the strong closure and the convex hull of E respectively, and define $|||E||| = \inf_{x \in E} \|x\|$ if $E \neq \phi$.

Let A be a multivalued operator from X into X, that is, to each $x \in X$ a subset Ax of X be assigned. We define $D(A) = \{x \in X; Ax \neq \phi\}$, $R(A) = \bigcup_{x \in X} Ax$, $G(A) = \{(x, x') \in X \times X; x' \in Ax\}$ and for a subset E of X, $A(E) = \bigcup_{x \in E} Ax$. For a point $z \in X$ the multivalued operator $A+z$ is defined by $(A+z)x = Ax + z = \{x' + z; x' \in Ax\}$. Then $D(A+z) = D(A)$.

In what follows an operator means a multivalued operator unless otherwise stated.

Let A and A' be operators from X into X. By $A \supset A'$ we mean that A is an extension of A', that is, $G(A) \supset G(A')$. We say that A is *demiclosed* if $(x_n, x_n') \in G(A)$, $n = 1, 2, \cdots$, $x_n \to x$ strongly and $x_n' \to x'$ weakly in X imply that $(x, x') \in G(A)$.

The duality mapping F from X into X^* is defined by

$$Fx = \{x^* \in X^*; \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

In general, F is multivalued and its domain is the whole of X. We know that if X^* is uniformly convex, then Fx consists of a single point for each $x \in X$ and F is strongly uniformly continuous on each bounded subset of X (see T. Kato [5]).

An operator A from X into X is called *accretive* if for any (x, x'), $(y, y') \in G(A)$ there is $f \in F(x-y)$ such that $\langle x' - y', f \rangle \geq 0$. An accretive operator A is called *maximal accretive* if there is no proper accretive extension of A, and called *m-accretive* if $R(I+A) (= \bigcup_{x \in X} (x + Ax)) = X$.

We use symbols "$\xrightarrow{s}$" (or "s-lim") and "$\xrightarrow{w}$" (or "w-lim") to denote the convergence in the strong and the weak topology, respectively.

§ 2. Lemmas

Four lemmas which will be used in the proof of our main theorems are stated below without proof.

LEMMA 1. *Suppose that X^* is uniformly convex. If A is an accretive operator from X into X, then,*

(i) *the operator $\tilde{A}$ given by $G(\tilde{A}) = \{(x, x') \in X \times X$; there is a sequence $\{(x_n, x_n')\} \subset G(A)$ such that $x_n \xrightarrow{s} x$ and $x_n' \xrightarrow{w} x'\}$ is accretive,*

(ii) *the operator $x \to \overline{co(Ax)}$ is accretive and its domain is $D(A)$,*

(iii) *if A is maximal accretive, then it is demiclosed,*
(iv) *if A is m-accretive, then it is maximal accretive,*
(v) *if A is demiclosed and if $\{(x_n, x_n')\}$ is a sequence in $G(A)$ such that $x_n \xrightarrow{s} x_0$ and $\{x_n'\}$ is bounded in X, then $x_0 \in D(A)$.*

Proofs of (i), (ii), (iii) and (v) are elementary. A proof of (iv) is found in T. Kato [6].

The following two lemmas are due to T. Kato ([5], [6]).

LEMMA 2. *Let $u(t)$ be an X-valued function on a real interval. Suppose that $u(t)$ has the weak derivative $u'(s)$ at $t=s$ and $\|u(t)\|$ is differentiable at $t=s$. Then,*

$$\frac{d}{ds}(\|u(s)\|^2)=2<u'(s), f> \qquad \text{for every } f \in Fu(s).$$

LEMMA 3. *Suppose that X is reflexive. Let $\{u_n\}$ be a sequence in $L^p(0, r; X)$, $1<p<\infty$, $0<r\leq\infty$, such that $\{u_n(t)\}$ is bounded for a.e. $t \in (0, r)$. Let $V(t)$ be the set of all weak cluster points of $\{u_n(t)\}$. If $u_n \xrightarrow{w} u$ in $L^p(0, r; X)$, then*

$$u(t) \in \overline{co(V(t))} \qquad \text{for a.e. } t \in (0, r).$$

Now we consider the initial value problem of the form

$$\text{(E)} \qquad u'(t)+Au(t) \ni 0, \qquad u(0)=a,$$

where A is an operator from X into X and the unknown $u(t)$ is an X-valued function on a real interval Ω. Let $\Omega=[0, r)$ or $[0, r]$, where $0<r\leq\infty$. Then $u(t)$ is called a *strong solution* of (E) on Ω if

(a) $u(t)$ is strongly absolutely continuous on any bounded closed interval contained in Ω and $u(0)=a$,
(b) the strong derivative $u'(t)$ exists, $u(t) \in D(A)$ and $u'(t)+Au(t) \ni 0$ for a.e. $t \in \Omega$.

