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# *Some Remarks to the Construction of Branching Markov Processes with Age and Sign*

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## § 1. Introduction

Some semi-linear equations are in close connection with branching Markov processes. Suppose we are given the infinitesimal generator $A$ of a Markov process $x_t$ on a topological space $S$ together with the following quantities: (i) a non-negative Borel function $k(x)$ on $S$, (ii) a sequence $\{q_n(x)\}_{n=0,2,3,\dots}$ of Borel functions on $S$, and (iii) a sequence $\{\pi_n(x, dy)\}_{n=0\ 2,3,\dots}$ of stochastic kernels from $S$ to the $n$-fold symmetric product $S^n$ of $S$. We consider the following equations:

$$\text{(I)}\qquad \frac{\partial u(t, x)}{\partial t} = Au(t, x) + k(x)\left\{\sum_{n\neq 1} q_n(x)\int_{S^n} \pi_n(x, d\mathrm{y})u(t, \mathrm{y}) - u(t, x)\right\},$$

$$\text{(II)}\qquad \frac{\partial u(t, x)}{\partial t} = Au(t, x) + k(x)\sum_{n\neq 1} q_n(x)\int_{S^n} \pi_n(x, d\mathrm{y})u(t, \mathrm{y}),\ x\in S,\ t\geq 0.$$

In the equation (I) it is assumed that $q_n(x)\geq 0$, $n=0, 2, 3,\dots$, $\Sigma_{n\neq 1} q_n(x)=1$, while in the equation (II) $q_n(x)$ can be negative but $\Sigma_{n\neq 1}|q_n(x)|=1$. Then, it is known that the equation (I) corresponds to the branching Markov process (abbreviated: BM-process) whose non-branching part and branching system are $\exp\left(-\int_0^t k(x_s)ds\right)$-subprocess of $x_t$ and $(q_n(x), \pi_n(x, d\mathrm{y}))_{n=0,2,3,\dots}$, respectively (see N. Ikeda-M. Nagasawa-S. Watanabe [2]). BM-processes of this type do not correspond to the equation (II) in a straightforward way. After a while, M. Nagasawa and T. Sirao ([3], [4], [6]) constructed another type of branching Markov process with age and sign (abbreviated: BMAS-process) corresponding to the equation (II).

The purpose of this paper is to remark that BMAS-processes can be constructed in a frame of the ordinary BM-processes due to Ikeda-Nagasawa-Watanabe, by introducing two extra states. More precisely, taking two extra points $a$ and $b$ not belonging to $S$, we extend the state space $S$ of the given Markov process $x_t$ to $S_0=S\cup\{a, b\}$ so that the new states $a$ and $b$ become traps. We then introduce new quantities $k^0(x)$, $q_n^0(x)$, $\pi_n^0(x, d\mathrm{y})$ for $x\in S_0$ and $d\mathrm{y}\subset S_0^n$ by the formulas (3.1), (3.2.$a$) and (3.2.$b$) in § 3. Let $\mathbf{X}$ be the BM-process determined by this extended system $\{X^0, k^0(x), (q_n^0(x), \pi_n^0(x, dy))_{n=0,1,2,\dots}\}$ where $X^0$ is the extended Markov process on the enlarged state space $S_0$. In the terminology of Sevast'yanov [5], $\{a, b\}$ is a final class. Our result is that the BMAS-process $\mathbf{Z}$ due to Nagasawa and Sirao is equivalent in law to a certain factor process of $\mathbf{X}$ (THEOREM in § 3).

## § 2. Preliminaries

We here introduce some notations following [2]. Let $S$ be a compact Hausdorff space with a countable open base, $S^n$ the $n$-fold symmetric product of $S$, $\mathbf{S}=\bigcup_{n=0}^{\infty} S^n$ the topological sum of $S^n$ and $\hat{\mathbf{S}}=\mathbf{S}\cup\{\Delta\}$ the one point compactification of $\mathbf{S}$ where $S^0=\{\partial\}$ and $\partial$ is an extra point. We put $\mathbf{N}=\{0, 1, 2,\dots\}$, $\mathbf{J}=\{0, 1\}$ and

$\mathscr{B}(S)$ = the topological Borel field of $S$ (also similar notations $\mathscr{B}(\mathbf{S})$, etc. will be used),

$\mathbf{B}(S)$ = the set of all bounded Borel functions on $S$,

$\mathbf{C}^*(S)=\{f: f \text{ is continuous on } S \text{ and } \sup_{x\in S}|f(x)|<1\}$,

$\mathbf{C}_0(\mathbf{S}\times\mathbf{N}\times\mathbf{J})=\{f: f \text{ is continuous on } \mathbf{S}\times\mathbf{N}\times\mathbf{J} \text{ and } \lim_{\mathbf{x}\to\Delta} f(\mathbf{x})=0\}$,

where $\Delta$ is an extra point added by the one-point compactification of $\mathbf{S}\times\mathbf{N}\times\mathbf{J}$, and define a function $\hat{f}(\mathbf{x})$ on $\hat{\mathbf{S}}$ for a function $f\in\mathbf{B}(S)$ as follows:

$$\hat{f}(\mathbf{x})=\begin{cases} 1 & \text{if } \mathbf{x}=\partial, \\ f(x_1)f(x_2)\dots f(x_n) & \text{if } \mathbf{x}=[x_1, x_2,\dots, x_n]\in S^n, \\ 0 & \text{if } \mathbf{x}=\Delta. \end{cases} \tag{2.1}$$

In this paper we are given a conservative strong Markov process $X=(W, \mathscr{B}_t, x_t, P_x, x\in S)$ on $S$ with right continuous sample paths having left limits such that $\mathscr{B}_t=\bar{\mathscr{B}}_{t+0}$, and also the following quantities (i), (ii) and (iii):

(i) a non-negative Borel function $k(x)$ on $S$,

(ii) Borel functions $q_n(x)$ on $S$, $n=0, 2, 3,\dots$, satisfying $\Sigma_{n\neq 1}|q_n(x)|=1$,

(iii) stochastic kernels $\pi_n(x, dy)$ on $S\times S^n$, $n=0, 2, 3,\dots$.

