HIROSHIMA MATH. J.
3 (1973), 37–60

Energy of Functions on a Self-adjoint Harmonic Space II

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(Received January 19, 1973)

Introduction

In the previous paper [13] under the same title, we introduced a notion of energy of functions on a self-adjoint harmonic space. By a self-adjoint harmonic space, we mean a Brelot's harmonic space possessing a symmetric Green function. We showed that a notion of energy which is given in terms of differentiation in the classical case can be defined on such an abstract harmonic space. In [13], however, we defined energy only for certain bounded functions and for harmonic functions. In the present paper, we shall extend the definition to more general functions, which correspond to BLD-functions (see [10] and [5]) or Dirichlet functions (see [9]) in the classical potential theory.

Here, let us review basic definitions and main results in [13].

The base space Ω is a connected, locally connected, noncompact, locally compact Hausdorff space with a countable base. We consider a structure of harmonic space $\mathfrak{H}=\{\mathscr{H}(\omega)\}_{\omega:\mathrm{open}\subset\Omega}$ on Ω satisfying Axioms 1, 2 and 3 of M. Brelot [4]. In addition to these axioms, we assume:

Axiom 4. The constant function 1 is superharmonic.

Axiom 5. There exists a positive potential on Ω.

Axiom 6. Two positive potentials with the same point (harmonic) support are proportional.

The pair $(\Omega, \mathfrak{H})$ is called a *self-adjoint harmonic space* if there exists a function $G(x, y)$: $\Omega\times\Omega\to(0, +\infty]$ such that $G(x, y)=G(y, x)$ for all $x, y\in\Omega$ and, for each $y\in\Omega$, $x\to G(x, y)$ is a potential on Ω and is harmonic on $\Omega-\{y\}$. Such $G(x, y)$ is uniquely determined up to a multiplicative constant and is called a *Green function* for $(\Omega, \mathfrak{H})$. In our theory, we assume that $(\Omega, \mathfrak{H})$ is a self-adjoint harmonic space and fix a Green function $G(x, y)$ throughout. For any domain ω in Ω, $\mathfrak{H}|\omega=\{\mathscr{H}(\omega')\}_{\omega'\subset\omega}$ is also a structure of self-adjoint harmonic space on ω satisfying Axioms $1\sim6$ and there is a Green function $G^{\omega}(x, y)$ for $(\omega, \mathfrak{H}|\omega)$ having the same singularity as $G(x, y)$ (see Proposition 1.2). For a non-negative measure (=Radon measure) μ on Ω (resp. on ω) $U^{\mu}(x)=\int_{\Omega}G(x, y)d\mu(y)$ (resp.

$U_\omega^\mu(x)=\int_\omega G^\omega(x, y)d\mu(y))$ gives a potential on Ω (resp. on ω) if it is not constantly infinite. Conversely, to any superharmonic function s on Ω, there corresponds a unique non-negative measure σ_s on Ω such that $s|\omega=U_\omega^{\sigma_s}+u_\omega$ with $u_\omega\in\mathscr{H}(\omega)$ for any relatively compact domain ω. We use the symbols: $\pi\equiv\sigma_1$ and $\mu_u\equiv\sigma_{-u^2}$ for $u\in\mathscr{H}(\Omega)$. If a function f on Ω is expressed as $f=s_1-s_2$ with finite-valued superharmonic functions s_1 and s_2, then $\sigma_f=\sigma_{s_1}-\sigma_{s_2}$ is determined by f as a signed measure on Ω. We consider the classes

$$\mathbf{M}_B(\Omega)=\{\mu;\ \text{non-negative measure on } \Omega,\ U^\mu \text{ is bounded and } \mu(\Omega)<+\infty\},$$

$$\mathbf{H}_{BE}(\Omega)=\{u\in\mathscr{H}(\Omega);\ \text{bounded and } \mu_u(\Omega)<+\infty\}$$

and

$$\mathbf{B}_E(\Omega)=\{u+U^\mu-U^\nu;\ u\in\mathbf{H}_{BE}(\Omega) \text{ and } \mu,\nu\in\mathbf{M}_B(\Omega)\}.$$

For $f, g\in\mathbf{B}_E(\Omega)$, their *mutual energy* is defined by

$$E_\Omega[f, g]=\frac{1}{2}\{\int_\Omega f d\sigma_g+\int_\Omega g\, d\sigma_f-\sigma_{fg}(\Omega)+\int_\Omega fg\, d\pi\},$$

which makes sense as a finite value. The *energy* of $f\in\mathbf{B}_E(\Omega)$ is defined by $E_\Omega[f]=E_\Omega[f, f]$. The main results in Chapter II are:

PROPOSITION 2.1. *If $u\in\mathbf{H}_{BE}(\Omega)$, then $E_\Omega[u]\geqq 0$.*

THEOREM 2.1. *If $\mu\in\mathbf{M}_B(\Omega)$, then $E_\Omega[U^\mu]=\int_\Omega U^\mu d\mu$.*

COROLLARY. *If $f_i=U^{\mu_i}-U^{\nu_i}$, $i=1, 2$, with $\mu_1, \mu_2, \nu_1, \nu_2\in\mathbf{M}_B(\Omega)$, then*

$$E_\Omega[f_1, f_2]=\int_\Omega f_1(d\mu_2-d\nu_2)=\int_\Omega f_2(d\mu_1-d\nu_1).$$

THEOREM 2.2. *If $u\in\mathbf{H}_{BE}(\Omega)$ and $\mu\in\mathbf{M}_B(\Omega)$, then $E_\Omega[u, U^\mu]=0$.*

For a harmonic function u, its energy is defined by

$$E_\Omega[u]=\frac{1}{2}\{\mu_u(\Omega)+\int_\Omega u^2 d\pi\} \qquad (0\leqq E_\Omega[u]\leqq+\infty).$$

We consider the space

$$\mathbf{H}_E(\Omega)=\{u\in\mathscr{H}(\Omega);\ E_\Omega[u]<+\infty\}$$

and the norm

$$\|u\|=\{E_\Omega[u]+|u(x_0)|^2\}^{1/2} \quad \text{if } 1\in\mathscr{H}(\Omega)\ (x_0\in\Omega\text{: fixed});$$

$$\|u\|=E_\Omega[u]^{1/2} \quad \text{if } 1\notin\mathscr{H}(\Omega)$$

for $u \in \mathbf{H}_E(\Omega)$. Then

THEOREM 3.3. *$\mathbf{H}_E(\Omega)$ is a Hilbert space with respect to the norm $\|\cdot\|$.*

COROLLARY 1 to PROPOSITION 3.5. *$\mathbf{H}_{BE}(\Omega)$ is dense in $\mathbf{H}_E(\Omega)$.*

It follows from Proposition 2.1 and Theorems 2.1 and 2.2 that $E_\Omega[f] \geqq 0$ for every $f \in \mathbf{B}_E(\Omega)$ if and only if $G(x, y)$ is a kernel of positive type. At present, we do not know whether this property follows from our assumptions on $(\Omega, \mathfrak{H})$. In Chapter IV, which is the first chapter of the present paper, we shall investigate this property and give several necessary and sufficient conditions; in fact, we shall see that $G(x, y)$ is of positive type if and only if any one of the domination principle, Frostman's maximum principle and the continuity principle holds for superharmonic functions on Ω. Assuming this property as an additional axiom (Axiom 7), we then make a functional completion of the space $\mathbf{B}_E(\Omega)$, or rather of its potential part, in the sense of N. Aronszajn-K.-T. Smith [1], and thus extend the class of functions for which the notion of energy is defined (Chapter V). The local investigation of energy leads to a notion of energy measure (Chapter VI), which is regarded as the measure $\{|\mathrm{grad}\, f|^2 + Pf^2\}dx$ in the case where $\mathfrak{H}$ is given by the solutions of $\Delta u = Pu$ on a Euclidean domain Ω. The notion of energy measure is useful in the study of lattice structures of the spaces of energy-finite functions.

We shall freely use the notation in [13] except for the reference numbers; references are rearranged in the present paper.

CHAPTER IV. Energy principle and its equivalent forms

§4.1. Properties of G-potentials.

LEMMA.4.1. *Given a non-negative measure μ on Ω such that U^μ is a potential, we can choose a sequence $\{\mu_n\}$ in $\mathbf{M}_B(\Omega)$ such that each $S(\mu_n)$ is compact, each U^{μ_n} is bounded continuous and $U^{\mu_n} \uparrow U^\mu$ as $n \to \infty$.*

PROOF. By [2; Satz 2.5.8], there is a sequence $\{p_n\}$ of potentials such that each $\sigma(p_n)$ is compact, each p_n is continuous and $p_n \uparrow U^\mu$. The boundedness of p_n follows from [11; Lemme 3.1]. If we write $p_n = U^{\mu_n}$, then $\{\mu_n\}$ is the required sequence.

LEMMA 4.2. *Let $\mathbf{C}_0(\Omega)$ be the space of all finite continuous functions with compact support in Ω and let*

$$\mathbf{P}_E(\Omega) = \{U^\mu - U^\nu;\ \mu, \nu \in \mathbf{M}_B(\Omega)\}.$$

Then, $\mathbf{P}_E(\Omega) \cap \mathbf{C}_0(\Omega)$ is dense in $\mathbf{C}_0(\Omega)$; in fact, given $f \in \mathbf{C}_0(\Omega)$, $\varepsilon > 0$ and a rela-

tively compact open set ω *containing the support* $S(f)$ *of* f, *there is* $g \in \mathbf{P}_E(\Omega) \cap \mathbf{C}_0(\Omega)$ *such that* $S(g) \subset \omega$ *and* $|g(x)-f(x)|<\varepsilon$ *for all* $x \in \Omega$.

PROOF. The space $\mathbf{P}_E(\Omega) \cap \mathbf{C}_0(\Omega)$ is obviously a linear subspace of $\mathbf{C}_0(\Omega)$. If $g \in \mathbf{P}_E(\Omega)$, i.e., $g=U^\mu - U^\nu$ with $\mu, \nu \in \mathbf{M}_B(\Omega)$, then $\min(g, 0)=\min(U^\mu, U^\nu) - U^\nu$. It follows that $\min(g, 0) \in \mathbf{P}_E(\Omega)$. Thus we see that $\mathbf{P}_E(\Omega) \cap \mathbf{C}_0(\Omega)$ is a vector lattice with respect to the max. and min. operations. For a regular domain ω and $y \in \omega$, let

$$p_y^\omega(x)=\begin{cases} G(x, y) & \text{if } x \notin \omega, \\ \int G(\xi, y)d\mu_x^\omega(\xi) & \text{if } x \in \omega. \end{cases}$$

Then p_y^ω is a continuous potential such that $\sigma(p_y^\omega) \subset \partial\omega$, so that it is also bounded by [11; Lemme 3.1]. If ω and ω' are regular domains such that $\bar{\omega} \subset \omega'$ and if $y \in \omega$, then $g \equiv p_y^\omega - p_y^{\omega'} \in \mathbf{P}_E(\Omega) \cap \mathbf{C}_0(\Omega)$ and $g(y)>0$. Then the present lemma follows from an argument similar to the proof of Stone's approximation theorem (see, e.g., [9; Hilfssatz 0.1]).

