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Harmonic Functions on Hermitian Hyperbolic Spaces

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Introduction

The classical theory on Dirichlet problem shows that certain classes of harmonic functions on the unit disc are given by the Poisson integral (cf. [1]). Recently S. Helgason proved in [5] that any eigenfunction of the Laplace-Beltrami operator corresponding to the Poincaré metric can be given as the Poisson transform of a hyperfunction. On the contrary, it was proved in [3] that, on the euclidean space, one should consider the space which properly contains the hyperfunctions on the sphere to obtain arbitrary eigenfunctions of the laplacian.

The present paper shows that the harmonic functions of the Laplace-Beltrami operator on the hermitian hyperbolic spaces are given as the Poisson transforms of the hyperfunctions on the boundary (Theorem 4.5 in §4). For the case of real hyperbolic spaces we shall discuss in [11].

The construction of this paper is as follows.

In §1, we show that on a compact riemannian manifold, there exists an isomorphism of the space of hyperfunctions onto the space of Fourier coefficients of hyperfunctions with respect to the Laplace-Beltrami operator. In §2, we show that any harmonic function can be expanded in an absolutely convergent series of K-finite harmonic functions, and in §3 we determine the K-finite harmonic functions by solving differential equations. In the final section we define the Poisson transform of hyperfunctions which is a natural generalization of Poisson integral. Then, making use of an isomorphism in §1, we prove Theorem 4.5.

§1. Hyperfunctions on compact real analytic riemannian manifolds

We shall show in this section that the hyperfunctions on a compact real analytic riemannian manifold can be characterized by the eigenvalues of the Laplace-Beltrami operator on the manifold.

Let M be a compact real analytic riemannian manifold, g a riemannian metric on M and Δ the Laplace-Beltrami operator corresponding to g.

Let $L^2(M)$ be the space of square integrable functions on M with respect to the measure $d\mu$ corresponding to g, $(\ ,\ )$ its unitary inner product and $\|\ \|$ its norm.

We denote by $\mathscr{A}(M)$ the space of analytic functions on M equipped with the usual topology.

As is well-known, the eigenvalues of Δ are non-negative and we can choose analytic functions $\phi_n (n \in \boldsymbol{N})$ so that they form a complete orthonormal base of $L^2(M)$ and the corresponding eigenvalues λ_n satisfy

$$0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n \leqq \cdots.$$

LEMMA 1.1. *For $s \in \boldsymbol{C}$ with $\mathrm{Re}(s) > \frac{1}{2} \dim M$, the series*

$$\sum_{n=1}^{\infty} (1+\lambda_n)^{-s}$$

is convergent and holomorphic in s.

For the proof of the lemma, see [10].

LEMMA 1.2. *For $t > 0$, the series*

$$\sum_{n=1}^{\infty} e^{-t\sqrt{\lambda_n}}$$

is convergent.

PROOF. Take an $s \in \boldsymbol{R}$ such that $s > \frac{1}{2} \dim M$. Then there exists a real number $m > 0$ satisfying

$$(1+\lambda_n)^s e^{-t\sqrt{\lambda_n}} < m$$

for any $n \geqq 1$. Therefore

$$\sum_{n=1}^{\infty} e^{-t\sqrt{\lambda_n}} \leqq m \sum_{n=1}^{\infty} (1+\lambda_n)^{-s},$$

which is convergent by Lemma 1.1.

Let $C^{\infty}(M)$ be the set of indefinitely differentiable functions on M. It is well-known that any $\phi \in C^{\infty}(M)$ has an absolutely and uniformly convergent expansion

$$\phi = \sum_{n=1}^{\infty} a_n \phi_n,$$

where $a_n = (\phi, \phi_n)$. Since $\Delta\phi = \sum_{n=1}^{\infty} a_n \lambda_n \phi_n$, the series

$$\sum_{n=1}^{\infty} a_n \sqrt{\lambda_n} \phi_n$$

is also absolutely and uniformly convergent and defines an element of $C^{\infty}(M)$. We denote it by $\Delta^{1/2}\phi$. It is easy to show

LEMMA 1.3. *For f and h in $C^{\infty}(M)$,*

$$(\Delta^{1/2}f, h)=(f, \Delta^{1/2}h).$$

Analogously, for any $t\geqq 0$, we can define a mapping $\exp(-t\Delta^{1/2})$ by

$$\exp(-t\Delta^{1/2})\phi=\sum_{n=1}^{\infty} a_n e^{-t\sqrt{\lambda_n}}\phi_n$$

for $\phi=\sum_{n=1}^{\infty} a_n\phi_n$ in $C^{\infty}(M)$. Then we have

LEMMA 1.4. *For f and h in $C^{\infty}(M)$,*

$$(\exp(-t\Delta^{1/2})f, h)=(f, \exp(-t\Delta^{1/2})h).$$

We introduce two systems of semi-norms $|\ \ |_H (H>0)$ and $\|\ \ \|_h (h>0)$ on $C^{\infty}(M)$ defined by

$$|\phi|_H=\sup_{k\in \boldsymbol{Z}^+}\frac{1}{(2k)!H^k}\|\Delta^k\phi\|$$

and

$$\|\phi\|_h=\sup_{m\in \boldsymbol{Z}^+}\frac{1}{m!h^m}\|(\Delta^{1/2})^m\phi\|,$$

where $\boldsymbol{Z}^+$ denotes the set of non-negative integers. For $H>0$ and $h>0$, we define

$$\mathscr{A}_{0,H}(M)=\{\phi\in C^{\infty}(M)\mid |\phi|_H<\infty\}$$

and

$$\mathscr{A}_h(M)=\{\phi\in C^{\infty}(M)\mid \|\phi\|_h<\infty\}.$$

Then we have

LEMMA 1.5. *For $\phi\in C^{\infty}(M)$, we have the following two inequalities.*

(i) $|\phi|_H\leqq \|\phi\|_{\sqrt{H}}$

(ii) $\|\phi\|_h\leqq\sqrt{2}\,|\phi|_{h^2}$

PROOF. (i) Taking the supremum of the equality

$$\frac{1}{(2k)!H^k}\|\Delta^k\phi\|=\frac{1}{(2k)!(\sqrt{H})^{2k}}\|(\Delta^{1/2})^{2k}\phi\|,$$

we obtain the required inequality

$$|\phi|_H \leqq \|\phi\|_{\sqrt{H}}.$$

(ii) In case of $m=2l$ where l is a non-negative integer,

$$\frac{1}{m!h^m}\|(\Delta^{1/2})^m\phi\|$$

$$=\frac{1}{(2l)!(h^2)^l}\|\Delta^l\phi\|.$$

Taking the supremum of the last term, we have

$$\frac{1}{m!h^m}\|(\Delta^{1/2})^m\phi\| \leqq |\phi|_{h^2}. \tag{1.1}$$

In case of $m=2l+1$,

$$\left(\frac{1}{m!h^m}\|(\Delta^{1/2})^m\phi\|\right)^2$$

$$=\frac{((\Delta^{1/2})^{2l+1}\phi,\ (\Delta^{1/2})^{2l+1}\phi)}{\{(2l+1)!\}^2h^{2(2l+1)}}$$

$$=\frac{(\Delta^{l+1/2}\phi,\ \Delta^{l+1/2}\phi)}{\{(2l+1)!\}^2h^{2(2l+1)}}.$$

Using Lemma 1.3 and Schwarz's inequality, we have

$$\left(\frac{1}{m!h^m}\|(\Delta^{1/2})^m\phi\|\right)^2$$

$$=\frac{(\Delta^{l+1}\phi,\ \Delta^l\phi)}{\{(2l+1)!\}^2h^{2(2l+1)}}$$

$$\leqq\frac{\|\Delta^{l+1}\phi\|\|\Delta^l\phi\|}{\{(2l+1)!\}^2h^{2(2l+1)}}$$

$$=\frac{\|\Delta^{l+1}\phi\|}{\{2(l+1)\}!(h^2)^{l+1}}\cdot\frac{\|\Delta^l\phi\|}{(2l)!(h^2)^l}\cdot\frac{\{2(l+1)\}!}{(2l+1)!}\cdot\frac{(2l)!}{(2l+1)!}.$$

As $0<\frac{2(l+1)}{2l+1}\leqq 2$, taking the supremum, we have

$$\left(\frac{1}{m!h^m}\|(\Delta^{1/2})^m\phi\|\right)^2 \leqq 2(|\phi|_{h^2})^2.$$

Therefore, we get

$$\frac{1}{m!h^m}\|(\Delta^{1/2})^m\phi\| \leqq \sqrt{2}\,|\phi|_{h^2}. \tag{1.2}$$

(1.1) together with (1.2) gives

$$\frac{1}{m!h^m}\|(\Delta^{1/2})^m\phi\|\leqq\sqrt{2}\,|\phi|_{h^2}$$

for $n\in \boldsymbol{Z}^+$. Taking the supremum of the above inequality, we obtain the required inequality

$$\|\phi\|_h\leqq\sqrt{2}\,|\phi|_{h^2},$$

which finishes the proof.

Lemma 1.5. implies that the inductive limit of $\mathscr{A}_{0,H}(M)$, denoted by $\varinjlim_{H\to\infty}\mathscr{A}_{0,H}(M)$ and that of $\mathscr{A}_h(M)$, denoted by $\varinjlim_{h\to\infty}\mathscr{A}_h(M)$ are identical with their topologies. On the other hand, $\mathscr{A}(M)=\varinjlim_{H\to\infty}\mathscr{A}_{0\,H}(M)$ with its topology (see [9]). Therefore we have the following proposition which will be useful for our purpose.

PROPOSITION 1.6. $\mathscr{A}(M)=\varinjlim_{h\to\infty}\mathscr{A}_h(M)$.

