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Groups of Self-equivalences of Certain Complexes

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Introduction

Throughout this note, all spaces, maps and homotopies are assumed to be based, and any map and its homotopy class are written by the same letter.

Let $\mathscr{E}(X)$ denote the group of self-equivalences of a topological space X. The member of $\mathscr{E}(X)$ is a homotopy class of homotopy equivalences of X into itself. The group operation of $\mathscr{E}(X)$ is given by the composition of maps. This group $\mathscr{E}(X)$ is a homotopy type invariant of X.

Several examples are known (see [5]-[10]). In particular, for a CW-complex $K=S^n\cup e^{n+k+1}$, $k\geqq -1$, having two cells, the group $\mathscr{E}(K)$ has been studied in the case $k=-1$, $n\geqq 2$ and the case $k=0$, $n\geqq 1$. The former case is treated in [9: Example 8], and the latter is due to P.Olum [7] for $n=1$ and the recent work of A.J. Sieradski [10] for arbitrary $n\geqq 1$.

The purpose of this note is to determine the group $\mathscr{E}(K)$ for a CW-complex $K=S^n\cup_\alpha e^{n+k+1}$, $k\geqq 1$, under the condition that the attaching class α is a double suspension, $\alpha=E^2\alpha''$, and both α and $E\alpha''$ have the same order. Our main result is stated as follows:

THEOREM 3.2. *Let $K=S^n\cup_\alpha e^{n+k+1}$, $k\geqq 1$, $n\geqq 2$. Suppose that there exists an element $\alpha''\in\pi_{n+k-2}(S^{n-2})$ such that $E^2\alpha''=\alpha$, and both $E\alpha''$ and α have the same order m. Let $i: S^n\to K$ and $p: K\to S^{n+k+1}$ be the inclusion and the projection, respectively, and set*

$$G=i_*p^*\pi_{n+k+1}(S^n),$$

which is a subgroup of the group $[K, K]$ with the track addition.

Define a two-sided action of the multiplicative group $Z_2=\{-1, 1\}$ on G by

$$(-1)g=i_*p^*(-\iota_n)\gamma,\quad g(-1)=-g\quad \textit{for}\quad g=i_*p^*\gamma\in G,$$

where $\iota_n\in\pi_n(S^n)$ is the class of the identity map of S^n.

Then, the group $\mathscr{E}(K)$ of self-equivalences of K is isomorphic to the multiplicative group whose entries are matrices

$$\begin{pmatrix} x & g\\ 0 & y\end{pmatrix},\quad x,\ y\in Z_2,\quad g\in G\qquad \textit{for } m=1, 2,$$

$$\begin{pmatrix} x & g\\ 0 & x\end{pmatrix},\quad x\in Z_2,\quad g\in G,\qquad \textit{for } m>2,$$

where the matrix multiplication is given as usual.

The procedure of the computation is as follows. We first calculate, in §1, the homotopy set $[K, K]$, which is an abelian group with respect to the track addition $+$, since K is a double suspension. The result is summarized in Thm. 1.3. We study secondly, in §2, the multiplicative structure of $[K, K]$ defined by the composition of maps. As is well known, the left distributive law $\beta(\gamma+\delta)=\beta\gamma+\beta\delta$ holds, but the right one does not in general. Introducing the homomorphisms $\varphi_a\colon [K, K]\to[K, K]$ defined by $\varphi_a(\theta)=(a\iota)\theta$ for arbitrary integer a(ι denotes the class of the identity map of K), the multiplicative structure of $[K, K]$ is determined in Thm. 2.2. In §3, the group $\mathscr{E}(K)$ is determined by making use of Thm. 2.2, where $\mathscr{E}(K)$ is regarded as the subset of $[K, K]$ of all invertible elements with respect to the multiplication. The result is, first, summarized in Thm. 3.2. by use of certain matrices. Next, we paraphrase the result as the form of certain semi-direct products of groups (Thm. 3.3). Finally, in Thm. 3.5, we treat especially the case when $\alpha=0$ or K is a wedge of two spheres. In §4, several examples are given.

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§ 1. Additive structure of $[K, K]$

Let K denote a CW-complex

(1.1) $$K=S^n\cup_\alpha e^{n+k+1}, \qquad n\geqq 2,\ k\geqq 1,$$

such that the attaching class $\alpha\in\pi_{n+k}(S^n)$ satisfies the following condition:

(1.2) *There is an element $\alpha''\in\pi_{n+k-2}(S^{n-2})$ such that α is the double suspension of α'': $\alpha=E^2\alpha''$, and both $\alpha'=E\alpha''$ and α have the same order m.*

Obviously, by (1.2), $n\geqq 4$ if $\alpha\neq 0$, and m is finite.

Set $K'=S^{n-1}\cup_{\alpha'}e^{n+k}$ the mapping cone of α'. Then, there is a sequence of cofibrations

$$S^{n-1}\xrightarrow{i'}K'\xrightarrow{p'}S^{n+k}\xrightarrow{\alpha}S^n\xrightarrow{i}K\xrightarrow{p}S^{n+k+1}.$$

We identify canonically K and the suspension EK' of K': $EK'=K$, and so

(1.3) $$Ei'=i, \qquad Ep'=-p, \qquad E\iota'=\iota,$$

where (ι' resp. ι) stands for the class of the identity map of K' (resp. K). Also we denote by $\iota_n\in\pi_n(S^n)$ the identity class of S^n.