LEMMA 4. *Let A be an accretive operator from X into X, $a \in D(A)$ and λ be a non-negative real number. Let $u(t)$ be a strong solution of*

$$(2.1) \qquad u'(t)+(\lambda I+A)u(t) \ni 0, \qquad u(0)=a$$

on $[0, r)$. Then,

(i) *$u(t)$ is uniquely determined by the intitial value a,*
(ii) $\|u'(t)\|=|||(\lambda I+A)u(t)|||\leq|||(\lambda I+A)a|||$ *for a.e. $t \in [0, r)$,*
(iii) *if $u(t)$ is strongly differentiable and satisfies (2.1) at $t=s, s'$, $0<s<s'<r$, then*

$$\|u'(s')\|\leq e^{\lambda(s-s')}\|u'(s)\|.$$

This lemma is a special case of LEMMA 6.2 in T. Kato [6]. In case $\lambda=0$, a simple proof of LEMMA 4 is also found in H. Brezis and A. Pazy [1].

§ 3. A necessary and sufficient condition for m-accretiveness

Throughout this section we assume that X^* is uniformly convex. Note that X is reflexive in this case. Our main result is the following.

THEOREM 1. *Let A be an accretive operator from X into X. Then A is m-accretive if and only if it is demiclosed and satisfies the following condition: for each $x \in D(A)$ and each $z \in X$, the initial value problem*

$$u'(t)+Au(t)+z \ni 0, \qquad u(0)=x \tag{3.1}$$

has a strong solution on $[0, \infty)$.

The "only if" part of the theorem is already known. In fact, if A is m-accretive, then it is demiclosed by (iii) and (iv) of LEMMA 2 and $A+z$ is also m-accretive for each $z \in X$. Now we recall the following result by T. Kato [6; THEOREM 7.1]:

THEOREM A. *Let B be an m-accretive operator from X into X. Then, for each $a \in D(B)$ the initial value problem*

$$u'(t)+Bu(t) \ni 0, \qquad u(0)=a$$

has a unique strong solution on $[0, \infty)$.

This theorem implies that for each $x \in D(A)$ and each $z \in X$ the problem (3.1) has a strong solution on $[0, \infty)$, if A is m-accretive. Therefore, to complete the proof of THEOREM 1 it is sufficient to show only the "if" part. We shall prove it by means of a sequence of lemmas which are valid under the assumptions that A is demiclosed and that for each $x \in D(A)$ and each $z \in X$ the problem (3.1) has a strong solution on $[0, \infty)$.

LEMMA 5. *For each $x \in D(A)$, Ax is closed and convex in X.*

PROOF. Let B be the operator $x \rightarrow \overline{co(Ax)}$. Then, by (ii) of LEMMA 1, B is accretive and $D(B)=D(A)$. Let (y, y') be an arbitrary point of $G(B)$. Then, by our assumption, there is a strong solution $u(t)$ of the problem

$$u'(t)+Au(t)-y' \ni 0, \qquad u(0)=y.$$

Since $B \supset A$, this function $u(t)$ is also a strong solution of

$$u'(t)+Bu(t)-y' \ni 0, \qquad u(0)=y.$$

Observe that this is a special case of (2.1) because $B-y'$ is an accretive operator. Hence from (i) of LEMMA 4 we infer that $u(t)=y$ for all $t\geq 0$. This implies that $(y, y')\in G(A)$. Thus $A=B$. *q.e.d.*

Now for any given $a\in D(A)$ we consider the initial value problem

$$u'(t)+Au(t)+u(t)\ni 0, \qquad u(0)=a, \tag{3.2}$$

and shall show the existence of a strong solution of (3.2) on $[0, \infty)$.

