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Characterizations of Radicals of Infinite Dimensional Lie Algebras

Dedicated to Professor Tôzirô Ogasawara
on the occasion of his retirement

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Introduction

Recently investigations have been made on the Lie algebras of infinite dimension. As the Lie analogues of the infinite group theory, B. Hartley [1] has considered the notions of subideals and ascendant subalgebras and studied the locally nilpotent radicals which reduce to the nilpotent radical in finite-dimensional case. In [4, 5] we have introduced and studied the locally solvable radicals which reduce to the solvable radical in finite-dimensional case. If $\mathfrak{X}$ is a coalescent (resp. an ascendantly coalescent) class of Lie algebras, for an arbitrary Lie algebra L we there defined the radical $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$) as the subalgebra generated by all the $\mathfrak{X}$ subideals (resp. all the ascendant $\mathfrak{X}$ subalgebras) of L. In particular, if the basic field is of characteristic 0, $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}\text{-si}}(L)$ and $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}\text{-asc}}(L)$ are respectively the Baer radical $\beta(L)$ and the Gruenberg radical $\gamma(L)$ which are locally nilpotent [1], and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)$ and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$ are locally solvable radicals [4, 5], where $\mathfrak{N}$, $\mathfrak{S}$ and $\mathfrak{F}$ denote respectively the classes of nilpotent, solvable and finite-dimensional Lie algebras.

The purpose of this paper is to investigate the radicals of Lie algebras, especially to present certain characterizations of $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ and $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$ and to study two new radicals.

For a class $\mathfrak{X}$ of Lie algebras, we denoted by $\mathrm{L}\mathfrak{X}$ the collection of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$ subalgebra of L [4]. In Section 2, in connection with $\mathrm{L}\mathfrak{X}$ we define $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) as the class of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$ subideal (resp. an ascendant $\mathfrak{X}$ subalgebra) of L and study their properties. In Section 3 we show that if $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent), any Lie algebra L has a unique maximal $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) ideal (Theorem 3.2) and $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$) is the subalgebra generated by all the $\mathrm{M}\mathfrak{X}$ subideals (resp. all the ascendant $\acute{\mathrm{M}}\mathfrak{X}$ subalgebras) of L and belongs to $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) (Theorem 3.5). Hence if furthermore $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$) is an ideal of L then it is the unique maximal $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) ideal of L (Theorem 3.6). In Section 4 we apply these results to $\beta(L)$, $\gamma(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{si}}(L)$ and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{asc}}(L)$ to get their characterizations. E.g., $\beta(L)$ is the unique maximal $\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ ideal and the unique maximal $\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ subideal of L (Theorem 4.1). In Section 5 we study the two new radicals $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})}(L)$ and $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)$. We show that each of them is an ideal but not necessarily a characteristic ideal of L, and that if the basic field is of characteristic 0 then $\beta(L)\subseteq\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})}(L)\subseteq\gamma(L)$ and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{si}}(L)\subseteq\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)\subseteq\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{asc}}(L)$ where the equalities do not hold in general (Theorems 5.1 and 5.3).

§1. Preliminaries

We shall be concerned with Lie algebras over a field Φ which are not necessarily finite-dimensional. Throughout this paper, L will be an arbitrary Lie algebra over a field Φ, and $\mathfrak{X}$ an arbitrary class of Lie algebras, that is, an arbitrary collection of Lie algebras over a field Φ such that $(0)\in\mathfrak{X}$ and if $H\in\mathfrak{X}$ and $H\simeq K$ then $K\in\mathfrak{X}$, unless otherwise specified.

We mainly employ the terminology and notations which were used in [4, 5]. $H\leq L$, $H\lhd L$, H si L and H asc L mean that H is respectively a subalgebra, an ideal, a subideal and an ascendant subalgebra of L. A Lie algebra (resp. a subalgebra, an ideal, a subideal and an ascendant subalgebra of L) belonging to $\mathfrak{X}$ is called an $\mathfrak{X}$ algebra (resp. an $\mathfrak{X}$ subalgebra, an $\mathfrak{X}$ ideal, an $\mathfrak{X}$ subideal and an ascendant $\mathfrak{X}$ subalgebra of L). $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent) provided H, K si L (resp. H, K asc L) and $H, K\in\mathfrak{X}$ imply $\langle H, K\rangle$ si L (resp. $\langle H, K\rangle$ asc L) and $\langle H, K\rangle\in\mathfrak{X}$. $\mathfrak{F}$, $\mathfrak{N}$, $\mathfrak{S}$ and $\mathfrak{G}$ denote respectively the classes of finite-dimensional, nilpotent, solvable, and finitely generated Lie algebras. Then both $\mathfrak{N}\cap\mathfrak{F}$ and $\mathfrak{S}\cap\mathfrak{F}$ are coalescent and ascendantly coalescent if the basic field Φ is of characteristic 0.

$\mathrm{L}\mathfrak{X}$ denotes the class of locally $\mathfrak{X}$ algebras, that is, the class of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$ subalgebra of L.

$\mathrm{N}\mathfrak{X}$ (resp. $\acute{\mathrm{N}}\mathfrak{X}$) denotes the class of Lie algebras generated by $\mathfrak{X}$ subideals (resp. ascendant $\mathfrak{X}$ subalgebras). $\mathfrak{X}$ is said to be N_0-closed provided the sum of any two $\mathfrak{X}$ ideals of any Lie algebra always belongs to $\mathfrak{X}$.

