Hiroshima Math. J.
2 (1972), 253–261

Ascendantly Coalescent Classes and Radicals of Lie Algebras

Shigeaki Tôgô and Naoki Kawamoto

(Received September 6, 1972)

Introduction

Recently the Lie algebras of infinite dimension have been investigated by several mathematicians in the papers [1]–[11]. By making use of the concepts of subideals, ascendant subalgebras, coalescency and ascendant coalescency, B. Hartley [1] and I. Stewart [7] have defined and studied locally nilpotent radicals of a Lie algebra as the Lie analogues of radicals in the infinite group theory. In [11] the senior author of the present paper has obtained more classes of coalescency and has mainly investigated locally solvable radicals of a Lie algebra which correspond to the solvable radical in finite-dimensional case. However, he there restricted himself to the radicals connected with ideals, subideals and coalescency, and he did not take up the radicals which may be defined in connection with ascendant subalgebras and ascendant coalescency.

The purpose of this paper is to study the ascendantly coalescent classes of Lie algebras and to add to the radicals in [11] two locally solvable radicals of a Lie algebra L, which reduce to the solvable radical in finite-dimensional case, corresponding to the two locally nilpotent radicals of L in [1], the Gruenberg radical $\gamma(L)$ and the Hirsch-Plotkin radical $\rho(L)$.

In Section 2, we first remark that if H is a finite-dimensional ascendant subalgebra of a Lie algebra L then $H^{(\omega)}$ and H^{ω} are characteristic ideals of L (Lemma 2.1) and show the results on ascendantly coalescent classes of Lie algebras corresponding to Theorems 4.1, 4.2 and 4.3 in [11], especially the ascendant coalescency of the class $\mathfrak{S}\cap\mathfrak{F}$ of finite-dimensional solvable Lie algebras and of the class $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$ of finite-dimensional Lie algebras K such that $K/K^{(\omega)}$ are nilpotent (Theorem 2.2 and Corollary 2.3). Furthermore by making use of a result in [10], we shall obtain several new coalescent and ascendantly coalescent classes of Lie algebras (Theorems 2.4 and 2.5). In Section 3, we define the radical $\mathrm{Rad}_{\mathfrak{X}}(L)$ of L for an N_0-closed class $\mathfrak{X}$, that is, a class $\mathfrak{X}$ such that the sum of any two $\mathfrak{X}$ ideals of any Lie algebra belongs to $\mathfrak{X}$ (Definition 3.1), and the radical $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$) of L for an ascendantly coalescent (resp. a coalescent) class $\mathfrak{X}$ (Definition 3.3). In Section 4, observing that $\gamma(L)$ can be written as $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}\text{-asc}}(L)$ by our notation, we consider the locally solvable radical $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$. $\rho(L)$ can be expressed as

$\mathrm{Rad}_{\mathrm{L}\mathfrak{N}}(L)$ by our notation, but it is not known whether $\mathrm{L}\mathfrak{S}$ is N_0-closed or not. However $\rho(L)$ can be shown to be written as $\mathrm{Rad}_{\mathrm{L}(\mathfrak{N}\cap\mathfrak{F})}(L)$ (Lemma 4.1) and $\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})$ is N_0-closed. Thus we consider $\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$ as the locally solvable radical corresponding to $\rho(L)$. It is known [1] that $\gamma(L)$ is not necessarily an ideal of L and that the ideal $\rho(L)$ is not necessarily a characteristic ideal of L. We show that $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{asc}}(L)$ and $\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$ have respectively the analogous properties (Theorems 4.2 and 4.3). Finally we show that these two radicals and the seven locally solvable radicals obtained in [11] are different from each other in general (Theorem 4.4).

§ 1. Preliminaries

We shall be concerned with Lie algebras over a field Φ which are not necessarily finite-dimensional. Throughout this paper, the basic field Φ will be of arbitrary characteristic and L will be an arbitrary Lie algebra over a field Φ, unless otherwise specified.

We mainly employ the terminology and notations used in [11].