Since α' is a suspension and of order m, we have $(m\iota_{n-1})\alpha'=\alpha'(m\iota_{n+k-1})=0$, hence there are elements $\xi'\in\pi_{n+k}(K')$ and $\eta'\in[K', S^{n-1}]$ satisfying $p'_*\xi'=m\iota_{n+k}$ and $i'^*\eta'=m\iota_{n-1}$. The element ξ' is a coextension (for the definition, see e.g. [11: p. 13]) of $m\iota_{n+k-1}$ and determined up to $i'_*\pi_{n+k}(S^{n-1})$. Also η' is an extension of $m\iota_{n-1}$ and determined up to $p'^*\pi_{n+k}(S^{n-1})$. We put

$$\lambda'=p'^*\xi' \qquad \text{and} \qquad \mu'=i'_*\eta' \qquad \text{in } [K', K'].$$

These are determined up to $i'_*p'^*\pi_{n+k}(S^{n-1})$. Since K' is a suspension, the set $[K', K']$ becomes a group by the track addition $+$. We shall show the relation $\lambda'+\mu'=m\iota'$ in $[K', K']$.

Since i' is a suspension, we have $i'^*(\lambda'+\mu'-m\iota')=i'^*\lambda'+i'^*\mu'-mi'^*\iota'=0$, and so $\lambda'+\mu'-m\iota'=p'^*\gamma$, $\gamma\in\pi_{n+k}(K')$, by the exactness of $\pi_{n+k}(K')\to[K', K']\to\pi_{n-1}(K')$. We have also $p'^*p'_*\gamma=p'_*(\lambda'+\mu'-m\iota')=0$ and $p'_*\gamma=0$ since $p'^*: \pi_{n+k}(S^{n+k})\to[K', S^{n+k}]$ is isomorphic. The homomorphism $\bar{p}'_*: \pi_j(K', S^{n-1})\to\pi_j(S^{n+k})$ is isomorphic for $j\leqq 2n+k-3$ and $n\geqq 3$ by Thm. II of [2], hence if $n\geqq 3$ we have $\gamma\in i'_*\pi_{n+k}(S^{n-1})$ from the exact sequence $\pi_{n+k}(S^{n-1})\to\pi_{n+k}(K')\to\pi_{n+k}(K', S^{n-1})$. Thus, $\lambda'+\mu'-m\iota'\in i'_*p'^*\pi_{n+k}(S^{n-1})$, and by a suitable choice of λ' up to $i'_*p'^*\pi_{n+k}(S^{n-1})$, we obtain $\lambda'+\mu'-m\iota'=0$ if $n\geqq 3$. If $n=2$, then $\alpha'=0$ by (1.2), K' is a wedge of S^1 and S^{k+2}: $K=S^1\vee S^{k+2}$, and ξ' and η' are unique since $\pi_{k+2}(S^1)=0$. So we can choose ξ' (resp. η') as the inclusion (resp. retraction). Thus, $\xi'p'+i'\eta'=\iota'$ and so $\lambda'+\mu'=\iota'$.

We have proved the following

LEMMA 1.1. *There exist elements λ' and μ' of $[K', K']$ satisfying*

$$\lambda'=p'^*\xi', \qquad p'_*\xi'=m\iota_{n+k} \qquad \textit{for some } \xi'\in\pi_{n+k}(K'),$$

$$\mu'=i'_*\eta', \qquad i'^*\eta'=m\iota_{n-1} \qquad \textit{for some } \eta'\in[K', S^{n-1}],$$

$$\lambda'+\mu'=m\iota'.$$

The track addition $+$ defines a group structure on $[K, K]$, which is abelian since K is a double suspension. We define two elements λ and μ of $[K, K]$ by

$$\lambda=E\lambda', \qquad \mu=E\mu'. \tag{1.4}$$

By (1.3) and Lemma 1.1, these elements satisfy

$$\begin{aligned} &\lambda=p^*\xi, && p_*\xi=m\iota_{n+k+1},\\ &\mu=i_*\eta, && i^*\eta=m\iota_n,\\ &\lambda+\mu=m\iota, \end{aligned} \tag{1.5}$$

where $\xi=-E\xi'$ and $\eta=E\eta'$.

LEMMA 1.2. *The images of the following two homomorphisms are equal.*

$$(E\alpha)^*: \pi_{n+1}(S^n)\to\pi_{n+k+1}(S^n),$$

$$\alpha_*: \pi_{n+k+1}(S^{n+k})\to\pi_{n+k+1}(S^n).$$

PROOF. If $\alpha=0$, the lemma is obviously true. So, we prove the lemma assuming $\alpha\neq 0$ and $n\geqq 4$.

As is well known, the group $\pi_{l+1}(S^l)$, $l\geqq 2$, is cyclic and the generator η_l satisfies $E\eta_{l-1}=\eta_l$ and $2\eta_l=0$ for $l\geqq 3$.

Applying the theorem of M.G. Barratt and P.J. Hilton [1: Thm. 3.2] to the reduced join $\eta_2\wedge\alpha''\in\pi_{n+k+1}(S^n)$ for an element α'' satisfying $E^2\alpha''=\alpha$, we obtain the commutativity

$$\alpha\eta_{n+k}=\eta_n E\alpha.$$

This proves the lemma. *q.e.d.*

Now, we calculate the group $[K, K]$.