Nagasawa [3], [4] and Sirao [6] constructed a BMAS-process corresponding to the equation (II) on the basis of the Markov process $X$, $k(x)$ and $(q_n(x), \pi_n(x, dy))_{n=0,2,3,\dots}$. In this section we list some properties of the BMAS-process $\mathbf{Z}=(\mathbf{Z}_t, \mathbf{P}^0_{(\mathbf{x},k,j)})$ with state space $\hat{\mathbf{Q}}=(\mathbf{S}\times\mathbf{N}\times\mathbf{J})\cup\{\Delta\}$ constructed in [3][1)]

1) In [3], $S^n$ is the $n$-fold Cartesian product of $S$, but for simplicity we assume here $S^n$ to be the symmetric product.

for later use. We put

$$\xi_t = n,\ a_t = k \quad \text{if} \quad \mathbf{Z}_t = (\mathbf{x}, k, j) \in \mathbf{S} \times \mathbf{N} \times \mathbf{J},\ \mathbf{x} \in S^n,$$

$$\tau = \inf\{t:\ \xi_t \neq \xi_0\}, \qquad = \infty \quad \text{if} \quad \{\quad\} = \emptyset,$$

$$\tau_0 = 0,\ \tau_1 = \tau,\ \tau_n = \tau_{n-1} + \tau \circ \theta_{\tau_{n-1}}, \qquad n = 2, 3, \ldots,$$

and introduce the following quantities:

$$\mathbf{U}_t^{(r)}(\cdot, \mathrm{B}) = \mathbf{P}^0_{\cdot}(\mathbf{Z}_t \in \mathrm{B},\ \tau_r \leq t < \tau_{r+1})$$

$$\mathbf{U}_t(\cdot, \mathrm{B}) = \mathbf{P}^0_{\cdot}(\mathbf{Z}_t \in \mathrm{B}),$$

$$\tilde{f}(\mathbf{x}, k, j) = (-1)^j \lambda^k \hat{f}(\mathbf{x}) \qquad \text{for } \lambda \geq 0, f \in \mathbf{B}(S).$$

Then

$$\mathbf{U}_t^{(r)} \tilde{f}(\mathbf{x}, k, j) = (-1)^j \lambda^k \sum^{(r,n)} \prod_{i=1}^{n} \mathbf{U}_t^{(r_i)} f(x_i), \qquad \mathbf{x} = [x_1, \ldots, x_n] \in S^n, \tag{2.2}$$

$$\mathbf{U}_t \tilde{f}(\mathbf{x}, k, j) = \widetilde{\mathbf{U}_t \tilde{f}|_S}(\mathbf{x}, k, j), \tag{2.3}$$

where $F|_S = F(x, 0, 0)$, $x \in S$, $F \in \mathbf{B}(\mathbf{Q})$ and $\Sigma^{(r,n)}$ denotes the sum over all $(r_1, r_2, \ldots, r_n)$ satisfying $\Sigma_{i=1}^n r_i = r$. Furthermore,

$$\begin{aligned} &\mathbf{E}^0_{(x,0,0)}[\tilde{f}(\mathbf{Z}_t):\ a_t = k,\ t < \tau] \\ &= \frac{1}{k!} E_x[f(x_t)\{\lambda \int_0^t k(x_s) ds\}^k \exp(-2\int_0^t k(x_s) ds)], \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\mathbf{P}^0_{(x,0,0)}(\tau \leq t,\ \mathbf{Z}_\tau \in d(\mathbf{y}, k, j)) \\ &= \frac{1}{k!} \int_0^t E_x\Big[\{\mu^+((x_s, k, 0), d(\mathbf{y}, k', j)) + \mu^-((x_s, k, 0), d(\mathbf{y}, k', j))\} \\ &\quad \cdot k(x_s) \Big\{\int_0^s k(x_u) du\Big\}^k \exp\Big(-2\int_0^s k(x_u) du\Big)\Big] ds, \end{aligned} \tag{2.5}$$

where $\mu^{\pm}((x, k, 0), d(\mathbf{y}, k', j)) = q_n^{\pm}(x) \pi_n(x, d\mathbf{y}) \delta_{kk'} \delta_j^{\pm}$, $d(\mathbf{y}, k', j) \subset S^n \times \mathbf{N} \times \mathbf{J}$, $n = 0, 2, 3, \ldots$ and $\delta_j^+ = \delta_{0j}$, $\delta_j^- = \delta_{1j}$. Finally, if we put

$$\begin{aligned} &u(t, x) = \mathbf{U}_t \tilde{f}(x, 0, 0), \qquad \lambda = 2, \\ &T_t f(x) = E_x[f(x_t)], \\ &K(x, ds, dy) = E_x[k(x_s) ds:\ x_s \in dy], \end{aligned} \tag{2.6}$$

then the following $S$-equation holds:

(2.7) $$u(t, x)=T_t f(x)+\int_0^t\int_S K(x, ds, dy)\sum_{n\neq 1} q_n(y)\int_{S^n}\pi_n(y, d\mathbf{z})u_{t-s}(\mathbf{z}),$$

which is an integral equation corresponding to the equation (II). The BMAS-process $\mathbf{Z}$ corresponding to the equation (II) is a strong Markov process with state space $\hat{\mathbf{Q}}$ characterized by the three properties (2.3), (2.4) and (2.5).