By $H\leq L$, $H\lhd L$ and H si L we mean respectively that H is a subalgebra, an ideal and a subideal of L. H is an ascendant subalgebra of L provided that there exists an ascending series $\{H_\alpha : \alpha\leq\lambda\}$ of subalgebras of L, indexed by the ordinals $\leq\lambda$, such that $H_0=H$, $H_\alpha\lhd H_{\alpha+1}$ for all $\alpha<\lambda$, $H_\mu=\bigcup_{\alpha<\mu}H_\alpha$ for all limit ordinals $\mu\leq\lambda$, and $H_\lambda=L$. We then write H asc L, more precisely H λ-asc L. Then the following fact is easily seen [1].

Lemma 1.1. (1) *If H λ-asc L and $K\leq L$, then $H\cap K$ λ-asc K.*

(2) *If H λ-asc L and $K\lhd L$, then $H+K$ λ-asc L.*

(3) *Let $\{H_\alpha : \alpha\leq\lambda\}$ be a tower of subalgebras of L, indexed by the ordinals $\alpha\leq\lambda$, with $H_\alpha\leq H_{\alpha+1}$ for $\alpha<\lambda$, $H_\mu=\bigcup_{\alpha<\mu}H_\alpha$ for limit ordinals $\mu\leq\lambda$, and $H_\lambda=L$. Suppose H_α asc $H_{\alpha+1}$ for each $\alpha<\lambda$. Then H_0 asc L.*

(4) *Let f be a homomorphism of L onto a Lie algebra $\bar{L}$. If H asc L, then $f(H)$ asc $\bar{L}$. If $\bar{H}$ asc $\bar{L}$, then $f^{-1}(\bar{H})$ asc L.*

By a class $\mathfrak{X}$ of Lie algebras we always mean a collection of Lie algebras over Φ such that $(0)\in\mathfrak{X}$ and if $H\in\mathfrak{X}$ and $H\simeq K$ then $K\in\mathfrak{X}$. A Lie algebra (resp. a subalgebra, a subideal of L) belonging to $\mathfrak{X}$ is called an $\mathfrak{X}$ algebra (resp. an $\mathfrak{X}$ subalgebra, an $\mathfrak{X}$ subideal of L).

A class $\mathfrak{X}$ of Lie algebras is ascendantly coalescent (resp. coalescent) provided that H, K asc L (resp. si L) and $H, K\in\mathfrak{X}$ imply $\langle H, K\rangle$ asc L (resp. si L) and $\langle H, K\rangle\in\mathfrak{X}$. $\mathfrak{F}$, $\mathfrak{N}$, $\mathfrak{S}$ and $\mathfrak{G}$ denote respectively the classes of finite-dimensional, nilpotent, solvable, and finitely generated Lie algebras. For a class $\mathfrak{X}$ of Lie algebras, $\mathfrak{X}_{(\omega)}$ (resp. $\mathfrak{X}_\omega$) is the class of Lie algebras L such that $L/L^{(\omega)}\in\mathfrak{X}$ (resp. $L/L^\omega\in\mathfrak{X}$), where $L^{(\omega)}=\bigcap_{n=0}^{\infty}L^{(n)}$ and $L^\omega=\bigcap_{n=1}^{\infty}L^n$. $\mathrm{L}\mathfrak{X}$ is the class of locally $\mathfrak{X}$ algebras.

We shall borrow some notations from [10]. s$\mathfrak{X}$ (resp. q$\mathfrak{X}$) is the collection of all subalgebras (resp. all quotient algebras) of $\mathfrak{X}$ algebras. $\mathfrak{X}$ is called A-closed for A=s, q provided $\mathfrak{X}=$A$\mathfrak{X}$. Then the q-closed classes are the same as the classes possessing the property (P) in Definition 3.1 in [11]. $\mathfrak{X}$ is called N_0-closed provided the sum of any two $\mathfrak{X}$ ideals of any Lie algebra belongs to $\mathfrak{X}$.

