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On the Ring of Endomorphisms of an Indecomposable Injective Module over a Prüfer Ring

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It is well known, in the theory of abelian groups, that the ring of endomorphisms of the p-torsion part of Q/Z is the p-adic integers; in modern language, the p-torsion part of Q/Z is nothing but the injective envelope of the additive group Z/p. The aim of this paper is to study the ring of endomorphisms of indecomposable injective modules over Prüfer rings, which generalizes the above fact.

In §1, we shall deal with a basic theory of indecomposable injective modules over commutative rings and this enables us to reduce the problem to the case where valuation rings act as operators. The notion of pseudo-convergence plays an essential role in our discussions; it is originally due to Ostrowsky and by employing it, *I.* Kaplansky succeeded in proving the uniqueness of a maximal immediate extension under some conditions. In §2, the relationship between the module structure and the breadth of a pseudo-cenvergent set is cleared up. In §5, it is shown that the ring of endomorphisms of an indecomposable injective module over a Prüfer ring is, rather unexpectedly, not commutative and the structure of its center is determined.

Throughout this paper a ring R will always be understood to be commutative, to have a unit and a module over R to be unitary.

§1. Preliminaries

We say that an R-module M is co-irreducible if M is not zero and has no non-zero submodules $N_i(i=1, 2)$ such that $N_1 \cap N_2=0$. A submodule N of an R-module M is irreducible in M if the quotient module M/N is co-irreducible; this is equivalent to saying that N is properly contained in M and if N is the intersection of submodules N_1 and N_2, then $N_1=N$ or $N_2=N$. It is clear that non-zero submodules and essential extensions of a co-irreducible module are co-irreducible. Therefore we see immediately that the injective envelope $E(M)$ of a co-irreducible module M is co-irreducible and the order ideal $0(x)$ of every non-zero element x of $E(M)$ is an irreducible ideal of R.

We denote by Σ the set of irreducible ideals in R. Let I be a member of

1) $0(x)$ means the ideal annihilating the element x.

Σ. Then the injective envelope $E(R/I)$ is co-irreducible. We introduce an equivalence relation in Σ as follows: $I \sim J$ if and only if $E(R/I) \simeq E(R/J)$. Since an injective module E is indecomposable if and only if E is co-irreducible, the set $\Sigma/\sim$ of classes modulo the relation $\sim$ can be identified with that of isomorphism classes of indecomposable injective modules over R.

Proposition 1.[2] *$I \sim J$ if and only if $(I: r)=(J: s)$ for some $r \in R-I$ and $s \in R-J$.*

Proof. We put $E=E(R/I)$. First we suppose that $E(R/I) \simeq E(R/J)$; then we can find elements x, y in E such that $0(x)=I$ and $0(y)=J$. Since E is co-irreducible, there is a non-zero element z in $Rx \cap Ry$; then $z=rx$ and $z=sy$ for some r, s in R. Therefore both $(I: r)$ and $(J: s)$ coincide with $0(z)$. Conversely the assumption $(I: r)=(J: s)$ implies that there are elements $z \in R/I$ and $z' \in R/J$ such that the order ideal of z in R/I is equal to that of z' in R/J. It is easy to see that the isomorphism $Rz \rightarrow Rz'$ can be extended to one of $E(R/I)$ onto $E(R/I)$.

Proposition 2. *Let $\mathfrak{R}$ be a class modulo $\sim$ and E the indecomposable injective module determined by $\mathfrak{R}$. Then the following statements concerning an element s of R are equivalent:*

(i) *The homothety $s: E \ni x \rightarrow sx \in E$ is an automorphism.*
(ii) *$(I: s)=I$ for every ideal I belonging to the class $\mathfrak{R}$.*
(iii) *$(I: s)=I$ for some ideal I belonging to the class $\mathfrak{R}$.*

Proof. It is clear that (i) implies (ii) and (ii) implies (iii). We prove that (iii) implies (i). We take an element x of E such that $0(x)=I$. Then we see that the homothety s induces a monomorphism on Rx. Since E is co-irreducible and $Rx \neq 0$, the homothety s is a monomorphism. Every monomorphism of E to E must be surjective, because E is indecomposable, and the proof is completed.

When an element s of R has the properties mentioned in Proposition 2, we say that s is an $\mathfrak{R}$-unit. The set S of $\mathfrak{R}$-units is multiplicatively closed and we can readily see that the complementary set P of S in R is a prime ideal. We call P the prime ideal associated to $\mathfrak{R}$ or to E.

Proposition 3. *The notation being as above, we have $P=\bigcup_{I \in \mathfrak{R}} I$ and $I_0=\bigcap_{I \in \mathfrak{R}} I$ is the annihilator of E; moreover every element of I_0 is a zero-divisor.*

Proof. Except the last assertion the proof follows readily from the definitions. Now let r be a non-zero divisor in R; then, since E is divisible,

2) $(I: r)$ means the ideal $\{x \in R;\ xr \in I\}$ and $R-I$ means the complementary set of I in R.

the homothety r is a surjective map and this implies that r can not be an element of I_0.