## § 3. Construction of branching Markov processes with age and sign

In this section we construct the BM-process $\mathbf{X}$ stated in § 1, and then prove that the BMAS-process $\mathbf{Z}$ in Nagasawa [3] is equivalent to a certain factor process of $\mathbf{X}$, that is, $\mathbf{Z}$ is obtained from $\mathbf{X}$ by means of a certain transformation on the state space.

Let $S_0=S\cup\{a, b\}$, where $a$ and $b$ are extra points outside $S$. Given a system $\{X, k, (q_n, \pi_n)_{n=0,2,3,\dots}\}$ as in § 2, we first introduce a new system $\{X^0, k^0, (q_n^0, \pi_n^0)_{n=0,1,2,\dots}\}$ on $S_0$ as follows. $X^0=(W^0, \mathscr{B}_t^0, x_t^0, P_x^0, x\in S_0)$ is a right continuous conservative strong Markov process with state space $S_0$ satisfying the following (i), (ii) and (iii):

(i) $W^0\supset W$, $\mathscr{B}_t^0=\{B: B\cap W\in\mathscr{B}_t\}$,

(ii) $x_t^0|_W=x_t$, $P_x^0(x_t^0\in A)=P_x(x_t\in A)$ for $x\in S$, $A\in\mathscr{B}_t$,

(iii) $a$ and $b$ are traps for the process $X^0$.

$k^0(x)$ is a non-negative Borel function on $S_0$ such that

(3.1) $$k^0|_S=2k,\ k^0(a)=k^0(b)=0,$$

and $(q_n^0(x), \pi_n^0(x, A))_{n=0,1,2,\dots}$ is a branching system on $S_0$ defined by the following (3.2.$a$) and (3.2.$b$).

(3.2.$a$) For $x\in S$, we put

$$q_2^0(x)=\frac{1}{2}(q_2^+(x)+1),$$

$$q_n^0(x)=\frac{1}{2}(q_n^+(x)+q_{n-1}^-(x)),\ n\neq 2,\ n\geq 0,$$

$$\pi_2^0(x, A\cdot\{a\})=\frac{1}{q_2^+(x)+1}\delta(x, A),\ A\in\mathscr{B}(S),$$

$$\pi_2^0(x, A)=\frac{q_2^+(x)}{q_2^+(x)+1}\pi_2(x, A),\ A\in\mathscr{B}(S^2),$$

$$\pi_n^0(x, A\cdot\{b\})=\frac{q_{n-1}^-(x)}{q_n^+(x)+q_{n-1}^-(x)}\pi_{n-1}(x, A),\ A\in\mathscr{B}(S^{n-1}),\ n\neq 2,\ n\geq 1,$$

$$\pi_n^0(x, A)=\frac{q_n^+(x)}{q_n^+(x)+q_{n-1}^-(x)}\pi_n(x, A),\ A\in\mathscr{B}(S^n),\ n\neq 2,\ n\geq 0,$$

where $q_1(x)=q_{-1}^-(x)=0$, $A\cdot\{a\}=\{[x_1,\dots,x_n,a]\in S^{n+1}: [x_1,\dots,x_n]\in A\}$ for $A\subset S^n$ and $A\cdot\{a\}\cdot\{b\}$ is defined similarly. If the form $\frac{0}{0}$ appears in the definition of $\pi_n^0$, it is interpreted as 0.

(3.2.$b$) For $x=a$ or $b$, $q_n^0(x)$ and $\pi_n^0(x, A)$ are defined arbitrarily but subject to the condition:

$$q_n^0(x)\geq 0,\ \Sigma_{n=0}^{\infty} q_n^0(x)=1,$$

$$\pi_n^0(x,\cdot) \text{ is a probability measure on } S_0^n, \qquad n=0, 1, 2,\dots.$$

REMARK. We can see immediately that for $x\in S$

$$\pi_n^0(x, A\cdot\{a\})=0,\ A\in\mathscr{B}(S^{n-1}),\ n\neq 2,\ n\geq 1,$$

$$\pi_n^0(x, A\cdot\{a\}\cdot\{b\})=0,\ A\in\mathscr{B}(S^{n-2}),\ n\geq 2,$$

where $\{\partial\}\cdot\{a\}=\{a\}$, $\{\partial\}\cdot\{a\}\cdot\{b\}=\{[a,b]\}\subset S_0^2$.
We next define a stochastic kernel $\pi^0(x, A)$ on $S_0\times\hat{\mathbf{S}}_0$[2)] by

(3.3) $$\pi^0(x, A)=q_n^0(x)\pi_n^0(x, A) \qquad \text{for } x\in S_0,\ A\in\mathscr{B}(S_0^n)$$

or what is the same, by

(3.4.$a$) $$\begin{cases} \pi^0(x, A\cdot\{a\})=\frac{1}{2}\delta(x, A) & \text{if } x\in S,\ A\in\mathscr{B}(S), \\ \pi^0(x, A\cdot\{b\})=\frac{1}{2}q_{n-1}^-(x)\pi_{n-1}(x, A) & \text{if } x\in S,\ A\in\mathscr{B}(S^{n-1}), \\ & \qquad n\neq 2,\ n\geq 1, \\ \pi^0(x, A)=\frac{1}{2}q_n^+(x)\pi_n(x, A) & \text{if } x\in S,\ A\in\mathscr{B}(S^n),\ n\neq 1, \end{cases}$$

(3.4.$b$) $\pi^0(x,\cdot)$ is an arbitrary probability measure on $\mathbf{S}_0$ if $x=a$, $b$.