For each positive integer n, we define an X-valued function $u_n(t)$ on $[0, 1]$ as follows. Let $u_n(t)$ be a strong solution of (3.1) with $z=x=a$ on $\left[0, \frac{1}{n}\right]$. Next, assume that for a positive integer k, $1\leq k<n$, $u_n(t)$ is already defined on $\left[0, \frac{k}{n}\right]$ in such a way that $u_n\left(\frac{k}{n}\right)\in D(A)$. Let $v(t)$ be a strong solution of (3.1) with $z=x=u_n\left(\frac{k}{n}\right)$, and define

$$u_n(t)=v\left(t-\frac{k}{n}\right) \qquad \text{for } t\in\left[\frac{k}{n}, \frac{k+1}{n}\right].$$

Then, by (ii) of LEMMA 4 we have

$$\|u_n'(t)\|\leq\left|\left|\left|Au_n\left(\frac{k}{n}\right)+u_n\left(\frac{k}{n}\right)\right|\right|\right| \qquad \text{a.e. on } \left(\frac{k}{n}, \frac{k+1}{n}\right),$$

and hence, $u_n\left(\frac{k+1}{n}\right)\in D(A)$ by (v) of LEMMA 1, since $u_n(t)\xrightarrow{s} u_n\left(\frac{k+1}{n}\right)$ as $t\nearrow\frac{k+1}{n}$. Thus $u_n(t)$ is defined on $[0, 1]$ by induction. Clearly $u_n(t)$ is strongly absolutely continuous on $[0, 1]$.

LEMMA 6. *Set $K=|||Aa+a|||$. Then,*

$$\|u_n'(t)\|\leq\left|\left|\left|Au_n\left(\frac{k}{n}\right)+u_n\left(\frac{k}{n}\right)\right|\right|\right|\leq\left(1+\frac{1}{n}\right)^k K \tag{3.3}$$

$$a.e. \text{ on } \left(\frac{k}{n}, \frac{k+1}{n}\right),\ k=0, 1,\cdots, n-1.$$

PROOF. By the above argument we have

$$\|u_n'(t)\|\leq\left|\left|\left|Au_n\left(\frac{k}{n}\right)+u_n\left(\frac{k}{n}\right)\right|\right|\right| \tag{3.4}$$

$$a.e. \text{ on } \left(\frac{k}{n}, \frac{k+1}{n}\right),\ k=0, 1,\cdots, n-1.$$

Furthermore we shall show that for $k=0, 1,\cdots, n-1$,

$$\left|\left|\left| Au_n\left(\frac{k+1}{n}\right)+u_n\left(\frac{k}{n}\right)\right|\right|\right| \leq \left|\left|\left| Au_n\left(\frac{k}{n}\right)+u_n\left(\frac{k}{n}\right)\right|\right|\right|. \tag{3.5}$$

In fact, by (3.4) there exists a sequence $\{t_j\}$ such that $t_j \nearrow \frac{k+1}{n}$, $-u_n'(t_j) \in Au_n(t_j) + u_n\left(\frac{k}{n}\right)$, $\|u_n'(t_j)\| \leq \left|\left|\left| Au_n\left(\frac{k}{n}\right)+u_n\left(\frac{k}{n}\right)\right|\right|\right|$ and $-u_n'(t_j) \xrightarrow{w} y$ in X for some $y \in X$ as $j \to \infty$. Since $u_n(t_j) \xrightarrow{s} u_n\left(\frac{k+1}{n}\right)$ and A is demiclosed, we have $y \in Au_n\left(\frac{k+1}{n}\right) + u_n\left(\frac{k}{n}\right)$, and hence,

$$\left|\left|\left| Au_n\left(\frac{k+1}{n}\right)+u_n\left(\frac{k}{n}\right)\right|\right|\right| \leq \|y\| \leq \left|\left|\left| Au_n\left(\frac{k}{n}\right)+u_n\left(\frac{k}{n}\right)\right|\right|\right|.$$

Obviously $\|u_n'(t)\| \leq K$ *a.e.* on $\left[0, \frac{1}{n}\right]$ by (ii) of LEMMA 4. Now assume that (3.3) holds for k-1. Then we have by (3.5)

$$\left|\left|\left| Au_n\left(\frac{k}{n}\right)+u_n\left(\frac{k-1}{n}\right)\right|\right|\right| \leq \left|\left|\left| Au_n\left(\frac{k-1}{n}\right)+u_n\left(\frac{k-1}{n}\right)\right|\right|\right| \leq \left(1+\frac{1}{n}\right)^{k-1} K.$$

and by (3.4)

$$\left\| u_n\left(\frac{k-1}{n}\right)-u_n\left(\frac{k}{n}\right)\right\| \leq \int_{\frac{k-1}{n}}^{\frac{k}{n}} \|u_n'(s)\|\, ds \leq \frac{1}{n}\left(1+\frac{1}{n}\right)^{k-1} K.$$