For non-negative measures μ, ν on Ω, let

$$I(\mu)=\int U^\mu d\mu \quad \text{and} \quad <\mu, \nu> = \int U^\mu d\nu = \int U^\nu d\mu.$$

The space of measures

$$\mathbf{M}_E(\Omega)=\{\mu;\ \text{non-negative measure such that } I(\mu)<+\infty\}$$

contains $\mathbf{M}_B(\Omega)$. For $\mu, \nu \in \mathbf{M}_E(\Omega)$,

$$I(\mu-\nu)=I(\mu)+I(\nu)-2<\mu, \nu>$$

has a definite value in $[-\infty, +\infty)$. We remark that if $\mu \in \mathbf{M}_E(\Omega)$ and ν is a non-negative measure such that $U^\nu \leqq U^\mu$, then $\nu \in \mathbf{M}_E(\Omega)$ and $I(\nu) \leqq I(\mu)$. Also, by a standard method we can easily show:

LEMMA 4.3. *If* $\mu_n, \nu_n, \mu, \nu \in \mathbf{M}_E(\Omega)$ $(n=1, 2, \ldots)$, $U^{\mu_n} \uparrow U^\mu$ *and* $U^{\nu_n} \uparrow U^\nu$, *then* $<\mu_n, \nu_n> \uparrow <\mu, \nu>$; *in particular*, $I(\mu_n) \uparrow I(\mu)$.

§4.2. Equivalence of various principles.

THEOREM 4.1. *The following statements are mutually equivalent*:

(i) $E_\Omega[f] \geqq 0$ *for all* $f \in \mathbf{B}_E(\Omega)$;

(ii) $G(x, y)$ *is a kernel of positive type, i.e., for any* $\mu, \nu \in \mathbf{M}_E(\Omega)$,

$$I(\mu-\nu) \geqq 0, \tag{4.1}$$

or, equivalently, for any $\mu, \nu \in \mathbf{M}_E(\Omega)$, (4.2) $$<\mu, \nu>^2 \leqq I(\mu)I(\nu);$$

(iii) *$G(x, y)$ satisfies the energy principle, i.e., it is of positive type and, in addition, the equality in* (4.1) (resp. (4.2)) *occurs only when $\mu=\nu$* (resp. *μ and ν are proportional*);

(iv) (*Cartan's maximum principle*) *If $\mu\in\mathbf{M}_E(\Omega)$ and if s is a non-negative superharmonic function on Ω such that $s\geqq U^\mu$ on $S(\mu)$, then $s\geqq U^\mu$ on Ω*;

(v) (*Domination principle*) *If p is a potential on Ω which is locally bounded on $\sigma(p)$ and if s is a non-negative superharmonic function such that $s\geqq p$ on $\sigma(p)$, then $s\geqq p$ on Ω*;

(vi) (*Frostman's maximum principle*) *If p is a potential on Ω, then*

$$\sup_{x\in\Omega} p(x) = \sup_{x\in\sigma(p)} p(x);$$

(vii) (*Continuity principle*) *If s is a non-negative superharmonic function on Ω and if $s|\sigma(s)$ is finite continuous, then s is continuous on Ω.*

PROOF. (i)⇔(ii): By Proposition 2.1, the corollary to Theorem 2.1 and Theorem 2.2, we see that $E_\Omega[f]\geqq 0$ for all $f\in\mathbf{B}_E(\Omega)$ if and only if $I(\mu-\nu)\geqq 0$ for all μ, $\nu\in\mathbf{M}_B(\Omega)$. Since $\mathbf{M}_B(\Omega)\subset\mathbf{M}_E(\Omega)$, the implication (ii)⇒(i) is trivial. Suppose now that $I(\mu-\nu)\geqq 0$, i.e.,

(4.3) $$I(\mu)+I(\nu)\geqq 2<\mu, \nu>$$

for all μ, $\nu\in\mathbf{M}_B(\Omega)$. Then, by virtue of Lemmas 4.1 and 4.3, we see that (4.3) also holds for any μ, $\nu\in\mathbf{M}_E(\Omega)$. Thus we obtain the implication (i)⇒(ii).

(ii)⇒(iii): By using Lemma 4.2, this implication is easily verified by a method due to H. Cartan [6; p. 86] (also cf. [7; p. 234] and [3; pp. 132–133]).

(iii)⇒(iv): The proof of this implication is again carried out by Cartan's method (see [6; Proposition 2]; also [3; p. 133]).

(iv)⇒(v): Let $p=U^\mu$ be locally bounded on $\sigma(p)=S(\mu)$. For an exhaustion $\{\Omega_n\}$ of Ω, let $\mu_n=\mu|\Omega_n$. Then $\mu_n\in\mathbf{M}_E(\Omega)$ and $U^{\mu_n}\leqq s$ on $S(\mu_n)$ for each n. Hence, by (iv), $U^{\mu_n}\leqq s$ on Ω. Since $U^{\mu_n}\uparrow U^\mu$, we have $U^\mu\leqq s$ on Ω.

(v)⇒(vi): The equality in (vi) is trivially true if $\alpha\equiv\sup_{x\in\sigma(p)} p(x)=+\infty$. In case $\alpha<+\infty$, we apply (v) with $s=\alpha$.

(vi)⇒(ii): This implication follows from a general theory by N. Ninomiya [14; Théorème 3] or by G. Choquet [8].

(vi)⇒(vii): To prove (vii), we may assume that s is a potential: $s=U^\mu$. Let $x_0\in\sigma(s)=S(\mu)$. Assuming that $s|\sigma(s)$ is finite continuous at x_0, we shall prove that s is continuous at x_0. Let $\mu_1=\mu|\Omega-\{x_0\}$ and $\mu_2=\mu|\{x_0\}$. Since $s=U^{\mu_1}+U^{\mu_2}$, $U^{\mu_1}|\sigma(s)$ is finite continuous at x_0. We can apply the proof of [14; Lemme 3] and see that U^{μ_1} is continuous at x_0, since $\mu_1(\{x_0\})=0$. (Note that the proof of [14; Lemme 3] fails to be valid if $K(\xi, \xi)<+\infty$ and $\lambda(\{\xi\})>0$.) On the other hand, since $s(x_0)<+\infty$, $\mu_2\neq 0$ if and only if $G(x_0, x_0)<+\infty$. In this case, $G_{x_0}\leqq G(x_0, x_0)$ on Ω ($G_{x_0}(x)\equiv G(x, x_0)$) by (vi). It follows from the lower semi-continuity G_{x_0} that G_{x_0} is of continuous at x_0. Hence $U^{\mu_2}=\mu_2(\{x_0\})G_{x_0}$ is continuous at x_0, and hence s is continuous at x_0.

(vii)⇒(v): As the proof of (iv)⇒(v) shows, it is enough to prove the case where $\sigma(p)$ is compact. Let $p=U^{\mu}$. By Kishi's lemma ([12]; also see [9; Hilfssatz 4.2] and [4; Part III, Proposition 4]), there exists a sequence $\{\mu_n\}$ of non-negative measures such that $S(\mu_n)\subset S(\mu)$ for each n, each U^{μ_n} is finite continuous on Ω and $U^{\mu_n}\uparrow U^{\mu}$ $(n\to\infty)$. For each n, $U^{\mu_n}\leqq s$ on $S(\mu_n)$, so that by [11; Lemme 3.1] this inequality holds on Ω. Letting $n\to\infty$, we have $U^{\mu}\leqq s$ on Ω.

REMARK 1. The domination principle (v) implies Axiom D of M. Brelot [4; Part IV]. Thus we may prove the implication (v)⇒(vii) in the following way: We may assume that s is a potential and $\sigma(s)$ is compact. Since $s|\sigma(s)$ is finite continuous by assumption, s is bounded on $\sigma(s)$. Hence, by (v) (or, rather by its immediate consequence (vi)), s is bounded on Ω. Then, by [4; Part IV, Theorem 26], we see that s is continuous on Ω.

REMARK 2. Kishi's lemma mentioned in the proof of the implication (vii) ⇒(v) is apparently an improvement of Lemma 4.1. However Kishi's lemma requires the continuity principle.

§4.3. Axiom 7 and its consequences.

In order to assure that energies of functions are non-negative, we shall assume any one of (i)～(vii) in the above theorem as our additional axiom. As an axiom on a harmonic space, either (vi) or (vii) may be the most preferable form:

Axiom 7. *Frostman's maximum principle* (vi) *holds.*

Hereafter we shall always assume this axiom. By considering the continuity principle and using the continuation theorem [4; Part IV, Theorem 14] (or [11; Théorème 13.1]), we can easily show

PROPOSITION 4.1. *For any domain* $\omega\subset\Omega$, $\mathfrak{H}|\omega$ *also satisfies Axiom* 7.

By virtue of Theorem 4.1, the following lemmas are proved by standard methods:

LEMMA 4.4. *For any* $f, g\in \mathbf{B}_E(\Omega)$,

$$E_\Omega[f, g]^2\leqq E_\Omega[f]E_\Omega[g]$$

and

$$E_\Omega[f+g]^{1/2}\leqq E_\Omega[f]^{1/2}+E_\Omega[g]^{1/2}.$$

If $f\in \mathbf{P}_E(\Omega)$ (see Lemma 4.2) *and* $E_\Omega[f]=0$, *then* $f=0$.

LEMMA 4.5. *If μ_n, $\mu \in \mathbf{M}_E(\Omega)$ and $U^{\mu_n} \uparrow U^{\mu}$, then $I(\mu_n - \mu) \to 0$.*

COROLLARY. *Given $\mu \in \mathbf{M}_E(\Omega)$, there is a sequence $\{\mu_n\}$ of measures in $\mathbf{M}_B(\Omega)$ such that each U^{μ_n} is finite continuous, each $S(\mu_n)$ is compact and $I(\mu_n - \mu) \to 0$.*

CHAPTER V. Functional completion

§5.1. Polar sets and *G*-capactity.

In order to obtain a functional completion in the sense of Aronszajn-Smith [1], it is necessary to introduce exceptional sets. As in the classical case, we let polar sets be our exceptional sets. In this connection we shall also introduce a capacity defined by $G(x, y)$.

By definition, a set $e \subset \Omega$ is *polar* if there is a positive superharmonic function (or a potential) s on Ω such that $s(x) = +\infty$ for all $x \in e$. We denote by $\mathcal{N}$ the set of all polar sets in Ω. If $e \in \mathcal{N}$ and $e' \subset e$, then $e' \in \mathcal{N}$; if $\{e_n\}$ is a countable collection of polar sets, then $\cup_n e_n \in \mathcal{N}$ (cf. [4; Part IV, § 32]). We say that a property holds quasi-everywhere, or simply, *q.e.* on a set A if it holds on $A - e$ with $e \in \mathcal{N}$. For any μ, $\nu \in \mathbf{M}_E(\Omega)$, $f = U^{\mu} - U^{\nu}$ is defined q.e. on Ω.

LEMMA 5.1. *Let s_1, s_2, s be superharmonic functions on an open set $\omega \subset \Omega$. If $s_1 \leqq s_2 + \varepsilon s$ on ω for any $\varepsilon > 0$, then $s_1 \leqq s_2$ on ω.*

PROOF. For any regular domain ω' such that $\bar{\omega}' \subset \omega$, $H_{s_1}^{\omega'} \leqq H_{s_2}^{\omega'} + \varepsilon H_s^{\omega'}$ for all $\varepsilon > 0$. It follows that $H_{s_1}^{\omega'} \leqq H_{s_2}^{\omega'}$. Since $s(x) = \lim_{\omega' \in \mathfrak{V}_x} H_s^{\omega'}$ for any superharmonic function s, where $\mathfrak{V}_x$ is the directed family of regular domains containing x, we have $s_1 \leqq s_2$ on ω.