Then we can get the branching Markov process $\mathbf{X}=(\Omega, \mathscr{M}_t, \mathbf{X}_t, \mathbf{P}_{\mathbf{x}}, \mathbf{x}\in\hat{\mathbf{S}}_0)$ determined by $(X^0, k^0, \pi^0)$, that is, the branching Markov process with the branching law $\pi^0$ and $\exp\left(-\int_0^t k^0(x_s^0)ds\right)$-subprocess of $X^0$ for the non-branching part (cf. Ikeda-Nagasawa-Watanabe [2]).

Now we shall make a transformation of $\mathbf{X}$ so that the transformed process is equivalent to the BMAS-process $\mathbf{Z}$. For $\mathbf{x}=[\mathbf{x}_0, a,\dots, a, b,\dots, b]\in\mathbf{S}_0$ we put $n(\mathbf{x})=n$ if $\mathbf{x}_0=[x_1,\dots,x_n]\in S^n$, $n^a(\mathbf{x})=$the number of $a$ in $\mathbf{x}$, $n^b(\mathbf{x})=$ the number of $b$ in $\mathbf{x}$, $j(\mathbf{x})=0$ (if $n^b(\mathbf{x})$ is even) and $=1$ (if $n^b(\mathbf{x})$ is odd), and in-

2) $\mathbf{S}_0=\bigcup_{n=0}^{\infty}\mathbf{S}_0^n$ and $\hat{\mathbf{S}}_0=\mathbf{S}_0\cup\{\Delta\}$

troduce the mapping $\gamma: \hat{\mathbf{S}}_0 \to \mathbf{S} \times \mathbf{N} \times \mathbf{J} \cup \{\Delta\}$ defined by $\gamma(\mathbf{x}) = (\mathbf{x}_0, n^a(\mathbf{x}), j(\mathbf{x}))$ and $\gamma(\Delta) = \Delta$.

LEMMA 1. $\gamma(\mathbf{X}) = (\tilde{\mathbf{X}}_t, \tilde{\mathcal{M}}_t, \tilde{\mathbf{P}}_{\tilde{\mathbf{x}}})$ is a strong Markov process on $\hat{\mathbf{Q}} = \mathbf{S} \times \mathbf{N} \times J \cup \{\Delta\}$, where $\tilde{\mathbf{X}}_t = \gamma \mathbf{X}_t$, $\tilde{\mathcal{N}}^0 = \sigma\{\{\mathbf{X}_t \in \Gamma\}: t \geq 0, \Gamma \in \mathscr{B}(\hat{\mathbf{Q}})\}$, $\tilde{\mathcal{M}}_t = \mathcal{M}_t \cap \tilde{\mathcal{N}}^0$ and $\tilde{\mathbf{P}}_{\gamma\mathbf{x}}(A) = \mathbf{P}_{\mathbf{x}}(A)$ for $A \in \tilde{\mathcal{N}}^0$, $\mathbf{x} \in \hat{\mathbf{S}}_0$.

PROOF. Put $\tilde{f}(\tilde{\mathbf{x}}) = (-1)^j \lambda^k \hat{f}(\mathbf{x}_0)$ and $\bar{f}(\tilde{\mathbf{x}}) = \lambda^k \hat{f}(\mathbf{x}_0)$ for $\tilde{\mathbf{x}} = (\mathbf{x}_0, k, j), f \in \mathbf{C}^*(S)$, $\lambda \geq 0$. Then the linear hull of the subset $\{\tilde{f}, \bar{f}: 0 \leq \lambda < 1, f \in \mathbf{C}^*(S)\}$ of $\mathbf{C}_0(\mathbf{S} \times \mathbf{N} \times \mathbf{J})$ is dense in $\mathbf{C}_0(\mathbf{S} \times \mathbf{N} \times \mathbf{J})$. Since

$$\hat{f}(\mathbf{x}) = \tilde{f}(\gamma\mathbf{x}) \quad \text{if} \quad f(a) = \lambda, f(b) = -1,$$

$$\hat{f}(\mathbf{x}) = \bar{f}(\gamma\mathbf{x}) \quad \text{if} \quad f(a) = \lambda, f(b) = 1,$$

we can see, by the branching property of $\mathbf{X}$, that

$$\mathbf{E}_{\mathbf{x}}[\tilde{f}(\gamma\mathbf{X}_t)] = \mathbf{E}_{\mathbf{x}}[\hat{f}(\mathbf{X}_t)] = (-1)^m \lambda^k \prod_{i=1}^{n} \mathbf{E}_{x_i}[\hat{f}(\mathbf{X}_t)] = \mathbf{E}_{\mathbf{x}'}[\tilde{f}(\gamma\mathbf{X}_t)],$$

$$\mathbf{E}_{\mathbf{x}}[\bar{f}(\gamma\mathbf{X}_t)] = \mathbf{E}_{\mathbf{x}'}[\bar{f}(\gamma\mathbf{X}_t)],$$

provided $\gamma\mathbf{x} = \gamma\mathbf{x}'$, $\mathbf{x} = [x_1, \ldots, x_n, a, \ldots, a, b, \ldots, b]$, $n^a(\mathbf{x}) = k$, $n^b(\mathbf{x}) = m$. Therefore, by Theorem 10.13 of Dynkin [1] $\gamma(\mathbf{X})$ is a strong Markov process.