For a coalescent (resp. an ascendantly coalescent) class $\mathfrak{X}$, the radical $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}-\mathrm{asc}}(L)$) of L is the subalgebra generated by all the $\mathfrak{X}$ subideals (resp. all the ascendant $\mathfrak{X}$ subalgebras) of L. For an N_0-closed class $\mathfrak{X}$, the radical $\mathrm{Rad}_{\mathfrak{X}}(L)$ of L is the sum of all the $\mathfrak{X}$ ideals of L. These three radicals belong to $\mathrm{L}\mathfrak{X}$. $\mathrm{Rad}_{\mathrm{L}\mathfrak{N}}(L)$ is the Hirsch-Plotkin radical $\rho(L)$. If the basic field Φ is of characteristic 0, then $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}-\mathrm{si}}(L)$ is the Baer radical $\beta(L)$, and $\mathrm{Rad}_{\mathfrak{N}\cap\mathfrak{F}-\mathrm{asc}}(L)$ is the Gruenberg radical $\gamma(L)$. These reduce to the nilpotent radical in finite-dimensional case. Corresponding to these radicals, $\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{si}}(L)$, and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{asc}}(L)$ have been investigated in [4, 5]. These reduce to the solvable radical in finite-dimensional case.

§2. Operations M, $\acute{\mathrm{M}}$, M_1 and $\acute{\mathrm{M}}_1$

We begin with introducing new closure operations M, $\acute{\mathrm{M}}$, M_1 and $\acute{\mathrm{M}}_1$ which are intimately connected with the operation L.

DEFINITION 2.1. *For any class $\mathfrak{X}$ of Lie algebras, we denote by $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) the class of Lie algebras L such that any finite subset of L lies inside an $\mathfrak{X}$ subideal (resp. an ascendant $\mathfrak{X}$ subalgebra) of L and by $\mathrm{M}_1\mathfrak{X}$ (resp. $\acute{\mathrm{M}}_1\mathfrak{X}$) the class of Lie algebras L such that any element of L lies inside an $\mathfrak{X}$ subideal (resp. an ascendant $\mathfrak{X}$ subalgebra) of L.*

Then these classes and $\mathrm{L}\mathfrak{X}$ are related to each other as in the following diagram:

$$\begin{array}{ccccccc} \mathfrak{X} & \subseteq & \mathrm{M}\mathfrak{X} & \subseteq & \acute{\mathrm{M}}\mathfrak{X} & \subseteq & \mathrm{L}\mathfrak{X} \\ & & \cap\vert & & \cap\vert & & \\ & & \mathrm{M}_1\mathfrak{X} & \subseteq & \acute{\mathrm{M}}_1\mathfrak{X} & & \end{array}$$

Generally these six classes are different from each other. This fact will be shown by examples in Section 6.

LEMMA 2.2. *If $\mathfrak{X}$ is a coalescent (resp. an ascendantly coalescent) class of Lie algebras, then*

$$\mathrm{M}\mathfrak{X}=\mathrm{M}_1\mathfrak{X}=\mathrm{N}\mathfrak{X} \qquad (\text{resp. } \acute{\mathrm{M}}\mathfrak{X}=\acute{\mathrm{M}}_1\mathfrak{X}=\acute{\mathrm{N}}\mathfrak{X}).$$

PROOF. For any class $\mathfrak{X}$ it is evident that

$$\mathrm{M}\mathfrak{X}\subseteq\mathrm{M}_1\mathfrak{X}\subseteq\mathrm{N}\mathfrak{X} \quad \text{and} \quad \acute{\mathrm{M}}\mathfrak{X}\subseteq\acute{\mathrm{M}}_1\mathfrak{X}\subseteq\acute{\mathrm{N}}\mathfrak{X}.$$

Now let $\mathfrak{X}$ be coalescent (resp. ascendantly coalescent) and assume that $L\in\mathrm{N}\mathfrak{X}$ (resp. $\acute{\mathrm{N}}\mathfrak{X}$). Let $\{x_1, \dots, x_n\}$ be any finite subset of L. Then for each i there exist H_{ij}'s such that

$$x_i\in \langle H_{i1}, \dots, H_{im_i}\rangle,\ H_{ij} \text{ si } L \text{ (resp. } H_{ij} \text{ asc } L) \text{ and } H_{ij}\in\mathfrak{X}.$$

Denote the join of all the H_{ij} by H. Since $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent),

$$H \text{ si } L \text{ (resp. } H \text{ asc } L) \quad \text{and} \quad H\in\mathfrak{X}.$$

Hence $L\in\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$). Therefore

$$\mathrm{N}\mathfrak{X}\subseteq\mathrm{M}\mathfrak{X} \qquad (\text{resp. } \acute{\mathrm{N}}\mathfrak{X}\subseteq\acute{\mathrm{M}}\mathfrak{X}),$$

which establishes the lemma.