COROLLARY 1. *If s_1, s_2 are superharmonic on an open set ω and $s_1 \leqq s_2$ q.e. on ω, then $s_1 \leqq s_2$ everywhere on ω.*

COROLLARY 2. (*Extended domination principle*) *If p is a potential on Ω which is locally bounded on $\sigma(p)$ and s is a non-negative superharmonic function on Ω such that $s \geqq p$ q.e. on $\sigma(p)$, then $s \geqq p$ on Ω.*

PROPOSITION 5.1. *If e is a polar set and $\mu \in \mathbf{M}_E(\Omega)$ (or $\mu | K \in \mathbf{M}_E(\Omega)$ for any compact set K), then $\mu(e) = 0$.*

This proposition can be proved in the same way as in the classical case (see, e.g., [9; Hilfssatz 5.1]).

The following lemma is a consequence of [4; Part IV, Definition 9, Proposition 10, Example a) in § 15 and Proposition 23]:

LEMMA 5.2. *Let A be a relatively compact set in Ω and let*

$$p_A = \inf\{s;\ \text{non-negative superharmonic on } \Omega,\ s \geqq 1 \text{ on } A\}.$$

Then the regularization $\hat{p}_A$ of p_A is a potential on Ω such that $\sigma(\hat{p}_A)\subset\bar{A}$, $\hat{p}_A=1$ q.e. on A and $\hat{p}_A=1$ on the interior of A.

Let λ_A be the associated measure of $\hat{p}_A$: $U^{\lambda_A}=\hat{p}_A$.

For a compact set K in Ω, the *G-capacity* $C(K)$ is defined by

$$C(K)=\sup\{\mu(K);\ U^\mu\leqq 1 \text{ on } \Omega\}$$

(cf. [4; Part III, Chap. IV]). By virtue of Corollary 2 to Lemma 5.1, we can apply the methods in the classical potential theory to our case; for instance, by the same methods as in [9; § 5], we can prove the following results.

LEMMA 5.3. *For any compact set K, $S(\lambda_K)\subset K$ and*

$$C(K)=\lambda_K(K)=I(\lambda_K).$$

For the proof, see [9; Satz 5.2].

PROPOSITION 5.2. *C is a Choquet capacity (or, a strong capacity, in the sense of* [4; Part II]).

See [9; Satz 5.3] for the proof. Also cf. [4; Part III, Theorems 7 and 8].

The (outer) capacity of an arbitrary set is defined in the usual way: for an open set ω in Ω,

$$C(\omega)=\sup\{C(K);\ K\text{: compact}\subset\omega\},$$

and for an arbitrary set A in Ω,

$$C(A)=\inf\{C(\omega);\ \omega\text{: open}\supset A\}.$$

It is known that C is then a true capacity in the sense of [4; Part III] (see Theorem 2 there). In particular, it is countably subadditive:

$$C\left(\bigcup_{n=1}^{\infty} A_n\right)\leqq\sum_{n=1}^{\infty} C(A_n).$$

LEMMA 5.4. *If ω is a relatively compact open set, then*

$$C(\omega)=\lambda_\omega(\Omega)=I(\lambda_\omega). \tag{5.1}$$

More generally, if ω is an open set with $C(\omega)<+\infty$, then

$$p_\omega=\sup\{U^{\lambda_K};\ K\text{: compact}\subset\omega\}$$

is a potential on Ω and its associated measure λ_ω satisfies (5.1).

The proof is the same as that of [9; Hilfssatz 5.5]. Note that Hilfssatz 5.2 and 5.3 in [9] are also valid in our case.

Obviously, if $C(\omega)<+\infty$ for an open set ω, then $U^{\lambda_\omega}\leqq 1$ on Ω, $U^{\lambda_\omega}=1$ on ω and $S(\lambda_\omega)\subset\bar{\omega}$. It also follows that

$$U^{\lambda_\omega} = \inf\{s;\ \text{non-negative superharmonic on } \Omega,\ s \geqq 1 \text{ on } \omega\}.$$

LEMMA 5.5. *A set e is polar if and only if $C(e)=0$.*

For the proof, see [9; Hilfssatz 5.6]. Note that we use Lemma 1.5 (in [13]) as well as the above lemma. Also, cf. [4; Part IV, the corollary to Theorem 10].

§ 5.2. Quasi-continuous functions.

Now that we obtain the G-capacity C, the notion of quasi-continuous functions is defined in terms of this capacity: An extended real valued function f on an open set ω in Ω is called *quasi-continuous* if for any $\varepsilon>0$ there is an open set $\omega_\varepsilon \subset \omega$ such that $f|(\omega-\omega_\varepsilon)$ is finite continuous and $C(\omega_\varepsilon)<\varepsilon$. A quasi-continuous function is finite q.e. (cf. Lemma 5.5). If f is quasi-continuous on ω and if $g=f$ q.e. on ω, then g is quasi-continuous on ω. If f_1, f_2 are quasi-continuous on ω and α_1, α_2 are real numbers, then $\alpha_1 f_1 + \alpha_2 f_2$ is defined to be quasi-continuous by assigning any value at every point where $+\infty-\infty$ or $-\infty+\infty$ occurs.

LEMMA 5.6. *For any $\mu \in \mathbf{M}_E(\Omega)$, U^μ is quasi-continuous on Ω; thus, for any μ, $\nu \in \mathbf{M}_E(\Omega)$, $U^\mu - U^\nu$ is defined as a quasi-continuous function on Ω.*

This lemma is proved in the same way as in the classical case (see [9; Satz 5.4] or [6; Proposition 5]).

For the later use we prove:

LEMMA 5.7. *Let f be a quasi-continuous function on an open set ω_0 in Ω. If f is μ_x^ω-summable and $\int f d\mu_x^\omega = 0$ for every regular domain ω such that $\bar{\omega} \subset \omega_0$ and for any $x \in \omega$, then $f=0$ q.e. on ω_0.*

PROOF. (Cf. the proof of [9; Hilfssatz 5.9]) We say that a set e in ω_0 is negligible (cf. [4; Part IV, Def. 8]) if $\mu_x^\omega(e)=0$ for any regular domain ω such that $\bar{\omega} \subset \omega_0$ and for any $x \in \omega$. The assumption that $\int f d\mu_x^\omega = 0$ for any such ω and x implies $\int |f| d\mu_x^\omega = 0$ for any such ω and x (see [4; Part IV, Proposition 16 and the proof of Basic Lemma 1 (pp. 103–104)]), and hence that $A=\{x \in \omega_0; f(x) \neq 0\}$ is negligible. Given $\varepsilon>0$, let ω_ε be an open set such that $C(\omega_\varepsilon)<\varepsilon$ and $f|(\omega_0-\omega_\varepsilon)$ is finite continuous. Then the set

$$\omega' = \{x \in \omega_0;\ \text{there is a neighborhood } U \text{ of } x \text{ such that } U-\omega_\varepsilon \text{ is negligible}\}$$

is an open set containing ω_ε. Since $A-\omega_\varepsilon$ is relatively open in $\omega_0-\omega_\varepsilon$, for each $x \in A-\omega_\varepsilon$, there is a neighborhood U of x such that $U-\omega_\varepsilon \subset A-\omega_\varepsilon$, so that $x \in \omega'$. Therefore $A \subset \omega'$. On the other hand, since ω' is covered by a countable number of open sets U such that $U-\omega_\varepsilon$ are negligible, $\omega'-\omega_\varepsilon$ is negligible. It follows that, for any compact set K in ω', $U^{\lambda_K}\leqq 1=U^{\lambda_\varepsilon}$ on ω' except on a negligible set, where $\lambda_\varepsilon=\lambda_{\omega_\varepsilon}$. Since U^{λ_K}, U^{λ_ε} are superharmonic, it then follows that $U^{\lambda_K}\leqq U^{\lambda_\varepsilon}$ on ω' (cf. the proof of Lemma 5.1). Hence, by the domination principle, $U^{\lambda_K}\leqq U^{\lambda_\varepsilon}$ everywhere on Ω. Thus, $C(K)\leqq C(\omega_\varepsilon)<\varepsilon$, and hence $C(\omega')<\varepsilon$. Therefore $C(A)=0$.

COROLLARY. *Let f be a quasi-continuous function on an open set ω in Ω. If f is μ-summable and $\int f d\mu=0$ for all $\mu\in\mathbf{M}_B(\Omega)$ such that $S(\mu)$ is compact and contained in ω, then $f=0$ q.e. on ω.*

§5.3. Functional completion of the potential part.

The space $\mathbf{B}_E(\Omega)$ is a direct sum of the spaces $\mathbf{H}_{BE}(\Omega)$ and $\mathbf{P}_E(\Omega)$. We know that $\mathbf{H}_E(\Omega)$ is complete and contains $\mathbf{H}_{BE}(\Omega)$ as a dense subspace (Theorem 3.3 and Corollary 1 to Proposition 3.5). Thus we shall now consider a functional completion of $\mathbf{P}_E(\Omega)$, or rather its subspace

$$\mathbf{P}_{EC}(\Omega)=\{U^\mu-U^\nu;\ \mu,\ \nu\in\mathbf{M}_B(\Omega),\ U^\mu \text{ and } U^\nu \text{ are continuous}\}.$$

By virtue of the corollary to Lemma 4.5 and the corollary to Theorem 2.1, $\mathbf{P}_{EC}(\Omega)$ is dense in $\mathbf{P}_E(\Omega)$ with respect to the norm $E_\Omega[\cdot]^{1/2}$.

LEMMA 5.8. *If $f\in\mathbf{P}_{EC}(\Omega)$, then $|f|\in\mathbf{P}_{EC}(\Omega)$ and $E_\Omega[|f|]=E_\Omega[f]$.*

PROOF. Let $f=U^\mu-U^\nu$ with μ, $\nu\in\mathbf{M}_B(\Omega)$ such that U^μ, U^ν are continuous. Then $|f|=U^\mu+U^\nu-2\min(U^\mu,\ U^\nu)$. Obviously $\min(U^\mu,\ U^\nu)$ is a continuous potential. Hence, we see that its associated measure λ belongs to $\mathbf{M}_B(\Omega)$ and that $|f|\in\mathbf{P}_{EC}(\Omega)$. Since f is continuous, $\Omega_+=\{x\in\Omega;\ f(x)>0\}$ and $\Omega_-=\{x\in\Omega;\ f(x)<0\}$ are open sets. It follows from Lemma 1.8 ([13]) that $\lambda|\Omega_+=\nu|\Omega_+$ and $\lambda|\Omega_-=\mu|\Omega_-$. Hence, by the corollary to Theorem 2.1,

$$\begin{aligned}E_\Omega[|f|]&=\int_\Omega |f|\,(d\mu+d\nu-2d\lambda)\\&=\int_{\Omega_+} f(d\mu-d\nu)-\int_{\Omega_-} f(d\nu-d\mu)\\&=\int_\Omega f(d\mu-d\nu)=E_\Omega[f].\end{aligned}$$

COROLLARY. *If $f\in\mathbf{P}_{EC}(\Omega)$ and $\mu\in\mathbf{M}_B(\Omega)$, then*

$$\left(\int_\Omega |f|\,d\mu\right)^2\leqq E_\Omega[f]\cdot I(\mu).$$

PROOF. $\left(\int |f|d\mu\right)^2=E_\Omega[|f|,\ U^\mu]^2\leqq E_\Omega[|f|]\cdot E_\Omega[U^\mu]=E_\Omega[f]\cdot I(\mu).$

LEMMA 5.9. *For any set A in Ω,*

$$C(A)\leqq\inf\{E_\Omega[f];\ f\in\mathbf{P}_{EC}(\Omega),\ |f(x)|\geqq 1 \text{ q.e. on } A\}.$$

PROOF. Let $f\in\mathbf{P}_{EC}(\Omega)$ and $|f(x)|\geqq 1$ q.e. on A. We shall show that $C(A)\leqq E_\Omega[f]$. For $\varepsilon>0$, $A_\varepsilon=\{x\in\Omega;\ |f(x)|>1-\varepsilon\}$ is an open set and $C(A-A_\varepsilon)=0$. For any compact set $K\subset A_\varepsilon$, using the above corollary and Lemma 5.3 we have

$$\begin{aligned}C(K)=\lambda_K(K)&\leqq\frac{1}{1-\varepsilon}\int_\Omega|f|d\lambda_K\\&\leqq\frac{1}{1-\varepsilon}E_\Omega[f]^{1/2}I(\lambda_K)^{1/2}=\frac{1}{1-\varepsilon}E_\Omega[f]^{1/2}C(K)^{1/2}.\end{aligned}$$

Hence $C(K)\leqq E_\Omega[f]/(1-\varepsilon)^2$. Therefore $C(A_\varepsilon)\leqq E_\Omega[f]/(1-\varepsilon)^2$. It then follows that $C(A)\leqq E_\Omega[f]$.