THEOREM 1.3. *Let K be a complex of* (1.1) *such that the attaching class α of K satisfies* (1.2). *Then, the group $[K, K]$ is the direct sum*

$$[K, K]=Z\oplus Z\oplus G, \qquad G=i_*p^*\pi_{n+k+1}(S^n),$$

where the elements ι of (1.3) *and λ of* (1.4) *generate the first and the second infinite cyclic factors, and G is isomorphic to $\pi_{n+k+1}(S^n)/\mathrm{Im}\alpha_*=\pi_{n+k+1}(S^n)/\mathrm{Im}(E\alpha)^*$.*

PROOF. By Thm. II of [2], the homomorphism $\bar{p}_*\colon \pi_j(K, S^n)\to\pi_j(S^{n+k+1})$ is isomorphic for $j=n+k+1$ and epimorphic for $j=n+k+2$. The boundary homomorphism $\partial\colon \pi_{j+1}(K, S^n)\to\pi_j(S^n)$ of the homotopy sequence of the pair (K, S^n) coincides with $\alpha_*E^{-1}\bar{p}_*\colon \pi_{j+1}(K, S^n)\to\pi_{j+1}(S^{n+k+1})\xleftarrow{\approx}\pi_j(S^{n+k})\to\pi_j(S^n)$ for $j=n+k+1$, $n+k+2$. Thus, we obtain an exact sequence

$$\pi_{n+k+1}(S^{n+k})\xrightarrow{\alpha_*}\pi_{n+k+1}(S^n)\xrightarrow{i_*}\pi_{n+k+1}(K)\xrightarrow{p_*}\pi_{n+k+1}(S^{n+k+1})$$
$$\xrightarrow{\alpha_*E^{-1}}\pi_{n+k}(S^n),$$

from which the group $\pi_{n+k+1}(K)$ is calculated: $\pi_{n+k+1}(K)=Z\oplus i_*\pi_{n+k+1}(S^n)$, the first factor is generated by ξ of (1.5).

By the cellular approximation theorem, $i_*\colon \pi_{n+1}(S^n)\to\pi_{n+1}(K)$ is epimorphic. Hence, by Lemma 1.2, the homomorphism $(E\alpha)^*\colon \pi_{n+1}(K)\to\pi_{n+k+1}(K)$ is trivial. Also it follows that $\pi_n(K)$ is isomorphic to Z with the generator i and $\alpha^*\colon \pi_n(K)\to\pi_{n+k}(K)$ is trivial.

We have, therefore, the following split exact sequence

$$0\to\pi_{n+k+1}(K)\xrightarrow{p^*}[K, K]\xrightarrow{i^*}\pi_n(K)\to 0. \tag{1.6}$$

A splitting homomorphism $s\colon \pi_n(K)\to[K, K]$ is given by $s(i)=\iota$, and so $[K, K]$ is the direct sum of $\mathrm{Im}p^*$ and $\mathrm{Im}s$, since $[K, K]$ is abelian. Thus, the theorem is established. *q.e.d.*

REMARK. The above discussion can be done for K' instead of K. Consequently, the group $[K', K']$ is a semi-direct product of $\mathrm{Im}\, p'^*$ and $\mathrm{Im}\, s'$ for a splitting $s': \pi_{n-1}(K') \to [K', K']$.

§2. Multiplicative structure of $[K, K]$

The composition of maps defines an associative multiplication of $[K, K]$ with the unit ι. Concerning with the addition $+$, the left distributive law $\beta(\gamma+\delta)=\beta\gamma+\beta\delta$ holds, but the right one does not hold in general. As is well known,

$$(2.1)\qquad (\gamma+\delta)E\beta=\gamma E\beta+\delta E\beta \qquad \textit{for } \beta \in [X, Y], \qquad \gamma, \delta \in [EY, Z],$$

and in particular

$$(2.2)\quad a\gamma=\gamma(a\iota),\ aE\gamma'=(a\iota)E\gamma' \quad \textit{for } \gamma \in [K, Y], \gamma' \in [X, K'] \textit{ and any integer } a.$$

For any integer a, we define a homomorphism

$$\varphi_a : [K, K] \to [K, K] \qquad \text{by } \varphi_a(\theta)=(a\iota)\theta.$$

Similarly we define $\psi_a: \pi_{n+k+1}(S^n) \to \pi_{n+k+1}(S^n)$ as the left translation by $a\iota_n$ instead of $a\iota$. Then, the following is obtained immediately.

LEMMA 2.1. *$\varphi_a i_* p^* = i_* p^* \psi_a$, $\varphi_a(\theta)=a\theta$ if $\theta=E\theta'$ and $\psi_a(g)=ag$ if $g=Eg'$.*

Especially we have

$$(2.3)\qquad \varphi_a(\lambda)=a\lambda, \qquad \varphi_a(\mu)=a\mu \qquad \textit{for } \lambda \textit{ and } \mu \textit{ of } (1.4).$$

By Lemma 2.1, the subgroup G of $[K, K]$ defined in Thm. 1.3 is closed with respect to φ_a, and $\varphi_a | G$ is determined by ψ_a.

REMARK. According to the theorem of P.J. Hilton [3], the homomorphism ψ_a is described by use of the (iterated) Whitehead products and the (higher) Hopf invariants.

(2.4) (Hilton [3: Thms. 6.7 and 6.9]) *For any integer a and any $g \in \pi_{n+k+1}(S^n)$,*

$$\psi_a(g)=ag+\frac{a(a-1)}{2}[\iota_n,\iota_n]H_0(g)+\frac{(a+1)a(a-1)}{3}[\iota_n, [\iota_n, \iota_n]]H_1(g),$$

where $H_0: \pi_i(S^n) \to \pi_i(S^{2n-1})$ and $H_1: \pi_i(S^n) \to \pi_i(S^{3n-2})$ are the Hopf invariants being generalized by P.J. Hilton [3: p. 165].