LEMMA 2.3. (1) $\mathrm{M}\mathfrak{N}=\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ *(resp.* $\acute{\mathrm{M}}\mathfrak{N}=\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})$*) and these classes are equal to the collection of* $L\in\mathrm{L}\mathfrak{N}$ *such that* $H\leq L$ *and* $H\in\mathfrak{F}$ *imply* H *si* L *(resp.* H *asc* L*).*

(2) $\mathrm{M}_1\mathfrak{N}=\mathrm{M}_1(\mathfrak{N}\cap\mathfrak{F})$ *and* $\acute{\mathrm{M}}_1\mathfrak{N}=\acute{\mathrm{M}}_1(\mathfrak{N}\cap\mathfrak{F})$.

(3) $\mathrm{N}\mathfrak{N}=\mathrm{N}(\mathfrak{N}\cap\mathfrak{F})$ *and* $\acute{\mathrm{N}}\mathfrak{N}=\acute{\mathrm{N}}(\mathfrak{N}\cap\mathfrak{F})$.

(4) *If the basic field* Φ *is of characteristic* 0, *then*

$$\mathrm{M}\mathfrak{N}=\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})=\mathrm{M}_1\mathfrak{N}=\mathrm{M}_1(\mathfrak{N}\cap\mathfrak{F})=\mathrm{N}\mathfrak{N}=\mathrm{N}(\mathfrak{N}\cap\mathfrak{F}),$$

$$\acute{\mathrm{M}}\mathfrak{N}=\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})=\acute{\mathrm{M}}_1\mathfrak{N}=\acute{\mathrm{M}}_1(\mathfrak{N}\cap\mathfrak{F})=\acute{\mathrm{N}}\mathfrak{N}=\acute{\mathrm{N}}(\mathfrak{N}\cap\mathfrak{F}).$$

PROOF. (1) Assume $L\in\mathrm{M}\mathfrak{N}$ (resp. $\acute{\mathrm{M}}\mathfrak{N}$). Let K be any finite subset of L. Then there exists H such that

$$K\subseteq H,\ H \text{ si } L \text{ (resp. } H \text{ asc } L) \text{ and } H\in\mathfrak{N}.$$

Since $H\in\mathfrak{N}$, $<K>$ si H and therefore $<K>$ si L (resp. $<K>$ asc L). Taking account of the fact that $\mathfrak{N}\cap\mathfrak{G}\subseteq\mathfrak{F}$, we have $<K>\in\mathfrak{N}\cap\mathfrak{F}$. Hence $L\in\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ (resp. $\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})$). Consequently

$$\mathrm{M}\mathfrak{N}\subseteq\mathrm{M}(\mathfrak{N}\cap\mathfrak{F}) \qquad (\text{resp. } \acute{\mathrm{M}}\mathfrak{N}\subseteq\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})).$$

Since the converse inclusion is evident, we have the first statement of (1).

Assume $L\in\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ (resp. $\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})$). Evidently $L\in\mathrm{L}\mathfrak{N}$. Let H be an $\mathfrak{F}$ subalgebra of L. Then $H=(x_1, \dots, x_n)$. By assumption, there exists K such that

$$\{x_1, \dots, x_n\}\subseteq K,\ K \text{ si } L \text{ (resp. } K \text{ asc } L) \text{ and } K\in\mathfrak{N}\cap\mathfrak{F}.$$

Since $K\in\mathfrak{N}$, H si K and therefore H si L (resp. H asc L). Conversely, assume that $L\in\mathrm{L}\mathfrak{N}$ and any $\mathfrak{F}$ subalgebra of L is a subideal (resp. an ascendant subalgebra). Let K be any finite subset of L. Since $\mathrm{L}\mathfrak{N}=\mathrm{L}(\mathfrak{N}\cap\mathfrak{F})$ by Lemma 4.1 in [5], there exists H such that

$$K\subseteq H,\ H\leq L \text{ and } H\in\mathfrak{N}\cap\mathfrak{F}.$$

Hence, by assumption, H si L (resp. H asc L). This shows that $L\in\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ (resp. $\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})$).

The statement in (2) can be proved in the same way as the first part of (1).

(3) Assume $L\in\mathrm{N}\mathfrak{N}$ (resp. $\acute{\mathrm{N}}\mathfrak{N}$). Let H be any one of $\mathfrak{N}$ subideals (resp. ascendant $\mathfrak{N}$ subalgebras) generating L. For any $x\in H$, (x) si H since $H\in\mathfrak{N}$. It follows that

$$(x) \text{ si } L \qquad (\text{resp. } (x) \text{ asc } L).$$

Hence H is a union of $\mathfrak{N}\cap\mathfrak{F}$ subideals (resp. ascendant $\mathfrak{N}\cap\mathfrak{F}$ subalgebras) of L. Therefore $L \in \mathrm{N}(\mathfrak{N} \cap \mathfrak{F})$ (resp. $\acute{\mathrm{N}}(\mathfrak{N} \cap \mathfrak{F})$). Consequently

$$\mathrm{N}\mathfrak{N} \subseteq \mathrm{N}(\mathfrak{N} \cap \mathfrak{F}) \qquad (\text{resp. } \acute{\mathrm{N}}\mathfrak{N} \subseteq \acute{\mathrm{N}}(\mathfrak{N} \cap \mathfrak{F})).$$

Since the converse inclusion is evident, we have the statement of (3).