Hence,

$$\left|\left|\left| Au_n\left(\frac{k}{n}\right)+u_n\left(\frac{k}{n}\right)\right|\right|\right| \leq \left|\left|\left| Au_n\left(\frac{k}{n}\right)+u_n\left(\frac{k-1}{n}\right)\right|\right|\right| + \left\| u_n\left(\frac{k-1}{n}\right)-u_n\left(\frac{k}{n}\right)\right\|$$
$$\leq \left(1+\frac{1}{n}\right)^{k} K.$$

Thus (3.3) is proved by induction. *q.e.d.*

For the sequence $\{u_n\}_{n=1}^{\infty}$ we prove

LEMMA 7. *The sequence* $\{u_n\}$ *is strongly uniformly convergent on* $[0, 1]$, *and the limit* $u(t)$ *is strongly continuous on* $[0, 1]$ *and satisfies* $u(0)=a$.

PROOF. From the definition of u_n it follows that

$$u_n'(t)+U_n(t)+u_n\left(\frac{[nt]}{n}\right)=0 \qquad a.e. \text{ on } [0, 1], \tag{3.6}$$

where $U_n(t)$ is an X-valued function on $[0, 1]$ such that $U_n(t) \in Au_n(t)$ *a. e.* on $[0, 1]$, and $[\,.\,]$ denotes the Gaussian bracket. For positive integers n, m we have by (3.6), LEMMA 2 and the accretiveness of A

$$\frac{d}{ds}(\|u_n(s)-u_m(s)\|^2)$$

$$=-2<U_n(s)+u_n\left(\frac{[ns]}{n}\right)-U_m(s)-u_m\left(\frac{[ms]}{m}\right),\ F(u_n(s)-u_m(s))>$$

$$\leq -2<u_n\left(\frac{[ns]}{n}\right)-u_m\left(\frac{[ms]}{m}\right),\ F(u_n(s)-u_m(s))-F\left(u_n\left(\frac{[ns]}{n}\right)-u_m\left(\frac{[ms]}{m}\right)\right)>$$

$$a.e.\ on\ [0,1].$$

Furthermore, by (3.3) we have $\|u_n(t)\|\leq\|a\|+eK$ for all $t\in[0,1]$ and all n. Hence, we obtain

$$\frac{d}{ds}(\|u_n(s)-u_m(s)\|^2)$$

$$\leq 4(\|a\|+eK)\left\|F(u_n(s)-u_m(s))-F\left(u_n\left(\frac{[ns]}{n}\right)-u_m\left(\frac{[ms]}{m}\right)\right)\right\|$$

a.e. on $[0,1]$. Integrating this inequality on $[0,t]$,

$$(3.7)\quad \|u_n(t)-u_m(t)\|^2$$

$$\leq 4(\|a\|+eK)\int_0^1\left\|F(u_n(s)-u_m(s))-F\left(u_n\left(\frac{[ns]}{n}\right)-u_m\left(\frac{[ms]}{m}\right)\right)\right\|ds$$

for all $t\in[0,1]$. On the other hand, we have by Lemma 6 again

$$\left\|u_n(s)-u_m(s)-u_n\left(\frac{[ns]}{n}\right)+u_m\left(\frac{[ms]}{m}\right)\right\|$$

$$\leq\left\|u_n(s)-u_n\left(\frac{[ns]}{n}\right)\right\|+\left\|u_m(s)-u_m\left(\frac{[ms]}{m}\right)\right\|$$

$$\leq\int_{\frac{[ns]}{n}}^{s}\|u_n'(r)\|dr+\int_{\frac{[ms]}{m}}^{s}\|u_m'(r)\|dr$$

$$\leq eK\left(\frac{1}{n}+\frac{1}{m}\right).$$

Hence, by the strong uniform continuity of F on bounded subsets of X, the right hand side of (3.7) converges to 0 as $n, m\to\infty$, that is, $\{u_n\}$ is strongly uniformly convergent on $[0,1]$. Then, it is easily seen that the limit $u(t)$ is strongly continuous on $[0,1]$ and $u(0)=a$. *q.e.d.*

Lemma 8. *The function $u(t)$ is a strong solution of* (3.2) *on* $[0,1]$ *and* $u(t)\in D(A)$ *for all* $t\in[0,1]$.