Now we consider the multiplication of $[K, K]$. Since $pi=0$, for the elements λ and μ of (1.4) and (1.5), we have easily

$$(2.5)\qquad \lambda\mu=(\xi p)(i\eta)=0, \qquad m\lambda-\lambda^2=\lambda(m\iota-\lambda)=\lambda\mu=0.$$

Since λ and μ are suspensions, we have

$$m\mu-\mu^2=(m\iota-\mu)\mu=\lambda\mu=0, \qquad \mu\lambda=(m\iota-\lambda)\lambda=m\lambda-\lambda^2=0,$$

by (2.1) and (2.2). For any elements $g=i_*p^*\gamma$ and $h=i_*p^*\delta$ of $G\subset[K, K]$, $\gamma, \delta \in \pi_{n+k+1}(S^n)$, we have also

$$\begin{aligned}
(2.6)\qquad & \lambda g=(\xi p)(i\gamma p)=0, \qquad g\mu=(i\gamma p)(i\eta)=0,\\
& g\lambda=g(m\iota-\mu)=mg-g\mu=mg,\\
& \mu g=(i\eta)(i\gamma p)=i(m\iota_n)\gamma p=(m\iota)i\gamma p=\varphi_m(g),\\
& gh=(i\gamma p)(i\delta p)=0.
\end{aligned}$$

THEOREM 2.2. *Let K be a complex of Thm.* 1.3 *and G be the subgroup of $[K, K]$ defined in Thm.* 1.3. *Then, the multiplication in $[K, K]$ is given by the formula*:

$$(a\iota+b\lambda+g)(a'\iota+b'\lambda+g')=aa'\iota+(ab'+a'b+mbb')\lambda+(a'+mb')g+\varphi_a(g'),$$

where $a, b, a', b' \in Z$ and $g, g' \in G$.

PROOF. Put $\theta=a\iota+b\lambda+g$ and $\theta'=a'\iota+b'\lambda+g'$. Then, by use of the right distributive law and (2.1–2.2), we have

$$\begin{aligned}
& \theta\theta'=\theta(a'\iota)+\theta(b'\lambda)+\theta g',\\
& \theta(a'\iota)=a'\theta=aa'\iota+a'b\lambda+a'g,\\
& \theta(b'\lambda)=b'\theta\lambda=ab'\lambda+bb'\lambda^2+b'g\lambda=(ab'+mbb')\lambda+mb'g \qquad \text{by (2.5–2.6).}
\end{aligned}$$

Set $g'=i\gamma p$. Then, since i is a suspension and $\lambda i=gi=0$, we have also

$$\theta g'=\theta i\gamma p=((a\iota)i+(b\lambda)i+gi)\gamma p=(a\iota)i\gamma p=(a\iota)g'=\varphi_a(g').$$

Thus, the theorem is established. *q.e.d.*

§3. The group $\mathscr{E}(K)$

The group $\mathscr{E}(K)$ consists of the elements θ of $[K, K]$ having two-sided inverses θ', that is to say

$$(3.1)\qquad \theta\theta'=\theta'\theta=\iota.$$

Put $\theta=a\iota+b\lambda+g$ and $\theta'=a'\iota+b'\lambda+g'$, $a, b, a', b' \in Z$, $g, g' \in G$. Then, by Thm. 2.2, we have

$$\begin{aligned}
& aa'=1, \qquad ab'+a'b+mbb'=0,\\
& (a'+mb')g+\varphi_a(g')=0, \qquad (a+mb)g'+\varphi_{a'}(g)=0.
\end{aligned}$$

From the first two equations, it follows that

$$a=a'=\pm 1,\ b=b'=0 \qquad \text{for arbitrary } m,$$

$$a=a'=\pm 1,\ b=b'=\mp(2/m) \qquad \text{for } m=1,\ 2.$$

Therefore the solutions of (3.1) are the following elements.

$$\begin{array}{lll} \theta=\iota+g, & \theta'=\iota-g & \text{for arbitrary } m, \\ \theta=-\iota+g, & \theta'=-\iota+(-\iota)g & \text{for arbitrary } m, \\ \theta=\theta'=\iota-\lambda+g & & \text{for } m=2, \\ \theta=-\iota+\lambda+g, & \theta'=-\iota+\lambda-(-\iota)g & \text{for } m=2, \\ \theta=\theta'=\iota-2\lambda+g & & \text{for } m=1, \\ \theta=-\iota+2\lambda+g, & \theta'=-\iota+2\lambda-(-\iota)g & \text{for } m=1. \end{array}$$

In the above, g runs over the whole of G.