Now, we define a subset $\mathscr{F}_a$ of $\boldsymbol{C}^{\mathrm{N}}$ by

$$\mathscr{F}_a=\{(a_n)_{n\geqq 1}\,|\,a_n\in \boldsymbol{C},\ \sum_{n=1}^{\infty}|a_n|e^{t\sqrt{\lambda_n}}<\infty \quad \text{for some } t>0\}$$

and a mapping Φ of $\mathscr{A}(M)$ into $\boldsymbol{C}^{\mathrm{N}}$ by

$$\Phi(\phi)=(a_n)_{n\geqq 1},$$

where $\phi\in\mathscr{A}(M)$ and $a_n=(\phi,\phi_n)$. $\mathscr{F}_a$ is a vector space over $\boldsymbol{C}$ and Φ is a $\boldsymbol{C}$-linear mapping of $\mathscr{A}(M)$ into $\boldsymbol{C}^{\mathrm{N}}$.

PROPOSITION 1.7. *Φ is an isomorphism of $\mathscr{A}(M)$ onto $\mathscr{F}_a$.*

PROOF. At first we prove that the image of Φ is contained in $\mathscr{F}_a$. Take and fix an arbitrary element ϕ in $\mathscr{A}(M)$ and put $a_n=(\phi,\phi_n)$. Then ϕ has an expansion

$$\phi=\sum_{n=1}^{\infty}a_n\phi_n$$

which converges absolutely and uniformly in M. On the other hand, Proposition 1.6 implies that there exists an $h>0$ such that $\mathscr{A}_h(M)$ contains ϕ. Therefore

$$\sup_{m\in Z^+}\frac{1}{m!h^m}\|\Delta^{m/2}\phi\|=\|\phi\|_h<\infty,$$

and

$$\|\phi\|_h=\sup_{m\in Z^+}\frac{1}{m!h^m}\|\sum_{n=1}^{\infty}a_n(\sqrt{\lambda_n})^m\phi_n\|$$

$$\geqq \frac{1}{m!h^m} \| \sum_{n=1}^{\infty} a_n(\sqrt{\lambda_n})^m \phi_n \|$$

for any $m \in \boldsymbol{Z}^+$. Hence, for any $m \in \boldsymbol{Z}^+$ and any $n \in \boldsymbol{N}$,

$$\frac{1}{m!h^m} (\sqrt{\lambda_n})^m |a_n| \leqq \|\phi\|_h.$$

Multiplying 2^{-m} and summing the above inequality with respect to m, we have

$$e^{\sqrt{\lambda_n}/2h} |a_n| \leqq 2\|\phi\|_h \tag{1.3}$$

for $n \in \boldsymbol{N}$. Putting here $t = 1/4h$, we obtain

$$\sum_{n=1}^{\infty} |a_n| e^{t\sqrt{\lambda_n}}$$

$$\leqq 2\|\phi\|_h \sum_{n=1}^{\infty} e^{-\sqrt{\lambda_n}/2h} e^{\sqrt{\lambda_n}/4h}$$

$$= 2\|\phi\|_h \sum_{n=1}^{\infty} e^{-\sqrt{\lambda_n}/4h}$$

which is finite by Lemma 1.2. This means that $\Phi(\phi)$ lies in $\mathscr{F}_a$.

Next, we show the surjectivity of Φ. Take and fix an arbitrary $(a_n)_{n \geqq 1}$ in $\mathscr{F}_a$. There exists a $t > 0$ such that

$$\sum_{n=1}^{\infty} |a_n| e^{\sqrt{\lambda_n}/t} < \infty.$$

On the other hand, for any $n \in \boldsymbol{N}$, $\phi_n \in \mathscr{A}(M)$ and

$$\|\phi_n\|_t = \sup_{m \in \boldsymbol{Z}^+} \frac{1}{m!t^m} (\sqrt{\lambda_n})^m$$

$$\leqq \sum_{m \in \boldsymbol{Z}^+} \frac{1}{m!t^m} (\sqrt{\lambda_n})^m$$

$$= e^{\sqrt{\lambda_n}/t}.$$

Hence, we obtain

$$\| \sum_{n=N}^{N+l} a_n \phi_n \|_t \leqq \sum_{n=N}^{N+l} |a_n| \, \|\phi_n\|_t$$

$$\leqq \sum_{n=N}^{N+l} |a_n| \, e^{\sqrt{\lambda_n}/t},$$

which implies that the sequence

$$\left(\sum_{n=1}^{N} a_n\phi_n\right)_{N\geqq 1}$$

is a Cauchy sequence in the Banach space $\mathscr{A}_t(M)$. Therefore there exists a unique element ϕ in $\mathscr{A}(M)$ such that $\phi=\sum_{n=1}^{\infty} a_n\phi_n$ in the topology of $\mathscr{A}(M)$. In particular, $\sum_{n=1}^{\infty} a_n\phi_n$ converges to ϕ absolutely and uniformly in M. So, we have $\Phi(\phi)=(a_n)_{n\geqq 1}$, which means the surjectivity of Φ.

Finally, we prove the injectivity of Φ. Assume $\Phi(\phi)=0$ for $\phi\in\mathscr{A}(M)$. Then $(\phi, \phi_n)=0$ for $n\in\boldsymbol{N}$. On the other hand, ϕ has an expansion

$$\phi=\sum_{n=1}^{\infty}(\phi, \phi_n)\phi_n$$

which is absolutely and uniformly convergent. So we have $\phi=0$. This completes the proof.

COROLLARY 1. *For $\phi\in\mathscr{A}(M)$, the series*

$$\sum_{n=1}^{\infty}(\phi, \phi_n)\phi_n$$

converges to ϕ in the topology of $\mathscr{A}(M)$.

PROOF. We have shown in the proof of the above proposition.

COROLLARY 2. *For $\phi\in\mathscr{A}_h(M)$ and t such that $1/2h>t\geqq 0$, the series*

$$\sum_{n=1}^{\infty}(\phi, \phi_n)e^{t\sqrt{\lambda_n}}\phi_n$$

converges in the topology of $\mathscr{A}(M)$ and defines an element of $\mathscr{A}(M)$, which we denote by $\exp(t\Delta^{1/2})\phi$. In addition,

$$\|\exp(t\Delta^{1/2})\phi\|\leqq 2\|\phi\|_h.$$

PROOF. From (1.3) in the proof of Proposition 1.7,

$$|(\phi, \phi_n)|\leqq 2\|\phi\|_h e^{-\sqrt{\lambda_n}/2h}$$

for $n\geqq 1$ and $\phi\in\mathscr{A}_h(M)$. So, taking an s such that $0<s<1/2h-t$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}|(\phi, \phi_n)e^{t\sqrt{\lambda_n}}|e^{s\sqrt{\lambda_n}}\\
&\leqq\sum_{n=1}^{\infty}(2\|\phi\|_h e^{-\sqrt{\lambda_n}/2h})e^{(t+s)\sqrt{\lambda_n}}\\
&=2\|\phi\|_h\sum_{n=1}^{\infty}{}^{(t+s-1/2h)\sqrt{\lambda_n}}
\end{aligned}$$

Since $t+s-1/2h<0$, Lemma 1.2 implies that $((\phi, \phi_n)e^{t\sqrt{\lambda_n}})_{n\geq 1}\in\mathscr{F}_a$. By Proposition 1.7, we deduce that the series

$$\sum_{n=1}^{\infty}(\phi, \phi_n)e^{t\sqrt{\lambda_n}}\phi_n$$

converges in $\mathscr{A}(M)$ and defines an element $\exp(t\Delta^{1/2})\phi$ of $\mathscr{A}(M)$ as $\mathscr{A}(M)$ is complete. Since $\exp(t\Delta^{1/2})\phi=\sum_{n=1}^{\infty}(\phi, \phi_n)e^{t\sqrt{\lambda_n}}\phi_n$ is convergent absolutely and uniformly in M,

$$\begin{aligned}
\|\exp(t\Delta^{1/2})\phi\| &\leqq \sum_{m=0}^{\infty}\frac{t^m}{m!}\|\Delta^{m/2}\phi\| \\
&= \sum_{m=0}^{\infty}\frac{(th)^m}{m!h^m}\|\Delta^{m/2}\phi\| \\
&\leqq \left(\sup_{m\in Z^+}\frac{1}{m!h^m}\|\Delta^{m/2}\phi\|\right)\sum_{m=0}^{\infty}(th)^m \\
&\leqq \frac{1}{1-th}\|\phi\|_h \\
&< 2\|\phi\|_h,
\end{aligned}$$

which finishes the proof.

Let $\mathscr{B}=\mathscr{B}(M)$ be the space of all continuous linear functionals of $\mathscr{A}(M)$ into $\boldsymbol{C}$. M being compact, $\mathscr{B}$ is identical with the space of Sato's hyperfunctionsons on M (for detail, see [12]), and henceforth we call the elements of $\mathscr{B}$ hyperfunctions on M.

We define a subset $\mathscr{F}_b$ of $\boldsymbol{C}^{\mathrm{N}}$ by

$$\mathscr{F}_b=\{(a_n)_{n\geqq 1}|a_n\in\boldsymbol{C}, \sum_{n=1}^{\infty}|a_n|e^{-t\sqrt{\lambda_n}}<\infty \qquad \text{for any } t>0\}$$

and a mapping Ψ of $\mathscr{B}(M)$ into $\boldsymbol{C}^N$ by

$$\Psi(T)=(a_n)_{n\geqq 1},$$

where $T\in\mathscr{B}(M)$ and $a_n=T(\bar{\phi}_n)$, $\bar{\phi}$ denoting the complex conjugate of ϕ. $\mathscr{F}_b$ is a vector space over $\boldsymbol{C}$ and Ψ is a $\boldsymbol{C}$-linear mapping of $\mathscr{B}(M)$ into $\boldsymbol{C}^{\mathrm{N}}$.

We can now state the theorem characterizing the hyperfunctions on M.