Proof. For a positive number p, $1<p<\infty$, $\{u_n'\}$ is bounded in $L^p(0,1;X)$ by Lemma 6. It follows that there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u'_{n_j} \xrightarrow{w} v$ in $L^p(0,1;X)$. Moreover, since $u_{n_j}(t) \xrightarrow{s} u(t)$ in X uniformly on $[0, 1]$ by LEMMA 7, it follows that $u'=v$ in the distribution sense, and hence, $u(t)$ is strongly absolutely continuous and $u'(t)=v(t)$ *a.e.* on $[0, 1]$. Let $V(t)$ be the set of all weak cluster points of $\{u'_{n_j}(t)\}$. Then, by LEMMAS 3 and 6,

$$u'(t) \in \overline{co(V(t))} \qquad \text{for a.e. } t \in [0, 1].$$

Since

$$u'_{n_j}(t)+Au_{n_j}(t)+u_{n_j}\left(\frac{[n_jt]}{n_j}\right) \ni 0 \qquad a.e. \text{ on } [0, 1],$$

$u_{n_j}\left(\frac{[n_jt]}{n_j}\right) \xrightarrow{s} u(t)$ for all $t \in [0, 1]$ as $j \to \infty$ and A is demiclosed, it follows that

$$V(t) \in -(Au(t)+u(t)) \qquad \text{for a.e. } t \in [0, 1].$$

By LEMMA 5 we have

$$\overline{co(V(t))} \subset -(Au(t)+u(t)) \qquad a.e. \text{ on } [0, 1],$$

and hence

$$u'(t) \in -(Au(t)+u(t)) \qquad a.e. \text{ on } [0, 1].$$

Thus u is a strong solution of (3.2) on $[0, 1]$. The fact that $u(t) \in D(A)$ for all $t \in [0, 1]$ follows easily from (v) of LEMMA 1. *q.e.d.*

Finally, to complete the proof of THEOREM 1 we prove

LEMMA 9. *A is m-accretive.*

PROOF. In LEMMA 8 we have shown that for any given $a \in D(A)$ the initial value problem (3.2) has a strong solution $u(t)$ on $[0, 1]$. Applying LEMMA 8 with the initial time $t=1$ and the initial value $u(1) \in D(A)$, we obtain a strong solution of (3.2) on $[0, 2]$. Thus, successively we obtain a strong solution $u(t)$ on $[0, \infty)$. By (iii) of LEMMA 4 there is a sequence $\{t_j\}$ such that $t_j \nearrow \infty$, $Au(t_j)+u(t_j) \ni -u'(t_j)$ and $u'(t_j) \xrightarrow{s} 0$ in X as $j \to \infty$. By (iii) of LEMMA 4 again, for a positive number t_0 we have

$$\begin{aligned} \|u(t_j)-u(t_{j'})\| &\leq \int_{t_j}^{t_{j'}} \|u'(s)\|\, ds \\ &\leq \|u'(t_0)\| e^{t_0} \int_{t_j}^{t_{j'}} e^{-s} ds \\ &= \|u'(t_0)\| e^{t_0}(-e^{-t_{j'}}+e^{-t_j}) \end{aligned}$$

for all t_j and $t_{j'}$, $t_0 < t_j \leq t_{j'}$, and hence, $\|u(t_j)-u(t_{j'})\| \to 0$ as $j, j' \to \infty$, that is, $s\text{-}\lim_{j\to\infty} u(t_j)=u_0$ exists. Since A is demiclosed, we have $0 \in Au_0+u_0$. Thus $R(I+A) \ni 0$.

For an arbitrary point $z \in X$, replacing A by $A-z$ in the above argument, we conclude that $z \in R(I+A)$. *q.e.d.*

§4. Contraction semigroups and their generators

Let X_0 be a subset of X and let $T=\{T(t);\ t\geq 0\}$ be a family of nonlinear singlevalued operators from X_0 into itself. We say that T is a *contraction semigroup* on X_0 if

(a) $T(t+t')x=T(t)T(t')x$ *for* $t,\ t'\geq 0$ *and* $x \in X_0$,
(b) $\|T(t)x-T(t)y\|\leq\|x-y\|$ *for* $t\geq 0$ *and* $x,\ y\in X_0$,
(c) $T(0)x=x$ *for* $x \in X_0$,
(d) *the function* $t\rightarrow T(t)x$ *is strongly continuous on* $[0, \infty)$ *for each* $x \in X_0$.