(4) If Φ is of characteristic 0, then $\mathfrak{N} \cap \mathfrak{F}$ is coalescent and ascendantly coalescent. Hence the statement is immediate from (1)–(3) and Lemma 2.2. The proof is complete.

§3. Characterizations of $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ and $\mathrm{Rad}_{\mathfrak{X}-\mathrm{asc}}(L)$

In this section, for any coalescent (resp. ascendantly coalescent) class $\mathfrak{X}$ we shall show the existence of a unique maximal $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) ideal of L and use it to give characterizations of the radical $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}-\mathrm{asc}}(L)$).

LEMMA 3.1. *If $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent), then the sum of any collection of $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) ideals of L belongs to $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$). In particular $\mathrm{M}\mathfrak{X}$ and $\acute{\mathrm{M}}\mathfrak{X}$ are N_0-closed.*

PROOF. Let $\mathfrak{C}$ be any collection of $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) ideals of L and R be the sum of ideals in $\mathfrak{C}$. Suppose $\{x_1, \dots, x_n\}$ is any finite subset of R. Then

$$x_i = \sum_{j=1}^{m_i} x_{ij}, \qquad x_{ij} \in N_{ij} \in \mathfrak{C}.$$

Since $N_{ij} \in \mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$), there exist H_{ij}'s such that

$$x_{ij} \in H_{ij}, \quad H_{ij} \text{ si } N_{ij} \text{ (resp. } H_{ij} \text{ asc } N_{ij}), \quad H_{ij} \in \mathfrak{X}.$$

It follows that

$$H_{ij} \text{ si } L \qquad (\text{resp. } H_{ij} \text{ asc } L).$$

Denote the join of all the H_{ij} by H. Then coalescency (resp. ascendant coalescency) of $\mathfrak{X}$ tells us that

$$H \text{ si } L \text{ (resp. } H \text{ asc } L), \quad H \in \mathfrak{X}.$$

Taking account of the fact that $H \subseteq R$, we have

$$H \text{ si } R \qquad (\text{resp. } H \text{ asc } R).$$

Since $H \supseteq \{x_1, \dots, x_n\}$, R belongs to $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$), and this completes the proof.

THEOREM 3.2. *If $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent), then $\mathrm{Rad}_{\mathrm{M}\mathfrak{X}}(L)$ (resp. $\mathrm{Rad}_{\acute{\mathrm{M}}\mathfrak{X}}(L)$) is the unique maximal $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) ideal of L.*

PROOF. Since $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) is N_0-closed by Lemma 3.1, $\mathrm{Rad}_{\mathrm{M}\mathfrak{X}}(L)$ (resp.

$\mathrm{Rad}_{\acute{\mathrm{M}}\mathfrak{X}}(L)$) can be defined. By Lemma 3.1 it belongs to $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$). Therefore it is the unique maximal $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) ideal of L.

LEMMA 3.3. *Every $\mathrm{M}\mathfrak{X}$ subideal (resp. ascendant $\acute{\mathrm{M}}\mathfrak{X}$ subalgebra) of L is a union of $\mathfrak{X}$ subideals (resp. ascendant $\mathfrak{X}$ subalgebras) of L.*

PROOF. Let H be an $\mathrm{M}\mathfrak{X}$ subideal (resp. an ascendant $\acute{\mathrm{M}}\mathfrak{X}$ subalgebra) of L. For any $x \in H$, there exists an $\mathfrak{X}$ subideal (resp. an ascendant $\mathfrak{X}$ subalgebra) of H containing x. It is then an $\mathfrak{X}$ subideal (resp. an ascendant $\mathfrak{X}$ subalgebra) of L. Therefore H is a union of $\mathfrak{X}$ subideals (resp. ascendant $\mathfrak{X}$ subalgebras) of L.

LEMMA 3.4. *If $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent), the subalgebra generated by any collection of $\mathfrak{X}$ subideals (resp. ascendant $\mathfrak{X}$ subalgebras) of L belongs to $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$).*

PROOF. Let $\mathfrak{C}$ be any collection of $\mathfrak{X}$ subideals (resp. ascendant $\mathfrak{X}$ subalgebras) of L and R be the subalgebra generated by all the subalgebras in $\mathfrak{C}$. Suppose $\{x_1, \dots, x_n\}$ is any finite subset of R. Then for each i there exist H_{ij}'s such that

$$x_i \in \langle x_{i1}, \dots, x_{im_i} \rangle, \qquad x_{ij} \in H_{ij} \in \mathfrak{C}.$$

Denote the join of all the H_{ij} by H. Since $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent),

$$H \text{ si } L \text{ (resp. } H \text{ asc } L), \qquad H \in \mathfrak{X}.$$

Taking account of the fact that $H \subseteq R$, we have

$$H \text{ si } R \qquad \text{(resp. } H \text{ asc } R).$$

Since $H \supseteq \{x_1, \dots, x_n\}$, R belongs to $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$), and this completes the proof.