LEMMA 5.10. *Let $\{f_n\}$ be a sequence in $\mathbf{P}_{EC}(\Omega)$ such that $E_\Omega[f_n-f_m]\to 0$ $(n, m\to\infty)$ and $f_n\to 0$ q.e. on Ω. Then $E_\Omega[f_n]\to 0$ $(n\to\infty)$.*

PROOF. Let $\mu\in\mathbf{M}_B(\Omega)$. Then, the corollary to Theorem 2.1, Proposition 5.1, Fatou's lemma and the corollary to Lemma 5.8 imply

$$\begin{aligned}|E_\Omega[f_n, U^\mu]|&=\left|\int_\Omega f_n d\mu\right|\\&\leqq\int_\Omega|f_n|d\mu\leqq\liminf_{m\to\infty}\int_\Omega|f_n-f_m|d\mu\\&\leqq\{\liminf_{m\to\infty}E_\Omega[f_n-f_m]^{1/2}\}I(\mu)^{1/2}.\end{aligned}$$

Since $E_\Omega[f_n-f_m]\to 0$ $(n, m\to\infty)$, it follows that $E_\Omega[f_n, U^\mu]\to 0$ $(n\to\infty)$. Hence

$$\lim_{n\to\infty}E_\Omega[f_n, f_m]=0 \tag{5.2}$$

for each m. Now, $\{E_\Omega[f_n]\}$ is bounded: $E_\Omega[f_n]\leqq M$ $(n=1, 2, \dots)$. Given $\varepsilon>0$, choose m so large that $n\geqq m$ implies $E_\Omega[f_n-f_m]<\varepsilon^2/M$. Then, for $n\geqq m$,

$$\begin{aligned}E_\Omega[f_n]&=E_\Omega[f_n, f_n-f_m]+E_\Omega[f_n, f_m]\\&\leqq M^{1/2}E_\Omega[f_n-f_m]^{1/2}+|E_\Omega[f_n, f_m]|\leqq\varepsilon+|E_\Omega[f_n, f_m]|.\end{aligned}$$

Hence, by (5.2), $\limsup_{n\to\infty}E_\Omega[f_n]\leqq\varepsilon$, and hence $E_\Omega[f_n]\to 0$ $(n\to\infty)$.

The space $\mathbf{P}_{EC}(\Omega)$ is a normed functional space in the sense of Aronszajn-Smith [1] with respect to the norm $\|f\|=E_\Omega[f]^{1/2}$. Lemma 5.9 shows that the G-capacity C is admissible with respect to $\mathbf{P}_{EC}(\Omega)$ and the exceptional class $\mathscr{N}$. Therefore, in view of Lemma 5.10, it follows from [1; § 6, Theorem I] that $\mathbf{P}_{EC}(\Omega)$

has a functional completion relative to $\mathcal{N}$; more precisely, we obtain (cf. also, [9] and [10]):

THEOREM 5.1. *Let*

$$\mathscr{E}_0(\Omega)=\left\{ f;\ \begin{array}{l} \text{there is a sequence } \{f_n\} \text{ in } \mathbf{P}_{EC}(\Omega) \text{ such that} \\ f_n \to f \text{ q.e. on } \Omega \text{ and } \|f_n-f_m\|\to 0\ (n, m\to\infty) \end{array} \right\}.$$

Then $\mathscr{E}_0(\Omega)$ has the following properties:

(a) *If $f\in\mathscr{E}_0(\Omega)$ and g is a function on Ω such that $g=f$ q.e. on Ω, then $g\in\mathscr{E}_0(\Omega)$.*

(b) *For any $f\in\mathscr{E}_0(\Omega)$, let $\{f_n\}$ be a sequence in $\mathbf{P}_{EC}(\Omega)$ such that $f_n\to f$ q.e. on Ω and $\|f_n-f_m\|\to 0$ $(n, m\to\infty)$. Then*

$$\|f\|=\lim_{n\to\infty}\|f_n\|$$

is well defined, i.e., it is independent of the choice of $\{f_n\}$. Furthermore, $\|f_n-f\|\to 0$ $(n\to\infty)$ for such $\{f_n\}$.

(c) *If we identify functions which are equal q.e. on Ω, then $\mathscr{E}_0(\Omega)$ is a Banach space with respect to the above norm, and contains $\mathbf{P}_{EC}(\Omega)$ as a dense subspace.*

(d) *If f_n, $f\in\mathscr{E}_0(\Omega)$ and $\|f_n-f\|\to 0$ $(n\to\infty)$, then there is a subsequence $\{f_{n_k}\}$ which converges to f q.e. on Ω.*

The *energy* of a function $f\in\mathscr{E}_0(\Omega)$ is defined by

$$E_\Omega[f]=\|f\|^2$$

and the *mutual energy* of f, $g\in\mathscr{E}_0(\Omega)$ by

$$E_\Omega[f, g]=\frac{1}{2}\{E_\Omega[f+g]-E_\Omega[f]-E_\Omega[g]\}.$$

If $\|f_n-f\|\to 0$ and $\|g_n-g\|\to 0$ with f_n, $g_n\in\mathbf{P}_{EC}(\Omega)$, then $E_\Omega[f_n, g_n]\to E_\Omega[f, g]$. Hence, we see that the mapping $(f, g)\to E_\Omega[f, g]$ is a symmetric bilinear form on $\mathscr{E}_0(\Omega)\times\mathscr{E}_0(\Omega)$. Obviously $E_\Omega[f, f]=E_\Omega[f]$. Therefore, by (c) of the above theorem we have

COROLLARY. *$\mathscr{E}_0(\Omega)$ is a Hilbert space with respect to the inner product $E_\Omega[f, g]$, identifying functions which are equal q.e. on Ω.*

PROPOSITION 5.3. *Any function in $\mathscr{E}_0(\Omega)$ is quasi-continuous.*

PROOF. Let $f\in\mathscr{E}_0(\Omega)$. There is a sequence $\{f_n\}$ in $\mathbf{P}_{EC}(\Omega)$ such that $f_n\to f$ q.e. on Ω and $E_\Omega[f_n-f_{n+1}]<1/2^{2n}$ $(n=1, 2,\ldots)$. Then, using Lemma 5.9, we can show by the same method as in the proof of [9; Hilfssatz 7.8] (also cf. the proof of [10; Théorème 3.11]) that given $\varepsilon>0$ there is a set B_ε such that $C(B_\varepsilon)<\varepsilon$ and $\{f_n\}$ converges uniformly on $\Omega-B_\varepsilon$. Then we immediately see that f is quasi-continuous.

LEMMA 5.11. *If $\mu\in\mathbf{M}_E(\Omega)$, then $U^\mu\in\mathscr{E}_0(\Omega)$ and $E_\Omega[U^\mu]=I(\mu)$.*

PROOF. By the corollary to Lemma 4.5, we can choose a sequence $\{\mu_n\}$ in $\mathbf{M}_B(\Omega)$ such that each U^{μ_n} is continuous and $I(\mu_n-\mu)\to 0$. Then $U^{\mu_n}\in\mathbf{P}_{EC}(\Omega)$ and, by the corollary to Theorem 2.1, $E_\Omega[U^{\mu_n}-U^{\mu_m}]=I(\mu_n-\mu_m)\to 0$ $(n, m\to\infty)$. Hence $U^\mu\in\mathscr{E}_0(\Omega)$. Furthermore, $E_\Omega[U^\mu]=\lim_{n\to\infty}E_\Omega[U^{\mu_n}]=\lim_{n\to\infty}I(\mu_n)=I(\mu)$.

COROLLARY. *If μ, $\nu\in\mathbf{M}_E(\Omega)$, then $E_\Omega[U^\mu-U^\nu]=I(\mu-\nu)$ and $E_\Omega[U^\mu, U^\nu]=\langle\mu, \nu\rangle$.*

LEMMA 5.12. *If $f\in\mathscr{E}_0(\Omega)$ and $\mu\in\mathbf{M}_E(\Omega)$, then f is μ-summable; in fact*

$$\left(\int_\Omega|f|d\mu\right)^2\leqq E_\Omega[f]\cdot E_\Omega[U^\mu], \tag{5.3}$$

and

$$\int_\Omega f d\mu=E_\Omega[f, U^\mu]. \tag{5.4}$$

PROOF. First suppose $f\in\mathbf{P}_{EC}(\Omega)$. By Lemma 5.8, $|f|\in\mathbf{P}_{EC}(\Omega)$, i.e., $|f|=U^{\lambda_1}-U^{\lambda_2}$ with λ_1, $\lambda_2\in\mathbf{M}_B(\Omega)$. Given $\mu\in\mathbf{M}_E(\Omega)$, choose $\mu_n\in\mathbf{M}_B(\Omega)$, $n=1, 2, \ldots$, such that $U^{\mu_n}\uparrow U^\mu$. Then, using the corollary to Lemma 5.8, we have

$$\begin{aligned}\int_\Omega|f|d\mu&=\int_\Omega(U^{\lambda_1}-U^{\lambda_2})d\mu=\int_\Omega U^\mu d\lambda_1-\int_\Omega U^\mu d\lambda_2\\&=\lim_{n\to\infty}\int_\Omega U^{\mu_n}d\lambda_1-\lim_{n\to\infty}\int_\Omega U^{\mu_n}d\lambda_2\\&=\lim_{n\to\infty}\int_\Omega|f|d\mu_n\leqq E_\Omega[f]^{1/2}\lim_{n\to\infty}I(\mu_n)^{1/2}=E_\Omega[f]^{1/2}\cdot I(\mu)^{1/2}.\end{aligned}$$

Similarly, we obtain

$$\int_\Omega f d\mu=\lim_{n\to\infty}\int_\Omega f d\mu_n=\lim_{n\to\infty}E_\Omega[f, U^{\mu_n}]=E_\Omega[f, U^\mu],$$

where the last equality follows from the fact that $E_\Omega[U^{\mu_n}-U^\mu]\to 0$ $(n\to\infty)$ (cf. the proof of the above lemma).