§ 2. Ascendantly coalescent classes

It is known [1] that if Φ is of characteristic 0 $\mathfrak{N}\cap\mathfrak{F}$ and $\mathfrak{F}$ are coalescent and ascendantly coalescent. Furthermore $\mathfrak{S}\cap\mathfrak{F}$, $\mathfrak{F}_{(\omega)}$, $\mathfrak{F}_{\omega}$, $(\mathfrak{N}\cap\mathfrak{F})_{(\omega)}$, $\mathfrak{S}\cap\mathfrak{F}_{\omega}$, $(\mathfrak{S}\cap\mathfrak{F}_{\omega})_{(\omega)}$, $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$, $\mathfrak{N}_{\omega}\cap\mathfrak{G}$, $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$, $(\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$, $(\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G})_{(\omega)}$ and their intersections are coalescent [11]. The two classes of them, $\mathfrak{S}\cap\mathfrak{F}$ and $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$, which are subclasses of $\mathfrak{F}$ can be shown to be ascendantly coalescent in a similar way as their coalescency has been shown in [11]. We begin with this verification.

Lemma 2.1. *If $H\in\mathfrak{F}$ and H* asc *L, then $H^{(\omega)}$ and H^{ω} are characteristic ideals of L.*

Proof. If $H\in\mathfrak{F}$ and H asc L, then for any finite-dimensional subspace M of L there exists an integer $t(M)>0$ such that $M(\mathrm{ad}\ H)^{t(M)}\subseteq H$. This can be shown by a modification of the proof of Lemma 4 in [1].

Now assume L asc K. Then H asc K. For any $x\in K$, let t be the integer $t(M)$ which exists for $M=(x)$ in K as above. Then by induction on k we obtain

$$[x, H^{(t+k)}]\subseteq H^{(k+1)} \quad\text{and}\quad [x, H^{t+k}]\subseteq H^{k+1}$$

for $k=0, 1, 2,\cdots$. It follows that

$$[x, H^{(\omega)}]\subseteq H^{(\omega)} \quad\text{and}\quad [x, H^{\omega}]\subseteq H^{\omega}.$$

Hence $H^{(\omega)}$ and H^{ω} are ideals of K and therefore ascendantly stable in L [10]. Especially, they are characteristic ideals of L.

It is to be noted that the part on H^{ω} in Lemma 2.1 is stated in [1] as a consequence of [4].

Theorem 2.2. *Let Φ be of characteristic* 0 *and let $\mathfrak{X}$ be a* q*-closed class of Lie algebras.*

(1) *If $\mathfrak{X}$ is ascendantly coalescent, so are $\mathfrak{X}_{(\omega)}\cap\mathfrak{F}$ and $\mathfrak{X}_{\omega}\cap\mathfrak{F}$.*
(2) *Let $\mathfrak{X}\subseteq\mathfrak{N}_{\omega}$. If $\mathfrak{X}$ and $\mathfrak{N}\cap\mathfrak{X}$ are ascendantly coalescent, so is $\mathfrak{S}\cap\mathfrak{X}\cap\mathfrak{F}$.*
(3) *If $\mathfrak{X}$ and $\mathfrak{N}\cap\mathfrak{X}$ are ascendantly coalescent, so is $\mathfrak{N}_{(\omega)}\cap\mathfrak{X}\cap\mathfrak{F}$.*

Proof. In the proofs of Theorems 4.1, 4.2 and 4.3 in [11], replace subideals by ascendant subalgebras, and use the ascendant coalescency of $\mathfrak{F}$, Lemmas 1.1 and 2.1 instead of Lemmas 1.4 and 1.6 there.

Corollary 2.3. *If Φ is of characteristic* 0, $\mathfrak{S}\cap\mathfrak{F}$ *and* $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$ *are ascendantly coalescent.*

Proof. Take $\mathfrak{X}=\mathfrak{F}$ in (2), (3) of Theorem 2.2. Then the ascendant coalescency of $\mathfrak{S}\cap\mathfrak{F}$ and $\mathfrak{N}_{(\omega)}\cap\mathfrak{F}$ follows from that of $\mathfrak{F}$ and $\mathfrak{N}\cap\mathfrak{F}$.

The ascendant coalescency of $\mathfrak{S}\cap\mathfrak{F}$ obtained above is a special case of the following result which has recently been shown by I. Stewart in [10, Theorem 3.4]:

If $\mathfrak{X}$ is N_0-closed, $\mathfrak{X}\cap\mathfrak{F}$ is coalescent and ascendantly coalescent.

We shall make use of this result in order to obtain more classes of coalescency and of ascendant coalescency in the following two theorems.