THEOREM 3.5. *If $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent), $\mathrm{Rad}_{\mathfrak{X}\text{-si}}(L)$ (resp. $\mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)$) is the subalgebra generated by all the $\mathrm{M}\mathfrak{X}$ subideals (resp. ascendant $\acute{\mathrm{M}}\mathfrak{X}$ subalgebras) of L and belongs to $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$).*

PROOF. Let R be the subalgebra generated by all the $\mathrm{M}\mathfrak{X}$ subideals (resp. all the ascendant $\acute{\mathrm{M}}\mathfrak{X}$ subalgebras) of L. Then by Lemma 3.3

$$R \subseteq \mathrm{Rad}_{\mathfrak{X}\text{-si}}(L) \qquad \text{(resp. } R \subseteq \mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)).$$

The converse inclusion is immediate from the fact that $\mathfrak{X} \subseteq \mathrm{M}\mathfrak{X}$. Therefore

$$R = \mathrm{Rad}_{\mathfrak{X}\text{-si}}(L) \qquad \text{(resp. } R = \mathrm{Rad}_{\mathfrak{X}\text{-asc}}(L)).$$

The other part of the statement follows from Lemma 3.4.

THEOREM 3.6. *Let $\mathfrak{X}$ be coalescent (resp. ascendantly coalescent). If* $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ *(resp.* $\mathrm{Rad}_{\mathfrak{X}-\mathrm{asc}}(L)$*) is a subideal (resp. an ascendant subalgebra) of L, then it is the unique maximal* $\mathrm{M}\mathfrak{X}$ *subideal (resp. ascendant* $\acute{\mathrm{M}}\mathfrak{X}$ *subalgebra) of L. If* $\mathrm{Rad}_{\mathfrak{X}-\mathrm{si}}(L)$ *(resp.* $\mathrm{Rad}_{\mathfrak{X}-\mathrm{asc}}(L)$*) is an ideal of L, then it is the unique maximal* $\mathrm{M}\mathfrak{X}$ *(resp.* $\acute{\mathrm{M}}\mathfrak{X}$*) ideal of L.*

PROOF. This is an immediate consequence of Theorems 3.2 and 3.5.

It is finally to be noted that by Lemma 2.2 the theorems and lemmas in this section are valid with $\mathrm{M}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}\mathfrak{X}$) replaced by each of $\mathrm{M}_1\mathfrak{X}$, $\mathrm{N}\mathfrak{X}$ (resp. $\acute{\mathrm{M}}_1\mathfrak{X}$, $\acute{\mathrm{N}}\mathfrak{X}$).

§4. Characterizations of $\beta(L)$, $\gamma(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{si}}(L)$ and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{asc}}(L)$

In this section we assume that the basic field Φ is of characteristic 0. We shall apply the results of the preceding section for $\beta(L)$, $\gamma(L)$, $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{si}}(L)$ and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{asc}}(L)$ to obtain their characterizations.

The Baer radical $\beta(L)$ of L is equal to the subalgebra generated by all the $\mathfrak{N}$ (resp. all the one-dimensional) subideals of L and to the set of $x \in L$ such that (x) si L [2, Theorem 10.4]. We have further characterizations of $\beta(L)$ in the following

THEOREM 4.1. *The Baer radical $\beta(L)$ of L is the unique maximal* $\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ *ideal, the unique maximal* $\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ *subideal and the unique maximal characteristic* $\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ *ideal of L.*

PROOF. It is shown in Corollary to Theorem 3 of [1] that $\beta(L)$ is a characteristic ideal of L. Hence the statement follows from Theorem 3.6.

L is called [2] a Baer algebra if $L=\beta(L)$. We call an ideal of L which is itself a Baer algebra a Baer ideal of L. Then the $\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ ideals of L are the Baer ideals of L, since $\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})=\mathrm{N}(\mathfrak{N}\cap\mathfrak{F})$ by Lemma 2.2. Therefore a part of the theorem may be expressed as in the following

COROLLARY 4.2. *The Baer radical of L is the sum of all the Baer ideals of L and is the unique maximal Baer ideal of L.*

The Gruenberg radical $\gamma(L)$ of L is equal to the subalgebra generated by all the ascendant $\mathfrak{N}$ (resp. one-dimensional) subalgebras of L and to the set of $x \in L$ such that (x) asc L. The proof may be carried out in the same way as that of the corresponding characterizations of $\beta(L)$ given in [2]. We have further characterizations of $\gamma(L)$ in the following statements.

THEOREM 4.3. *The Gruenberg radical $\gamma(L)$ of L is the subalgebra generated by the ascendant* $\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})$ *subalgebras of L and belongs to* $\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})$.

PROOF. This follows from Theorem 3.5.

COROLLARY 4.4. *The Gruenberg radical of L is the subalgebra generated by all the ascendant* $\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ *subalgebras of L.*

PROOF. This follows from Theorem 4.3 and the fact that

$$\mathfrak{N}\cap\mathfrak{F}\subseteq\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})\subseteq\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F}).$$

THEOREM 4.5. *The radical* $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-si}(L)$ *of L is the unique maximal* $\mathrm{M}(\mathfrak{S}\cap\mathfrak{F})$ *ideal, the unique maximal* $\mathrm{M}(\mathfrak{S}\cap\mathfrak{F})$ *subideal and the unique maximal characteristic* $\mathrm{M}(\mathfrak{S}\cap\mathfrak{F})$ *ideal of L.*

PROOF. It is shown in Theorem 8.3 of [4] that $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-si}(L)$ is a characteristic ideal of L. Hence the statement follows from Theorem 3.6.