THEOREM 1.8. *Ψ is an isomorphism of $\mathscr{B}$ onto $\mathscr{F}_b$.*

PROOF. At first we prove that the image of Ψ is contained in $\mathscr{F}_b$. It is enough to show that for every $t>0$

$$\sum_{n=1}^{\infty}|a_n|e^{-t\sqrt{\lambda_n}}<\infty,$$

where $a_n = T(\bar{\phi}_n)$ and $T \in \mathscr{B}(M)$. Take an $h_0 > 0$ such that $1/h_0 < t$. As $\bar{\phi}_n \in \mathscr{A}_h(M)$ for every $h > 0$ and T is continuous on $\mathscr{A}_h(M)$, there exists a constant c such that

$$\begin{aligned} |a_n| &= |T(\bar{\phi}_n)| \\ &\leqq c \|\bar{\phi}_n\|_{h_0} \\ &= c \sup_{m \in Z^+} \frac{1}{m! h_0^m} \|\Delta^{m/2} \bar{\phi}_n\| \\ &= c \sup_{m \in Z^+} \frac{(\sqrt{\lambda_n})^m}{m! h_0^m} \\ &\leqq c e^{\sqrt{\lambda_n}/h_0} . \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{n=1}^{\infty} |a_n| e^{-t\sqrt{\lambda_n}} \\ &\quad \leqq c \sum_{n=1}^{\infty} e^{\sqrt{\lambda_n}/h_0} e^{-t\sqrt{\lambda_n}} \\ &\quad = c \sum_{n=1}^{\infty} e^{(1/h_0 - t)\sqrt{\lambda_n}} \end{aligned}$$

Since $1/h_0 - t < 0$, by Lemma 1.2, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} |a_n| e^{-t\sqrt{\lambda_n}} \\ &\quad \leqq c \sum_{n=1}^{\infty} e^{(1/h_0 - t)\sqrt{\lambda_n}} \\ &\quad < \infty . \end{aligned}$$

Next, take and fix an arbitrary $(a_n)_{n \geqq 1}$ in $\mathscr{F}_b$ and an arbitrary $h > 0$. Then by Corollary 2 to Proposition 1.7, $\exp\left(\frac{1}{4h} \Delta^{1/2}\right) \phi \in \mathscr{A}(M)$ for $\phi \in \mathscr{A}_h(M)$. Using Lemma 1.4, we have

$$\begin{aligned} (\phi_n, \bar{\phi}) &= \left(\phi_n, \exp\left(-\frac{1}{4h} \Delta^{1/2}\right) \exp\left(\frac{1}{4h} \Delta^{1/2}\right) \bar{\phi}\right) \\ &= \left(\exp\left(-\frac{1}{4h} \Delta^{1/2}\right) \phi_n, \exp\left(\frac{1}{4h} \Delta^{1/2}\right) \bar{\phi}\right) \\ &= \left(\exp\left(-\frac{1}{4h} \Delta^{1/2}\right) \phi_n, \left(\exp\left(\frac{1}{4h} \Delta^{1/2}\right) \phi\right)^{-}\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}|a_n|\,|(\phi_n, \bar{\Phi})| \\
&\leqq \sum_{n=1}^{\infty}|a_n|\left|\left(\exp\left(-\frac{1}{4h}\Delta^{1/2}\right)\phi_n, \left(\exp\left(\frac{1}{4h}\Delta^{1/2}\right)\phi\right)^{-}\right|\right. \\
&\leqq \sum_{n=1}^{\infty}|a_n|\left\|\exp\left(-\frac{1}{4h}\Delta^{1/2}\right)\phi_n\right\|\left\|\left(\exp\left(\frac{1}{4h}\Delta^{1/2}\right)\phi\right)^{-}\right\| \\
&\leqq\left(\sum_{n=1}^{\infty}|a_n|e^{-\sqrt{\lambda_n}/4h}\right).2\|\phi\|_h, \qquad (1.4)
\end{aligned}$$

which means that the series

$$\sum_{n=1}^{\infty}a_n(\phi_n, \bar{\Phi})$$

is absolutely convergent. We put

$$T(\phi)=\sum_{n=1}^{\infty}a_n(\phi_n, \bar{\Phi}).$$

It is clear that T is a $\boldsymbol{C}$-linear mapping of $\mathscr{A}(M)$ into $\boldsymbol{C}$ and $T(\bar{\Phi}_n)=a_n$. Furthermore (1.4) shows that T is continuous on $\mathscr{A}_h(M)$. Since h is arbitrary, T is continuous on $\mathscr{A}(M)$, which proves the surjectivity of Ψ.

Finally we prove the injectivity of Ψ. Assume that $\Psi(T)=0$ for $T\in\mathscr{B}$. That is, $T(\bar{\Phi}_n)=0$ for $n\geqq 1$. Since $\sum_{n=1}^{\infty}(\bar{\Phi}, \phi_n)\phi_n$ converges to $\bar{\Phi}$ in the topology of $\mathscr{A}(M)$ for any ϕ in $\mathscr{A}(M)$ (Corollary 1 to Proposition 1.7),

$$\begin{aligned}
T(\phi)&=\sum_{n=1}^{\infty}(\bar{\Phi}, \phi_n)^{-}T(\bar{\Phi}_n) \\
&=0,
\end{aligned}$$

which means that $T=0$. This completes the proof of the theorem.

REMARK. The following two conditions are equivalent.

(i) $\sum_{n=1}^{\infty}|a_n|e^{-t\sqrt{\lambda_n}}<\infty$ *for any* $t>0$.

(ii) $\sum_{n=1}^{\infty}|a_n|^2e^{-s\sqrt{\lambda_n}}<\infty$ *for any* $s>0$.

In fact, assume that (i) is satisfied. Since $\sum_{n=1}^{\infty}|a_n|e^{-s\sqrt{\lambda_n}/2}<\infty$, there exists an integer N such that for $n\geqq N$,

$$|a_n|e^{-s\sqrt{\lambda_n}/2}<1.$$

Then, for such n, we have

$$|a_n|^2 e^{-s\sqrt{\lambda_n}} < |a_n| e^{-s\sqrt{\lambda_n}/2}$$

which implies that $\sum_{n=1}^{\infty} |a_n|^2 e^{-s\sqrt{\lambda_n}} < \infty$.

Conversely, using the Schwarz's inequality,

$$\sum_{n=1}^{\infty} |a_n| e^{-t\sqrt{\lambda_n}} = \sum_{n=1}^{\infty} (|a_n| e^{-t\sqrt{\lambda_n}/2}) e^{-t\sqrt{\lambda_n}/2}$$

$$\leqq \left(\sum_{n=1}^{\infty} |a_n|^2 e^{-t\sqrt{\lambda_n}}\right) \left(\sum_{n=1}^{\infty} e^{-t\sqrt{\lambda_n}}\right),$$

which is finite by Lemma 1.2.

Therefore, $\mathscr{F}_b = \{(a_n)_{n\geqq 1} | a_n \in \boldsymbol{C}, \sum_{n=1}^{\infty} |a_n|^2 e^{-t\sqrt{\lambda_n}}$ for any $t>0\}$.

§2. Poisson transforms of K-finite functions

In this section we assume that G is a connected real semisimple Lie group with finite center and of real rank one. Let $\mathfrak{g}_0$ be the Lie algebra of G, $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ a Cartan decomposition, θ the corresponding Cartan involution and $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. Let $\mathfrak{a}_{\mathfrak{p}_0}$ be a maximal abelian subspace of $\mathfrak{p}_0$ and extend $\mathfrak{a}_{\mathfrak{p}_0}$ to a Cartan subalgebra $\mathfrak{a}_0$ of $\mathfrak{g}_0$. Then $\mathfrak{a}_0 = \mathfrak{a}_{\mathfrak{k}_0} + \mathfrak{a}_{\mathfrak{p}_0}$ where $\mathfrak{a}_{\mathfrak{k}_0} = \mathfrak{a}_0 \cap \mathfrak{k}_0$. On account of our assumption on G, $\mathfrak{a}_{\mathfrak{p}_0}$ is one-dimensional. Complexify $\mathfrak{k}_0$, $\mathfrak{p}_0$, $\mathfrak{a}_0$, $\mathfrak{a}_{\mathfrak{p}_0}$ and $\mathfrak{a}_{\mathfrak{k}_0}$ to $\mathfrak{k}$, $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}_{\mathfrak{k}}$ in $\mathfrak{g}$ respectively and introduce compatible orders in the spaces of real-valued linear functions on $\mathfrak{a}_{\mathfrak{p}_0} + \sqrt{-1}\mathfrak{a}_{\mathfrak{k}_0}$ and $\mathfrak{a}_{\mathfrak{p}_0}$. Let P be the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$ under this ordering. For a root α, let $\mathfrak{g}^{\alpha}$ denote the root subspace of α. Put P_+ be the set of $\alpha \in P$ with $\alpha \circ \theta \neq \alpha$, $\mathfrak{n} = \sum_{\alpha \in P_+} \mathfrak{g}^{\alpha}$, $\mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}_0$ and $\rho = \frac{1}{2} \sum_{\alpha \in P_+} \alpha$. Then $G = KAN$ is an Iwasawa decomposition, where K, A and N are the analytic subgroups of G with Lie alge bras $\mathfrak{k}_0$, $\mathfrak{a}_{\mathfrak{p}_0}$ and $\mathfrak{n}_0$ respectively. For $x \in G$, let $H(x)$ be the unique element such that $x \in K(\exp H(x))N$. Let $X = G/K$ and $B = K/M$, where M is the centralizer of A in K. We define a real analytic function $P(xK, kM)$ on the manifold $X \times B$ by

$$P(xK, kM) = e^{-2\varrho H(x^{-1}k)}.$$

We denote by $\mathscr{E}_K$ the set of equivalence classes of irreducible unitary representation of K and by $\mathscr{E}_K^0$ the subset of $\mathscr{E}_K$ which consists of the representation of class one with respect to M. For each $\gamma \in \mathscr{E}_K$, we take and fix a representative $(\tau^{\gamma}, W^{\gamma}) \in \gamma$ and choose an orthonormal base $\{w_1^{\gamma}, \dots, w_{\deg\gamma}^{\gamma}\}$ of W^{γ} so that w_1^{γ} is an M-fixed vector when $\gamma \in \mathscr{E}_k^0$, where $\deg \gamma$ is the dimension of W^{γ}. Since rank $(G/K) = 1$, w_1^{γ} is unique up to a scalar for $\gamma \in \mathscr{E}_k^0$. Put $\tau_{ij}^{\gamma}(k) = (\tau^{\gamma}(k) w_j^{\gamma}, w_i^{\gamma})$ and $\phi_{ij}^{\gamma} =$