Theorem 2.4. *Let Φ be of characteristic* 0 *and let* $\mathfrak{X}$, $\mathfrak{X}'$ *and* $\mathfrak{X}''$ *be any classes of Lie algebras. If* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *are* $\{\mathrm{N}_0, \mathrm{Q}\}$*-closed and* $\mathfrak{X}''$ *is* N_0*-closed, then the classes*

$$\mathfrak{X}_{(\omega)}\cap\mathfrak{F},\ \mathfrak{X}_{\omega}\cap\mathfrak{F},\ \mathfrak{X}_{(\omega)}\cap\mathfrak{X}'_{\omega}\cap\mathfrak{F},$$

$$\mathfrak{X}_{(\omega)}\cap\mathfrak{X}''\cap\mathfrak{F},\ \mathfrak{X}_{\omega}\cap\mathfrak{X}''\cap\mathfrak{F},\ \mathfrak{X}_{(\omega)}\cap\mathfrak{X}'_{\omega}\cap\mathfrak{X}''\cap\mathfrak{F}$$

are coalescent and ascendantly coalescent.

Proof. Since $\mathfrak{F}\subseteq\mathfrak{F}_{(\omega)}\subseteq\mathfrak{F}_{\omega}$, it is evident that

$$(\mathfrak{X}\cap\mathfrak{F})_{(\omega)}\cap\mathfrak{F}=\mathfrak{X}_{(\omega)}\cap\mathfrak{F} \text{ and } (\mathfrak{X}\cap\mathfrak{F})_{\omega}\cap\mathfrak{F}=\mathfrak{X}_{\omega}\cap\mathfrak{F}.$$

Therefore by Theorem 2.2 (1) $\mathfrak{X}_{(\omega)}\cap\mathfrak{F}$ and $\mathfrak{X}_{\omega}\cap\mathfrak{F}$ are ascendantly coalescent. They are also coalescent, since $(\mathfrak{X}\cap\mathfrak{F})_{(\omega)}$ and $(\mathfrak{X}\cap\mathfrak{F})_{\omega}$ are coalescent by Theorem 4.1 in [11] and $\mathfrak{F}$ is coalescent. The other classes are coalescent and ascendantly coalescent as the intersections of such classes.

Theorem 2.5. *Let Φ be of characteristic* 0 *and let* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *be any* $\{\mathrm{N}_0, \mathrm{Q}\}$*-closed classes of Lie algebras. Then the classes*

$$(\mathfrak{X}\cap\mathfrak{F})_{(\omega)},\ (\mathfrak{X}\cap\mathfrak{F})_{\omega},\ (\mathfrak{X}\cap\mathfrak{F})_{\omega}\cap\mathfrak{S},\ (\mathfrak{X}\cap\mathfrak{F})_{\omega}\cap\mathfrak{S}_{(\omega)},$$

$$(\mathfrak{X}\cap\mathfrak{F})_{(\omega)}\cap\mathfrak{G},\ (\mathfrak{X}\cap\mathfrak{F})_{\omega}\cap\mathfrak{G},\ (\mathfrak{X}\cap\mathfrak{F})_{\omega}\cap\mathfrak{S}\cap\mathfrak{G},\ (\mathfrak{X}\cap\mathfrak{F})_{\omega}\cap\mathfrak{S}_{(\omega)}\cap\mathfrak{G},$$

$$(\mathfrak{X}\cap\mathfrak{F})_{\omega}\cap\mathfrak{G}_{(\omega)},\ (\mathfrak{X}\cap\mathfrak{F})_{\omega}\cap\mathfrak{S}_{(\omega)}\cap\mathfrak{G}_{(\omega)},$$

$$(\mathfrak{X}\cap\mathfrak{F})_{(\omega)}\cap(\mathfrak{X}'\cap\mathfrak{F})_{\omega},\ (\mathfrak{X}\cap\mathfrak{F})_{(\omega)}\cap(\mathfrak{X}'\cap\mathfrak{F})_{\omega}\cap\mathfrak{G}$$

are coalescent.