Summarizing the above, we have proved the following

PROPOSITION 3.1. *As a subset of* $[K, K]$ *the group* $\mathscr{E}(K)$ *is as follows:*

$$\mathscr{E}(K)=\begin{cases} \{\pm\iota+g \mid g\in G\} & \text{for } m>2, \\ \{\pm\iota+g,\ \pm(\iota-\lambda)+g \mid g\in G\} & \text{for } m=2, \\ \{\pm\iota+g,\ \pm(\iota-2\lambda)+g \mid g\in G\} & \text{for } m=1. \end{cases}$$

When $m=1,2$, we define an element $\sigma\in\mathscr{E}(K)$ by

$$\sigma=\begin{cases} -\iota+\lambda & \text{for } m=2, \\ -\iota+2\lambda & \text{for } m=1. \end{cases}$$

Then, from Prop. 3.1, any entry of $\mathscr{E}(K)$ is written as

$$\begin{array}{llll} \sigma^{\varepsilon}(-\sigma)^{\varepsilon'}+g, & \varepsilon,\ \varepsilon'=0 \text{ or } 1, & g\in G & \text{for } m=1,\ 2, \\ (-\iota)^{\varepsilon}+g, & \varepsilon=0 \text{ or } 1, & g\in G & \text{for } m>2. \end{array}$$

By Thm.2.2, we have easily

$$\begin{aligned} &(\sigma^{\varepsilon}(-\sigma)^{\varepsilon'}+g)(\sigma^{\eta}(-\sigma)^{\eta'}+h) \\ &\qquad =\sigma^{\varepsilon+\eta}(-\sigma)^{\varepsilon'+\eta'}+(-1)^{\eta'}g+(-\iota)^{\varepsilon}h \qquad \text{for } m=1,\ 2, \\ &((-\iota)^{\varepsilon}+g)((-\iota)^{\eta}+h)=(-\iota)^{\varepsilon+\eta}+(-1)^{\eta}g+(-\iota)^{\varepsilon}h \qquad \text{for } m>2. \end{aligned}$$

This suggests us to describe the group $\mathscr{E}(K)$ as a matrix form.

THEOREM 3.2. *Let K be the complex of* (1.1) *satisfying* (1.2), *and G be the* *(additive) subgroup* $i_* p^* \pi_{n+k+1}(S^n)$ *of* $[K, K]$. *Define a two-sided action of the multiplicative group* $Z_2 = \{-1, 1\}$ *on* G *by*

$$(-1)g = (-\iota)g, \qquad g(-1) = -g \qquad \text{for} \ \ g \in G.$$

Then the group $\mathscr{E}(K)$ *of self-equivalences of* K *is isomorphic to the multiplicative group whose members are matrices*

$$\begin{pmatrix} x & g \\ 0 & y \end{pmatrix}, \qquad x,\ y \in Z_2, \qquad g \in G \qquad \text{for} \ \ m=1, 2,$$

$$\begin{pmatrix} x & g \\ 0 & x \end{pmatrix}, \qquad x \in Z_2, \qquad g \in G \qquad \text{for} \ \ m>2,$$

where the matrix multiplication is given as usual:

$$\begin{pmatrix} x & g \\ 0 & y \end{pmatrix} \begin{pmatrix} x' & g' \\ 0 & y' \end{pmatrix} = \begin{pmatrix} xx' & xg' + gy' \\ 0 & yy' \end{pmatrix},$$

and the elements xg' *and* gy' *are given by the above action.*

PROOF. The isomorphism is given by the following correspondence:

$$\sigma^{\varepsilon}(-\sigma)^{\varepsilon'} + g \to \begin{pmatrix} (-1)^{\varepsilon} & g \\ 0 & (-1)^{\varepsilon'} \end{pmatrix}, \qquad \varepsilon,\ \varepsilon' = 0 \text{ or } 1 \qquad \text{for} \ \ m=1, 2,$$

$$(-\iota)^{\varepsilon} + g \quad \to \begin{pmatrix} (-1)^{\varepsilon} & g \\ 0 & (-1)^{\varepsilon} \end{pmatrix}, \qquad \varepsilon = 0 \text{ or } 1 \qquad \text{for} \ \ m>2.$$

q.e.d.

Now we set

(3.2) $$\mathscr{E}_0(K) = \{\iota + g \mid g \in G\}.$$

We see easily that

(3.3) $\mathscr{E}_0(K)$ *is a normal subgroup of* $\mathscr{E}(K)$, *and is isomorphic to the (additive) group* G *by corresponding* $\iota + g \in \mathscr{E}_0(K)$ *and* $g \in G$.

Consider the quotient group $\Gamma = \mathscr{E}(K)/\mathscr{E}_0(K)$. Then, we see easily

$$\Gamma \approx \begin{cases} Z_2 \times Z_2 & \text{with the generators } <\sigma> \text{ and } <-\sigma> \qquad \text{for} \ \ m=1, 2, \\ Z_2 & \text{with the generator } <-\iota> \qquad \text{for} \ \ m>2, \end{cases}$$

where $<\theta>$ stands for the coset $\theta\mathscr{E}_0(K)$.

Set

$$\Gamma' = \begin{cases} \{\pm\iota, \ \pm\sigma\} & \text{for } m=1, 2, \\ \{\pm\iota\} & \text{for } m>2. \end{cases}$$

Then, Γ' is a subgroup of $\mathscr{E}(K)$ and isomorphic to Γ, and so, by (3.3), we have a split exact sequence

$$\{e\} \to G \to \mathscr{E}(K) \to \Gamma \to \{e\}.$$

Thus, we obtain the following

THEOREM 3.3. *Let K and G be as above Thm. 3.2. Then, if $m=1, 2$, the group $\mathscr{E}(K)$ is the split extension*

$$\{e\} \to G \to \mathscr{E}(K) \to Z_2 \times Z_2 \to \{e\},$$

where the generators of $Z_2 \times Z_2$ act on G by the following two automorphisms of G:

$$G \ni g \to (-\iota)g \in G, \qquad G \ni g \to -g \in G.$$

If $m>2$, the group $\mathscr{E}(K)$ is the split extension

$$\{e\} \to G \to \mathscr{E}(K) \to Z_2 \to \{e\},$$

where the operation of Z_2 on G is given by the automorphism of G sending $g \in G$ to $-(-\iota)g \in G$.