$\sqrt{\deg\gamma}\ \bar{\tau}_{ij}^{\gamma}$ for $\gamma\in\mathcal{E}_K$ and $\phi_i^{\gamma}=\phi_{i1}^{\gamma}$ for $\gamma\in\mathcal{E}_K^0$, (,) denoting the unitary inner product of W^{γ}. We denote by V_{γ} the space of elements in $C^{\infty}(K)$ which transform according to γ by the left regular representation $\pi(k)$ of K. It is easy to see that

$$\pi(k)\phi_{ij}^{\gamma}=\sum_{l=1}^{\deg\gamma}\tau_{li}^{\gamma}(k)\phi_{lj}^{\gamma}$$

for $\gamma\in\mathcal{E}_K$ and $k\in K$, and

$$\phi_i^{\gamma}(km)=\phi_i^{\gamma}(k)$$

for $\gamma\in\mathcal{E}_k^0$, $k\in K$ and $m\in M$. Hence for $\gamma\in\mathcal{E}_K$, $\phi_{ij}^{\gamma}\in V_{\gamma}$ and for $\gamma\in\mathcal{E}_K^0$, we can regard V_{γ} as a subspace of $C^{\infty}(B)$. As is well-known, $\{\phi_{ij}^{\gamma}|1\leqq i,\ j\leqq\deg\gamma\}$ is an orthonormal base of V_{γ} for $\gamma\in\mathcal{E}_K$.

Let g be the G-invariant riemannian metric on X induced by the Killing form of $\mathfrak{g}_0$ and Δ be the Laplace-Beltrami operator corresponding to g. We identify the functions on X with those on G which are right K-invariant. Let $\mathfrak{B}$ be the universal enveloping algebra of $\mathfrak{g}$. We regard elements of $\mathfrak{B}$ as left G-invariant differential operators on G. Then, as is well-known, Δ can be identified with the Casimir operator Ω on G by

$$(\Delta f)\ (xK)=(\Omega f)\ (x)$$

for $x\in G$.

We put

$$\mathcal{H}=\{f\in C^{\infty}(X)\,|\,\Delta f=0\}$$

and

$$\mathcal{H}_{\gamma}=\{f\in\mathcal{H}\,|\,f\ \ \textit{transforms according to}\ \gamma\}.$$

Now, we define the Poisson transform $\mathcal{P}\phi$ of $\phi\in C^{\infty}(B)$. Put

$$(\mathcal{P}\phi)(x)=\int_K P(xK,\ kM)\ \phi(k)\ dk,$$

where $x\in G$, $k\in K$ and dk is the normalized Haar measure on K. On this mapping $\mathcal{P}$, the following results hold.

PROPOSITION 2.1. (1) *The image of $C^{\infty}(B)$ by $\mathcal{P}$ is contained in $\mathcal{H}$ and for $\gamma\in\mathcal{E}_K^0$, the restriction of $\mathcal{P}$ on V_{γ} is an isomorphism onto $\mathcal{H}_{\gamma}$.*

(2) *If $\mathcal{H}_{\gamma}\neq\{0\}$, then $\gamma\in\mathcal{E}_K^0$.*

For the proof of the above proposition, see Lemma 1.2 and Theorem 1.4 in Chap. IV in [5].

We put $f_i^\gamma=\mathscr{P}\phi_i^\gamma$. Then we have

PROPOSITION 2.2. (1) *For $f\in\mathscr{H}$, there exists a unique complex number a_i^γ for every $\gamma\in\mathscr{E}_K^0$ and $1\leqq i\leqq\deg\gamma$ such that*

$$f(z)=\sum_{\gamma\in\mathscr{E}_K^0}\sum_{i=1}^{\deg\gamma}a_i^\gamma f_i^\gamma(z),$$

which is absolutely convergent for any z in X.

(2) *Put $\phi_f^z(k)=f(kz)$. Then*

$$\phi_f^z=\sum_{\gamma\in\mathscr{E}_K^0}\frac{1}{\sqrt{\deg\gamma}}\sum_{i,j=1}^{\deg\gamma}a_i^\gamma f_j^\gamma(z)\phi_{ij}^\gamma,$$

which is absolutely and uniformly convergent in K.

(3) $$\|\phi_f^z\|^2=\sum_{\gamma\in\mathscr{E}_K^0}\frac{1}{\deg\gamma}\Big(\sum_{1=i}^{\deg\gamma}|a_i^\gamma|^2\Big)\Big(\sum_{j=1}^{\deg\gamma}|f_j^\gamma(z)|\Big)^2,$$

where $\|\quad\|$ denotes the norm of $L^2(K)$.

PROOF. By the theory of Fourier expansion of smooth functions on compact Lie groups (see [14]),

$$\phi_f^z=\sum_{\gamma\in\mathscr{E}_K}\sum_{i,j}b_{ij}^\gamma(z)\phi_{ij}^\gamma, \tag{2.1}$$

where the series converges absolutely and uniformly in K and

$$b_{ij}^\gamma(z)=\int_K f(kz)\overline{\phi_{ij}^\gamma(k)}dk.$$

Since

$$\begin{aligned}(L_k b_{ij}^\gamma)(z)&=b_{ij}^\gamma(k^{-1}z)\\&=\sum_l\tau_{lj}^\gamma(k)b_{il}^\gamma(z),\end{aligned}$$

b_{ij}^γ lies in $\mathscr{H}_\gamma$. Putting $k=$identity in (2.1), we have an absolutely convergent series

$$f(z)=\sum_{\gamma\in\mathscr{E}_K}\sum_i b_{ii}^\gamma(z)\sqrt{\deg\gamma}. \tag{2.2}$$

If $\sum_{i=1}^{\deg\gamma}b_{ii}^\gamma\neq0$, it follows from Proposition 2.1 that $\gamma\in\mathscr{E}_K^0$ and

$$\sqrt{\deg\gamma}\sum_{i=1}^{\deg\gamma}b_{ii}^\gamma=\sum_{i=1}^{\deg\gamma}a_i^\gamma f_i^\gamma \tag{2.3}$$

for some a_i^γ. Since z is arbitrary, replacing z by kz in (2.3), we have

$$\sqrt{\deg\gamma}\sum_i b_{ii}^\gamma(kz)=\sqrt{\deg\gamma}\sum_{i,l}\tau_{li}^\gamma(k^{-1})b_{il}^\gamma(z)$$
$$=\sum_{i,l}b_{il}^\gamma(z)\phi_{il}^\gamma(k)$$

and

$$\sum_i a_i^\gamma f_i^\gamma(kz)=\sum a_i^\gamma\int_K P(kz,\ k_0M)\phi_i^\gamma(k_0)dk_0$$
$$=\sum_i a_i^\gamma\int_K P(z,\ k^{-1}k_0M)\phi_i^\gamma(k_0)dk_0$$
$$=\sum_{i,l}\frac{1}{\sqrt{\deg\gamma}}a_i^\gamma\int_K P(z,\ k_0M)\phi_{il}^\gamma(k)\phi_l^\gamma(k_0)dk_0$$
$$=\sum_{i,l}\frac{1}{\sqrt{\deg\gamma}}a_i^\gamma f_l^\gamma(z)\phi_{il}^\gamma(k). \tag{2.4}$$

As ϕ_{il}^γ are linearly independent,

$$\frac{1}{\sqrt{\deg\gamma}}a_i^\gamma f_l^\gamma=b_{il}^\gamma.$$

Putting $i=l$ in the above equality, we obtain from (2.2) an absolutely convergent series

$$f(z)=\sum_{\gamma\in\mathcal{E}_K^0}\sum_i a_i^\gamma f_i^\gamma(z),$$

which proves (1).

Next, from (1) and (2.4) we have

$$\phi_f^z(k)=f(kz)=\sum_{\gamma\in\mathcal{E}_K^0}\sum_i a_i^\gamma f_i^\gamma(kz)$$
$$=\sum_{\gamma\in\mathcal{E}_K^0}\sum_{i,j}\frac{1}{\sqrt{\deg\gamma}}a_i^\gamma f_i^\gamma(z)\phi_{ij}^\gamma(k),$$

which proves (2) and (3) immediately. This completes the proof.

Now we transform the Casimir operator Ω. For $\lambda\in\mathfrak{a}^*$, the dual space of $\mathfrak{a}$, let $\bar{\lambda}$ denote the restriction of λ on $\mathfrak{a}_\mathfrak{p}$. Let P_+ denote the set of $\alpha\in P$ such that $\bar{\alpha}\neq 0$. For each root α, select $X_\alpha\in\mathfrak{g}^\alpha$ and normalize it in such a way that $<X_\alpha, X_{-\alpha}>=1$ where $<\ ,\ >$ is the Killing form of $\mathfrak{g}$. Then $[X_\alpha, X_{-\alpha}]=H_\alpha$ where H_α is the element in $\mathfrak{a}$ such that $<H, H_\alpha>=\alpha(H)$ for any $H\in\mathfrak{a}$. Choose bases H_1 and $H_2, \ldots, H_m$ for $\mathfrak{a}_\mathfrak{p}$ and $\mathfrak{a}_\mathfrak{k}$ respectively so that $<H_i, H_j>=\delta_{ij}$ for

$1 \leqq i, j \leqq m$. Then $H_1, \ldots, H_m$ together with $X_\alpha, X_{-\alpha}$ $(\alpha \in P)$ form a base for $\mathfrak{g}$. It is easy to see that $uf=0$ on X for $u \in \mathfrak{B}\mathfrak{k}$ and $f \in C^\infty(X)$. Hence we can transform Ω modulo $\mathfrak{B}\mathfrak{k}$.