We define the *strong infinitesimal generator* G_s of T by

$$G_s x=s-\lim_{t\searrow 0}\frac{T(t)x-x}{t}$$

and the *weak infinitesimal generator* G_w of T by

$$G_w x=w-\lim_{t\searrow 0}\frac{T(t)x-x}{t}$$

whenever the right sides exist. It is easy to see that if X^* is uniformly convex, then $-G_s$ and $-G_w$ are accretive and $G_s\subset G_w$.

By using THEOREM 1 we shall prove

THEOREM 2. *Suppose that X^* is uniformly convex. Let A be an accretive operator from X into X. Then the following statements are equivalent to each other*:

(i) *A is m-accretive.*
(ii) *For each $z \in X$, there is a contraction semigroup $T^{(z)}=\{T^{(z)}(t);\ t\geq 0\}$ on $\overline{D(A)}$ such that $-G_s^{(z)}\subset A+z$ and* $D(A)\subset\left\{x \in \overline{D(A)};\ \liminf_{t\searrow 0}\frac{\|T^{(z)}(t)x-x\|}{t}<\infty\right\}$.
(iii) *For each $z \in X$, there is a contraction semigroup $T^{(z)}=\{T^{(z)}(t);\ t\geq 0\}$ on $\overline{D(A)}$ such that $-G_w^{(z)}\subset A+z$ and* $D(A)\subset\left\{x \in \overline{D(A)};\ \liminf_{t\searrow 0}\frac{\|T^{(z)}(t)x-x\|}{t}<\infty\right\}$.

Here, $G_s^{(z)}$ and $G_w^{(z)}$ are the strong and the weak infinitesimal generators of $T^{(z)}$, *respectively.*

The assertion (iii) $\rightarrow$ (ii) of THEOREM 2 immediately follows from the fact that $G_s^{(z)} \subset G_w^{(z)}$ for each $z \in X$.

Now we recall results on generation of semigroups by M.G. Crandall and T.M. Liggett [3] and I. Miyadera [10].

THEOREM *B.* (M.G. Crandall and T.M. Liggett [3; THEOREMS I and II]) *Let B be an m-accretive operator from X into X. Then,*

(a) *there exists a contraction semigroup $T = \{T(t); t \geq 0\}$ on $\overline{D(B)}$ such that*

$$T(t)x = s - \lim_{n\to\infty}\left(I + \frac{t}{n}B\right)^{-n} x \qquad \textit{for } x \in \overline{D(B)}$$

uniformly on every bounded interval in $[0, \infty)$,

(b) *if X is reflexive, then for each $x \in D(B)$ the function $T(t)x$ is a strong solution of*

$$u'(t) + Bu(t) \ni 0, \qquad u(0) = x.$$

For an operator B from X into X we define B^0 by

$$B^0 x = \{x' \in Bx;\ \|x'\| = |||Bx|||\}$$

and call it the *canonical restriction* of B.

THEOREM *C.* (I. Miyadera [10; COROLLARY 1 and THEOREM 3]) *Let B be an m-accretive operator from X into X, and let $T = \{T(t); t \geq 0\}$ be the contraction semigroup on $\overline{D(B)}$ given by* THEOREM *B. Then we have*

(a) *if $x \in D(B)$ and if for some sequence $\{t_n\}$ with $t_n \searrow 0$*

$$x' = w - \lim_{n\to\infty} \frac{T(t_n)x - x}{t_n},$$

then $(x, x') \in G(B^0)$, where B^0 is the canonical restriction of B,

(b) *if X is reflexive, then*

$$D(B) = \left\{x \in \overline{D(B)};\ \liminf_{t\searrow 0} \frac{\|T(t)x - x\|}{t} < \infty\right\},$$

(c) *if X is reflexive and if B^0 is singlevalued, then $D(B^0) = D(B)$ and $-B^0$ is the weak infinitesimal generator of T,*

(d) *if X is reflexive and X and X^* are strictly convex, then B^0 is singlevalued and $-B^0$ is the weak infinitesimal generator of T.*

Proof of the assertion (i) $\rightarrow$ (iii) of THEOREM 2. Since A is m-accretive, $A + z$ is also m-accretive for each $z \in X$. Therefore, there is a contraction semigroup $T^{(z)}=\{T^{(z)}(t);\ t\geq 0\}$ generated by $B=A+z$ in the sense of THEOREM B. We see from (a) of THEOREM C that $-G_w^{(z)}\subset A+z$, and from (b) of THEOREM C that

$$D(A)=D(A+z)=\left\{x\in \overline{D(A)};\ \liminf_{t\searrow 0}\frac{\|T^{(z)}(t)x-x\|}{t}<\infty\right\}.$$

Thus we have (iii). *q.e.d.*

To prove that (ii) implies (i), we use the following lemma that is due to M. G. Crandall and A. Pazy [4; LEMMA 1.1 and LEMMA 6.1].