THEOREM 4.6. *The radical* $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-asc}(L)$ *of L is the subalgebra generated by all the ascendant* $\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})$ *subalgebras of L and belongs to* $\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})$.

PROOF. This follows from Theorem 3.5.

COROLLARY 4.7. *The radical* $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-asc}(L)$ *of L is the subalgebra generated by all the ascendant* $\mathrm{M}(\mathfrak{S}\cap\mathfrak{F})$ *subalgebras of L.*

PROOF. This follows from Theorem 4.6 and the fact that

$$\mathfrak{S}\cap\mathfrak{F}\subseteq\mathrm{M}(\mathfrak{S}\cap\mathfrak{F})\subseteq\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F}).$$

It is to be noted that, by Lemma 2.3, Theorems 4.1, 4.3 and Corollary 4.4 are valid with $\mathrm{M}(\mathfrak{N}\cap\mathfrak{F})$ (resp. $\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})$) replaced by each of $\mathrm{M}\mathfrak{N}$, $\mathrm{M}_1\mathfrak{N}$, $\mathrm{M}_1(\mathfrak{N}\cap\mathfrak{F})$, $\mathrm{N}\mathfrak{N}$, $\mathrm{N}(\mathfrak{N}\cap\mathfrak{F})$ (resp. $\acute{\mathrm{M}}\mathfrak{N}$, $\acute{\mathrm{M}}_1\mathfrak{N}$, $\acute{\mathrm{M}}_1(\mathfrak{N}\cap\mathfrak{F})$, $\acute{\mathrm{N}}\mathfrak{N}$, $\acute{\mathrm{N}}(\mathfrak{N}\cap\mathfrak{F})$) and, by Lemma 2.2, Theorems 4.5, 4.6 and Corollary 4.7 are valid with $\mathrm{M}(\mathfrak{S}\cap\mathfrak{F})$ (resp. $\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})$) replaced by each of $\mathrm{M}_1(\mathfrak{S}\cap\mathfrak{F})$, $\mathrm{N}(\mathfrak{S}\cap\mathfrak{F})$ (resp. $\acute{\mathrm{M}}_1(\mathfrak{S}\cap\mathfrak{F})$, $\acute{\mathrm{N}}(\mathfrak{S}\cap\mathfrak{F})$).

§5. $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})}(L)$ and $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)$

$\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})}(L)$ and $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)$ are respectively locally nilpotent and locally solvable radicals of L whose existence was shown in Theorem 3.2. This section is devoted to investigation of the properties of these two new radicals. We first show the following

THEOREM 5.1. (1) $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})}(L)$ *is not necessarily a characteristic ideal of L and*

$$\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})}(L)\subseteq\rho(L).$$

(2) *If the basic field* Φ *is of characteristic* 0, *then*

$$\beta(L) \subseteq \mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L) \subseteq \gamma(L)$$

and these are generally different from each other.

PROOF. Since $\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F}) \subseteq \mathrm{L}\mathfrak{N}$, we have $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L) \subseteq \rho(L)$. Assume that the basic field Φ is of characteristic 0. Then by Theorem 4.1 $\beta(L)$ is an $\mathrm{M}(\mathfrak{N} \cap \mathfrak{F})$ ideal of L and therefore an $\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})$ ideal of L. Hence $\beta(L) \subseteq \mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L)$. By Theorem 4.3, $\gamma(L)$ is the subalgebra generated by all the ascendant $\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})$ subalgebras of L. Hence $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L) \subseteq \gamma(L)$. $\beta(L)$ is a characteristic ideal of L and $\gamma(L)$ is not necessarily an ideal of L. Since $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L)$ is an ideal of L, it only remains to show that it is not necessarily a characteristic ideal of L.

Let L be the Lie algebra in Example C in [4]. That is, L is the semi-direct sum of an infinite-dimensional abelian Lie algebra $A=(e_0, e_1, e_2, \dots)$ and a nilpotent Lie algebra (x, y, z) of derivations of A with $[x, y]=z$, $[x, z]=[y, z]=0$, where

$$x: e_i \to e_{i+1} \qquad (i \geq 0),$$

$$y: e_0 \to 0, \qquad e_i \to i e_{i-1} \qquad (i \geq 1),$$

$$z: e_i \to e_i \qquad (i \geq 0).$$

Let $L_1=A+(y, z)$. Then the $\mathfrak{N} \cap \mathfrak{F}$ subalgebras of L_1 containing z are (z) and (y, z). The idealizers of (z) and (y, z) in L_1 are (y, z). Hence neither (z) nor (y, z) is an ascendant subalgebra of L_1. This shows that $L_1 \notin \acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})$. On the other hand, any finite subset of $A+(y)$ lies inside some ascendant $\mathfrak{N} \cap \mathfrak{F}$ subalgebra $A_n+(y)$ where $A_n=(e_0, e_1, \dots, e_n)$. Hence $A+(y)$ is an $\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})$ ideal of L_1. Therefore $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L_1)=A+(y)$. $\mathrm{ad}_L x$ induces the derivation D of L_1 sending y to $-z$. Hence $A+(y)$ is not invariant under D. Thus $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L_1)$ is not a characteristic ideal of L_1.

The proof is completed.