It is clear that

$$\begin{aligned}\Omega &= H_1^2 + \cdots + H_m^2 + \sum_{\alpha \in P} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) \\ &\equiv H_1^2 + \sum_{\alpha \in P_+} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) \bmod \mathfrak{B}\mathfrak{k},\end{aligned} \tag{2.5}$$

since X_α, $X_{-\alpha}$ and H_i lie in k for $\alpha \in P - P_+$ and $i > 1$. For $\alpha \in P_+$, let $X_\alpha = Z_\alpha + Y_\alpha$ where $Z_\alpha \in \mathfrak{k}$ and $Y_\alpha \in \mathfrak{p}$ and put $X_\alpha^a = Ad(a) X_\alpha^a$ where $a = \exp H$ and $H \in \mathfrak{a}'_{\mathfrak{p}_0} = \mathfrak{a}_{\mathfrak{p}_0} - \{0\}$. Then

$$X_\alpha^a = Z_\alpha^a + Y_\alpha^a.$$

On the other hand

$$X_\alpha^a = e^{\alpha(H)} Z_\alpha + e^{\alpha(H)} Y_\alpha,$$

and we have

$$Z_\alpha^a + Y_\alpha^a = e^{\alpha(H)} Z_\alpha + e^{\alpha(H)} Y_\alpha. \tag{2.6}$$

Since $\theta(Z_\alpha^a + Y_\alpha^a) = Z_\alpha^{a^{-1}} - Y_\alpha^{a^{-1}}$, we have also

$$Z_\alpha^{a^{-1}} - Y_\alpha^{a^{-1}} = e^{\alpha(H)} Z_\alpha - e^{\alpha(H)} Y_\alpha. \tag{2.7}$$

In (2.6), replacing H by $-H$, we have

$$Z_\alpha^{a^{-1}} + Y_\alpha^{a-1} = e^{-\alpha(H)} Z_\alpha + e^{-\alpha(H)} Y_\alpha. \tag{2.8}$$

(2.8) together with (2.7) gives

$$Y_\alpha = (\coth \alpha(H)) Z_\alpha - (\sinh \alpha(H))^{-1} Z_\alpha^{a^{-1}}. \tag{2.9}$$

By the way, since

$$\begin{aligned}X_\alpha X_{-\alpha} &= X_\alpha (Z_{-\alpha} + Y_{-\alpha}) \\ &= X_\alpha Z_{-\alpha} + X_\alpha Y_{-\alpha} \\ &\equiv X_\alpha Y_{-\alpha} \\ &= (Z_\alpha + Y_\alpha) Y_{-\alpha},\end{aligned}$$

we get from (2.9) that

$$X_\alpha X_{-\alpha} \equiv \{(1 + \coth \alpha(H)) Z_\alpha - (\sinh \alpha(H))^{-1} Z_\alpha^{a^{-1}}\} Y_{-\alpha}$$

$$=(1+\coth\alpha(H))[Z_\alpha,\ Y_{-\alpha}]+(1+\coth\alpha(H))Y_{-\alpha}Z_\alpha$$

$$-(\sinh\alpha(H))^{-1}Z_\alpha^{a^{-1}}Y_{-\alpha}$$

$$\equiv(1+\coth\alpha(H))\ [Z_\alpha,\ Y_{-\alpha}]-(\sinh\alpha(H))^{-1}Z_\alpha^{a^{-1}}Y_{-\alpha}. \tag{2.10}$$

Hence we have

$$X_{-\alpha}X_\alpha\equiv(1-\coth\alpha(H))[Z_{-\alpha},\ Y_\alpha]+(\sinh\alpha(H))^{-1}Z_{-\alpha}^{a^{-1}}Y_\alpha.$$

Replacing H by $-H$, we find that

$$X_{-\alpha}X_\alpha\equiv(1+\coth\alpha(H))[Z_{-\alpha},\ Y_\alpha]-(\sinh\alpha(H))^{-1}Z_{-\alpha}^{a}Y_\alpha.$$

Therefore,

$$\theta(X_{-\alpha}X_\alpha)\equiv-(1+\coth\alpha(H))[Z_{-\alpha},\ Y_\alpha]$$
$$+(\sinh\alpha(H))^{-1}Z_{-\alpha}^{a}Y_\alpha, \tag{2.11}$$

since $[Z_{-\alpha},\ Y_\alpha]\in\mathfrak{p}$. From (2.10) and (2.11), we have

$$X_\alpha X_{-\alpha}+\theta(X_{-\alpha}X_\alpha)$$
$$\equiv(1+\coth\alpha(H))([Z_\alpha,\ Y_{-\alpha}]+[Y_\alpha,\ Z_{-\alpha}])$$
$$-(\sinh\alpha(H))^{-1}(Z_\alpha^{a^{-1}}Y_{-\alpha}-Z_{-\alpha}^{a^{-1}}Y_\alpha).$$

Since

$$[Z_\alpha,\ Y_{-\alpha}]+[Y_\alpha,\ Z_{-\alpha}]=\frac{1}{2}\{[X_\alpha,\ X_{-\alpha}]-[\theta X_\alpha,\ \theta X_{-\alpha}]\}$$

$$=\frac{1}{2}\{H_\alpha-\theta H_\alpha\}=H_{\bar{\alpha}},$$

We have

$$X_\alpha X_{-\alpha}+\theta(X_{-\alpha}X_\alpha)$$
$$\equiv(+\coth\alpha(H))H_{\bar{\alpha}}-(\sinh\alpha(H))^{-1}(Z_\alpha^{a^{-1}}Y_{-\alpha}+Z_{-\alpha}^{a^{-1}}Y_\alpha). \tag{2.12}$$

It is easy to see $\theta\Omega=\Omega$. Therefore from (2.5) and (2.12)

$$\Omega=\frac{1}{2}(\Omega+\theta\Omega)$$

$$\equiv H_1^2+\frac{1}{2}\sum_{\alpha\in P_+}\{(X_\alpha X_{-\alpha}+\theta(X_{-\alpha}X_\alpha))+(X_{-\alpha}X_\alpha+\theta(X_\alpha X_{-\alpha}))\}$$

$$=H_1^2+\frac{1}{2}\sum_{\alpha\in P_+}\{(1+\coth\alpha(H))H_{\bar\alpha}-(\sinh\alpha(H))^{-1}(Z_\alpha^{a^{-1}}Y_{-\alpha}-Z_{-\alpha}^{a^{-1}}Y_\alpha)$$

$$+(1-\coth\alpha(H))H_{-\bar\alpha}+(\sinh\alpha(H))^{-1}(Z_{-\alpha}^{a^{-1}}Y_\alpha-Z_\alpha^{a^{-1}}Y_{-\alpha})\}.$$

Noticing $H_{-\bar\alpha}=-H_{\bar\alpha}$, we get

$$\Omega\equiv H_1^2+\sum_{\alpha\in P_+}\{(\coth\alpha(H))H_{\bar\alpha}+(\sinh\alpha(H))^{-1}(Z_{-\alpha}^{a^{-1}}Y_\alpha-Z_\alpha^{a^{-1}}Y_{-\alpha})\}.$$

Since $Y_\alpha=(\coth\alpha(H))Z_\alpha-(\sinh\alpha(H))^{-1}Z_\alpha^{a^{-1}}$ from (2.9), we find that

$$\Omega\equiv H_1^2+\sum_{\alpha\in P_+}(\coth\alpha(H))H_{\bar\alpha}-\sum_{\alpha\in P_+}(\sinh\alpha(H))^{-2}(Z_\alpha^{a^{-1}}Z_{-\alpha}^{a^{-1}}+Z_{-\alpha}^{a^{-1}}Z_\alpha^{a^{-1}}).$$

Let $L_X(X\in g)$ be the differential of the left regular representation of G and extend it to the representation of $\mathfrak{B}$. Then

$$(X^{x^{-1}}f)\,(x)=(L_{-X}f)\,(x)$$

for $x\in G, f\in C^\infty(G)$ and $X\in\mathfrak{g}$. Hence

$$(\Omega f)\,(a)=[\{H_1^2+\sum_{\alpha\in P_+}(\coth\alpha(H))H_{\bar\alpha}$$

$$-\sum_{\alpha\in P_+}(\sinh\alpha(H))^{-2}L_{Z_\alpha Z_{-\alpha}+Z_{-\alpha}Z_\alpha}\}f]\,(a).$$

Let μ_0 be the restricted root such that $\frac{1}{2}\mu_0$ is not a restricted root. Let P_{μ_0} (resp. $P_{2\mu_0}$) be the set of positive root α such that $\bar\alpha$ is equal to μ_0 (resp. $2\mu_0$). Let p and q denote the number of roots in P_{μ_0} and $P_{2\mu_0}$ respectively. We normalize H_0 in $\mathfrak{a}_{\mathfrak{p}_0}$ so that $\mu_0(H_0)=1$. Then $<H_0, H_0>=2(p+4q)$ and $H_1=(2p+8q)^{-1/2}H_0$. Put $a_t=\exp tH_0$ for $t\in\boldsymbol{R}$. Then t can be regarded as the coordinate function on the one-dimensional Lie group A. It is evident that $H_{\mu_0}=(2p+8q)^{-1}H_0$ and

$$(\Omega f)\,(a_t)=\Big\{\frac{1}{2(p+4q)}\frac{d^2}{dt^2}+\frac{p\coth t}{2(p+4q)}\frac{d}{dt}$$

$$+\frac{q\coth 2t}{2(p+4q)}\cdot 2\frac{d}{dt}\Big\}f(a_t)$$

$$-\Big[\Big\{\frac{1}{(\sinh t)^2}\sum_{\alpha\in P_{\mu_0}}L_{Z_\alpha Z_{-\alpha}+Z_{-\alpha}Z_\alpha}$$

$$+\frac{1}{(\sinh 2t)^2}\sum_{\alpha\in P_{2\mu_0}}L_{Z_\alpha Z_{-\alpha}+Z_{-\alpha}Z_\alpha}\Big\}f\Big]\,(a_t).$$