Next, let $f\in\mathscr{E}_0(\Omega)$. Choose $\{f_n\}$ in $\mathbf{P}_{EC}(\Omega)$ such that $f_n\to f$ q.e. on Ω and $E_\Omega[f_n-f]\to 0$ $(n\to\infty)$. By the above result, Proposition 5.1 and Fatou's lemma, we have

$$\begin{aligned}\int_\Omega|f|d\mu\leqq\liminf_{n\to\infty}\int_\Omega|f_n|d\mu&\leqq(\liminf_{n\to\infty}E_\Omega[f_n]^{1/2})\cdot E_\Omega[U^\mu]^{1/2}\\&=E_\Omega[f]^{1/2}\cdot E_\Omega[U^\mu]^{1/2}.\end{aligned}$$

Applying this result to $f-f_n$, we also have

$$\int_\Omega |f-f_n| d\mu \leqq E_\Omega[f-f_n]^{1/2} \cdot E_\Omega[U^\mu]^{1/2} \to 0 \quad (n\to\infty).$$

Hence,

$$\int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu = \lim_{n\to\infty} E_\Omega[f_n, U^\mu] = E_\Omega[f, U^\mu].$$

LEMMA 5.13. *If $f\in\mathscr{E}_0(\Omega)$ and $\alpha>0$, then*

$$C(\{x\in\Omega;\ |f(x)|\leqq\alpha\}) \leqq \frac{E_\Omega[f]}{\alpha^2}.$$

We can prove this lemma in a way similar to the proof of [9; Hilfssatz 7.6], using Proposition 5.3 and the above lemma (also, cf. the proof of Lemma 5.9).

By means of this lemma, we obtain the following proposition in the same way as [9; Hilfssatz 7.7]:

PROPOSITION 5.4. *For any $f\in\mathscr{E}_0(\Omega)$, there is a potential p on Ω such that $|f|\leqq p$ on Ω.*

COROLLARY. *$\mathscr{E}_0(\Omega)\cap\mathscr{H}(\Omega)=\{0\}$; in particular, $\mathscr{E}_0(\Omega)\cap\mathbf{H}_E(\Omega)=\{0\}$.*

§5.4. The space of energy-finite functions.

Now we consider the vector sum of two function spaces $\mathbf{H}_E(\Omega)$ and $\mathscr{E}_0(\Omega)$:

$$\mathscr{E}(\Omega)=\mathbf{H}_E(\Omega)+\mathscr{E}_0(\Omega).$$

This is a direct sum by virtue of the corollary to Proposition 5.4, so that each $f\in\mathscr{E}(\Omega)$ is uniquely expressed as $f=u+f_0$ with $u\in\mathbf{H}_E(\Omega)$ and $f_0\in\mathscr{E}_0(\Omega)$. We define the energy of f by

$$E_\Omega[f]=E_\Omega[u]+E_\Omega[f_0]$$

and the mutual energy of f and $g\in\mathscr{E}(\Omega)$ by

$$E_\Omega[f, g]=E_\Omega[u, v]+E_\Omega[f_0, g_0],$$

where $g=v+g_0$ with $v\in\mathbf{H}_E(\Omega)$ and $g_0\in\mathscr{E}_0(\Omega)$.

By definition, $\mathbf{B}_E(\Omega)\subset\mathscr{E}(\Omega)$ and the notion of energy for functions in $\mathscr{E}(\Omega)$ is compatible with that for functions in $\mathbf{B}_E(\Omega)$ defined in Chapter II. By Proposition 5.3, any function in $\mathscr{E}(\Omega)$ is quasi-continuous. As immediate consequences of Theorem 5.1, its corollary and Theorem 3.3, we obtain

THEOREM 5.2. (a) *If $f\in\mathscr{E}(\Omega)$ and $g=f$ q.e. on Ω, then $g\in\mathscr{E}(\Omega)$.*

(b) *$\mathscr{E}(\Omega)$ is a linear space (identifying functions which are equal q.e.)*

and $E_\Omega[f, g]$ *is a symmetric bilinear form on* $\mathscr{E}(\Omega)\times\mathscr{E}(\Omega)$; *in case* $1\in\mathscr{H}(\Omega)$, $E_\Omega[f]^{1/2}$ *defines a semi-norm on* $\mathscr{E}(\Omega)$ *such that* $E_\Omega[f]=0$ *if and only if* $f=const.$ *q.e. on* Ω; *in case* $1\notin\mathscr{H}(\Omega)$, $E_\Omega[f]^{1/2}$ *defines a norm on* $\mathscr{E}(\Omega)$; $\mathscr{E}(\Omega)$ *is complete with respect to the semi-norm* $E_\Omega[f]^{1/2}$ *in any case.*

(c) *For any* $f\in\mathscr{E}(\Omega)$, *there is a sequence* $\{f_n\}$ *in* $\mathbf{B}_E(\Omega)$ (or, in $\mathbf{H}_E(\Omega)+\mathbf{P}_{EC}(\Omega)$) *such that* $E_\Omega[f_n-f]\to 0$ *and* $f_n\to f$ *q.e. on* Ω.

(d) *If* $E_\Omega[f_n-f]\to 0$ $(n\to\infty)$ *for* $f_n, f\in\mathscr{E}(\Omega)$, *then there are a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *and a sequence* $\{c_k\}$ *of constants such that* $f_{n_k}+c_k\to f$ *q.e. on* Ω; *we can choose* $c_k=0$, $k=1, 2, \ldots$, *if* $1\notin\mathscr{H}(\Omega)$.

The following lemma will be used in the next chapter:

LEMMA 5.14. *If* $f\in\mathscr{E}(\Omega)$ *and* μ *is a non-negative measure such that* $\mu|K\in\mathbf{M}_E(\Omega)$ *for any compact set* K, *then* f *is locally* μ*-summable. If* $\{f_n\}$ *is a sequence in* $\mathscr{E}(\Omega)$ *such that* $f_n\to f$ *q.e. on* Ω *and* $E_\Omega[f_n-f]\to 0$ $(n\to\infty)$, *then* $\int_K|f_n-f|d\mu\to 0$ $(n\to\infty)$ *for each compact set* K.

PROOF. Let $f=u+g$ with $u\in\mathbf{H}_E(\Omega)$ and $g\in\mathscr{E}_0(\Omega)$. Since u is locally bounded and g is $\mu|K$-summable for any compact set K by Lemma 5.12, f is locally μ-summable. Let $f_n=u_n+g_n$ with $u_n\in\mathbf{H}_E(\Omega)$ and $g_n\in\mathscr{E}_0(\Omega)$ for each n. Then $E_\Omega[u_n-u]\to 0$ and $E_\Omega[g_n-g]\to 0$ $(n\to\infty)$. By the corollary to Theorem 3.2 ([13]), there are constants c_n, $n=1, 2, \ldots$, such that $u_n+c_n\to u$ locally uniformly in Ω. We shall show that $c_n\to 0$. Supposing the contrary, we find $\varepsilon_0>0$ and a subsequence $\{c_{n_j}\}$ of $\{c_n\}$ such that $|c_{n_j}|\geqq\varepsilon_0$ for all j. Since $E_\Omega[g_{n_j}-g]\to 0$ $(j\to\infty)$ and g_{n_j}, $g\in\mathscr{E}_0(\Omega)$, Theorem 5.1, d) implies that there is a subsequence $\{g_{n'_j}\}$ of $\{g_{n_j}\}$ converging to g q.e. on Ω. Since $f_{n'_j}\to f$ q.e. on Ω, $u_{n'_j}\to u$ q.e. on Ω. This is impossible, since $u_{n'_j}+c_{n'_j}\to u$ and $|c_{n'_j}|\geqq\varepsilon_0$. Thus we have shown that $u_n\to u$ locally uniformly on Ω. Hence, for each compact set K, $\int_K|u_n-u|d\mu\to 0$ $(n\to\infty)$. On the other hand, by Lemma 5.12, $\int_K|g_n-g|d\mu\to 0$ $(n\to\infty)$. Hence we have the lemma.

CHAPTER VI. Energy measures and lattice structures

§6.1. Energy measures for locally bounded functions.

Let us consider the space

$$\mathbf{B}_{\mathrm{loc}}(\Omega)=\{f;\ \text{for any relatively compact domain } \omega,\ f|\omega\in\mathbf{B}_E(\omega)\}.$$

First we observe

LEMMA 6.1. *If* $u\in\mathscr{H}(\Omega)$ *and* U^μ, U^ν *are locally bounded potentials, then* $f=u+U^\mu-U^\nu$ *belongs to* $\mathbf{B}_{\mathrm{loc}}(\Omega)$.

PROOF. For any relatively compact domain ω,

$$f|\omega = u_\omega + U^\mu_\omega - U^\nu_\omega$$

with $u_\omega \in \mathscr{H}(\omega)$. Obviously, U^μ_ω and U^ν_ω are bounded. Furthermore, $\mu(\omega) < +\infty$ and $\nu(\omega) < +\infty$, so that $\mu|\omega$, $\nu|\omega \in \mathbf{M}_B(\omega)$. Thus, what remains to prove is $u_\omega \in \mathbf{H}_{BE}(\omega)$. Since there is another relatively compact domain ω' such that $\bar{\omega} \subset \omega'$, we may assume that μ, $\nu \in \mathbf{M}_B(\Omega)$. Now

$$u_\omega = u|\omega + (U^\mu|\omega - U^\mu_\omega) + (U^\nu|\omega - U^\nu_\omega).$$

Since $u|\omega$ is bounded and $\mu_{u|\omega}(\omega) = \mu_u(\omega) < +\infty$, $u|\omega \in \mathbf{H}_{BE}(\omega)$. Next we consider $v = U^\mu|\omega - U^\mu_\omega$. Then $v \in \mathscr{H}(\omega)$ and is bounded. By Lemma 2.3 (in [13]), $(U^\mu)^2 = U^{\mu_1} - U^{\mu_2}$ with μ_1, $\mu_2 \in \mathbf{M}_B(\Omega)$. Thus

$$v^2 = h + U^{\mu_1}_\omega - U^{\mu_2}_\omega + (U^\mu_\omega)^2 - 2U^\mu U^\mu_\omega$$

on ω with $h \in \mathscr{H}(\omega)$. It follows that

$$U^{\mu_v}_\omega = -U^{\mu_1}_\omega + U^{\mu_2}_\omega - (U^\mu_\omega)^2 + 2U^\mu U^\mu_\omega \leqq U^{\mu_2}_\omega + 2MU^\mu_\omega$$

on ω, where $M = \sup_\omega U^\mu$. Hence $\mu_v(\omega) \leqq \mu_2(\omega) + 2M\mu(\omega) < +\infty$, and hence $v \in \mathbf{H}_{BE}(\omega)$. Similarly, we see that $U^\nu|\omega - U^\nu_\omega \in \mathbf{H}_{BE}(\omega)$. Therefore $u_\omega \in \mathbf{H}_{BE}(\omega)$.

By this lemma, we see that $\mathbf{B}_E(\Omega) \subset \mathbf{B}_{\mathrm{loc}}(\Omega)$, $\mathscr{H}(\Omega) \subset \mathbf{B}_{\mathrm{loc}}(\Omega)$ and constant functions belong to $\mathbf{B}_{\mathrm{loc}}(\Omega)$.