PROOF. It suffices to investigate the operation of Γ' on $\mathscr{E}_0(K)$. This is checked by virtue of the following equalities in $\mathscr{E}(K)$.

$$\sigma^{-1}(\iota+g)\sigma = \iota+(-\iota)g, \qquad (-\sigma)^{-1}(\iota+g)(-\sigma) = \iota-g \qquad \text{for } m=1, 2,$$

$$(-\iota)^{-1}(\iota+g)(-\iota) = \iota-(-\iota)g \qquad \text{for arbitrary } m.$$

Then the theorem follows. *q.e.d.*

COROLLARY 3.4. *Suppose that $(-\iota_n)\gamma = -\gamma$ holds for arbitrary element $\gamma \in \pi_{n+k+1}(S^n)$. Then, the group $\mathscr{E}(K)$ is isomorphic to $D(G) \times Z_2$ for $m=1, 2$ and to $G \times Z_2$ for $m>2$, and the second factor Z_2 is generated by $-\iota$. The group $D(G)$ is the split extension*

$$\{e\} \to G \to D(G) \to Z_2 \to \{e\},$$

where the operation of Z_2 on G is given by the automorphism

$$G \ni g \to -g \in G.$$

REMARK. The assumption of Cor.3.4 is satisfied in the following each case.

(i) $E: \pi_{n+k}(S^{n-1}) \to \pi_{n+k+1}(S^n)$ *is epimorphic.*

(ii) $n=3$ *or* 7.

PROOF OF COROLLARY 3.4. By Lemma 2.1, the assumption of Cor. 3.4 implies $(-\iota)g=-g$ for any $g\in G$. So, the corollary is an easy consequence of Thm. 3.3. *q.e.d.*

The group $D(G)$ of the above is a generalization of the dihedral group. Indeed, $D(Z_t)$ is the usual dihedral group, written as D_t, of order $2t$. In general, we define the group $D(A)$ for any abelian group A, written multiplicatively, by the split extension

$$\{e\}\to A\to D(A)\to Z_2\to\{e\},$$

where the generator of Z_2 acts on A as the automorphism

$$A\ni a\to a^{-1}\in A.$$

The following isomorphism is verified easily.

$$D(A\times Z_2)\approx D(A)\times Z_2. \tag{3.4}$$

From Thms. 1.3 and 3.3, we see that the group $\mathscr{E}(K)$ depends on the compositions $\alpha\eta_{n+k}$ and $(-\iota_n)\gamma$ for $\gamma\in\pi_{n+k+1}(S^n)$ as well as the isomorphism class of the group $\pi_{n+k+1}(S^n)$. We give, however, a particular case that $\mathscr{E}(K)$ does not depend on the compositions in $\pi_{n+k+1}(S^n)$.

THEOREM 3.5. *Let $K=S^n\vee S^{n+k+1}$, $k\geqq 0$, $n\geqq k+3$. Then, the group $\mathscr{E}(K)$ is isomorphic to*

$$D(\pi_{n+k+1}(S^n))\times Z_2,$$

and the second factor Z_2 is generated by $-\iota$.

PROOF. The homomorphism $E\colon \pi_{n+k}(S^{n-1})\to\pi_{n+k+1}(S^n)$ is epimorphic by the suspension theorem of Freudenthal, and the subgroup G of $[K, K]$ is isomorphic to $\pi_{n+k+1}(S^n)$ since $\alpha=0$. So, the theorem for $k\geqq 1$ follows from Cor. 3.4.

For the case $k=0$ and $\alpha=0$, the discussions in §§1–3 are done quite similarly. Let $K=S^n\vee S^{n+1}$, $i\colon S^n\to K$ and $\xi\colon S^{n+1}\to K$ be inclusions, and $p\colon K\to S^{n+1}$ and $\eta\colon K\to S^n$ be retractions. Then, similarly as Thm. 1.3, we have

$$[K, K]=Z\oplus Z\oplus Z_2,$$

where the generators of each factor are $\lambda=\xi p$, $\mu=i\eta$ and $g=i\eta_n p$, $\eta_n\in\pi_{n+1}(S^n)$, which are suspensions. We have also $\lambda+\mu=\iota$, and so the multiplication of $[K, K]$ is given by $\lambda^2=\lambda$, $\mu^2=\mu$, $\lambda\mu=\mu\lambda=0$, $\lambda g=g\mu=0$, $g\lambda=\mu g=g$, $g^2=0$ and by the two-sided distributive law. Hence, from the similar discussions as in Prop. 3.1 and Thm. 3.3, we have the desired result

$$\mathscr{E}(S^n \vee S^{n+1}) = Z_2 \times Z_2 \times Z_2 \quad (\text{generators} \quad \sigma = \lambda - \mu, \quad -\sigma, \quad \iota + g),$$

where $D(\pi_{n+1}(S^n)) = D(Z_2) = Z_2 \times Z_2$ by (3.4). *q.e.d.*

§ 4. Examples

In the following examples, the group operation of $[K, K]$ (resp. $\mathscr{E}(K)$) is written additively (resp. multiplicatively). And so, the group G is written additively as a subgroup $i_* p^* \pi_{n+k+1}(S^n)$ of $[K, K]$, and multiplicatively as a subgroup $\mathscr{E}_0(K)$ of $\mathscr{E}(K)$ as in (3.2). Indexing K, we write K_n instead of K, when K is $(n-1)$-connected.