Corollary. *If R is a domain, then R operates faithfully on E.*

Now S being the set of $\mathfrak{R}$-units as above, we consider the localization R_S (or R_P) of R by S. It is easy to see that the kernel of the canonical map: $R \to R_S$ is contained in the ideal I_0. Therefore we know that E becomes naturally an R_S-module and E is an indecomposable injective module over R_S. Moreover, every endomorphism of the R-module E is naturally R_S-linear. Thus we have the following

Proposition 4. *Let E be an indecomposable injective module over R. Then E has naturally an R_P-module structure and* $\mathrm{End}_R(E)$ *can be identified with* $\mathrm{End}_{R_P}(E)$, *where P is the associated prime ideal to E.*

Examples. 1) Let R be a noetherian ring. Then an irreducible ideal is an irreducible primary ideal. It is well known that every irreducible primary ideal is equivalent to the associated prime ideal in our sense. Therefore $\Sigma/\sim$ is the set of prime ideals in R.

2) Let R be a valuation ring with the value group Γ. Every ideal of R is irreducible and, for ideals I and J, $I\sim J$ if and only if $I=rJ$ or $rI=J$ for some $r \in R$. When Γ is the additive group $\boldsymbol{R}$ of reals, then $\Sigma/\sim=\{0, 1, 2\}$, where 0 means the zero ideal, 1 the class of principal ideals and 2 the class of not principal ideals; when Γ is the additive group $\boldsymbol{Q}$ of rationals, then $\Sigma=\{\boldsymbol{R}/\boldsymbol{Q}, 1, 2\}$.

Finally we give one more notion which will be used later.

Definition 1. *Let E be an indecomposable injective module. An endomorphism f of E is called a local homothety if, for every $x \in E$, $f(x)=r_x x$ for some $r_x \in R$ which may depend on x.*

Proposition 5. *Let E be an indecomposable injective module determined by the class $\mathfrak{R}$. Then the following statements are equivalent:*

(*i*) *Every endomorphim of E is a local homothety.*

(*ii*) *If $0(x)$ is contained in $0(y)$, then $y \in Rx$.*

(*iii*) *For every irreducible ideal I belonging to $\mathfrak{R}$, the submodule annihilated by I is cyclic.*

The proof is easy and is omitted.

§2. Immediate extensions

Let v be a valuation in a field and R its valuation ring. An extension of

v, or its valuarion ring T, is said to be immediate, if the value group and the residue class field coincide with those of v respectively. The extension of v will be also written by the same symbol v. We denote by Σ and Σ' the set of irreducible indeals in R and T respectively. It is easy to show that $IT \cap R=I$ for every $I \in \Sigma$ and conversely, $(I' \cap R)\cdot T=I'$ for every $I' \in \Sigma'$. This implies that we can set up an one to one correspondence between Σ and Σ' by the map: $I \rightarrow I'=IT$. Moreover, $I \sim J$ in Σ if and only if $I' \sim J'$ in Σ'; in fact $I=aJ$ implies $I'=aJ'$ and conversely, since every element a' of T is of the form $a'=u'a$, $a \in R$ and u' a unit of T, $I=I' \cap R=a'J' \cap R=aJ' \cap R=aJ$. Furthermore, let P be the associated prime ideal of I. Then it is easy to see that, by using Proposition 3, the associated prime ideal of I' is P'. Thus we can identify the set of classes $\Sigma/\sim$ with $\Sigma'/\sim$.

We put $E=E(R/I)$ and $E'=E(T/I')$, where I' is the ideal obtained by lifting I to T and $E(T/I')$ is the injective envelope of the T-module T/I'. In order to investigate connections between E and E', we need some definitions. First we borrow from I. Kaplansky [3] the following two definitions.

Definition 2. *A well-ordered set $\{a_\rho\}$ of elements in R, without a last term, is said to be pseudo-convergent if and only if $v(a_\sigma - a_\rho) < v(a_\tau - a_\sigma)$ for all $\rho < \sigma < \tau$.*

If $\{a_\rho\}$ is pseudo-convergent, then $v(a_\sigma - a_\rho)=v(a_{\rho+1}-a_\rho)$ for all $\rho<\sigma$; therefore, for fixed ρ, $v(a_\sigma - a_\rho)$ is independent on the ohoice of $\sigma>\rho$ ([3], Lemma 2). We denote it by γ_ρ; $\{\gamma_\rho\}$ is a monotone increasing set of elements in the value group Γ.

Definition 3. *An element x' of T is a limit of the pseudo-convergent set $\{a_\rho\}$ in R if and only if $v(x'-a_\rho)=\gamma_\rho$ for every ρ.*

We now give the notion of breadth in a slightly modified from.

Definition 4. *Let T be an immediate extension of R. Let a' be an element in T but not in R; then the ideal $B(a')=\{r' \in T;\ v(r')>v(a'-a)$ for every $a \in R\}$ is called the breadth of a'. For an element a of R the breadth $B(a)$ of a is zero.*

Remark 1. We also employ the notion of breadth of a pseudo-convergent set $\{a_\rho\}$ in R, which is given in [3] as the ideal in R cosisting of elements a of R such that $v(a)>\gamma_\rho$ for every ρ (cf. Def. 3). For an element a' in T but not in R, T being an immediate extension of R, our definition of the breadth $B(a')$ coincides with the above one in the following sense. We can find a pseudo-convergent set $\{a_\rho\}$ in R, which has a' as a limit but no limits in R ([3], Theorem 1); then $B(a')$ is equal to the breadth of $\{a_\rho\}$[3]. In fact, it is clear that $B(a')$ is contained in the breadth of $\{a_\rho\}$ and conversely, if there is an

3) We say that an ideal I' in T is equal to an ideal I in R if $I'=IT$ or equivalently $I' \cap R=I$. In what follows we shall often use this convension.

element r in the breadth of $\{a_\rho\}$ but not in $B(a')$, then $v(r)\leq v(a'-a)$ for some $a\in R$ and $v(a-a_\rho)=v(a-a'+a'-a_\rho)=v(a'-a_\rho)$, which implies that a is a limit of $\{a_\rho\}$.