Proof. By Theorem 4.1 in [11] $(\mathfrak{X}\cap\mathfrak{F})_{(\omega)}$ and $(\mathfrak{X}\cap\mathfrak{F})_{\omega}$ are coalescent. $\mathfrak{N}\cap(\mathfrak{X}\cap\mathfrak{F})_{\omega}$ is coalescent, since it equals $\mathfrak{N}\cap\mathfrak{X}\cap\mathfrak{F}$ by Lemma 3.3 in [11]. Since $\mathfrak{F}\subseteq\mathfrak{N}_{\omega}$, it follows from Lemma 3.5 in [11] that

$$(\mathfrak{X}\cap\mathfrak{F})_{\omega}\subseteq\mathfrak{N}_{\omega\omega}=\mathfrak{N}_{\omega(\omega)}=\mathfrak{N}_{\omega}.$$

Therefore by Theorem 4.2 in [11] $(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{S}$ is coalescent. Owing to Lemmas 3.2 and 3.5 in [11] we have

$$((\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{S})_{(\omega)}=(\mathfrak{X}\cap\mathfrak{F})_{\omega(\omega)}\cap\mathfrak{S}_{(\omega)}=(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{S}_{(\omega)}.$$

Therefore we use Theorem 4.1 in [11] to see that $(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{S}_{(\omega)}$ is coalescent.

Now by making use of the fact that $\mathfrak{N}_\omega\cap\mathfrak{G}$, $\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}$ and $(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}$ are all coalescent (Theorem 4.4 in [11]), we see that the other classes in the statement are coalescent as the intersections of coalescent classes. In fact, by Lemma 3.5 in [11]

$$(\mathfrak{X}\cap\mathfrak{F})_{(\omega)}\subseteq\mathfrak{N}_{\omega(\omega)}=\mathfrak{N}_\omega$$

and therefore we obtain

$$\begin{aligned}
&(\mathfrak{X}\cap\mathfrak{F})_{(\omega)}\cap(\mathfrak{N}_\omega\cap\mathfrak{G})=(\mathfrak{X}\cap\mathfrak{F})_{(\omega)}\cap\mathfrak{G},\\
&(\mathfrak{X}\cap\mathfrak{F})_\omega\cap(\mathfrak{N}_\omega\cap\mathfrak{G})=(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{G},\\
&(\mathfrak{X}\cap\mathfrak{F})_\omega\cap(\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G})=(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{S}\cap\mathfrak{G},\\
&(\mathfrak{X}\cap\mathfrak{F})_\omega\cap(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}=(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{N}_{\omega(\omega)}\cap\mathfrak{G}_{(\omega)}=(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{G}_{(\omega)},\\
&(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{S}_{(\omega)}\cap(\mathfrak{N}_\omega\cap\mathfrak{G})=(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{S}_{(\omega)}\cap\mathfrak{G},\\
&(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{S}_{(\omega)}\cap(\mathfrak{N}_\omega\cap\mathfrak{G})_{(\omega)}=(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{S}_{(\omega)}\cap\mathfrak{N}_{\omega(\omega)}\cap\mathfrak{G}_{(\omega)}\\
&\qquad\qquad=(\mathfrak{X}\cap\mathfrak{F})_\omega\cap\mathfrak{S}_{(\omega)}\cap\mathfrak{G}_{(\omega)}.
\end{aligned}$$

Thus the proof is complete.

§ 3. Radicals

In [11] the Fitting radical of L was denoted by $\mathrm{Rad}_{\mathfrak{N}}(L)$ and the $\mathfrak{S}$ radical $\mathrm{Rad}_{\mathfrak{S}}(L)$ of L was also introduced. Observing the fact that $\mathfrak{N}$, $\mathfrak{S}$ are N_0-closed, we furthermore define the radicals of L for any N_0-closed class.

DEFINITION 3.1. *For any* N_0*-closed class* $\mathfrak{X}$ *of Lie algebras, we denote by* $\mathrm{Rad}_{\mathfrak{X}}(L)$ *the sum of all the* $\mathfrak{X}$ *ideals of* L *and call it the* $\mathfrak{X}$ *radical of* L.