We refer the notations and the relations of $\pi_i(S^n)$ to Toda's book [11].

EXAMPLE 1. $\alpha = \eta_n \in \pi_{n+1}(S^n)$, $K_n = S^n \cup_\alpha e^{n+2}$.

$$\mathscr{E}(K_n) = Z_2 \times Z_2, \qquad n \geqq 3.$$

Generators $\sigma, -\sigma$.

$$\mathscr{E}(K_2) = Z_2.$$

Generator a,

$$Ea = \sigma.$$

Since $\pi_{n+2}(S^n)/\eta_n \pi_{n+2}(S^{n+1}) = 0$, the above holds for $n \geqq 4$ by Thm. 3.3. The element η_3 is not a double suspension. But E: $[K_3, K_3] \to [K_4, K_4]$ is isomorphic, and the above holds for $n=3$. For $n=2$, K_2 is the complex projective plane $CP(2)$, hence we have $[K_2, K_2] = [CP(2), CP(\infty)] = H^2(CP(2)) = Z$, and the above is established.

EXAMPLE 2. $\alpha = \eta_n \eta_{n+1} \in \pi_{n+2}(S^n)$, $K_n = S^n \cup_\alpha e^{n+3}$.

$$\mathscr{E}(K_n) = D_{12} \times Z_2, \qquad n \geqq 5.$$

Generators of D_{12} $\quad a = \iota + i_* p^*(\nu_n + \alpha_1(n))$, σ.

Relations in D_{12} $\quad a^{12} = \iota$, $\sigma^2 = \iota$, $\sigma a = a^{-1} \sigma$.

Generator of Z_2 $\quad -\iota$.

$\mathscr{E}(K_4)$

Generators $\quad b = \iota + i_* p^* \nu_4$, $c = \iota + i_* p^*(E\nu' + \alpha_1(4))$, σ, $-\sigma$.

Relations $\quad c^6 = \sigma^2 = (-\sigma)^2 = \iota$, $cb = bc$, $\sigma c = c^{-1} \sigma$,

$$(-\sigma)c = c^{-1}(-\sigma), \quad \sigma(-\sigma) = (-\sigma)\sigma (= -\iota),$$

$$\sigma b = bc^{-3} \sigma, \quad (-\sigma) b = b^{-1}(-\sigma).$$

$$Eb=a^9,\ Ec=a^4.$$

$\mathscr{E}(K_3)=D_6\times Z_2$.

Generators of D_6 $\quad d=\iota+i_*p^*(\nu'+\alpha_1(3)),\quad \sigma.$

Relations in D_6 $\quad d^6=\sigma^2=\iota,\quad \sigma d=d^{-1}\sigma.$

Generator of Z_2 $\quad -\iota.$

$$Ed=c.$$

We have $G\approx Z_{12}$ (generator $i_*p^*(\nu_n+\alpha_1(n))$) for $n\geqq 5$ and $G\approx Z\oplus Z_6$ (generators $i_*p^*\nu_4$ and $i_*p^*(E\nu'+\alpha_1(4))$) for $n=4$. All elements of G are supensions for $n\geqq 5$ and the above follows from Cor. 3.4. For $n=4$, the element $E\nu'+\alpha_1(4)$ is a suspension but ν_4 is not. In $\pi_7(S^4)$, we have

$$\begin{aligned}(-\iota_4)\nu_4&=-\nu_4+[\iota_4,\iota_4]H(\nu_4) \qquad (H \text{ is the Hopf invariant, cf. (2.4)})\\ &=-\nu_4+(2\nu_4-E\nu')\\ &\qquad\qquad \text{by Lemma 5.4 and (5.8) of [11] and (6.1) of [4]}\\ &=\nu_4-E\nu'.\end{aligned}$$

So, the relation $\sigma b=bc^{-3}\sigma$ in $\mathscr{E}(K_4)$ is proved, and the above follows for $n=4$. For $n=3$, $[K_3, K_3]$ is abelian since $E\colon [K_3, K_3]\to[K_4, K_4]$ is monomorphic. Similar discussions of Thm. 2.2 and Prop. 3.1 can be done for K_3, and the above follows from Cor. 3.4 for $n=3$ since $(-\iota_3)\gamma=-\gamma$ for any $\gamma\in\pi_i(S^3)$ (see the remark after Cor. 3.4).

EXAMPLE 3. $\alpha=\nu_n\in\pi_{n+3}(S^n)$, $K_n=S^n\cup_\alpha e^{n+4}$.

$\mathscr{E}(K_n)=Z_2,\qquad n\geqq 5.$

Generator $\quad -\iota.$

For $n\geqq 6$, $\pi_{n+4}(S^n)=0$ and the above follows from Cor. 3.4. For $n=5$, $E\colon [K_5, K_5]\to[K_6, K_6]$ is isomorphic, and the above follows.

As is well known, K_4 is the quaternion projective plane. So, according to P.J. Kahn [5],

$$\mathscr{E}(K_4)=Z_2$$

and the generator a of $\mathscr{E}(K_4)$ satisfies $Ea=\iota$.

EXAMPLE 4. $\alpha=0\in\pi_{n+4}(S^n)$, $K_n=S^n\vee S^{n+5}$.