Therefore, we have

PROPOSITION 2.3. *For* $f\in C^\infty(X)$,

$$(\Omega f)(a_t)=Df(a_t)$$
$$-\frac{1}{(\sinh t)^2}(L_{\omega_1}f)(a_t)$$
$$-\left\{\frac{1}{(\sinh 2t)^2}-\frac{1}{(\sinh t)^2}\right\}(L_{\omega_2}f)(a_t),$$

where $a_t=\exp tH_0$,

$$D=\frac{1}{2(P+4q)}\left\{\frac{d^2}{dt^2}+(p\coth t+2q\coth 2t)\frac{d}{dt}\right\},$$

$$\omega_1=\sum_{\alpha\in P_+}(Z_\alpha Z_{-\alpha}+Z_{-\alpha}Z_\alpha)$$

and

$$\omega_2=\sum_{\alpha\in P_{2\mu_0}}(Z_\alpha Z_{-\alpha}+Z_{-\alpha}Z_\alpha).$$

§3. Hermitian hyperbolic spaces

From now on, we deal with the case that $X=G/K$ is a hermitian hyperbolic space. That is, we deal with the case of $G=SU(n, 1)$. We compute ω_1 and ω_2 defined in section 2. At first we review the structure of the Lie algebra $\mathfrak{g}_0=\mathfrak{su}(n, 1)$. Put

$$\mathfrak{k}_0=\left\{\begin{pmatrix} Z & 0 \\ 0 & z\end{pmatrix}\middle|\ \begin{matrix} Z\in\mathfrak{u}(n),\ z\in\mathfrak{u}(1) \\ \mathrm{Tr}(Z)+z=0\end{matrix}\right\}$$

and

$$\mathfrak{p}_0=\left\{\begin{pmatrix} 0 & \eta \\ {}^t\bar{\eta} & 0\end{pmatrix}\middle|\ \eta\in\boldsymbol{C}^n\right\}.$$

Then $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0$ is a Cartan decomposition and negative conjugate transpose is the corresponding Cartan involution. Lie algebra $\mathfrak{k}=\mathfrak{k}_0^C$ and $\mathfrak{p}=\mathfrak{p}_0^C$ in $\mathfrak{g}=\mathfrak{g}_0^C=\mathfrak{sl}(n+1, \boldsymbol{C})$ are given as follows:

$$\mathfrak{k}=\left\{\begin{pmatrix} Z & 0 \\ 0 & z\end{pmatrix}\middle|\ \begin{matrix} Z\ n\times n \text{ complex matrix},\ z\in\boldsymbol{C}^n \\ \mathrm{Tr}(Z)+z=0\end{matrix}\right\},$$

$$\mathfrak{p}=\left\{\begin{pmatrix} 0 & \eta \\ {}^t\xi & 0\end{pmatrix}\middle|\ \xi,\ \eta\in\boldsymbol{C}^n\right\}.$$

Let $\mathfrak{h}_0$ be the set of diagonal elements of $\mathfrak{k}_0$. Then $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ and $\mathfrak{h}=\mathfrak{h}_0^c$ which consists of the diagonal elements of $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{k}$. Let $e_j(1\leqq j\leqq n+1)$ be the linear functional on $\mathfrak{h}$ whose value on a diagonal matrix is the j-th diagonal entry. Then roots of $(\mathfrak{g}, \mathfrak{h})$ are the differences $e_i-e_j(1\leqq i,\ j\leqq n+1)$. Choose an order so that the positive roots are $e_i-e_j(1\leqq i<j\leqq n+1)$. Let Q, Q_k and Q_n be the sets of positive, compact positive and non-compact positive roots respectively. Then, putting $\beta_{ij}=e_i-e_j$ $(1\leqq i, j\leqq n+1)$,

$$Q=\{\beta_{ij}|1\leqq i<j\leqq n+1\},$$

$$Q_k=\{\beta_{ij}|1\leqq i<j\leqq n\}$$

and

$$Q_n=\{\beta_{i,n+1}|1\leqq i\leqq n\}.$$

The root subspace $\mathfrak{g}^{\beta_{ij}}$ of β_{ij} is equal to $\boldsymbol{C}E_{ij}$ where $E_{ij}(1\leqq i, j\leqq n+1)$ is the matrix unit. Hence we have the following decompositions:

$$\mathfrak{g}=\mathfrak{h}+\sum_{\pm\beta\in Q}\mathfrak{g}^{\beta},$$

$$\mathfrak{k}=\mathfrak{h}+\sum_{\pm\beta\in Q_k}\mathfrak{g}^{\beta},$$

$$\mathfrak{p}=\sum_{\pm\beta\in Q_n}\mathfrak{g}^{\beta}.$$

The Killing form $<\ ,\ >$ in $\mathfrak{g}$ is given by

$$<X,\ Y>=2(n+1)\mathrm{Tr}(XY),\ X \text{ and } Y\in\mathfrak{g},$$

where Tr denotes the trace of the matrix of order $n+1$. For $\lambda\in\mathfrak{h}^*$, let H_λ be the element in $\mathfrak{h}$ such that $<H_\lambda,\ H>=\lambda(H)$ for $H\in\mathfrak{h}$. If $\lambda,\ \mu\in\mathfrak{h}^*$, put $<\lambda,\ \mu>$ $=<H_\lambda,\ H_\mu>$. For simplicity, we write β_0 for $\beta_{1,n+1}$.

Put $\mathfrak{h}_+=\sqrt{-1}\boldsymbol{R}H_{\beta_0}$ and $\mathfrak{h}_-=\{H\in\mathfrak{h}_0|<H_{\beta_0},\ H>=0\}$. Then $\mathfrak{h}_0=\mathfrak{h}_++\mathfrak{h}_-$ (direct sum). Put $E'_{\beta_0}=E_{1,n+1}$ and $E'_{-\beta_0}=E_{n+1,1}$. Then $<E'_{\beta_0},\ E'_{-\beta_0}>=2<\beta_0,$ $\beta_0>^{-1}$, $E'_{\beta_0}-E'_{-\beta_0}\in\sqrt{-1}\mathfrak{k}_0$ and $\sqrt{-1}(E'_{\beta_0}+E'_{-\beta_0})\in\sqrt{-1}\mathfrak{p}_0$. Put $\mathfrak{a}_{\mathfrak{p}_0}=\boldsymbol{R}(E'_{\beta_0}$ $+E'_{-\beta_0})$, $\mathfrak{a}_{\mathfrak{k}_0}=\mathfrak{h}_-$, $\mathfrak{a}_0=\mathfrak{a}_{\mathfrak{k}_0}+\mathfrak{a}_{\mathfrak{p}_0}$, $\mathfrak{a}=\mathfrak{a}_0^c$ and $u=\exp\frac{\pi}{4}(E'_{\beta_0}-E'_{-\beta_0})$. Then $Ad(u)$ is the identity on $\mathfrak{a}_{\mathfrak{k}_0}$, $Ad(u)\ \mathfrak{a}_{\mathfrak{p}_0}=\sqrt{-1}\mathfrak{h}_+$, $Ad(u)\mathfrak{a}=\mathfrak{h}$ and $\mathfrak{a}_0$ is a θ-stable Cartan subalgebra of $\mathfrak{g}_0$ ([7], [13]). It is easy to see that $\mathfrak{a}_{\mathfrak{p}_0}$, $\mathfrak{a}_0$ and $\mathfrak{a}_{\mathfrak{k}_0}$ satisfy the conditions in section 2. Hence we can take the above subalgebras as those defined in section 2. We can assume that P, the set of positive roots of $(\mathfrak{g}, \mathfrak{a})$ defined in section 2, is ${}^tAd(u)Q$. It is easy to see that μ_0 is equal to the half of the restriction of $\alpha_0={}^tAd(u)\beta_0$ on $\mathfrak{a}_{\mathfrak{p}_0}$. Putting $\alpha_{ij}={}^tAd(u)\beta_{ij}$, we have

$$P_+ = \{\alpha_0 = \alpha_{1,n+1}, \alpha_{1i}(1<i<n+1), \alpha_{j,n+1}(1<j<n+1)\},$$

$$P_{\mu_0} = \{\alpha_{1i}, \alpha_{j,n+1}(1<i, j<n+1)\}$$

and

$$P_{2\mu_0} = \{\alpha_0\}.$$

Put $E_{\beta_{ij}} = (2n+2)^{-1/2}E_{ij}$ and $X_{\alpha_{ij}} = Ad(u^{-1})E_{\beta_{ij}}(1 \leqq i, j \leqq n+1)$. Since $E_{\beta_{ij}} \in \mathfrak{g}^{\beta_{ij}}$ and $<E_{\beta_{ij}}, E_{-\beta_{ij}}> = 1$, $X_{\alpha_{ij}} \in \mathfrak{g}^{\alpha_{ij}}$ and $<X_{\alpha_{ij}}, X_{-\alpha_{ij}}> = 1$. Therefore for calculation of ω_1 and ω_2, we have only to see the $\mathfrak{k}$-component Z_α of X_α for any root α. Practising the above calculation, we have

LEMMA 3.1.