PROPOSITION 6. *Let r' and s' be elements of T. Then*

(1) *If $B(s')=0$, then $B(r'+s')=B(r')$; in particular, $B(r'+s)=B(r')$ for $s\in R$.*

(2) *$B(sr')=sB(r')$ for $s\in R$.*

These relations can be obtained easily from definitions.

Now let I be an ideal in R whose associated prime ideal is maximal; let I' be the corresponding ideal in T (namely $I'=IT$). Then the associated prime ideal of I' is also maximal. We put $E=E(R/I)$ and $E'=E(T/I')$. We are now giong to prove that E is isomorphic to E' as R-modules.

PROPOSITION 7. *Let x be a non-zero element in E', and r' an element in T. We denote by Rx the R-submodule, generated by x, of the R-module E'. Then $r'x\in Rx$ if and only if $B(r')\subsetneqq 0(x)$.*

PROOF. Suppose first $r'x\in Rx$. Then $r'x=ax$ for some $a\in R$ and therefore $r'-a\in 0(x)$. Since every element a' in $B(r')$ satisfies the relation $v(a')>v(r'-a)$, we have $B(r')\subsetneqq 0(x)$. Conversely, if $B(r')$ is properly contained in $0(x)$, then we can find an element a' in $0(x)$ but not in $B(r')$. From the definition of $B(r')$, $v(a')\leq v(r'-a)$ for some $a\in R$. Therefore $r'-a\in 0(x)$ and $r'x=ax\in Rx$.

PROPOSITION 8. *The notation being as above, we suppose $y=r'x\notin Rx$, namely $B(r')\supseteq 0(x)$. Let $\{a_\rho\}$ be a pseudo-convergent set, having r' as a limit but no limits in R; we put $b_\rho=a_{\rho+1}-a_\rho$ (the value of b_ρ is γ_ρ, see Definition 2). We denote by $0(\bar{y})$ the order ideal of y mod Rx, i.e. $0(\bar{y})=\{r\in R; ry\in Rx\}$. Then we have*

$$0(\bar{y})=\bigcup_\rho b_\rho^{-1}0(x).$$

PROOF. Let r be a non-zero element of $0(\bar{y})$; then $rr'x\in Rx$, whence $B(rr')=rB(r')\subsetneqq 0(x)$ by Proposition 6 and Proposition 7. Therefore, $r^{-1}0(x)$ $\supsetneqq B(r')$ and, since $B(r')$ coincides with the breadth of $\{a_\rho\}$, $r^{-1}0(x)$ contains b_ρ for some ρ, namely r belongs to $b_\rho^{-1}0(x)$. Conversely, since $B(r')$ is equal to the breadth of $\{a_\rho\}$, it is clear that $b_\rho^{-1}0(x)\cdot B(r')\subset 0(x)$ for every ρ. If $b_\rho^{-1}0(x)\cdot B(r')=0(x)$, then $b_\sigma^{-1}0(x)\cdot B(r')$ is also equal to $0(x)$ for $\sigma>\rho$; this implies that $b_\rho^{-1}b_\sigma\cdot 0(x)=0(x)$, which contradicts to the assumption that the associated prime ideal is maximal. Therefore, $b_\rho^{-1}0(x)\cdot B(r')$ is properly contained in $0(x)$. Now let r be any element in $b_\rho^{-1}0(x)$; we take an element $s\in 0(x)$ which is not in $rB(r')=B(rr')$. Then $v(s)\leq v(rr'-a)$ for some element a in R. Hence, $rr'-a\in 0(x)$ and $rr'x=ax$, which completes the proof.

Remark 2. Let I be an ideal in R. When I is a principal ideal: $I=(a)$, then we denote by I^0 the ideal $\{r \in R; v(r)>v(a)\}$ and otherwise we put $I^0=I$. Then we can show that the ideal $0(\bar{y})$ is also equal to $(0(x)\cdot B(r'))^0$ in a similar way.

Corollary. $0(\bar{y}) \supsetneqq 0(x)$.

Now we can state the following

Theorem 1. *The notation being as above, E and E' are isomorphic to each other as R-modules.*

Proof. It is easy to see that, since T is R-flat, every injective T-module is also R-injective; in particular, E' is R-injective. Therefore, it suffices to prove that E' is co-irreducible as an R-module. Let x and y be non-zero elements in E'. Since E' is co-irreducible as a T-module, we can find $t_i'(i=1, 2)$ in T so that $t_1'x=t_2'y\neq 0$. We put $t_i'=u_i'\cdot r_i$ for $i=1, 2$, where $u_i'(i=1, 2)$ are units in T and $r_i(i=1, 2)$ elements in R; then $u_2'^{-1}u_1'\cdot r_1x=r_2y\neq 0$. Applying the above Corollary to the unit $u_2'^{-1}u_1'$ and the non-zero element r_1x, we see that $Rx\cap Ry\neq 0$.

Theorem 1 means the the ring T is naturally embedded in the ring $\mathrm{End}_R(E)$ of endomorphisms of the R-module E.