For each $f \in \mathbf{B}_{\mathrm{loc}}(\Omega)$, its associated measure σ_f is well-defined by the following condition: for any relatively compact domain ω, $f|\omega = u_\omega + U^\mu_\omega - U^\nu_\omega$ with $u_\omega \in \mathscr{H}(\omega)$ and $\sigma_f|\omega = \mu - \nu$. Lemma 2.3 ([13]) implies that if f, $g \in \mathbf{B}_{\mathrm{loc}}(\Omega)$, then $fg \in \mathbf{B}_{\mathrm{loc}}(\Omega)$. Therefore,

$$\varepsilon_{[f,g]} = \frac{1}{2}(f\sigma_g + g\sigma_f - \sigma_{fg} + fg\pi)$$

defines a signed measure on Ω for f, $g \in \mathbf{B}_{\mathrm{loc}}(\Omega)$. Here, in general, $f\sigma$ means the signed measure defined by $d(f\sigma) = fd\sigma$ for a signed measure σ on Ω and a locally $|\sigma|$-summable function f in Ω. The measure $\varepsilon_{[f,g]}$ may be called the *mutual energy measure* of f and g. The mapping $(f, g) \rightarrow \varepsilon_{[f,g]}$ is symmetric and bilinear on $\mathbf{B}_{\mathrm{loc}}(\Omega) \times \mathbf{B}_{\mathrm{loc}}(\Omega)$. The measure

$$\varepsilon_f \equiv \varepsilon_{[f,f]} = \frac{1}{2}(2f\sigma_f - \sigma_{f^2} + f^2\pi)$$

will be called the *energy measure* of $f \in \mathbf{B}_{\mathrm{loc}}(\Omega)$.

We shall write $E_\omega[f]$ for $E_\omega[f|\omega]$. Obviously, if $f \in \mathbf{B}_{\mathrm{loc}}(\Omega)$, then $\varepsilon_f(\omega) = E_\omega[f]$ for any relatively compact domain ω and if $f \in \mathbf{B}_E(\Omega)$, then $\varepsilon_f(\Omega) = E_\Omega[f] < +\infty$.

PROPOSITION 6.1. *For any* $f \in \mathbf{B}_{loc}(\Omega)$, ε_f *is a non-negative measure.*

PROOF. Since $\mathfrak{H}|\omega$ satisfies Axiom 7 (Proposition 4.1), $\varepsilon_f(\omega)=E_\omega[f]\geqq 0$ for any relatively compact domain ω. It follows that $\varepsilon_f(\omega)\geqq 0$ for any open set ω, and hence that ε_f is a non-negative measure.

Proposition 6.1 implies that if f, $g \in \mathbf{B}_{loc}(\Omega)$ then

$$|\varepsilon_{[f,g]}(A)| \leqq \varepsilon_f(A)^{1/2} \cdot \varepsilon_g(A)^{1/2} \tag{6.1}$$

for any relatively compact Borel set A and

$$\varepsilon_{f+g}(A)^{1/2} \leqq \varepsilon_f(A)^{1/2} + \varepsilon_g(A)^{1/2} \tag{6.2}$$

for any Borel set A in Ω.

§6.2. Locally energy-finite functions.

Next we consider

$$\mathscr{E}_{loc}(\Omega)=\{f;\text{ for each relatively compact domain } \omega,\ f|\omega \in \mathscr{E}(\omega)\}.$$

$\mathscr{E}_{loc}(\Omega)$ is a linear space if we identify functions which are equal q.e. Each $f \in \mathscr{E}_{loc}(\Omega)$ is quasi-continuous on Ω. Obviously, $\mathbf{B}_{loc}(\Omega) \subset \mathscr{E}_{loc}(\Omega)$.

LEMMA 6.2. $\mathscr{E}(\Omega) \subset \mathscr{E}_{loc}(\Omega)$.

PROOF. Let $f \in \mathscr{E}(\Omega)$. Then there is a sequence $\{f_n\}$ in $\mathbf{B}_E(\Omega)$ such that $f_n \to f$ q.e. on Ω and $E_\Omega[f_n - f_m] \to 0$ $(n, m \to \infty)$. For any domain ω,

$$E_\omega[f_n - f_m] = \varepsilon_{f_n - f_m}(\omega) \leqq \varepsilon_{f_n - f_m}(\Omega) = E_\Omega[f_n - f_m] \to 0 \quad (n, m \to \infty).$$

Hence, $f|\omega \in \mathscr{E}(\omega)$. Therefore $f \in \mathscr{E}_{loc}(\Omega)$.

THEOREM 6.1. *For each* $f \in \mathscr{E}_{loc}(\Omega)$, *there exists a unique non-negative measure* ε_f *such that*

$$\varepsilon_f(\omega) = E_\omega[f] \tag{6.3}$$

for every relatively compact domain ω. *If* $f \in \mathscr{E}(\Omega)$, *then* $\varepsilon_f(\Omega) = E_\Omega[f]$.

PROOF. Uniqueness immediately follows from (6.3). Let $f \in \mathscr{E}_{loc}(\Omega)$ and ω be a relatively compact domain. (In case $f \in \mathscr{E}(\Omega)$, ω may be equal to Ω.) Choose $\{f_n\}$ in $\mathbf{B}_E(\omega)$ such that $f_n \to f$ q.e. on ω and $E_\omega[f_n - f] \to 0$ $(n \to \infty)$. For any Borel set A in ω,

$$\begin{aligned}|\varepsilon_{f_n}(A)^{1/2} - \varepsilon_{f_m}(A)^{1/2}| &\leqq \varepsilon_{f_n - f_m}(A)^{1/2} \leqq \varepsilon_{f_n - f_m}(\omega)^{1/2} \\ &= E_\omega[f_n - f_m] \to 0 \qquad (n, m \to \infty).\end{aligned}$$

It follows that a set function ε_f^ω is defined for all Borel sets A in ω by

$$\varepsilon_f^\omega(A)=\lim_{n\to\infty}\varepsilon_{f_n}(A)$$

and that it is a non-negative measure on ω. It is easy to see that if ω' is another relatively compact domain containing ω, then $\varepsilon_f^{\omega'}|\omega=\varepsilon_f^\omega$. Therefore, there is a non-negative measure ε_f on Ω such that $\varepsilon_f|\omega=\varepsilon_f^\omega$. Obviously, $\varepsilon_f(\omega)=\varepsilon_f^\omega(\omega)=\lim_{n\to\infty}\varepsilon_{f_n}(\omega)=\lim_{n\to\infty}E_\omega[f_n]=E_\omega[f]$ for each relatively compact domain ω and for $\omega=\Omega$ if $f\in\mathscr{E}(\Omega)$.

The measure ε_f in the above theorem will be called the energy measure of $f\in\mathscr{E}_{\mathrm{loc}}(\Omega)$. For f, $g\in\mathscr{E}_{\mathrm{loc}}(\Omega)$, their mutual energy measure is defined by

$$\varepsilon_{[f,g]}=\frac{1}{2}(\varepsilon_{f+g}-\varepsilon_f-\varepsilon_g).$$

It is easily verified that the mapping $(f, g)\to\varepsilon_{[f,g]}$ is symmetric bilinear on $\mathscr{E}_{\mathrm{loc}}(\Omega)\times\mathscr{E}_{\mathrm{loc}}(\Omega)$ and $\varepsilon_{[f,g]}(\omega)=E_\omega[f, g]$ for each relatively compact domain ω. Furthermore, $\varepsilon_{[f,g]}(\Omega)$ exists and equals $E_\Omega[f, g]$ if f, $g\in\mathscr{E}(\Omega)$. Also, (6.1) and (6.2) hold for f, $g\in\mathscr{E}_{\mathrm{loc}}(\Omega)$.

PROPOSITION 6.2. *If $f\in\mathscr{E}_{\mathrm{loc}}(\Omega)$ and α is a real constant, then*

$$\varepsilon_{[f,\alpha]}=\alpha f\pi;$$

in particular, $\varepsilon_\alpha=\alpha^2\pi$.

PROOF. By considering locally, we may assume that $f\in\mathscr{E}(\Omega)$. Choose $\{f_n\}$ in $\mathbf{B}_E(\Omega)$ such that $f_n\to f$ q.e. on Ω and $E_\Omega[f_n-f]\to 0$ $(n\to\infty)$. Since $\sigma_\alpha=\alpha\pi$,

$$\varepsilon_{[f_n,\alpha]}=\frac{1}{2}(f_n\alpha\pi+\alpha\sigma_{f_n}-\sigma_{\alpha f_n}+\alpha f_n\pi)=\alpha f_n\pi.$$

As in the proof of the above theorem, we see that $\varepsilon_{[f_n,\alpha]}(A)\to\varepsilon_{[f,\alpha]}(A)$ for any relatively compact Borel set A. On the other hand, Lemma 5.14 implies that $\int_A\alpha f_n d\pi\to\int_A\alpha f d\pi$ for such A. Hence we have the proposition.

LEMMA 6.3. *Let $f\in\mathscr{E}_{\mathrm{loc}}(\Omega)$ and ω be a relatively compact domain. If*

$$E_\omega[f, U^{\mu_1}-U^{\mu_2}]=0,$$

where $\mu_1=\mu_x^{\omega_1}$ and $\mu_2=\mu_x^{\omega_2}$, for any regular domains ω_1 and ω_2 such that $\bar{\omega}_1\subset\omega_2\subset\bar{\omega}_2\subset\omega$ and for any $x\in\omega_1$, then there is $u\in\mathbf{H}_E(\omega)$ such that $f|\omega=u$ q.e. on ω.

PROOF. Let $f|\omega=u+g$ with $u\in\mathbf{H}_E(\omega)$ and $g\in\mathscr{E}_0(\omega)$. Since $U^{\mu_1}=U^{\mu_2}$ on $\Omega-\bar{\omega}_2$, we have $U^{\mu_1}-U^{\mu_2}=U_\omega^{\mu_1}-U_\omega^{\mu_2}$ on ω. Hence, by (5.4) in Lemma 5.12,

$$E_\omega[f, U^{\mu_1}-U^{\mu_2}]=E_\omega[g, U^{\mu_1}_\omega-U^{\mu_2}_\omega]=\int g\,d\mu_1-\int g\,d\mu_2.$$

Thus, by assumption, $\int g\,d\mu_x^{\omega_1}=\int g\,d\mu_x^{\omega_2}$ for any regular domains ω_1, ω_2 such that $\bar{\omega}_1\subset\omega_2\subset\bar{\omega}_2\subset\omega$ and for any $x\in\omega_1$. Therefore, if we define $v(x)=\int g\,d\mu_x^{\omega'}$ for $x\in\omega'$, where ω' is a regular domain such that $\bar{\omega}'\subset\omega$, then v is defined as a harmonic function on ω. Since $\int(v-g)d\mu_x^{\omega'}=0$ for any such ω' and $x\in\omega'$, Lemma 5.7 implies that $g=v$ q.e. on ω. It follows that $v=0$, since $g\in\mathscr{E}_0(\omega)$. Hence $f|\omega=u$ q.e. on ω.

COROLLARY *Let $f\in\mathscr{E}_{\mathrm{loc}}(\Omega)$ and ω be a relatively compact domain. If $E_\omega[f, g]=0$ for any $g\in\mathscr{E}_0(\omega)$ (or, for any $g\in\mathbf{P}_E(\omega)$), then $f|\omega=$ a harmonic function q.e. on ω.*

THEOREM 6.2. $\mathscr{E}(\Omega)=\{f\in\mathscr{E}_{\mathrm{loc}}(\Omega);\ \varepsilon_f(\Omega)<+\infty\}$.