$$Z_{\alpha_0} = Z_{-\alpha_0} = -\{(n+1)/2\}^{1/2}H_{\beta_0},$$

$$Z_{\alpha_{1i}} = \frac{\sqrt{2}}{2}E_{\beta_{1i}}(1<i<n+1),$$

$$Z_{-\alpha_{1i}} = \frac{\sqrt{2}}{2}E_{-\beta_{1i}}(1<i<n+1),$$

$$Z_{\alpha_{j,n+1}} = -\frac{\sqrt{2}}{2}E_{-\beta_{1j}}(1<j<n+1),$$

$$Z_{-\alpha_{j,n+1}} = -\frac{\sqrt{2}}{2}E_{\beta_{1j}}(1<j<n+1).$$

Let $\mathfrak{m}$ be the Lie algebra of M which is the centralizer of A in K. Then

$$\mathfrak{m} = \sum_{\pm\alpha \in P-P_+} \mathfrak{g}^\alpha = \sum_{1<i,j<n+1} \mathfrak{g}^{\beta_{ij}},$$

because $Ad(u)$ is the identity on $\mathfrak{a}_{\mathfrak{k}_0}$. Let $\mathfrak{M}$ be the subalgebra of $\mathfrak{B}$ generated by $\mathfrak{m}$. By Lemma 3.1, we have that

$$\omega_2 = \sum_{\alpha \in P_{2\mu_0}} (Z_\alpha Z_{-\alpha} + Z_{-\alpha}Z_\alpha)$$

$$= 2\left(\frac{n+1}{2}\right)H_{\beta_0}^2 = (n+1)H_{\beta_0}^2$$

and

$$\omega_1 = \sum_{\alpha \in P_+} (Z_\alpha Z_{-\alpha} + Z_{-\alpha}Z_\alpha)$$

$$= \omega_2 + \sum_{\alpha \in P_{\mu_0}} (Z_\alpha Z_{-\alpha} + Z_{-\alpha}Z_\alpha)$$

$$= \omega_2 + \sum_{1<i<n+1} (E_{\beta_{1i}}E_{-\beta_{1i}} + E_{-\beta_{1i}}E_{\beta_{1i}})$$

$$\equiv \omega_2 + \sum_{\beta \in Q_k} (E_\beta E_{-\beta} + E_{-\beta} E_\beta) \bmod \mathfrak{M}.$$

Let ν be the negative of the restriction of $<\ ,\ >$ on $\mathfrak{k}_0$. Then ν is an $Ad(K)$-invariant inner product of $\mathfrak{k}_0$. Let ω be the Laplace-Beltrami operator corresponding to the riemannian metric induced by ν. Since $<E_\beta, E_{-\beta}>=1$ for $\beta \in Q_k$, $<\mathfrak{h}_+, \mathfrak{h}_->=0$ and $<\sqrt{n+1}H_{\beta_0}, \sqrt{n+1}H_{\beta_0}>=1$, we have

$$\omega_K = \sum_{\beta \in Q_k} (E_\beta E_{-\beta} + E_{-\beta} E_\beta) + (n+1) H_{\beta_0}{}^2 \bmod \mathfrak{M}.$$

Therefore $\omega_1 \equiv \omega_K \bmod \mathfrak{M}$ and $\omega_2 = (n+1) H_{\beta_0}{}^2$.

As M normalizes A, $f(a \exp tY) = f((\exp tY)a)$ for $a \in A$, $t \in \boldsymbol{R}$ and $Y \in \mathfrak{m}$. Therefore we have

$$(L_u f)(a) = 0 \tag{3.1}$$

for $u \in \mathfrak{M}$ and $f \in C^\infty(G/K)$. Let $Z_c = (n+1)^{-1}(\sum_{i=1}^n E_{ii} - nE_{n+1,n+1})$ and $Z_m = (n+1)^{-1}\{(n-1)E_{ii} + (n-1)E_{n+1,n+1} - 2\sum_{i=2}^n E_{ii}\}$. Then Z_c lies in the center of $\mathfrak{k}$, Z_m lies in $\mathfrak{m}$ and $H_{\beta_0} = (2n+2)^{-1}(2Z_c + Z_m)$, as $H_{\beta_0} = (2n+2)^{-1}(E_{11} - E_{n+1,n+1})$. Hence

$$\begin{aligned} L_{\omega_2} &= (n+1) L_{H_{\beta_0}^2} \\ &= (4n+4)^{-1}(4L_{Z_c^2} + 4L_m L_{Z_c} + L_{Z_m^2}). \end{aligned}$$

By (3.1), we conclude that

$$(L_{\omega_1} f)(a) = (L_{\omega_K} f)(a) \tag{3.2}$$

and

$$(L_{\omega_2} f)(a) = (n+1)^{-1}(L_{Z_c^2} f)(a). \tag{3.3}$$

Let L be the set of dominant integral form of $(\mathfrak{k}, \mathfrak{h})$. Then

$$L = \{\Lambda = l_1 e_1 + l_2 e_2 + \cdots + l_n e_n\},$$

where $l_i (1 \leqq i \leqq n)$ are integers such that $l_1 \geqq l_2 \geqq \cdots \geqq l_n$. As is easily seen, for $G = SU(n, 1)$, there exists a bijection $\gamma \leftrightarrow \Lambda_\gamma$ of $\mathscr{E}_K$ onto L. Let L^0 denote the image of $\mathscr{E}_K^0$ by this bijection. Since $\Lambda_\gamma (\in \mathscr{E}_K^0)$ vanishes on $\mathfrak{h}_-$, we have

$$L^0 = \{\Lambda_l = l(2e_1 + e_2 + \cdots + e_n) \mid l \in \mathbf{Z}^+\}.$$

From now on, we identify L^0 with $\mathbf{Z}^+$ and write τ_l, V_l, $\mathscr{H}_l$, ϕ_i^l, f_i^l and $\deg(l)$ instead of τ_γ, V_γ, $\mathscr{H}_\gamma$, ϕ_i^γ, f_i^γ and $\deg\gamma$. Put $\rho_k = 2^{-1} \sum_{\beta \in Q_k} \beta$. Then, as is well-known, for $f \in \mathscr{H}_l$ we have

$$L_{\omega_K} f = <\Lambda_l + 2\rho_K, \Lambda_l> f \tag{3.4}$$

and

$$L_{Z_c} f = \Lambda_l(Z_c) f. \tag{3.5}$$

Put $H_l = H_{\Lambda_l}$. Then $H_l = l(2n+2)^{-1}(E_{11} - E_{n+1,n+1})$. Since $2\rho = \sum_{1 \leq i < j \leq n} (e_i - e_j)$, we have

$$\begin{aligned} <2\rho_k, \Lambda_l> &= 2\rho_k(H_l) \\ &= (n-1)e_1(H_l) \\ &= (n-1)\,(2n+2)^{-1} l, \end{aligned}$$

and

$$\begin{aligned} <\Lambda_l, \Lambda_l> &= \Lambda_l(H_l) \\ &= l(2e_1 + e_2 + \cdots + e_n)(H_l) \\ &= (n+1)^{-1} l^2. \end{aligned}$$

Hence, we have

$$<\Lambda_l + 2\rho_k, \Lambda_l> = (2n+2)^{-1}\{2l^2 + (n-1)l\}.$$

On the other hand, since $Z_c = (n+1)^{-1}(\sum_{i=1}^{n} E_{ii} - nE_{n+1,n+1})$, we have

$$\begin{aligned} \Lambda_l(Z_c) &= l(2e_1 + e_2 + \cdots + e_n)\,(Z_c) \\ &= l. \end{aligned}$$

Therefore, from (3.2), (3.3), (3.4) and (3.5) we have

LEMMA 3.2. *For $f_l \in \mathscr{H}_l$, the following equations hold.*

$$(L_{\omega_1} f)\,(a) = (2n+2)^{-1}\{2l^2 + (n-1)l\} f(a)$$

$$(L_{\omega_2} f)\,(a) = (n+1)^{-1} l^2 f(a).$$

The above lemma together with Proposition 2.3 gives

PROPOSITION 3.3. *Let $l \in L^0$, $f \in \mathscr{H}_l$ and F be the restriction of f on A. Then F satisfies the following differential equation*

$$DF = 0,$$

where

$$D=\frac{d^2}{dt^2}+2\{(n-1)\coth t+\coth 2t\}\frac{d}{dt}$$

$$-\left[\frac{4l^2+2(n-1)l}{(\sinh t)^2}-4l^2\left\{\frac{1}{(\sinh t)^2}-\frac{1}{(\sinh 2t)^2}\right\}\right].$$

PROOF. Since $\Omega f=0$ and $p=2(n-1)$, $q=1$ in case of $SU(n, 1)$, we have this proposition immediately from Proposition 2.3 and Lemma 3.2. This completes the proof.