§3. Pre-maximal valuation rings

A valuation ring R is said to be maximal if it admits no proper immediate extensions; R is maximal if and only if every pseudo-convergent set in R has a limit in R ([3], Theorem 4). It is also well known that maximality is equivalent to the linear compactness[4)] of R (Zelinsky [4]), and that a valuation ring admits at least one maximal immediate extension. We introduce here a weaker condition as follows:

Definition 5. *A valuation ring R is said to be premaximal if every pseudo-convergent set in R, whose breadth is not zero, has a limit in R.*

For instance, a discrete valuation ring of rank 1 is always pre-maximal. Let T be a proper immediate extension of a pre-maximal valuation ring R and a' be an element in T but not in R; then the breadth $B(a')$ of a' must be zero. It is not so difficult to see that a valuation ring R is pre-maximal if and only if R is linearly pre-compact[5)]; however we do not need this fact.

Now, as in the preceding §, we put $E=E(R/I)$, where we assume that the

4), 5) For the definitions of linear compactness and of linear pre-compactness, see [2].

associated prime ideal of I is maximal in R (R may not be pre-maximal). Let x and y be non-zero elements in E such that $0(x)=0(y)$. Suppose now that $y \notin Rx$; we denote by $0(\bar{y})$ the order of y mod Rx, i.e. $0(\bar{y})=\{r \in R; ry \in Rx\}$. Then $0(\bar{y})$ is a proper ideal in R and, since E is co-irreducible, $0(x)$ is properly contained in $0(\bar{y})$. For every r in $0(\bar{y})$ but not in $0(x)$, $ry=sx\neq 0$ for some $s \in R$; this equality implies $r^{-1}0(x)=s^{-1}0(x)$ and therefore, by the maximality of the associated prime ideal, we can obtain $s=ru$, u a unit in R. If $0(\bar{y})$ is finitely generated, then $0(\bar{y})$ is principal, namely $0(\bar{y})=(r)$, $r \notin 0(x)$; the above discussion shows that $0(\bar{y})=0(y-ux)$ and this implies that $Rx\cap R(y-ux)=0$, which contradicts to co-irreducibility of E. Hence $0(\bar{y})$ is not finitely generated and, for every r in $0(\bar{y})$ but not in $0(x)$, we can find a unit u in R so that $r \in 0(y-ux)\subsetneqq 0(\bar{y})$.

Thus we can obtain a well-ordered set $\{u_\rho\}$ of units in R so that $0(\bar{y})=\bigcup_\rho 0(y-u_\rho x)$, $0(y-u_\rho x)\subsetneqq 0(y-u_\sigma x)$ for $\rho<\sigma$. The $\{u_\rho\}$ is a pseudo-convergent set without a limit in R. To show this, for $\rho<\sigma<\tau$, we choose an element r_σ in $0(y-u_\sigma x)$ but not in $0(y-u_\rho x)$; then $r_\sigma y=r_\sigma u_\sigma x=r_\sigma u_\tau x$ whence

$$u_\tau\equiv u_\sigma \pmod{r_\sigma^{-1}0(x)};$$

on the other hand, if $u_\sigma\equiv u_\rho \pmod{r_\sigma^{-1}0(x)}$, then $r_\sigma u_\sigma x=r_\sigma u_\rho x$, which implies $r_\sigma y=r_\sigma u_\rho x$, a contradiction; therefore

$$u_\sigma\not\equiv u_\rho \pmod{r_\sigma^{-1}0(x)}$$

and this implies $v(u_\sigma-u_\rho)<v(u_\tau-u_\sigma)$. It is clear, from the construction of $\{u_\rho\}$, that $\{u_\rho\}$ has not a last term. It remains to show that $\{u_\rho\}$ has no limits in R; if $\{u_\rho\}$ had a limit u in R, then the relation $(u_\sigma-u_\tau)r_\sigma x=0$ would yield the relation $(u_\sigma-u)r_\sigma x=0$, which implies $r_\sigma y=r_\sigma u_\sigma x=r_\sigma ux$ and $0(\bar{y})=0(y-ux)$, a contradiction. It is also clear that $u_\tau\not\equiv u_\sigma \pmod{0(x)}$, for otherwise $0(y-u_\sigma x)=0(y-u_\tau x)$, which contradicts to the assumption; this implies that the breadth of $\{u_\rho\}$ contains $0(x)$ and, therefore, is not zero.

Now suppose that R is pre-maximal. Then, the above discussions show that, for non-zero elements x, y in E with the same order ideals, $Rx=Ry$; more generally, if $0(x)$ is contained in $0(y)$, then $0(x)=r0(y)$ for some r in R and $Rx\supset Ry$. Therefore, if R is a pre-maximal valuation ring, then E satisfies the condition (2) of Proposition 5 and every endomorphism of E is a local homothety.