THEOREM. The Markov process $\gamma(\mathbf{X})$ and the BMAS-process $\mathbf{Z}$ in Nagasawa [3] are equivalent.

COROLLARY. We extend a function $f \in \mathbf{B}(S)$ to a function $f$ on $S_0$ so that $f(a) = 2$, $f(b) = -1$. If $u(t, x) = \mathbf{E}_x[\hat{f}(\mathbf{X}_t)]$, $x \in S$, has definite value, $u(t, x)$ is a solution of

$$u(t, x) = T_t f(x) + \int_0^t \int_S K(x, ds, dy) \sum_{n \neq 1} q_n(y) \int_{S^n} \pi_n(y, d\mathbf{z}) u_{t-s}(\mathbf{z}),$$

where $T_t(x) = E_x[f(x_t)]$ and $K(x, ds, dy) = E_x[k(x_s)ds: x_s \in dy]$.

For the proof of the theorem we shall prepare some lemmas. Let $Z_t^0$, $n_t^a$ and $n_t^b$ denote the number of particles in $S$, in $\{a\}$ and in $\{b\}$, respectively. More precisely, we put $Z_t^0 = n(\mathbf{X}_t)$, $n_t^a = n^a(\mathbf{X}_t)$ and $n_t^b = n^b(\mathbf{X}_t)$. We define some Markov times of $\mathbf{X}$ as follows.

$$\tau^0 = \begin{cases} \inf\{t: Z_t^0 \neq Z_0^0 \text{ or } n_t^a \neq n_0^a \text{ or } n_t^b \neq n_0^b\}, \\ \infty \quad \text{if } \{\quad\} = \varnothing, \end{cases}$$

$$\tau_0^0=0,\ \tau_1^0=\tau^0,\ \tau_n^0=\tau_{n-1}^0+\tau^0\circ\theta_{\tau_{n-1}^0} \text{ for } n=2, 3,\dots,$$

$$\sigma=\begin{cases}\inf\{t:\ Z_t^0\neq Z_0^0 \text{ or } n_t^b\neq n_0^b\},\\ \infty \qquad \text{if } \{\ \ \}=\emptyset,\end{cases}$$

$$\sigma_0=0,\ \sigma_1=\sigma,\ \sigma_n=\sigma_{n-1}+\sigma\circ\theta_{\sigma_{n-1}} \quad \text{for } n=2, 3,\dots.$$

Since $q_1=0$ by the assumption, the condition $n_t^b\neq n_0^b$ in the definition of $\tau^0$ and $\sigma$ is not necessary here. The Markov time $\sigma$ corresponds to the Markov time $\tau$ of the BMAS-process $\mathbf{Z}$. In fact, $\mathbf{P}_x(\sigma\le t,\ \gamma\mathbf{X}_\sigma\in E)=\mathbf{P}^0_{(x,0,0)}(\tau\le t,\ \mathbf{Z}_\tau\in E)$, $E\in\mathscr{B}(\mathbf{Q})$, as will be seen later by comparing (2.5) with LEMMA 4.

LEMMA 2. For $\mathbf{x}=[x_1,\dots, x_n, a,\dots, a, b,\dots, b]$ with $n^a(\mathbf{x})=k$, $n^b(\mathbf{x})=m$ and for $f\in\mathbf{B}(S_0)$, we have

$$\mathbf{E}_\mathbf{x}[\hat{f}(\mathbf{X}_t);\ \sigma_r\le t<\sigma_{r+1}]=\{f(a)\}^k\{f(b)\}^m\sum^{(r,n)}\prod_{i=1}^n\mathbf{E}_{x_i}[\hat{f}(\mathbf{X}_t);\ \sigma_{r_i}\le t<\sigma_{r_i+1}].$$

PROOF. Since

$$\mathbf{E}_\mathbf{x}[\hat{f}(\mathbf{X}_t);\ \sigma_r\le t<\sigma_{r+1}]=\{f(a)\}^k\{f(b)\}^m\mathbf{E}_{\mathbf{x}_0}[\hat{f}(\mathbf{X}_t);\ \sigma_r\le t<\sigma_{r+1}]$$

where $\mathbf{x}_0=[x_1,\dots, x_n]$, it is sufficient to prove that

$$\mathbf{E}_{\mathbf{x}_0}[\hat{f}(\mathbf{X}_t);\ \sigma_r\le t<\sigma_{r+1}]=\sum^{(r,n)}\prod_{i=1}^n\mathbf{E}_{x_i}[\hat{f}(\mathbf{X}_t);\ \sigma_{r_i}\le t<\sigma_{r_i+1}], \mathbf{x}_0\in\mathbf{S}.$$