We introduce a new parameter z. Put $z=(\tanh t)^2$. Then the differential equation in Proposition 3.3 turns into

$$z(1-z)^2\frac{d^2F}{dz^2}+(1-z)\,(n-z)\frac{dF}{dz}$$

$$-\frac{1-z}{4z}\{l^2(1-z)+2(n-1)l\}F=0.$$

A fundamental system of solutions of the above differential equation is given by

$$z^{l/2} \text{ and } z^{-l/2-(n-1)}F(-(n-1),\ -l-(n-1),\ -l-n+2;z),$$

where $F(-(n-1),\ -l-(n-1),\ -l-n+2;\ z)$ is the hypergeometric function. Since $F(z)$ must be a C^∞-function in t, there exists a complex number c such that $F(z)=cz^{l/2}$. Thus we have

LEMMA 3.4. *For $f\in\mathscr{H}_l$, there exists a complex number c such that for $t\in\boldsymbol{R}$,*

$$f(a_t)=c(\tanh t)^l.$$

By Lemma 3.4, there exists a complex number c_i^l for $l\in L^0$ and $1\leqq i\leqq \deg(l)$ such that

$$f_i^l(a_t)=c_i^l(\tanh t)^l.$$

On the other hand, A. W. Knapp proved ([6], Theorem 1.1) that in case of rank(X) $=1$

$$\lim_{t\to\infty}(\mathscr{P}\phi)\,(ka_t)=\phi(k)\quad \text{a.e. } k\in K$$

where ϕ is an integrable function on $B=K/M$. Therefore we have

$$c_i=\lim_{t\to\infty}c_i(\tanh t)^l=\phi_i^l(e)$$
$$=\sqrt{\deg(l)}\,\delta_{i1}.$$

Thus we obtain

PROPOSITION 3.5. *For* $l \in L^0$,

$$f_1^l(a_t)=\sqrt{\deg(l)}\,(\tanh t)^l,$$

$$f_i^l(a_t)\equiv 0 \quad (2\leqq i\leqq \deg(l)).$$

§4. Poisson transforms of hyperfunctions

In this section we keep to the notation in the previous sections. Let

$$\mathscr{F}_b=\{(a_i^l)_{1\leqq i\leqq \deg(l)}^{l\in L^0} \mid a_i^l\in \boldsymbol{C},\ \sum_{l\in L^0}\sum_i |a_i^l| e^{-t\sqrt{\lambda_l}}<\infty \qquad \text{for any } t>0\},$$

where $\lambda_l=<\Lambda_l+2\rho_k,\ \Lambda_l>=(2n+2)^{-1}\{2l^2+(n-1)l\}$. By an easy computation, we have the following

LEMMA 4.1. *For every non-negative integer* l,

$$\frac{l}{\sqrt{n+1}}\leqq \sqrt{\lambda_l}\leqq l.$$

For $s>0$, put

$$U_s=\{z=ka_tK\in X \mid k\in K,\ |\tanh t|\leqq e^{-2s}\}.$$

We assume that $z=ka_tK\in U_s$ and consider the series $S=\sum_{l\in L^0}\sum_i |a_i^l|\,|f_i^l(z)|$ in U_s for $(a_i^l)\in\mathscr{F}_b$. Since $f_i^l(ka_t)=\sum_j f_j^l(a_t)\overline{\tau_{ij}^l(k)}$ and $|\tau_{ij}^l(k)|\leqq 1$, we have

$$S\leqq \sum_{l\in L^0}\sum_{i,j}|a_i^l|\,|f_j^l(a_t)|.$$

Using Proposition 3.5, we have

$$\begin{aligned} S&\leqq \sum_{l\in L^0}\sum_i |a_i^l|\sqrt{\deg(l)}\,r^l \\ &=\sum_{l\in L^0}\sum_i |a_i^l|\{(\sqrt{\deg(l)})^{1/2l}r\}^l, \end{aligned} \tag{4.1}$$

where $r=|\tanh t|$.

Since $\deg(l)$ is a polynomial function in l (Weyl's dimension formula), $\lim_{l\to\infty}(\deg(l))^{1/2l}=1$. Therefore there exists an integer l_0 such that

$$(\deg(l))^{1/2l}e^{-2s}\leqq e^{-s}$$

for any $l>l_0$. Then from (4.1) we have

$$S\leqq \sum_{l=0}^{l_0}\sum_i |a_i^l|\sqrt{\deg(l)}\,r^l$$

$$+\sum_{l=l_0+1}^{\infty}\sum_i |a_i^l|\{\deg(l)^{1/2l}r\}^l$$
$$\leqq \sum_{l=0}^{l_0}\sum_i |a_i^l|\sqrt{\deg(l)}$$
$$+\sum_{l=l_0+1}^{\infty}\sum_i |a_i^l|e^{-sl},$$

for $z\in U_s$. On the other hand, from Lemma 4.1, $l\geqq\sqrt{\lambda_l}$. Therefore, we obtain an inequality

$$S\leqq\sum_{l=0}^{l_0}\sum_i |a_i^l|\sqrt{\deg(l)}+\sum_{l=l_0+1}^{\infty}\sum_i |a_i^l|e^{-s\sqrt{\lambda_l}}$$

for $z\in U_s$ and $(a_i^l)\in\mathscr{F}_b$. This implies that the series

$$\sum_{l\in L^0}\sum_{i=1}^{\deg(l)} a_i^l f_i^l(z)$$

converges absolutely and uniformly in U_s. Since $f_i^l\in\mathscr{H}=\mathscr{H}(X)$ $(l\in L^0,\ 1\leqq i\leqq\deg(l))$ and every compact subset is contained in U_s for some $s>0$, it follows that $\sum_{l\in L^0}\sum_i a_i^l f_i^l(z)$ lies in $\mathscr{H}$. Thus we have

LEMMA 4.2. *For $(a_i^l)\in\mathscr{F}_b$, the series $\sum_{l\in L^0}\sum_i a_i^l f_i^l(z)$ converges absolutely and uniformly in every compact set of X and defines a harmonic function on X.*

Conversely, if $f\in\mathscr{H}$, by Proposition 2.2, we have an expansion

$$f(z)=\sum_{l\in L^0}\sum_i a_i^l f_i^l(z)$$

and obtain

LEMMA 4.3. *The sequence (a_i^l) in the above expansion lies in $\mathscr{F}_b$.*

PROOF. From Proposition 2.2 in §2, we have

$$\|\phi_f^z\|^2=\sum_{l\in L^0}\frac{1}{\deg(l)}\Big(\sum_i |a_i^l|^2\Big)\Big(\sum_j |f_j^l(z)|^2\Big)$$
$$\geqq\sum_{l\in L^0}\sum_i |a_i^l|^2\frac{1}{\deg(l)}|f_1^l(z)|^2.$$

Putting $z=a_t$ and using Proposition 3.5, we have

$$\|\phi_f^z\|^2\geqq\sum_{l\in L^0}\sum_i |a_i^l|^2\frac{1}{\deg(l)}\deg(l)r^{2l}$$

$$= \sum_{l\in L^0} \sum_i |a_i^l|^2 r^{2l}$$

where $r = |\tanh t|$. Since $0<r<1$ and $l \leqq \sqrt{n+1}\sqrt{\lambda_l}$ (Lemma 4.1), we have

$$\|\phi_f^z\|^2 \geqq \sum_{l\in L^0} \sum_i |a_i^l|^2 (r^{2\sqrt{n+1}})^{\sqrt{\lambda_l}} \tag{4.2}$$

For an arbitrary $s>0$, choose a $t\in\boldsymbol{R}$ so that $r^{2\sqrt{n+1}} = e^{-s}$. Then from (4.2) we obtain

$$\|\phi_f^z\|^2 \geqq \sum_{l\in L^0} \sum_i |a_i^l|^2 e^{-s\sqrt{\lambda_l}}$$

which means, by the remark following Theorem 1.8 in §1, that (a_i^l) lies in $\mathscr{F}_b$. This completes the proof.

Now we define the Poisson transform of a hyperfunction on B. Let $T\in\mathscr{B}$. Since $P(z, b)$ is a real analytic function in b, we can operate T on $P(z, b)$ and $T(P(z, b))$ is a function on X. We denote this function by $\mathscr{P}(T)$ and call it the Poisson transform of T. By Theorem 1.8, there exists an isomorphism Ψ of $\mathscr{B}$ onto $\mathscr{F}_b$. Then we have

LEMMA 4.4. *Let $T\in\mathscr{B}$ and $(a_i^l) = \Psi(T)$. Then, for any $z\in X$,*

$$\mathscr{P}(T)(z) = \sum_{l\in L^0} \sum_{i=1}^{\deg l} a_i^l f_i^l(z).$$

PROOF. Fix an arbitrary z in X. Then from Corollary 1 to Proposition 1.7, $P(z, b)$ has an expansion

$$P(z, b) = \sum_{l\in L^0} \sum_{i=1}^{\deg(l)} \phi_i^l(b) \int_K P(z, kM)\overline{\phi_i^l}(k)dk, \tag{4.3}$$

which converges in $\mathscr{A}(B)$. Since $P(z, b)$ is real-valued and $f_i^l(z) = \int_K P(z, kM)\phi_i^l(k)dk$, taking complex conjugate of (4.3), we have

$$P(z, b) = \sum_{l\in L^0} \sum_i f_i^l(z)\overline{\phi_i^l}(b),$$

which also converges in $\mathscr{A}(B)$. Therefore

$$T(P(z, b)) = \sum_{l\in L^0} \sum_i f_i^l(z) T(\overline{\phi}_i^l).$$

From the definition of $\Psi(T)$, $a_i^l = T(\overline{\phi}_i^l)$, which finishes the proof.

Now we are in position to state the main

THEOREM 4.5. *Poisson transform $\mathscr{P}$ is an isomorphism of $\mathscr{B}(B)$ onto $\mathscr{H}(X)$, where X is a hermitian hyperbolic space.*

PROOF. Lemma 4.2 together with Lemma 4.4 implies that the image of $\mathscr{P}$

is contained in $\mathscr{H}$. Lemma 4.3 implies the surjectivity of $\mathscr{P}$. Let T satisfy $\mathscr{P}(T)=0$. Then putting $\Psi(T)=(a_i^l)$, we have

$$\sum_{l\in L^0}\sum_i a_i^l f_i^l(z)=0$$

for any $z\in X$. Replacing z by ka_t, we have from (2.4) and Proposition 3.5,

$$\sum_{l\in L^0}\sum_i (\tanh t)^l a_i^l \phi_i^l(k)=0$$

for $k\in K$. Since ϕ_i^l are linearly independent, we can deduce that $a_i^l=0$ for $l\in L^0$ and $1\leqq i\leqq \deg(l)$. Hence $T=0$. This completes the proof of the theorem.

REMARK. We can identify a C^∞-function ϕ on B with the hyperfunction defined by

$$\mathscr{A}(B)\ni\psi\to\int_K \psi(k)\phi(k)dk.$$

Then the Poisson transform of a hyperfunction ϕ coincides with the Poisson transform of a C^∞-function ϕ defined in §2.

Added in proof.

Theorem 4.5 is valid although one needs two parameters of integers to characterize $\mathscr{E}_K^0$, which contains L^0 properly. The proof in general case involves some technical skill and will be found in the forthcoming paper of the second author.

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