Next, we determine the ring $\mathrm{End}_R(E)$ of endomorphisms of E under the same condition that R is pre-maximal. Let f be a non-zero endomorphism of E which is not a homothety. We show that we can construct a pseudo-convergent set attached to f. First choose a non-zero element x_1 in E; then, since f is a local homothety, $f(x_1)=r_1x_1$ for r_1 in R, and if we put $M_1=\{x \in E; f(x)=r_1x\}$, then M_1 is a submodule containing x_1. By the assumption that f is not a homothety, M_1 is a proper submodule; we can choose an element x_2

of E, which does not belong to M_1, and obtain an element r_2 of R and a submodule M_2 in a similar manner. Continuing this process, we can obtain r_ρ, x_ρ and M_ρ for each ordinal ρ so that $E=\bigcup_\rho M_\rho$; it should be observed that the sequence of M_ρ's is monotonously increasing, namely $M_\rho \subsetneqq M_\sigma$ for $\rho<\sigma$. Therefore, for $\rho<\sigma<\tau$, we have $f(x_\rho)=r_\rho x_\rho=r_\sigma x_\rho$; hence $r_\sigma \equiv r_\rho \pmod{0(x_\rho)}$ and similarly $r_\tau \equiv r_\sigma \pmod{0(x_\sigma)}$. Since $0(x_\rho)\supsetneqq 0(x_\sigma)$ and $r_\sigma \not\equiv r_\rho \pmod{0(x_\sigma)}$, we have $v(r_\sigma - r_\rho)<v(r_\tau - r_\sigma)$. Again from the assumption that f is not a homothety, the set $\{r_\rho\}$ can not have a last term; thus the set $\{r_\rho\}$ is a pseudo-convergent set, whose breadth is clearly zero.

It is well known that any valuation ring has at least one maximal immediate extension which we denote by $\tilde{R}$. The pseudo-convergent set $\{r_\rho\}$ has a limit $\tilde{r}$ in $\tilde{R}$, which is uniquely determined since the breadth is zero, and it should be observed that the limit $\tilde{r}$ can not be an element in R, for otherwise f would become a homothety. As we discussed at the end of §2, the ring $\tilde{R}$ can be embedded naturally in $\mathrm{End}_R(E)$; in this sense we know that $f=\tilde{r}$. Now we can state the following

Proposition 9. *If R is a pre-maximal valuation ring, then* $\mathrm{End}_R(E)=\tilde{R}$.

Generally a maximal immediate extension of a valuation ring need not be unique; however the above proposition implies that it is unique for a pre-maximal valuation ring.

We add here one more corollary. Let R be a valution ring (not necessarily pre-maximal) and $\tilde{R}$ be its maximal immediate extension. As before, we may consider $\tilde{R}$ as a subring of $\mathrm{End}_R(E)$. We denote by $V(\tilde{R})$ the subalgebra of $\mathrm{End}_R(E)$ consisting of endomorphisms which commute with every element in $\tilde{R}$. Then, Proposition 9, combining with discussions in §2, leads to the following

Corollary. $V(\tilde{R})=\tilde{R}$.

§4. General valuation rings

We consider a valuation ring with a value group Γ; generally the rank of Γ, i. e. the cardinality of the set of isolated subgroups of Γ, may not be finite. Such one is never artificial but could appear in natural objects; for instance K. Aoyama treated the case of rank $2^{\aleph_1}$ in the course of studying the ring of entire funitions (see [1]). As in §3, let E be an indecomposable injective module over R, whose associated prime ideal is maximal, and let $\tilde{R}$ be a maximal immediate extension of R; then E has naturally an $\tilde{R}$-module structure.

We can choose a well-ordered set $\{\tilde{I}_\rho\}$ of ideals in $\tilde{R}$, belonging to the class determined by E, such that $\tilde{I}_\rho \supsetneqq \tilde{I}_\sigma$ for $\rho<\sigma$ and $\bigcap_\rho \tilde{I}_\rho=0$. We are going to construct a transfinite set $\{x_\rho\}$ of points of E and a set $\{t_{\rho,\sigma};\ \rho<\sigma\}$ of elements of R with the following properties:

(i) $0(x_\rho)=I_\rho$, where I_ρ is the ideal in R corresponding to $\tilde{I}_\rho$, i.e. $I_\rho=\tilde{I}_\rho \cap R$ and $Rx_\rho \subsetneqq Rx_\sigma$ for $\rho<\sigma$;

(ii) $x_\rho=t_{\rho,\sigma}x_\sigma$ and $t_{\rho,\tau}=t_{\rho,\sigma}t_{\sigma,\tau}$ for $\rho<\sigma<\tau$.

For x_1, we choose any point of E with the order ideal I_1, and for $t_{1,2}$, we choose any element of R such that $I_2=t_{1,2}I_1$, and so on. Suppose, for an ordinal λ, x_ρ and $t_{\rho,\sigma}$ have been chosen for all ρ, $\sigma<\lambda$ so as to satisfy the above conditions. First when λ is not a limit ordinal, then $x_{\lambda-1}$ is given; we choose an element $t_{\lambda-1,\lambda}$ of R so that $I_\lambda=t_{\lambda-1,\lambda}I_{\lambda-1}$ and choose a point x_λ in E such that $x_{\lambda-1}=t_{\lambda-1,\lambda}x_\lambda$. For $\rho<\lambda$, $x_\rho=t_{\rho,\lambda-1}x_{\lambda-1}=t_{\rho,\lambda-1}t_{\lambda-1,\lambda}x_\lambda$; therefore, if we put $t_{\rho,\lambda}=t_{\rho,\lambda-1}t_{\lambda-1,\lambda}$, then $t_{\rho,\lambda}$'s clearly satisfy the condition (ii). Now we proceed to the case when λ is a limit ordinal. Let $t_{1,\lambda}$ be an element of R such that $I_\lambda=t_{1,\lambda}I_1$ and y be a point of E such that $x_1=t_{1,\lambda}y$. For any $\rho<\lambda$, there is a point y' of E such that $x_\rho=\frac{t_{1,\lambda}}{t_{1,\rho}}y'$. Since the order ideals of y and y' are I_λ, we have $\tilde{R}y'=\tilde{R}y$; this implies that $y'=\tilde{u}_\rho y$ for some unit $\tilde{u}_\rho$ in $\tilde{R}$, namely there is a unit $\tilde{u}_\rho$ such that $x_\rho=\frac{t_{1,\lambda}}{t_{1,\rho}}\tilde{u}_\rho y$ for every $\rho<\lambda$. For $\rho<\sigma$, $x_\sigma=\frac{t_{1,\lambda}}{t_{1,\sigma}}\tilde{u}_\sigma y$ and hence $x_\rho=t_{\rho,\sigma}x_\sigma=\frac{t_{\rho,\sigma}t_{1,\lambda}}{t_{1,\sigma}}\tilde{u}_\sigma y=\frac{t_{1,\lambda}}{t_{1,\rho}}\tilde{u}_\sigma y$; therefore $\tilde{u}_\sigma\equiv\tilde{u}_\rho(\mathrm{mod}\,\tilde{I}_\rho)$. Thus we have a system of congruence equations:

$$(*)\qquad \Xi\equiv\tilde{u}_\rho(\mathrm{mod}\,\tilde{I}_\rho),$$

which is finitely solvable. Since $\tilde{R}$ is linearly compact, $(*)$ has the unique solution $\tilde{u}$ (the uniqueness comes from the fact that $\bigcap_\rho \tilde{I}_\rho=0$.) Clearly $x_\rho=\frac{t_{1,\lambda}}{t_{1,\rho}}\tilde{u}\,y$ for every $\rho<\lambda$; we put $x_\lambda=\tilde{u}\,y$ and $t_{\rho,\lambda}=\frac{t_{1,\lambda}}{t_{1,\rho}}$. The $t_{\rho,\lambda}$'s and x_λ satisfy the conditions (i) and (ii).

We are going to determine the ring $\mathrm{End}_R(E)$ of endomorphisms of the R-module E. Let φ be an endomorphism of E. Since the order ideal of $\varphi(x_\rho)$ contains that of x_ρ, namely I_ρ, we see that $\varphi(\tilde{R}x_\rho)$ is a submodule of $\tilde{R}x_\rho$. Therefore, for any but fixed element $\tilde{r}$ of $\tilde{R}$, $\varphi(\tilde{r}x_\rho)=\tilde{r}_\rho x_\rho$, $\tilde{r}_\rho\in\tilde{R}$. For $\rho<\sigma$, muitiplying $t_{\rho,\sigma}$ to both sides of the equatity $\varphi(\tilde{r}x_\sigma)=\tilde{r}_\sigma x_\sigma$, we have $\varphi(\tilde{r}x_\rho)=\tilde{r}_\sigma x_\rho$; therefore $\tilde{r}_\sigma\equiv\tilde{r}_\rho(\mathrm{mod}\ \tilde{I}_\rho)$ for $\rho<\sigma$. Similarly as the construchion of the set $\{x_\rho\}$, the system of congruence equations : $\Xi\equiv\tilde{r}_\rho(\mathrm{mod}\ \tilde{I}_\rho)$ has the unique solution, which we denote by $\tilde{\varphi}(\tilde{r})$; the map $\tilde{\varphi}:\tilde{R}\ni\tilde{r}\to\tilde{\varphi}(\tilde{r})\in\tilde{R}$ is chatacterized by the formula:

$$(**)\qquad \varphi(\tilde{r}x_\rho)=\tilde{\varphi}(\tilde{r})x_\rho \text{ for every } \rho.$$

By $(**)$ we see immediately that $\tilde{\varphi}$ is an endomorphism of the R-module $\tilde{R}$. We then obtain the map θ: $\mathrm{End}_R(E)\ni\varphi\to\tilde{\varphi}\in\mathrm{End}_R(\tilde{R})$. Again, by virtue of

(**), it is easy to see that θ is an algebra homomorphism over R, namely $\theta(\varphi+\psi)=\theta(\varphi)+\theta(\psi)$, $\theta(\psi\varphi)=\theta(\psi)\theta(\varphi)$ for φ, $\psi \in \mathrm{End}_R(E)$ and $\theta(r\varphi)=r\theta(\varphi)$ for $r \in R$, $\varphi \in \mathrm{End}_R(E)$. Now let φ be an endomorphism of E belonging to the kernel of θ; this implies that $\varphi(\tilde{r}x_\rho)=\tilde{\varphi}(\tilde{r})x_\rho=0$ for every $\tilde{r} \in \tilde{R}$ and ρ and, since $\bigcup_\rho \tilde{R}x_\rho=E$, $\varphi(E)=0$. Thus θ is a monomorphism. Finally we show that θ is an epimorphism. Let ξ be an endomorphism of the R-module $\tilde{R}$. Since every element $\tilde{r}$ of $\tilde{R}$ is of the form: $\tilde{r}=r\tilde{u}$, $r \in R$, $\tilde{u}$ *a* unit of $\tilde{R}$, we see that $\xi(\tilde{I}_\rho)\subset\tilde{I}_\rho$ for all ρ. We define a map $\varphi: E\to E$ by putting $\varphi(\tilde{r}x_\rho)=\xi(\tilde{r})x_\rho$; if $\tilde{r}x_\rho=\tilde{s}x_\sigma$ for $\rho<\sigma$, then $\tilde{r}x_\rho=\tilde{r}t_{\rho,\sigma}x_\sigma$, whence $\tilde{r}t_{\rho,\sigma}\equiv\tilde{s}$ (mod $\tilde{I}_\sigma$), which implies that $\xi(\tilde{s})x_\sigma=\xi(\tilde{r})\cdot t_{\rho,\sigma}x_\sigma=\xi(\tilde{r})x_\rho$. Thus the map is well-defined. The same technique can be applied to show that φ is an endomorphism of the R-module E.