By imposing certain conditions on L we have the following

PROPOSITION 5.2. *Let L be a Lie algebra of countable dimension. Then* $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L)=\rho(L)$. *If furthermore the basic field* Φ *is of characteristic* 0 *and* $L \in \mathrm{L}\mathfrak{F}$, *then* $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L)=\rho(L)=\gamma(L)$.

PROOF. Let H be any $\mathrm{L}\mathfrak{N}$ ideal of L and K be any finite subset of H. If $\{e_1, e_2, \dots\}$ denotes a basis of H, $K \subseteq H_n = \langle e_1, e_2, \dots, e_n \rangle$ for some n. Since $H \in \mathrm{L}\mathfrak{N}$, $H_k \in \mathfrak{N} \cap \mathfrak{F}$ and therefore H_k si H_{k+1} for any k. It follows that H_n asc H. Hence $H \in \acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})$. Thus the $\mathrm{L}\mathfrak{N}$ ideals of L are the $\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})$ ideals of L. Therefore $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L)=\rho(L)$. If Φ is of characteristic 0 and $L \in \mathrm{L}\mathfrak{F}$, it is shown in Corollary 3.9 of [3] that $\gamma(L) \subseteq \rho(L)$. Hence $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N} \cap \mathfrak{F})}(L) \subseteq \gamma(L) \subseteq$

$\rho(L)$ and therefore $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{N}\cap\mathfrak{F})}(L)=\gamma(L)=\rho(L)$.

THEOREM 5.3. (1) $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)$ *is not necessarily a characteristic ideal of* L,

$$\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)\subseteq\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$$

and these are generally different.

(2) *If the basic field* Φ *is of characteristic* 0, *then*

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)\subseteq\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)\subseteq\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$$

and these are generally different from each other.

PROOF. Since $\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})\subseteq\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})$, we have $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)\subseteq\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$. Assume that the basic field Φ is of characteristic 0. Then by Theorem 4.5 $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)$ is an $\mathrm{M}(\mathfrak{S}\cap\mathfrak{F})$ ideal of L and therefore an $\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})$ ideal of L. Hence $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)\subseteq\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)$. By Theorem 3.2 $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)$ is the unique maximal $\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})$ ideal of L and by Theorem 4.6 $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$ is the subalgebra generated by all the ascendant $\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})$ subalgebras of L. Hence $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)\subseteq\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$.

By Theorem 8.3 in [4] $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)$ is a characteristic ideal of L and by Theorem 4.2 in [5] $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$ is not necessarily an ideal of L. To show that $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-si}}(L)$, $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)$ and $\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)$ are generally different from each other, it therefore suffices to show that $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)$ is not necessarily a characteristic ideal of L.

Let L_1 be the Lie algebra as in the proof of Theorem 5.1. The $\mathfrak{S}\cap\mathfrak{F}$ subalgebras of L_1 containing z are

$$(z),\quad (y,z),\quad B+(z),\quad A_n+(y,z)$$

where B is any $\mathfrak{F}$ subalgebra of A. The idealizer of (z) is (y, z) and that of $B+(z)$ is either $B+(z)$ or $A_n+(y,z)$. (y, z) and $A_n+(y,z)$ are equal to their idealizers in L_1. Hence any $\mathfrak{S}\cap\mathfrak{F}$ subalgebra of L_1 containing z is not an ascendant subalgebra of L_1. Thus $L_1\notin\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})$. On the other hand any finite subset of $A+(y)$ lies inside some ascendant $\mathfrak{S}\cap\mathfrak{F}$ subalgebra $A_n+(y)$ of $A+(y)$. Hence $A+(y)$ is an $\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})$ ideal of L_1. Therefore $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L_1)=A+(y)$. It is not characteristic since it is not invariant under the derivation of L_1 induced by $\mathrm{ad}_L x$.

Thus it only remains to show that $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)$ and $\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$ are different in general. Let L be the Lie algebra as in the proof of Theorem 5.1. Then it is shown in the proofs of Theorems 4.2 and 4.3 in [5] that

$$\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}\text{-asc}}(L)=A+(y)\ \text{and}\ \mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)=A+(y,z).$$

Since $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)\subseteq\mathrm{Rad}_{\mathfrak{S}\cap\mathfrak{F}-\mathrm{asc}}(L)$, it follows that $\mathrm{Rad}_{\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})}(L)\neq\mathrm{Rad}_{\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})}(L)$. This completes the proof.

§6. Examples

This section is devoted to showing by examples that the six classes $\mathfrak{X}$, $\mathrm{M}\mathfrak{X}$, $\acute{\mathrm{M}}\mathfrak{X}$, $\mathrm{M}_1\mathfrak{X}$, $\acute{\mathrm{M}}_1\mathfrak{X}$ and $\mathrm{L}\mathfrak{X}$ are generally different from each other as announced in Section 2.

EXAMPLE 6.1. $\mathfrak{X}\neq\mathrm{M}\mathfrak{X}$ generally. Take $\mathfrak{X}=\mathfrak{N}$ and let L be the Lie algebra over a field of characteristic 0 in Theorem 12.1 in [2]. Then it is known that $L\notin\mathfrak{N}$ and $L=\beta(L)$. Hence $L\in\mathrm{N}(\mathfrak{N}\cap\mathfrak{F})$ and therefore by Lemms 2.3 $L\in\mathrm{M}\mathfrak{N}$.