If $f_0$ is a function in $\mathbf{B}(S_0)$ such that $f_0=f$ on $S\cup\{b\}$ and $f_0(a)=\lambda f(a)$, $0\le\lambda<1$, then it is known [2: I, p. 271] that

$$\mathbf{E}_{\mathbf{x}_0}[\hat{f}_0(\mathbf{X}_t);\ \tau_r^0\le t<\tau_{r+1}^0]=\sum^{(r,n)}\prod_{i=1}^n\mathbf{E}_{x_i}[\hat{f}_0(\mathbf{X}_t);\ \tau_{r_i}^0\le t<\tau_{r_i+1}^0].$$

On the other hand

$$\mathbf{E}_{\mathbf{x}_0}[\hat{f}_0(\mathbf{X}_t);\ \tau_r^0\le t<\tau_{r+1}^0]=\sum_{k=0}^r\lambda^k\mathbf{E}_{\mathbf{x}_0}[\hat{f}(\mathbf{X}_t);\ n_t^a=k,\ \tau_r^0\le t<\tau_{r+1}^0], \tag{3.5}$$

and

$$\begin{aligned}&\sum^{(r,n)}\prod_{i=1}^n\mathbf{E}_{x_i}[\hat{f}_0(\mathbf{X}_t):\ \tau_{r_i}^0\le t<\tau_{r_i+1}^0]\\ &=\sum^{(r,n)}\prod_{i=1}^n\sum_{k_i=0}^{r_i}\mathbf{E}_{x_i}[\hat{f}_0(\mathbf{X}_t):\ n_t^a=k_i,\ \tau_{r_i}^0\le t<\tau_{r_i+1}^0]\\ &=\sum_{k=0}^r\lambda^k\sum^{(k,n)}\sum^{(r,n)'}\prod_{i=1}^n\mathbf{E}_{x_i}[\hat{f}(\mathbf{X}_t):\ n_t^a=k_i,\ \tau_{r_i}^0\le t<\tau_{r_i+1}^0]\end{aligned} \tag{3.6}$$

where $\sum^{(r,n)'}$ denotes the sum over all $(r_1,\dots, r_n)$ satisfying $\sum_{i=1}^n r_i=r$ and $r_i\ge k_i$. Putting $r-k=r'$, $r_i-k_i=r_i'$ and comparing the coefficients of $\lambda^k$ in (3.5) with those of (3.6), we have

$$(3.7)\qquad \mathbf{E}_{\mathbf{x}_0}[\hat{f}(\mathbf{X}_t);\, n_t^a=k,\ \tau^0_{r'+k}\le t<\tau^0_{r'+k+1}]$$

$$=\sum^{(r',n)}\sum^{(k,n)}\prod_{i=1}^{n}\mathbf{E}_{x_i}[\hat{f}(\mathbf{X}_t);\, n_t^a=k_i,\ \tau^0_{r_i'+k_i}\le t<\tau^0_{r_i'+k_i+1}].$$

Noting the definition of $\tau_n^0$ and $\sigma_n$, we have

$$(3.8)\qquad \mathbf{E}_{\mathbf{x}_0}[\hat{f}(\mathbf{X}_t);\, n_t^a=k,\ \tau^0_{r'+k}\le t<\tau^0_{r'+k+1}]$$

$$=\mathbf{E}_{\mathbf{x}_0}[\hat{f}(\mathbf{X}_t);\, n_t^a=k,\ \sigma_{r'}\le t<\sigma_{r'+1}]$$

and hence, using (3.7) we obtain

$$\mathbf{E}_{\mathbf{x}_0}[\hat{f}(\mathbf{X}_t);\, \sigma_{r'}\le t<\sigma_{r'+1}]=\sum_{k=0}^{\infty}\mathbf{E}_{\mathbf{x}_0}[\hat{f}(\mathbf{X}_t);\, n_t^a=k,\ \sigma_{r'}\le t<\sigma_{r'+1}]$$

$$=\sum^{(r',n)}\prod_{i=1}^{n}\mathbf{E}_{x_i}[\hat{f}(\mathbf{X}_t);\, \sigma_{r_i'}\le t<\sigma_{r_i'+1}],$$

completing the proof of the lemma.

In the next two lemmas, we use the following formulas [2: III, p. 99]: for $f\in\mathbf{B}(S)$, $x\in S$,

$$(3.9)\qquad \mathbf{E}_x[\hat{f}(\mathbf{X}_t);\, t<\tau^0]=E_x^0\Big[f(x_t^0)\exp\Big(-\int_0^t k^0(x_s^0)ds\Big)\Big]$$

$$=E_x\Big[f(x_t)\exp\Big(-2\int_0^t k(x_s)ds\Big)\Big],$$

$$(3.10)\qquad \int_0^t\int_{S_0}\mathbf{P}_x(\tau^0\in ds,\, \mathbf{X}_{\tau^0}\in dy)f(y)=E_x^0\Big[\int_0^t f(x_s^0)k^0(x_s)$$

$$\cdot\exp\Big(-\int_0^s k^0(x_u^0)du\Big)ds\Big]=2\int_0^t E_x\Big[f(x_s)k(x_s)\exp\Big(-2\int_0^s k(x_u)du\Big)\Big]ds.$$