PROOF. Let $\mathscr{E}'=\{f\in\mathscr{E}_{\mathrm{loc}}(\Omega);\ \varepsilon_f(\Omega)<+\infty\}$. By Lemma 6.2 and Theorem 6.1, $\mathscr{E}(\Omega)\subset\mathscr{E}'$. So, we shall prove the converse inclusion. If $f, g\in\mathscr{E}'$, then $|\varepsilon_{[f,g]}|(\Omega)<+\infty$. Hence $<f, g>\equiv\varepsilon_{[f,g]}(\Omega)$ gives a symmetric bilinear form on $\mathscr{E}'$ and $<f, f>\geqq 0$. Let $f\in\mathscr{E}'$ be given. Since $\mathscr{E}_0(\Omega)$ is complete with respect to the norm $<f,f>^{1/2}=E_\Omega[f]^{1/2}$ (Corollary to Theorem 5.1), by the usual method of orthogonal projection, we find $f_0\in\mathscr{E}_0(\Omega)$ such that $<f-f_0, g>=0$ for all $g\in\mathscr{E}_0(\Omega)$. Let ω be a relatively compact domain and ω_1, ω_2 be regular domains such that $\bar{\omega}_1\subset\omega_2\subset\bar{\omega}_2\subset\omega$. Let $g=U^{\mu_1}-U^{\mu_2}$, where $\mu_1=\mu_x^{\omega_1}$ and $\mu_2=\mu_x^{\omega_2}$ with $x\in\omega_1$. Then, $g\in\mathscr{E}_0(\Omega)$. Since $g=0$ on $\Omega-\bar{\omega}_2$, $\varepsilon_{[f-f_0,g]}|(\Omega-\bar{\omega}_2)=0$. Hence

$$E_\omega[f-f_0, g]=\varepsilon_{[f-f_0,g]}(\omega)=\varepsilon_{[f-f_0,g]}(\Omega)=<f-f_0, g>=0.$$

Therefore, by the above lemma, there is $u\in\mathbf{H}_E(\omega)$ such that $f-f_0=u$ q.e. on ω. Since ω is arbitrary, modifying the values of f_0 on a polar set (i.e., re-defining f_0 by $f-u$ on ω), we have $f=u+f_0$ on Ω with $u\in\mathscr{H}(\Omega)$ and $f_0\in\mathscr{E}_0(\Omega)$. Since $f, f_0\in\mathscr{E}'$, $u\in\mathscr{E}'$. It follows from the definition of $\mathbf{H}_E(\Omega)$ that $u\in\mathbf{H}_E(\Omega)$. Hence $f\in\mathscr{E}(\Omega)$.

§6.3. Energy of superharmonic functions.

LEMMA 6.4. *Let μ be a non-negative measure such that U^μ is a potential. Then*

(i) *$U^\mu\in\mathscr{E}(\Omega)$ if and only if $\mu\in\mathbf{M}_E(\Omega)$;*

(ii) *$U^\mu\in\mathscr{E}_{\mathrm{loc}}(\Omega)$ if and only if $\mu|K\in\mathbf{M}_E(\Omega)$ for every compact set K in Ω.*

PROOF. The "if" part of (i) is already shown (Lemma 5.11). If $\mu|K\in\mathbf{M}_E(\Omega)$ for every compact set K, then, for each relatively compact domain ω, $\mu|\omega\in\mathbf{M}_E(\omega)$, and hence $U^{\mu}_{\omega}\in\mathscr{E}_0(\omega)\subset\mathscr{E}_{\text{loc}}(\omega)$. Since $U^{\mu}|\omega-U^{\mu}_{\omega}$ is harmonic, it belongs to $\mathscr{E}_{\text{loc}}(\omega)$. Hence, $U^{\mu}|\omega\in\mathscr{E}_{\text{loc}}(\omega)$. Since this is true for any relatively compact domain ω, we see that $U^{\mu}\in\mathscr{E}_{\text{loc}}(\Omega)$. Thus the "if" part of (ii) is proved.

Next, suppose $\mu(\Omega)<+\infty$ and $U^{\mu}\in\mathscr{E}(\Omega)$. Then $U^{\mu}\in\mathscr{E}_0(\Omega)$ by Proposition 5.4. Let $U^{\mu_m}=\min(U^{\mu}, m)$ for $m>0$. Then $\mu_m\in\mathbf{M}_E(\Omega)$, so that $U^{\mu_m}\in\mathscr{E}_0(\Omega)$. Using (5.4) of Lemma 5.12, we have

$$0\leqq E_{\Omega}[U^{\mu}-U^{\mu_m}]=E_{\Omega}[U^{\mu}]-\int_{\Omega}U^{\mu}d\mu_m+\int_{\Omega}(U^{\mu_m}-U^{\mu})d\mu_m$$

$$\leqq E_{\Omega}[U^{\mu}]-\int_{\Omega}U^{\mu_m}d\mu.$$

Hence, $\int U^{\mu_m}d\mu\leqq E_{\Omega}[U^{\mu}]$ for all $m>0$. Therefore, $I(\mu)\leqq E_{\Omega}[U^{\mu}]<+\infty$, i.e., $\mu\in\mathbf{M}_E(\Omega)$. Now let $U^{\mu}\in\mathscr{E}_{\text{loc}}(\Omega)$ and let $\{\Omega_n\}$ be an exhaustion of Ω. Let $U^{\mu}|\Omega_n=u_n+U^{\mu}_{\Omega_n}$ with $u_n\in\mathscr{H}(\Omega_n)$. Since $U^{\mu}|\Omega_n\in\mathscr{E}(\Omega_n)$, we have $u_n\in\mathbf{H}_E(\Omega_n)$, $U^{\mu}_{\Omega_n}\in\mathscr{E}_0(\Omega_n)$ and $E_{\Omega_n}[U^{\mu}_{\Omega_n}]\leqq E_{\Omega_n}[U^{\mu}]$. The above result implies that $\mu|\Omega_n\in\mathbf{M}_E(\Omega_n)$, since $\mu(\Omega_n)<+\infty$. Hence the "only if" part of (ii) follows. Furthermore, by Lemma 5.11,

$$\int_{\Omega_n}U^{\mu}_{\Omega_n}d\mu=E_{\Omega_n}[U^{\mu}_{\Omega_n}]\leqq E_{\Omega_n}[U^{\mu}].$$

Hence, if $U^{\mu}\in\mathscr{E}(\Omega)$, then $I(\mu)=\lim_{n\to\infty}\int_{\Omega_n}U^{\mu}_{\Omega_n}d\mu\leqq E_{\Omega}[U^{\mu}]<+\infty$. This means that the "only if" part of (i) holds.

PROPOSITION 6.3. *Let s be a superharmonic function on Ω.*

(i) *$s\in\mathscr{E}(\Omega)$ if and only if it has a harmonic minorant, its greatest harmonic minorant belongs to $\mathbf{H}_E(\Omega)$ and $\sigma_s\in\mathbf{M}_E(\Omega)$;*

(ii) *$s\in\mathscr{E}_{\text{loc}}(\Omega)$ if and only if $\sigma_s|K\in\mathbf{M}_E(\Omega)$ for every compact set K in Ω.*

PROOF. The "if" part of (i) is obvious. Suppose $s\in\mathscr{E}(\Omega)$. Then $s=u+p$ with $u\in\mathbf{H}_E(\Omega)$ and $p\in\mathscr{E}_0(\Omega)$. By Proposition 5.4, we see that p is a potential, so that u is the greatest harmonic minorant of s. Furthermore, $\sigma_p=\sigma_s$. Hence, by the above lemma, $\sigma_s\in\mathbf{M}_E(\Omega)$, and (i) is proved. Next, let s be any superharmonic function and ω be a relatively compact domain. Then s has a harmonic minorant on ω; in fact $s=u_{\omega}+U^{\sigma_s}_{\omega}$ on ω with $u_{\omega}\in\mathscr{H}(\omega)$. By the above lemma, $s|\omega\in\mathscr{E}_{\text{loc}}(\omega)$ if and only if $\sigma_s|K\in\mathbf{M}_E(\omega)$ for any compact set K in ω. Since ω is arbitrary, we obtain (ii).

§ 6.4. Lattice structures.

In this section, we first study the lattice structures of the following spaces:

$$\mathbf{Q}_E(\Omega)=\{f\,;\,f=U^\mu-U^\nu \text{ q.e. on } \Omega \text{ with } \mu,\ \nu\in\mathbf{M}_E(\Omega)\},$$

$$\mathbf{S}_E(\Omega)=\mathbf{H}_E(\Omega)+\mathbf{Q}_E(\Omega)$$

and

$$\mathbf{S}_{E,\mathrm{loc}}(\Omega)=\{f\,;\text{ for any relatively compact domain } \omega,\ f|\omega\in\mathbf{S}_E(\omega)\}.$$

Obviously, $\mathbf{P}_E(\Omega)\subset\mathbf{Q}_E(\Omega)\subset\mathscr{E}_0(\Omega)$, $\mathbf{B}_E(\Omega)\subset\mathbf{S}_E(\Omega)\subset\mathscr{E}(\Omega)$ and $\mathbf{B}_{\mathrm{loc}}(\Omega)\subset\mathbf{S}_{E,\mathrm{loc}}(\Omega)$ $\subset\mathscr{E}_{\mathrm{loc}}(\Omega)$. Furthermore, from Lemma 6.2 we can show that $\mathbf{S}_E(\Omega)\subset\mathbf{S}_{E,\mathrm{loc}}(\Omega)$.

LEMMA 6.5. *If* $u\in\mathbf{H}_E(\Omega)$, *then* $\min(u\vee 0, (-u)\vee 0)\in\mathbf{Q}_E(\Omega)$.

PROOF. Let $u^+=\max(u, 0)$ and $u^-=\max(-u, 0)$. Then $u^+=u\vee 0-U^\tau$ and $u^-=(-u)\vee 0-U^\tau$ with a non-negative measure τ on Ω. By Theorem 3.1 ([13]), $u\vee 0, (-u)\vee 0\in\mathbf{H}_E(\Omega)$. Since U^τ is locally bounded, $U^\tau\in\mathbf{B}_{\mathrm{loc}}(\Omega)$ by Lemma 6.1. Hence u^+, $u^-\in\mathbf{B}_{\mathrm{loc}}(\Omega)$. Since $u^+u^-=0$ and $S(\tau)\subset\{x\in\Omega;\ u(x)=0\}$, we have

$$\varepsilon_{[u^+,u^-]}=\frac{1}{2}(-u^+\tau-u^-\tau)=0.$$

It follows that $\varepsilon_u=\varepsilon_{u^+}+\varepsilon_{u^-}$. Hence $\varepsilon_{u^+}(\Omega)\leqq\varepsilon_u(\Omega)=E_\Omega[u]<+\infty$, so that $u^+\in\mathscr{E}(\Omega)$ by Theorem 6.2. Therefore, $U^\tau\in\mathscr{E}_0(\Omega)$, and by Lemma 6.4, $U^\tau\in\mathbf{Q}_E(\Omega)$. Since $U^\tau=\min(u\vee 0, (-u)\vee 0)$, we have the lemma.