EXAMPLE 6.2. $\mathrm{M}\mathfrak{X}\neq\acute{\mathrm{M}}\mathfrak{X}$ and $\mathrm{M}_1\mathfrak{X}\neq\acute{\mathrm{M}}_1\mathfrak{X}$ generally. Take $\mathfrak{X}=\mathfrak{N}$ and let $L=A+(y)$ be a subalgebra of the Lie algebra $A+(x, y, z)$ in the proof of Theorem 5.1. Suppose that there exists an $\mathfrak{N}$ subideal H of L containing y. Then $H\neq L$ and $H\neq(y)$. Therefore H contains

$$u=\sum_{i=0}^{k}a_ie_i+by, \qquad a_k\neq 0.$$

But

$$u(\mathrm{ad}\ y)^k=k!a_ke_0.$$

Hence $e_0\in H$. Considering $u-a_0e_0$ and $(\mathrm{ad}\ y)^{k-1}$ instead of u and $(\mathrm{ad}\ y)^k$, we obtain $e_1\in H$. By induction we see that $H\supseteq A_k+(y)$. It follows that $H=A_n+(y)$ for some n and H is not a subideal of L. Thus no $\mathfrak{N}$ subideals of L contain y. Hence $L\notin\mathrm{M}_1\mathfrak{N}$ and therefore $L\notin\mathrm{M}\mathfrak{N}$. On the other hand, any finite subset of L is obviously contained in a subalgebra $A_n+(y)$ for some n which is an ascendant $\mathfrak{N}$ subalgebra of L. Hence $L\in\acute{\mathrm{M}}\mathfrak{N}$ and therefore $L_1\in\acute{\mathrm{M}}\mathfrak{N}$.

EXAMPLE 6.3. $\acute{\mathrm{M}}\mathfrak{X}\neq\mathrm{L}\mathfrak{X}$, $\mathrm{M}_1\mathfrak{X}\neq\mathrm{L}\mathfrak{X}$ and $\acute{\mathrm{M}}_1\mathfrak{X}\neq\mathrm{L}\mathfrak{X}$ generally. Take $\mathfrak{X}=\mathfrak{S}\cap\mathfrak{F}$ and let $L=A+(z)$ be a subalgebra of the Lie algebra in the proof of Theorem 5.1. Suppose that H is an ascendant $\mathfrak{S}\cap\mathfrak{F}$ subalgebra of L containing z. Then $H\neq L$, $H\neq A$ and $H\neq(z)$. It follows that H is the sum of (z) and a subalgebra of A. But H is then its own idealizer in L, which contradicts our supposition on H. Thus there exist no ascendant $\mathfrak{S}\cap\mathfrak{F}$ subalgebras of L containing z. Hence $L\notin\acute{\mathrm{M}}_1(\mathfrak{S}\cap\mathfrak{F})$. It follows that $L\notin\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})$ and $L\notin\mathrm{M}_1(\mathfrak{S}\cap\mathfrak{F})$. On the other hand, any finite subset of L obviously lies inside some $A_n+(z)$. Hence $L\in\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})$. Thus we conclude that each of $\acute{\mathrm{M}}(\mathfrak{S}\cap\mathfrak{F})$, $\mathrm{M}_1(\mathfrak{S}\cap\mathfrak{F})$ and $\acute{\mathrm{M}}_1(\mathfrak{S}\cap\mathfrak{F})$ is different from $\mathrm{L}(\mathfrak{S}\cap\mathfrak{F})$.

EXAMPLE 6.4. $\mathrm{M}\mathfrak{X} \neq \mathrm{M}_1\mathfrak{X}$ generally. Take $\mathfrak{X}=\mathfrak{A}$ and let $L=(x, y, z)$ be a subalgebra of the Lie algebra in the proof of Theorem 5.1. For any element $u=ax+by+cz$ of L,

$$(u) \lhd (ax+by, z) \lhd L.$$

Hence (u) is an $\mathfrak{A}$ subideal of L. Therefore $L \in \mathrm{M}_1\mathfrak{A}$. However $L \notin \mathrm{M}\mathfrak{A}$, since the subalgebra containing $\{x, y\}$ is not abelian.

EXAMPLE 6.5. $\acute{\mathrm{M}}\mathfrak{X} \neq \acute{\mathrm{M}}_1\mathfrak{X}$ generally. Take $\mathfrak{X}=\mathfrak{A}$ and let L be a subalgebra $A+(y)$ of the Lie algebra in the proof of Theorem 5.1. Let u be any non-zero element of L. If $u=ay$, (u) asc L. Otherwise we have

$$u=\sum_{i=0}^{n} a_i e_i + by, \qquad a_n \neq 0.$$

Then

$$(u) \lhd (e_0, \sum_{i=1}^{n} a_i e_i + by) \lhd (e_0, e_1, \sum_{i=2}^{n} a_i e_i + by) \lhd \dots \lhd A_n+(y).$$

Since $A_n+(y)$ asc L, it follows that (u) asc L. Therefore $L \in \acute{\mathrm{M}}_1\mathfrak{A}$. It is however obvious that $L \notin \acute{\mathrm{M}}\mathfrak{A}$.

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