Lemma 3. For $f\in\mathbf{B}(S)$ and $x\in S$,

$$(3.11)\qquad \mathbf{E}_x[\hat{f}(\mathbf{X}_t);\ t<\sigma,\ n_t^a=k]=\frac{1}{k!}E_x\Big[f(x_t)\Big\{f(a)\int_0^t k(x_u)du\Big\}^k$$

$$\cdot\exp\Big(-2\int_0^t k(x_u)du\Big)\Big],\qquad k=0,1,2,\dots.$$

Proof. Put

$$\Phi(\mathbf{x}, t, r, k)=\mathbf{E}_{\mathbf{x}}[\hat{f}(\mathbf{X}_t);\, \tau_r^0\le t<\tau_{r+1}^0,\ \mathbf{X}_{\tau_r^0}\in S\cdot\{a\}^k],\qquad \mathbf{x}\in\mathbf{S}_0.$$

Then $\mathbf{E}_x[\hat{f}(\mathbf{X}_t);\, t<\sigma,\ n_t^a=k]=\Phi(x, t, k, k)$ for $x\in S$. If $k=0$, (3.11) is nothing but (3.9). Now assume that (3.11) is true for $k$. Then using the strong Markov property of $\mathbf{X}$, (3.10) and the induction hypothesis successively, we have

$$\begin{aligned}\Phi(x, t, k+1, k+1) &= \mathbf{E}_x[\Phi(\mathbf{X}_{\tau^0}, t-\tau^0, k, k+1) : \mathbf{X}_{\tau^0}\in S\cdot\{a\},\ \tau^0\leq t]\\ &= \frac{1}{2}\int_0^t\int_S \mathbf{P}(\tau^0\in ds,\ \mathbf{X}_{\tau^0_-}\in dy)\Phi([y, a], t-s, k, k+1)^{3)}\\ &= f(a)\int_0^t E_x[k(x_s)\exp\Big(-2\int_0^s k(x_u)du\Big)\Phi(x_s, t-s, k, k)]ds^{4)}\\ &= f(a)\int_0^t E_x[k(x_s)\exp\Big(-2\int_0^s k(x_u)du\Big)\\ &\qquad\cdot\frac{1}{k!}E_{x_s}[f(x_{t-s})\Big\{f(a)\int_0^{t-s}k(x_u)du\Big\}^k\exp\Big(-2\int_0^{t-s}k(x_u)du\Big)]]ds.\end{aligned}$$

By the Markov property of $x_t$ the last term is equal to

$$\begin{aligned}&\frac{1}{k!}\{f(a)\}^{k+1}\int_0^t E_x[f(x_t)k(x_s)\Big\{\int_s^t k(x_u)du\Big\}^k\\ &\qquad\qquad\cdot\exp\Big(-2\int_0^s k(x_u)du\Big)\exp\Big(-2\int_s^t k(x_u)du\Big)]ds\\ &=\frac{1}{k!}\{f(a)\}^{k+1}E_x[f(x_t)\exp\Big(-2\int_0^t k(x_u)du\Big)\int_0^t k(x_s)\Big\{\int_s^t k(x_u)du\Big\}^k ds]\\ &=\frac{1}{(k+1)!}\{f(a)\}^{k+1}E_x[f(x_t)\Big\{f(a)\int_0^t k(x_u)du\Big\}^{k+1}\exp\Big(-2\int_0^t k(x_u)du\Big)],\end{aligned}$$

and hence the proof is finished.

LEMMA 4. For $x\in S$, $A\in\mathscr{B}(\mathbf{S}_b\cup\{\Delta\})^{5)}$ and $k=0, 1, 2,\ldots$,

$$\mathbf{P}_x(\sigma\leq t,\ \mathbf{X}_\sigma\in A\cdot\{a\}^k)=\frac{2}{k!}\int_0^t E_x[k(x_s)\Big\{\int_0^s k(x_u)du\Big\}^k\cdot\exp\Big(-2\int_0^s k(x_u)du\Big)\pi^0(x_s, A)]\,ds. \tag{3.12}$$

PROOF. Put

$$\Psi(\mathbf{x}, t, r, k)=\mathbf{P}_{\mathbf{x}}(\tau_{r+1}^0\leq t, \mathbf{X}_{\tau_{r+1}^0}\in A\cdot\{a\}^k,\ \mathbf{X}_{\tau_r^0}\in\mathbf{S}_b\cdot\{a\}^k)\ \text{for}\ \mathbf{x}\in\mathbf{S}_0.$$

3) Use $\mathbf{P}_x(\tau^0\in ds,\ \mathbf{X}_{\tau^0}\in A\cdot\{a\})=\int_S\mathbf{P}_x(\tau^0\in ds, \mathbf{X}_{\tau^0_-}\in dy)\delta(y,A), A\in\mathscr{B}(S)$.

4) Use $\Phi([x, a], t, k, k+1)=f(a)\Phi(x, t, k, k)$.

5) $\mathbf{S}_b=\cup_{n=0}^\infty(S\cup\{b\})^n$ where $(S\cup\{b\})^0=\{\partial\}$.