THEOREM 6.3. $\mathbf{Q}_E(\Omega)$ *and* $\mathbf{S}_E(\Omega)$ *are vector lattices with respect to the max. and min. operations and*

$$E_\Omega[|f|]=E_\Omega[f] \qquad \textit{for any } f\in\mathbf{S}_E(\Omega);$$

$$E_\Omega[\max(f, g)]+E_\Omega[\min(f, g)]=E_\Omega[f]+E_\Omega[g] \quad \textit{for any } f,\ g\in\mathbf{S}_E(\Omega).$$

PROOF. It is enough to prove that if $f\in\mathbf{S}_E(\Omega)$ (resp. $\in\mathbf{Q}_E(\Omega)$), then $\max(f, 0)$, $\min(f, 0)\in\mathbf{S}_E(\Omega)$ (resp. $\in\mathbf{Q}_E(\Omega)$) and

$$E_\Omega[\max(f, 0), \min(f, 0)]=0 \tag{6.4}$$

(cf. the proof of Theorem 3.1 in [13]). Let $f=u+U^\mu-U^\nu$ q.e. on Ω with $u\in\mathbf{H}_E(\Omega)$ and μ, $\nu\in\mathbf{M}_E(\Omega)$. By the above lemma, $\min(u\vee 0, (-u)\vee 0)\in\mathbf{Q}_E(\Omega)$. It then follows that $\min\{u\vee 0+U^\mu, (-u)\vee 0+U^\nu\}$ is a potential belonging to $\mathbf{Q}_E(\Omega)$. Let

$$U^\lambda=\min\{u\vee 0+U^\mu, (-u)\vee 0+U^\nu\},\ \lambda\in\mathbf{M}_E(\Omega).$$

Since $\max(f, 0)=u\vee 0+U^\mu-U^\lambda$, $\min(f, 0)=u\wedge 0+U^\lambda-U^\mu$ q.e. on Ω and $u\vee 0$, $(-u)\vee 0\in\mathbf{H}_E(\Omega)$, we see that $\max(f, 0)$, $\min(f, 0)\in\mathbf{S}_E(\Omega)$. Furthermore, if

$f\in \mathbf{Q}_E(\Omega)$, then $u=0$, so that $\max(f, 0)$, $\min(f, 0)\in \mathbf{Q}_E(\Omega)$.

To obtain (6.4), we first suppose that U^μ, U^ν are continuous. Then $f\in \mathbf{B}_{\mathrm{loc}}(\Omega)$ and the above observations show that $\max(f, 0)$, $\min(f, 0)\in \mathbf{B}_{\mathrm{loc}}(\Omega)$. Let $\Omega_+ =\{x\in\Omega;\ f(x)>0\}$ and $\Omega_- =\{x\in\Omega;\ f(x)<0\}$. Since Ω_+, Ω_- are open sets, it follows from Lemma 1.8 ([13]) that $\lambda|\Omega_+ =\nu|\Omega_+$ and $\lambda|\Omega_- =\mu|\Omega_-$. Hence, noting that $\max(f, 0)\cdot\min(f, 0)=0$, we have

$$\varepsilon_{[\max(f,0),\min(f,0)]}=\frac{1}{2}\{\max(f, 0)(\lambda-\nu)+\min(f, 0)(\mu-\lambda)\}$$

$$=\frac{1}{2}\{f\chi_{\Omega_+}(\lambda-\nu)+f\chi_{\Omega_-}(\mu-\lambda)\}=0,$$

where χ_A means the characteristic function of a set A. Therefore, we obtain (6.4) in case U^μ, U^ν are continuous. In the general case, we choose μ_n and ν_n, $n=1, 2, \ldots$, such that U^{μ_n}, U^{ν_n} are continuous, $U^{\mu_n}\uparrow U^\mu$ and $U^{\nu_n}\uparrow U^\nu$. Let $f_n=u+U^{\mu_n}-U^{\nu_n}$ and

$$U^{\lambda_n}=\min\{u\vee 0+U^{\mu_n},\ (-u)\vee 0+U^{\nu_n}\}.$$

Then, $f_n\in \mathbf{S}_E(\Omega)$, $U^{\lambda_n}\uparrow U^\lambda$, $\max(f_n, 0)=u\vee 0+U^{\mu_n}-U^{\lambda_n}$ and $\min(f_n, 0)=u\wedge 0 +U^{\lambda_n}-U^{\nu_n}$. By Lemma 4.5 and the corollary to Lemma 5.11, $E_\Omega[U^{\mu_n}-U^\mu]\to 0$, $E_\Omega[U^{\nu_n}-U^\nu]\to 0$ and $E_\Omega[U^{\lambda_n}-U^\lambda]\to 0$ $(n\to\infty)$. Hence $E_\Omega[\max(f_n, 0)-\max(f, 0)]\to 0$ and $E_\Omega[\min(f_n, 0)-\min(f, 0)]\to 0$ $(n\to\infty)$. Since $E_\Omega[\max(f_n, 0), \min(f_n, 0)]=0$ for each n, we obtain (6.4).

COROLLARY. *$\mathbf{S}_{E,\mathrm{loc}}(\Omega)$ is a vector lattice with respect to the max. and min. operations and, for f, $g\in \mathbf{S}_{E,\mathrm{loc}}(\Omega)$,*

$$\varepsilon_{|f|}=\varepsilon_f, \qquad \varepsilon_{[\max(f,0),\min(f,0)]}=0$$

and

$$\varepsilon_{\max(f,g)}+\varepsilon_{\min(f,g)}=\varepsilon_f+\varepsilon_g.$$

REMARK. The proof of Theorem 6.3 shows that $\mathbf{B}_{\mathrm{loc}}(\Omega)$ is also closed under max. and min. operations.

THEOREM 6.4. *$\mathscr{E}(\Omega)$ and $\mathscr{E}_0(\Omega)$ are vector lattices with respect to the max. and min. operations; if f, $g\in\mathscr{E}(\Omega)$, then*

$$E_\Omega[|f|]\leqq E_\Omega[f]$$

and

$$E_\Omega[\max(f, g)]+E_\Omega[\min(f, g)]\leqq E_\Omega[f]+E_\Omega[g].$$

PROOF. Let $f\in\mathscr{E}(\Omega)$, i.e., $f=u+p$ with $u\in \mathbf{H}_E(\Omega)$ and $p\in\mathscr{E}_0(\Omega)$. By Theorem 5.1, there is $\{p_n\}$ in $\mathbf{P}_E(\Omega)$ such that $E_\Omega[p_n-p]\to 0$ and $p_n\to p$ q.e. on Ω $(n\to\infty)$. It follows from Lemma 5.14 that

(6.5) $$\int_{\Omega}|u+p_n|d\mu\to\int_{\Omega}|f|d\mu \qquad (n\to\infty)$$

for any $\mu\in\mathbf{M}_E(\Omega)$ with compact support. Let $g_n=|u+p_n|-u\vee(-u)$. Since $u+p_n\in\mathbf{S}_E(\Omega)$, Theorem 6.3 implies that $g_n\in\mathbf{S}_E(\Omega)$. Furthermore, since $|g_n| \leqq\{u\vee(-u)-|u|\}+|p_n|$, we see that $g_n\in\mathbf{Q}_E(\Omega)$ by using Proposition 5.4. Therefore, using Theorem 6.3 again, we have

$$E_\Omega[g_n]\leqq E_\Omega[|u+p_n|]=E_\Omega[u+p_n]=E_\Omega[u]+E_\Omega[p_n].$$

Hence $\{E_\Omega[g_n]\}$ is bounded. Regarding $\mathscr{E}_0(\Omega)$ as a Hilbert space, we can choose a subsequence $\{g_{n_j}\}$ of $\{g_n\}$ which converges to a function $g\in\mathscr{E}_0(\Omega)$ weakly in $\mathscr{E}_0(\Omega)$. It follows from Lemma 5.12 that $\int_{\Omega}g_{n_j}d\mu\to\int_{\Omega}gd\mu$ for any $\mu\in\mathbf{M}_E(\Omega)$. Hence

(6.6) $$\int_{\Omega}|u+p_{n_j}|d\mu\to\int_{\Omega}\{g+u\vee(-u)\}d\mu$$

for any $\mu\in\mathbf{M}_E(\Omega)$ with compact support. By (6.5) and (6.6),

$$\int_{\Omega}|f|d\mu=\int_{\Omega}\{g+u\vee(-u)\}d\mu$$

for any $\mu\in\mathbf{M}_E(\Omega)$ with compact support. Hence, by the corollary to Lemma 5.7, we conclude that $|f|=u\vee(-u)+g$ q.e. on Ω. Hence, $|f|\in\mathscr{E}(\Omega)$. Furthermore, if $f\in\mathscr{E}_0(\Omega)$, then $u=0$, so that $|f|\in\mathscr{E}_0(\Omega)$. Since $g_{n_j}\to g$ weakly in $\mathscr{E}_0(\Omega)$, we see that $|u+p_{n_j}|\to|f|$ weakly in $\mathscr{E}_0(\Omega)$. It then follows that

$$E_\Omega[|f|]\leqq\liminf_{j\to\infty}E_\Omega[|u+p_{n_j}|]\leqq\lim_{n\to\infty}E_\Omega[u+p_n]=E_\Omega[f].$$

COROLLARY. *$\mathscr{E}_{\mathrm{loc}}(\Omega)$ is a vector lattice with respect to the max. and min. operations; for any $f, g\in\mathscr{E}_{\mathrm{loc}}(\Omega)$,*

$$\varepsilon_{\max(f,g)}+\varepsilon_{\min(f,g)}\leqq\varepsilon_f+\varepsilon_g.$$

Finally we give

PROPOSITION 6.4. *If $f\in\mathscr{E}(\Omega)$ (resp. $\in\mathscr{E}_0(\Omega)$) and $\alpha\geqq0$, then $\min(f,\alpha)\in\mathscr{E}(\Omega)$ (resp. $\in\mathscr{E}_0(\Omega)$) and*

$$E_\Omega[\min(f,\alpha)]\leqq E_\Omega[f].$$

PROOF. Since $\alpha\in\mathscr{E}_{\mathrm{loc}}(\Omega)$, the above corollary implies

$$\varepsilon_{\max(f,\alpha)}+\varepsilon_{\min(f,\alpha)}\leqq\varepsilon_f+\varepsilon_\alpha.$$

Now, $\max(f,\alpha)=\{f-\min(f,\alpha)\}+\alpha$. Hence

$$\varepsilon_{\max(f,\alpha)}=\varepsilon_{f-\min(f,\alpha)}+2\varepsilon_{[f-\min(f,\alpha),\alpha]}+\varepsilon_\alpha,$$

so that we have

$$\varepsilon_f - \varepsilon_{\min(f,\alpha)} \geqq \varepsilon_{f-\min(f,\alpha)} + 2\varepsilon_{[f-\min(f,\alpha),\alpha]},$$

or, by Proposition 6.2,

$$\varepsilon_f - \varepsilon_{\min(f,\alpha)} \geqq \varepsilon_{f-\min(f,\alpha)} + 2\alpha\{f-\min(f,\alpha)\}\pi. \tag{6.7}$$

Since the right-hand side is a non-negative measure, $\varepsilon_{\min(f,\alpha)} \leqq \varepsilon_f$. Hence $\varepsilon_{\min(f,\alpha)}(\Omega) \leqq \varepsilon_f(\Omega) = E_\Omega[f] < +\infty$. Therefore, by Theorem 6.2, $\min(f, \alpha) \in \mathscr{E}(\Omega)$, and $E_\Omega[\min(f, \alpha)] = \varepsilon_{\min(f,\alpha)}(\Omega) \leqq E_\Omega[f]$. Furthermore, if $f \in \mathscr{E}_0(\Omega)$, then Proposition 5.4 and the inequality $|\min(f, \alpha)| \leqq |f|$ imply that $\min(f, \alpha) \in \mathscr{E}_0(\Omega)$.

REMARK. In view of the corollary to Theorem 6.3, the equality holds in (6.7) if $f \in \mathbf{S}_{E,\mathrm{loc}}(\Omega)$.

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