We can now state the following

Theorem 2. *The notation being as above, θ is an isomorphism of the R-algebra $\mathrm{End}_R(E)$ onto the R-algebra $\mathrm{End}_R(\tilde{R})$.*

The above theorem tells us that, although there are many indecomposable injective modules, not isomorphic to each other, whose associated prime ideals are maximal, their rings of endomorphisms are the same; on ther hand $\tilde{R}$, a maximal immediate extension of R, is not necessarily unique, but the algebra $\mathrm{End}_R(\tilde{R})$ does not depend on the choice of $\tilde{R}$.

A domain R is called Prüfer if every localization R_P, P being a prime ideal, becomes a valuation ring. Combining Theorem 2 with discussions in §1, we can obtain our main theorem.

Theorem 3. *Let R be a prüfer domain and E an indecomposable injective module over R with the associated prime ideal P. Let $\tilde{R}_P$ be a maximal immediate extension of R_P. Then the ring $End_R(E)$ of endomorphisms of E is isomorphic to the ring $End_{R_p}(\widetilde{R_P})$ of endomorphisms of the R_P-module $\widetilde{R_P}$.*

In case when R is the ring Z of rational integers, E is the p-component of the additive group Q/Z for some prime p and $\widetilde{Z_{(p)}}$ is the ring of p-adic integers. It is easy to see that $\mathrm{End}_{Z_{(p)}}(\widetilde{Z_{(p)}})$ is isomorphic to $\widetilde{Z_{(p)}}$ itself; therefore $\mathrm{End}(E)$ is isomorphic to $\widetilde{Z_{(p)}}$; this is a well-known theorem in the theory of abelian groups.

§ 5. Local homotheties

Let R be a valuation ring and $\tilde{R}$ be a maximal immediate extension of R; let E be an indecomposable injective module over R, whose associated prime ideal is maximal.

We consider an endomorphism φ of the R-module E and denote by $\tilde{\varphi}$ the corresponding one of the R-module $\tilde{R}$, which is defined in the preceding §. Let $\tilde{r}$ be a non-zreo element of $\tilde{R}$; then $\tilde{r}=r\tilde{u}$, where $r \in R$ and $\tilde{u}$ a unit of $\tilde{R}$. We then have $\frac{\tilde{\varphi}(\tilde{r})}{\tilde{r}}=\frac{\tilde{\varphi}(\tilde{u})}{\tilde{u}}$ and this implies that $\frac{\tilde{\varphi}(\tilde{r})}{\tilde{r}}$ is an element of $\tilde{R}$.

PROPOSITION 10. *For an endomorphism φ of E, the following statements are equivalent:*

(*i*) *φ is a local homothety.*

(*ii*) $B\left(\frac{\tilde{\varphi}(\tilde{r})}{\tilde{r}}\right)=0$ *for every non-zero element $\tilde{r}$ of $\tilde{R}$.*

(*iii*) $B\left(\frac{\tilde{\varphi}(\tilde{u})}{\tilde{u}}\right)=0$ *for every unit $\tilde{u} \in \tilde{R}$.*

PROOF. Suppose first that φ is a local homothety. Then, the notation being as in §5, we have $\varphi(\tilde{r}x_\rho)=\tilde{\varphi}(\tilde{r})x_\rho=r_\rho\cdot\tilde{r}x_\rho$ for some $r_\rho \in R$ and therefore $\tilde{\varphi}(\tilde{r})\equiv r_\rho\tilde{r} \pmod{\tilde{I}_\rho}$, which leads to the assertion (ii).

Conversely suppose that the breadth $B\left(\frac{\tilde{\varphi}(\tilde{r})}{\tilde{r}}\right)$ of $\frac{\tilde{\varphi}(\tilde{r})}{\tilde{r}}$ is zero for every $\tilde{r} \in \tilde{R}$. We can take an element r_ρ of R so that $\tilde{\varphi}(\tilde{r})\equiv r_\rho\tilde{r} \pmod{\tilde{I}_\rho}$ for each ρ. Then $\varphi(\tilde{r}x_\rho)=\tilde{\varphi}(\tilde{r})x_\rho=r_\rho\tilde{r}x_\rho$. This completes the proof.

COROLLARY. *An element $\tilde{r}$ of $\tilde{R}$ is a local homothety if and only if $B(\tilde{r})=0$.*

THEOREM 4. *A local homothety is an element of $\tilde{R}$ whose breadth is zero.*

PROOF. Let φ be a local homothety. Then, Proposition 10 shows that $\tilde{\varphi}(\tilde{r})=\tilde{a}\tilde{r}$, where $\tilde{a}$ is an element of $\tilde{R}$ whose breadth is zero. We have to show that $\tilde{a}$ does not depend on the choice of $\tilde{r} \in \tilde{R}$.