LEMMA 10. *Let $T=\{T(t);\ t\geq\}$ be a contraction semigroup on a subset X_0 of X and B be an accretive operator such that $-G_s\subset B$, where G_s is the strong infinitesimal generator of T. If $x\in D(B)\cap X_0$ and*

$$\liminf_{t\searrow 0}\frac{\|T(t)x-x\|}{t}=L<\infty,$$

then $L\leq |||Bx|||$ and $\|T(t)x-T(t')x\|\leq L|\ t-t'|$ *for* $t,\ t'\geq 0$.

Proof of the assertion (ii)$\rightarrow$(i) of THEOREM 2. Let $\tilde{A}$ be the operator given by

$$G(\tilde{A})=\{(x,\ x')\in X\times X;\ \textit{there is a sequence}\ \{(x_n,\ x_n')\}\subset G(A)$$

$$\textit{such that}\ x_n\xrightarrow{s}x\ \text{and}\ x_n'\xrightarrow{w}x'\ \text{in}\ X\},$$

and let z be an arbitrary point of X. Put

$$D_z=\left\{x\in\overline{D(A)};\ \liminf_{t\searrow 0}\frac{\|T^{(z)}(t)x-x\|}{t}<\infty\right\}.$$

Then we first have

$$D(\tilde{A})\subset D_z. \tag{4.1}$$

In fact, let $(x,\ x')$ be any element of $G(\tilde{A}+z)$. Then, there is a sequence $\{(x_n,\ x_n')\}\subset G(A+z)$ such that $x_n\xrightarrow{s}x$ and $x_n'\xrightarrow{w}x'$ in X as $n\rightarrow\infty$. Since $D(A+z)=D(A)\subset D_z$ by our assumption, $x_n\in D_z$ for each n. Hence we infer from LEMMA 10 that

$$\|T^{(z)}(t)x_n-x_n\|\leq\|x_n'\|t\qquad \textit{for}\ t\geq 0.$$

Since $\{\|x_n'\|\}$ is bounded, letting $n\rightarrow\infty$ in the above inequality, we have for some $M>0$

$$\|T^{(z)}(t)x-x\|\leq Mt\qquad \textit{for}\ t\geq 0,$$

and hence, $x\in D_z$. Thus (4.1) holds true. From (4.1) and LEMMA 10 it follows that the function $T^{(z)}(t)x$ on $[0,\infty)$ is Lipschitz continuous for each $x\in D(\tilde{A})$, and hence, it is strongly differentiable *a.e.* on $[0, \infty)$. Therefore,

$$\frac{d}{dt}T^{(z)}(t)x - G_s^{(z)}(T^{(z)}(t)x) = 0 \qquad a.e. \text{ on } [0, \infty).$$

By our assumption we have

$$\frac{d}{dt}T^{(z)}(t)x + A(T^{(z)}(t)x) + z \ni 0 \qquad a.e. \text{ on } [0, \infty).$$

Thus we have seen that for each $z \in X$ and each $x \in D(\tilde{A})$ the function $T^{(z)}(t)x$ is a strong solution of (3.1) on $[0, \infty)$. Now, let (x, x') be any element of $G(\tilde{A})$. Then $T^{(-x')}(t)x$ is a strong solution of (3.1) with $z = -x'$. Since $\tilde{A} \supset A$, it is also a strong solution of

$$u'(t) + \tilde{A}u(t) - x' \ni 0, \qquad u(0) = x.$$

By the uniqueness of a strong solution ((i) of LEMMA 4), we have

$$T^{(-x')}(t)x = x \qquad \text{for all } t \geq 0.$$

This means that $u(t) \equiv x$ is a strong solution of (3.1) with $z = -x'$. Therefore, $x \in D(A)$ and $x' \in Ax$. Thus $\tilde{A} = A$, and hence, A is demiclosed. Therefore from THEOREM 1 we obtain the m-accretiveness of A. *q.e.d.*