All the classes in Lemma 6.2 in [11] are shown to be N_0-closed. Further examples are given in the following

LEMMA 3.2. (1) *All the classes of Lie algebras stated in Theorems* 2.4 *and* 2.5 *are* N_0*-closed.*

(2) *The class* $\mathrm{L}\mathfrak{X}$, *where* $\mathfrak{X}$ *is* $\{\mathrm{N}_0, \mathrm{S}\}$*-closed and any subalgebra of an* $\mathfrak{X}$ *algebra is finitely generated, is* N_0*-closed.*

Proof. The statement (1) follows immediately from Lemma 6.1 in [11] and the statement (2) has been shown in Section 4.1 in [1].

In [11], the Baer radical was denoted by $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}\text{-si}}(L)$ and the radical $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ was introduced for certain kind of coalescent classes $\mathfrak{X}$. This leads us to the following

Definition 3.3. *If a class $\mathfrak{X}$ is ascendantly coalescent (resp. coalescent), we denote by* $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$ *(resp.* $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$*) the subalgebra generated by all the ascendant $\mathfrak{X}$ subalgebras (resp. all the $\mathfrak{X}$ subideals) of L, and call it the* $\mathfrak{X}$-asc *(resp. the* $\mathfrak{X}$-si*) radical of L.*

Now we have the following properties of the radicals which have been shown for special cases in Theorem 6.3 in [11].

Theorem 3.4. *For any finite subset of* $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$ *(resp.* $\mathrm{Rad}_{\mathfrak{X}}(L)$, $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$*), there exists an ascendant $\mathfrak{X}$ subalgebra (resp. an $\mathfrak{X}$ ideal, an $\mathfrak{X}$ subideal) of L containing the set. Especially,* $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$ *(resp.* $\mathrm{Rad}_{\mathfrak{X}}(L)$, $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$*) is the union of all the ascendant $\mathfrak{X}$ subalgebras (resp. all the $\mathfrak{X}$ ideals, all the $\mathfrak{X}$ subideals) of L and belongs to* $\mathrm{L}\mathfrak{X}$.

Proof. The statement can be proved in the same way as Theorem 6.3 has been proved in [11]. So we omit the proof.

§4. Locally solvable radicals

In [11], several radicals of L, which reduce to the solvable radical if $L \in \mathfrak{F}$, have been studied. Namely, by making use of the fact that $\mathfrak{S}\cap\mathfrak{F}$, $\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}$, $\mathfrak{S}\cap\mathfrak{F}_{\omega}$ (resp. $\mathfrak{S}$) are N_0-closed and coalescent (resp. N_0-closed) if Φ is of characteristic 0, they introduced the following locally solvable radicals $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}}(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}}(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_{\omega}}(L)$, $\mathrm{Rad}_{\mathfrak{S}}(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_{\omega}\cap\mathfrak{G}\text{-si}}(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_{\omega}\text{-si}}(L)$ and showed some of their properties. However, the locally solvable radicals corresponding to two other known locally nilpotent radicals, the Gruenberg radical $\gamma(L)$ and the Hirsch-Plotkin radical $\rho(L)$, were not considered there. In this section we shall study these two radicals.

By Corollary 2.3 $\mathfrak{S}\cap\mathfrak{F}$ is ascendantly coalescent. Hence $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$ exists by Definition 3.3 and is locally solvable by Theorem 3.4. This corresponds to the locally nilpotent radical $\gamma(L)=\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}\text{-asc}}(L)$.

We know $\rho(L)=\mathrm{Rad}_{\mathrm{L}\mathfrak{N}}(L)$. Although $\mathrm{L}\mathfrak{N}$ is N_0-closed, we do not know whether or not $\mathrm{L}\mathfrak{S}$ is N_0-closed. However $\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})$ is N_0-closed by Lemma 3.2 and therefore $\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$ exists by Definition 3.1 and is a unique maximal $\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})$ ideal of L. This may be considered to correspond to the locally nilpotent radical $\rho(L)$, since $\rho(L)$ can be expressed as $\mathrm{Rad}_{\mathrm{L}(\mathfrak{N}\cap\mathfrak{F})}(L)$ by the following

LEMMA 4.1. $\mathrm{L}\mathfrak{N}=\mathrm{L}(\mathfrak{N}\cap\mathfrak{F})$.