$\mathscr{E}(K_n)=Z_2\times Z_2,\qquad n\geqq 7.$

Generators $\quad \sigma,\quad -\sigma.$

$\mathscr{E}(K_6)=D_\infty \times Z_2$.

Generators of D_∞ $\quad a=\iota+i_*p^*[\iota_6, \iota_6], \quad -\sigma$.

Relations in D_∞ $\quad (-\sigma)^2=\iota, \quad (-\sigma)a=a^{-1}(-\sigma)$.

Generator of Z_2 $\quad \sigma$.

$Ea=\iota$.

$\mathscr{E}(K_5)=Z_2\times Z_2\times Z_2$.

Generators $\quad b=\iota+i_*p^*\nu_5\eta_8\eta_9, \quad \sigma, \quad -\sigma,$

$\mathscr{E}(K_4)=D_4\times Z_2$.

We put $c=\iota+i_*p^*\nu_4\eta_7\eta_8$ and $d=\iota+i_*p^*(E\nu')\eta_7\eta_8$.

Generators of D_4 $\quad c'=c\sigma, \quad \sigma$.

Relations in D_4 $\quad c'^4=\sigma^2=\iota, \quad \sigma c'=c'^{-1}\sigma(=cd)$.

Generator of Z_2 $\quad -\sigma$.

$Ec=b, \quad Ed=\iota$.

$\mathscr{E}(K_3)=Z_2\times Z_2\times Z_2$.

Generators $\quad f=\iota+i_*p^*\nu'\eta_6\eta_7, \quad \sigma, \quad -\sigma$.

$Ef=d_\iota=c'^2$.

$\mathscr{E}(K_2)=Z_2\times Z_2\times Z_2$.

Generators $\quad g=\iota+i_*p^*\eta_2\nu'\eta_6, \quad \sigma, \quad -\sigma$.

$Eg=\iota$.

For $n\geqq 7$, $\pi_{n+5}(S^n)=0$ and the above follows from Thm. 3.5. For $n=6$, $\pi_{11}(S^6)=Z$ is generated by $[\iota_6, \iota_6]$, and we have $(-\iota_6)[\iota_6, \iota_6]=[\iota_6, \iota_6]$. So, the above follows. For $n=5$, $\pi_{10}(S^5)=E\pi_9(S^4)=Z_2$ (generator $\nu_5\eta_8\eta_9$), hence the above follows from Cor. 3.4. For $n=4$, we have $\pi_9(S^4)=Z_2\oplus Z_2$ with the generators $\nu_4\eta_7\eta_8$ and $(E\nu')\eta_7\eta_8=E(\nu'\eta_6\eta_7)$ and $\nu_4\eta_7\eta_8 \notin \mathrm{Im}E$. By the computation in Example 2, $(-\iota_4)\nu_4\eta_7\eta_8=\nu_4\eta_7\eta_8+(E\nu')\eta_7\eta_8$. So, $\mathscr{E}(K_4)$ has four generators $c, d, \sigma, -\sigma$ with the relations $\sigma^2=(-\sigma)^2=c^2=d^2=\iota$, $\sigma(-\sigma)=(-\sigma)\sigma$, $c(-\sigma)=(-\sigma)c$, $d(-\sigma)=(-\sigma)d$, $cd=dc$, $d\sigma=\sigma d$, $\sigma c=cd\sigma$. Then the above result is an easy consequence. For $n=3$, $\pi_8(S^3)=Z_2$ with the generator $\nu'\eta_6\eta_7$, and $(-\iota_3)\nu'\eta_6\eta_7=\nu'\eta_6\eta_7$. So, the above follows. For $n=2$, $\pi_7(S^2)=Z_2$ with the generator $\eta_2\nu'\eta_6$ satisfying $E(\eta_2\nu'\eta_6)=0$ and $(-\iota_2)\eta_2\nu'\eta_6=\eta_2\nu'\eta_6$ since $(-\iota_2)\eta_2=\eta_2$. Hence, the desired result follows.

References

[1] M. G. Barratt and P. J. Hilton, *On join operations of homotopy groups*, Proc. London Math. Soc., **3** (1953), 430–445.

[2] A. L. Blakers and W. S. Massey, *The homotopy groups of a triad II*, Ann. of Math., **55** (1952), 192–201.

[3] P. J. Hilton, *On the homotopy groups of the union of spheres*, J. of London Math. Soc., **30** (1955), 154–172.

[4] P. J. Hilton and J. H. C. Whitehead, *Note on the Whitehead product*, Ann. of Math., **58** (1953), 429–442.

[5] P. J. Kahn, *Self-equivalences of $(n-1)$-connected $2n$-manifolds*, Math. Ann., **180** (1969), 26–47.

[6] G. Liebermann and D. L. Smallen, *Localization and self-equivalences*, to appear in J. Math. Soc. Japan.

[7] P. Olum, *Self-equivalences of pseudo-projective planes*, Topology, **4** (1965), 109–127.

[8] J. W. Rutter, *Groups of self-homotopy equivalences of induced spaces*, Comment. Math. Helv., **45** (1970), 236–255.

[9] A. J. Sieradski, *Twisted self-homotopy equivalences*, Pacific J. Math., **34** (1970), 789–802.

[10] A. J. Sieradski, *Stabilization of self-equivalences of the pseudoprojective spaces*, Michigan Math. J., **19** (1972), 109–119.

[11] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. **49**, Princeton, (1962).

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