REMARK. Assume that X^* is uniformly convex. Let A be an m-accretive operator from X into X. Then, for each $z \in X$, the contraction semigroup given by (ii) (or (iii)) of THEOREM 2 coincides with the contraction semigroup generated by $B = A + z$ in the sense of THEOREM B. In fact, let denote the former by $T^{(z)}$ and the latter by $\tilde{T}^{(z)}$. Then, as we have seen in the above proof, for each $x \in D(A)$ the function $T^{(z)}(t)x$ is a strong solution of (3.1) on $[0, \infty)$, and by (b) of THEOREM B the function $\tilde{T}^{(z)}(t)x$ is also a strong solution of (3.1) on $[0, \infty)$. Hence, by the uniqueness of a strong solution,

$$T^{(z)}(t)x = \tilde{T}^{(z)}(t)x \qquad \text{for all } t \geq 0 \text{ and all } x \in D(A).$$

Furthermore, by the strong continuity of $T^{(z)}(t)$ and $\tilde{T}^{(z)}(t)$,

$$T^{(z)}(t) = \tilde{T}^{(z)}(t) \qquad \text{on } \overline{D(A)} \qquad \text{for all } t \geq 0.$$

Thus $T^{(z)} = \tilde{T}^{(z)}$.

The next two corollaries are obtained from THEOREMS B and C and our THEOREM 2.

COROLLARY 1. (F.E. Browder [2]) *Suppose that X^* is uniformly convex. Let A be a singlevalued accretive operator from X into X. Then A is m-accretive if and only if for each $z \in X$ there is a contraction semigroup on $\overline{D(A)}$ whose weak infinitesimal generator is $-(A+z)$.*

PROOF. The "only if" part immediately follows from the assertion (iii) $\rightarrow$ (i) of THEOREM 2. Next, assume that A is m-accretive. Then, by (i) $\rightarrow$ (iii) of THEOREM 2, for each $z \in X$ there exists a contraction semigroup $T^{(z)}$ on $\overline{D(A)}$ such that $-G_w^{(z)} \subset A+z$ and $D(A) \subset D_z$. By the above REMARK this semigroup $T^{(z)}$ is the contraction semigroup generated by $B=A+z$ in the sense of THEOREM B. Therefore from (c) of THEOREM C it follows that $-G_w^{(z)}=A+z$. *q.e.d.*

COROLLARY 2. *Suppose that X is strictly convex and X^* is uniformly convex. Let A be an accretive operator from X into X. Then A is m-accretive if and only if for each $z \in X$, the canonical restriction $(A+z)^\circ$ is singlevalued, $D((A+z)^\circ)=D(A)$ and there is a contraction semigroup on $\overline{D(A)}$ whose weak infinitesimal generator is $-(A+z)^\circ$.*

PROOF. The "only if" part immediately follows from the assertion (iii) $\rightarrow$ (i) of THEOREM 2. The "if" part is also proved just as in the proof of COROLLARY 1 by using (d) of THEOREM C.

References

[1] H. Brezis and A. Pazy, Accretive sets and differential equations in Banach spaces, Israel J. Math., **8** (1970), 367–383.

[2] F. E. Browder, Nonlinear equations of evolution and nonlinear accretive operators in Banach spaces, Bull. Amer. Math. Soc., **73** (1967), 867–874.

[3] M. G. Crandall and T. M. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces, Amer. J. Math., **93** (1971), 265–298.

[4] M. G. Crandall and A. Pazy, Semigroups of nonlinear contractions and dissipative sets, J. Functional Analysis, **3** (1969), 376–418.

[5] T. Kato, Nonlinear semigroups and evolution equations, J. Math. Soc. Japan, **19** (1967), 508–520.

[6] T. Kato, Accretive operators and nonlinear evolution equations in Banach spaces, Proc. Symp. Nonlinear Functional Analysis, Amer. Math. Soc., Part **1** (1970), 138–161.

[7] N. Kenmochi, Accretive mappings in Banach spaces, Hiroshima Math. J., **2** (1972), 163–177.

[8] Y. Kōmura, Nonlinear semigroups in Hilbert space, J. Math. Soc. Japan, **19** (1967), 493–507.

[9] R. H. Martin, Jr., A global existence theorem for autonomous differential equations in a Banach space, Proc. Amer. Math. Soc., **26** (1970), 307–314.

[10] I. Miyadera, Some remarks on semigroups of nonlinear operators, Tōhoku Math. J., **23** (1971), 245–258.

[11] I. Miyadera and S. Ōharu, Approximation of semigroups of nonlinear operators, Tōhoku Math. J., **22** (1970), 24–47.

[12] G. F. Webb, Continuous nonlinear perturbations of linear accretive operators in Banach spaces, J. Functional Analysis, **10** (1972), 191–203.

Department of Mathematics,
Faculty of Science,
Hiroshima University