PROOF. Assume that $L\in\mathrm{L}\mathfrak{N}$ and let K be any finite subset of L. Then K lies in an $\mathfrak{N}$ subalgebra of L. It follows that

$$<K>\ \in \mathfrak{N}\cap\mathfrak{G}=\mathfrak{N}\cap\mathfrak{F}.$$

Therefore $L\in\mathrm{L}(\mathfrak{N}\cap\mathfrak{F})$. It follows that $\mathrm{L}\mathfrak{N}=\mathrm{L}(\mathfrak{N}\cap\mathfrak{F})$, completing the proof.

It is known [1] that $\gamma(L)$ is not necessarily an ideal of L. We show an analogue of this fact for $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$ in the following

THEOREM 4.2. *$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$ and $\mathrm{Rad}_{\mathfrak{N}_{(\omega)}\cap\mathfrak{F}\text{-asc}}(L)$ are not necessarily ideals of L.*

PROOF. Let L be the Lie algebra in Example C in [11] (see [1]). That is, L is the semi-direct sum of an infinite-dimensional abelian Lie algebra $A=(e_0, e_1, e_2,\cdots)$ and a nilpotent Lie algebra (x, y, z) of derivations of A with $[x, y]=z$, $[x, z]=[y, z]=0$, where

$$x:\ e_i\rightarrow e_{i+1} \qquad (i\geq 0),$$

$$y:\ e_0\rightarrow 0, \qquad e_i\rightarrow ie_{i-1} \qquad (i\geq 1),$$

$$z:\ e_i\rightarrow e_i \qquad (i\geq 0).$$

Then (e_i) is an $\mathfrak{S}\cap\mathfrak{F}$ subideal of L and (y) is an ascendant $\mathfrak{S}\cap\mathfrak{F}$ subalgebra of L. Hence $A+(y)\subseteq \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$. Now assume that $A+(y)\neq\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$. Then there exists $ax+by+cz+\Sigma d_ie_i$ in $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)\backslash A+(y)$. If $a\neq 0$, then

$$[ax+by+cz+\Sigma d_ie_i,\ y]=az+\Sigma id_ie_{i-1}$$

and therefore $z\in\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$. If $a=0$, then $c\neq 0$ and therefore $z\in\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$. Consequently in any case there exists an ascendant $\mathfrak{S}\cap\mathfrak{F}$ subalgebra H of L containing z by Theorem 3.4. Let $\{H_\alpha : \alpha\leq\lambda\}$ be an ascending series from H to L. That is, $H_0=H$, $H_\alpha\lhd H_{\alpha+1}$ if $\alpha<\lambda$, $H_\mu=\bigcup_{\alpha<\mu} H_\alpha$ if μ is a limit ordinal $\leq\lambda$, and $H_\lambda=L$. We assert that $H_\alpha\cap A=H\cap A$ for any $\alpha\leq\lambda$. Assume this for any $\beta<\alpha$. If α is not a limit ordinal, then $\alpha=\beta+1$. $\Sigma d_ie_i\in H_\alpha$ implies $\Sigma d_ie_i=[\Sigma d_ie_i,\ z]\in H_\beta$. Therefore $H_\alpha\cap A=H_\beta\cap A$. By induction hypothesis we have $H_\alpha\cap A=H\cap A$. If α is a limit ordinal, using induction hypothesis we have

$$H_\alpha\cap A=(\bigcup_{\gamma<\alpha}H_\gamma)\cap A=\bigcup_{\gamma<\alpha}(H_\gamma\cap A)=H\cap A.$$

Therefore by transfinite induction we see that $H_\alpha\cap A=H\cap A$ for $\alpha\leq\lambda$, as was asserted. Now take $\alpha=\lambda$. Then $A=H\cap A$ and therefore $A\subseteq H$, which contradicts the fact that $H\in\mathfrak{F}$. Thus we conclude that $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)=A+(y)$.

By replacing $\mathfrak{S}$ by $N_{(\omega)}$ in the preceding paragraph we see that

$\mathrm{Rad}_{\mathfrak{N}_{(\omega)}\cap\mathfrak{F}\text{-asc}}(L)=A+(y)$.