Let $\tilde{b}$ be an element of $\tilde{R}$ whose breadth is zero and ξ be any endomorphism of the R-module $\tilde{R}$. Then, for every ρ, we can find an element b_ρ of R such that $\tilde{b}\equiv b_\rho \pmod{\tilde{I}_\rho}$. Since $\xi(\tilde{I}_\rho)$ is contained in $\tilde{I}_\rho$, $\xi(\tilde{b}\tilde{r})\equiv b_\rho\xi(\tilde{r}) \pmod{\tilde{I}_\rho}$; therefore $\xi(\tilde{b}\tilde{r})\equiv \tilde{b}\xi(\tilde{r}) \pmod{\tilde{I}_\rho}$) for every ρ and this implies that $\tilde{b}$ commute with any endomorphism ξ.

Let $\tilde{s}$ be an element of $\tilde{R}$; then $\tilde{\varphi}(\tilde{s})=\tilde{b}\tilde{s}$ and $\tilde{\varphi}(\tilde{r}+\tilde{s})=\tilde{c}(\tilde{r}+\tilde{s})$, where $\tilde{b}$ and $\tilde{c}$ are elements of $\tilde{R}$ whose breadths are zero. Since $\tilde{\varphi}$ is linear, we see that $(\tilde{a}-\tilde{c})\tilde{r}+(\tilde{b}-\tilde{c})\tilde{s}=0$; here we should note that $B(\tilde{a}-\tilde{c})=B(\tilde{b}-\tilde{c})=0$. By what we have shown above, $\tilde{\varphi}$ commutes with $\tilde{a}-\tilde{c}$ and $\tilde{b}-\tilde{c}$; hence $(\tilde{a}-\tilde{c})\tilde{a}\tilde{r}+(\tilde{b}-\tilde{c})\tilde{b}\tilde{s}=0$. From these two relations we can easily show that $\tilde{a}=\tilde{b}$. Thus we can conclude that the set of local homotheties coincide with that of elements of $\tilde{R}$ whose breadths are zero.

We are now going to determine the center C of the algebra $\mathrm{End}_R(E)$. By Corollary at the end of §3, we see that C is contained in $\tilde{R}$, and also, by the proof of Theorem 4, every element of $\tilde{R}$ whose breadth is zero belongs to the center C.

Let $\tilde{u}$ be a unit of $\tilde{R}$ such that $B(\tilde{u})$ is not zero. We choose a point x of E so that $0(x)\subsetneqq B(\tilde{u})$. Then Proposition 7 shows that $\tilde{u}x \notin Rx$. We define an R-homomorphism of the submodule $Rx+R\tilde{u}x$, generated by x and $\tilde{u}x$, into E by putting that $\varphi(x)=x$, $\varphi(\tilde{u}x)=\tilde{u}x+\tilde{\lambda}x$ and $\varphi(rx+s\tilde{u}x)=r\varphi(x)+s\varphi(\tilde{u}x)$, where r, $s \in R$ and $\tilde{\lambda}$ is any fixed element in $B(\tilde{u})$ but not in $0(x)$. We note that φ is well-defined; in fact, if $rx+s\tilde{u}x=r'x+s'\tilde{u}x$, then $(r-r')x=(s'-s)\tilde{u}x$ and $s'-s$ belongs to the order ideal $0(\overline{\tilde{u}x})$ of $\tilde{u}x$ modulo Rx; by Proposition 8, $(s-s')\tilde{\lambda} \in 0(x)$, whence $s\tilde{\lambda}x=s'\tilde{\lambda}x$. Since E is an injective module, φ can be extended to an endomorphism of E, which we denote by the same symbol φ. From the definition of φ, $\varphi(\tilde{u}x)=\tilde{u}x+\tilde{\lambda}x$ and $\tilde{u}\varphi(x)=\tilde{u}x$; since $\tilde{\lambda}x\neq 0$, $\varphi(\tilde{u}x)\neq\tilde{u}\varphi(x)$. This means that φ does not commute with $\tilde{u}$, namely $\tilde{u} \notin C$. Now, let $\tilde{r}$ be an element of $\tilde{R}$ such that $B(\tilde{r})\neq 0$. We can write $\tilde{r}$ in the form $\tilde{r}=r\tilde{u}$, $r \in R$ and $\tilde{u}$ a unit in $\tilde{R}$. We take an endomorphism φ so that $\varphi\tilde{u}\neq\tilde{u}\varphi$. Then $\varphi\tilde{r}-\tilde{r}\varphi=r(\varphi\tilde{u}-\tilde{u}\varphi)$ and, noting that the homothety r is surjective, we have $\varphi\tilde{r}\neq\tilde{r}\varphi$. Thus we have shown that the set of elements of $\tilde{R}$ whose breadths are zero coincide with the center C. It is easy to see that C is an immediate extension of R contained in $\tilde{R}$. We can now state the following

THEOREM 5. *Let R be a valuation ring and E an indecomposable injective module whose associated prime is maximal. Then the center C of* $\mathrm{End}_R(E)$ *coincides with the set of elements in $\tilde{R}$ whose breadths are zero; moreover C is a valuation ring which is immediate over R and also C coincides with the set of local homotheties.*

If we call the immediate extension, consisting of elements of $\tilde{R}$ whose breadths are zero, the π-completion of R, then Theorem 5 shows that the π-completion of R is unique and is realized as the center of $\mathrm{End}_R(E)$.

COROLLARY. *The notation being as above, the following statements are equivalent:*

(*i*) *R is pre-maximal*
(*ii*) $\mathrm{End}_R(E)=\tilde{R}$
(*iii*) $\mathrm{End}_R(E)$ *is commutative*
(*iv*) *the π-completion of R coincides with a maximal immediate extension.*

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