Then $\mathbf{P}_x(\sigma \leq t, \mathbf{X}_\sigma \in A \cdot \{a\}^k) = \Psi(x, t, k, k)$. For $k=0$, (3.12) is true by (3.10). Let us assume that (3.12) is true for $k$. Then using the strong Markov property of $\mathbf{X}$, (3.10) and induction hypothesis successively, we have

$$\Psi(x, t, k+1, k+1) = \mathbf{E}_x[\Psi(\mathbf{X}_{\tau^0}, t-\tau^0, k, k+1) : \mathbf{X}_{\tau^0} \in S \cdot \{a\}, \tau^0 \leq t]$$

$$= \frac{1}{2}\int_0^t \int_S \mathbf{P}_x(\tau^0 \in du, \mathbf{X}_{\tau^0_-} \in dy) \Psi([y, a], t-u, k, k+1)$$

$$= \int_0^t E_x[k(x_u) \exp(-2\int_0^u k(x_v)dv) \Psi(x_u, t-u, k, k)]du$$

$$= 2\int_0^t E_x[k(x_u) \exp(-2\int_0^u k(x_v)dv)$$

$$\cdot \frac{1}{k!}\int_0^{t-u} E_{x_u}[k(x_s)\left\{\int_0^s k(x_v)dv\right\}^k \exp\left(-2\int_0^s k(x_v)dv\right)\pi^0(x_s, A)]ds]du.$$

Then, by the Markov property of $x_t$ the last term is equal to

$$\frac{2}{k!}\int_0^t \int_0^{t-u} E_x[k(x_u)k(x_{s+u})\left\{\int_u^{s+u} k(x_v)dv\right\}^k \exp\left(-2\int_0^{s+u} k(x_v)dv\right)$$

$$\cdot \pi^0(x_{s+u}, A)]dsdu$$

$$= \frac{2}{k!}\int_0^t E_x[k(x_s)\exp\left(-2\int_0^s k(x_v)dv\right)\pi^0(x_s, A)\int_0^s k(x_u)\left\{\int_u^s k(x_v)dv\right\}^k du]ds$$

$$= \frac{2}{(k+1)!}\int_0^t E_x[k(x_s)\left\{\int_0^s k(x_u)du\right\}^{k+1} \exp\left(-2\int_0^s k(x_v)dv\right)\pi^0(x_s, A)]ds,$$

and the proof is finished.

PROOF OF THEOREM. Put $f_1^* = \tilde{f}$ and $f_2^* = \bar{f}$ for a function $f$ on $S$. Since the linear hull of $\{\tilde{f}, \bar{f}: 0 \leq \lambda < 1, f \in \mathbf{C}^*(S)\}$ is dense in $\mathbf{C}_0(\mathbf{S} \times \mathbf{N} \times \mathbf{J})$ and

$$\tilde{\mathbf{E}}_{\tilde{\mathbf{x}}}[f_i^*(\gamma \mathbf{X}_t)] = \sum_{r=0}^{\infty} \tilde{\mathbf{E}}_{\tilde{\mathbf{x}}}[f_i^*(\gamma \mathbf{X}_t) : \sigma_r \leq t < \sigma_{r+1}],$$

it is sufficient to show that

$$\mathbf{E}^0_{\tilde{\mathbf{x}}}[f_i^*(\mathbf{Z}_t) : \tau_r \leq t < \tau_{r+1}] = \tilde{\mathbf{E}}_{\tilde{\mathbf{x}}}[f_i^*(\gamma \mathbf{X}_t) : \sigma_r \leq t < \sigma_{r+1}] \tag{3.13}$$

for $r=0, 1, 2, \ldots, i=1, 2, \tilde{\mathbf{x}} \in \mathbf{S} \times \mathbf{N} \times \mathbf{J}$ and $f \in \mathbf{C}^*(S)$. Let $f_i$ be a function in $\mathbf{B}(S_0)$ such that $f_i = f$ on $S$, $f_i(a) = \lambda$ and $f_i(b) = (-1)^i$ for $i=1, 2$. Then right hand side of (3.13) is equal to

$$\mathbf{E}_{\mathbf{x}}[\hat{f}_i(\mathbf{X}_t) : \sigma_r \leq t < \sigma_{r+1}]$$

where $\tilde{\mathbf{x}}=\gamma\mathbf{x}$, and therefore it is sufficient to prove that

$$\text{(3.14)} \qquad \mathbf{E}^0_{\tilde{\mathbf{x}}}[f^*_i(\mathbf{Z}_t): \tau_r \leq t < \tau_{r+1}] = \mathbf{E}_{\mathbf{x}}[\hat{f}_i(\mathbf{X}_t): \sigma_r \leq t < \sigma_{r+1}],$$
$$r=0, 1, 2,\dots, i=1, 2.$$

When $r=0$, Lemma 3 and (2.4) imply

$$\mathbf{E}_x[\hat{f}_i(\mathbf{X}_t): t<\sigma] = \sum_{k=0}^{\infty} \mathbf{E}_x[\hat{f}_i(\mathbf{X}_t): t<\sigma, n^a_t = k]$$
$$= \mathbf{E}^0_{(x,0,0)}[f^*_i(\mathbf{Z}_t): t<\tau] \quad \text{for } x\in S,$$

and therefore (3.14) for $r=0$ is obtained by Lemma 2 and (2.2). Since

$$\mathbf{E}_x[\hat{f}_i(\mathbf{X}_t): \ \sigma_r \leq t < \sigma_{r+1}] = \int_0^t \int_{S_0} \mathbf{E}_x[\sigma \in ds, \mathbf{X}_\sigma \in d\mathbf{y}]$$
$$\cdot E_{\mathbf{y}}[\hat{f}_i(\mathbf{X}_{t-s}): \sigma_{r-1} \leq t-s < \sigma_r],$$

we can prove (3.14) by induction in $r$ using Lemma 4, (2.5), Lemma 2 and (2.2). Thus the proof of the theorem is completed.

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