Since $[x, y]=z$, $A+(y)$ is not an ideal of L. Hence both $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$ and $\mathrm{Rad}_{\mathfrak{N}_{(\omega)}\cap\mathfrak{F}\text{-asc}}(L)$ are not ideals of L, and the theorem is proved.

It is known [1] that $\rho(L)$ need not be a characteristic ideal of L. Corresponding to this we show the following

THEOREM 4.3. *$\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$ is not necessarily a characteristic ideal of L.*

PROOF. Let L be the Lie algebra in Example C in [11] as in the proof of the preceding theorem. Since $\langle e_0, x\rangle \notin \mathfrak{F}$, we have $L \notin \mathrm{L}(\mathfrak{S}\cap\mathfrak{F})$. $A+(y, z)$ is an $\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})$ ideal of L, since any finite subset of $A+(y, z)$ lies in an $\mathfrak{S}\cap\mathfrak{F}$ subalgebra $(e_0, e_1,\cdots, e_n, y, z)$. Hence $\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)=A+(y, z)$.

Now we consider (x, x^2, y, z). Then it is a nilpotent Lie algebra of derivations of A such that

$$[x^2, x]=0, \qquad [x^2, y]=2x, \qquad [x^2, z]=0.$$

Let M be the semi-direct sum of A and (x, x^2, y, z). Then L is an ideal of M and therefore $\mathrm{ad}_M x^2$ induces a derivation D of L. $\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$ is not invariant under D, since $yD=-2x$. Therefore $\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$ is not a characteristic ideal of L and the proof is complete.

We note that

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)\subseteq \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L) \text{ and } \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)\subseteq \mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L).$$

In fact, the first inclusion is evident. By Theorem 8.3 in [11] $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)$ is an $\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})$ ideal of L, from which the second inclusion follows.

Finally in connection with Theorem 8.5 in [11] we obtain the following

THEOREM 4.4. *If Φ is of characteristic 0, the locally solvable radicals of L*

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}}(L),\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}}(L),\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega}(L),\ \mathrm{Rad}_{\mathfrak{S}}(L),$$

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L),\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{N}_\omega\cap\mathfrak{G}\text{-si}}(L),\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}_\omega\text{-si}}(L),$$

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L),\ \mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$$

are different from each other in general.

PROOF. Let L be the Lie algebra in Example C in [11]. Then by the proofs of Theorems 4.2 and 4.3

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)=A+(y) \text{ and } \mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)=A+(y, z).$$

By the first part of the proof of Theorem 8.5 in [11]

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}}(L)=(0),\ \mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)=A$$

and all the other radicals equal L. By taking account of the statement of

Theorem 8.5 in [11], we conclude that these nine radicals are generally different from each other.

References

[1] B. Hartley, Locally nilpotent ideals of a Lie algebra, Proc. Cambridge Philos. Soc., **63** (1967), 257–272.

[2] O. Maruo, Pseudo-coalescent classes of Lie algebras, Hiroshima Math. J., **2** (1972), 205–214.

[3] E. Schenkman, Infinite Lie algebras, Duke Math. J., **19** (1952), 529–535.

[4] L. A. Simonjan, Two radicals of Lie algebras, Dokl. Akad. Nauk SSSR, **157** (1964), 281–283 (Russian).

[5] I. Stewart, The minimal condition for subideals of Lie algebras, Math. Z., **111** (1969), 301–310.

[6] I. Stewart, A property of locally finite Lie algebras, J. London Math. Soc. (2), **3** (1971), 334–340.

[7] I. Stewart, Lie Algebras, Lecture Notes in Mathematics 127, Springer, Berlin-Heidelberg-New York, 1970.

[8] I. Stewart, An algebraic treatment of Mal'cev's theorems concerning nilpotent Lie groups and their Lie algebras, Compositio Math., **22** (1970), 289–312.

[9] I. Stewart, Infinite-dimensional Lie algebras in the spirit of infinite group theory, Compositio Math., **22** (1970), 313–331.

[10] I. Stewart, Structure theorems for a class of locally finite Lie algebras, Proc. London Math. Soc. (3), **24** (1972), 79–100.

[11] S. Tôgô, Radicals of infinite dimensional Lie algebras, Hiroshima Math. J., **2** (1972), 179–203.

Department of Mathematics,
Faculty of Science,
Hiroshima University