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A Semigroup Treatment of the Hamilton-Jacobi Equation in One Space Variable

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1. Introduction

This paper has been motivated by a recent paper [2] by M. G. Crandall, in which the Cauchy problem for the first order quasilinear equation

$$(*)\qquad u_t+\sum_{i=1}^{n}(\phi_i(u))_{x_i}=0,\qquad x\in R^n,\ t>0,$$

is treated from the point of view of the theory of semigroups of nonlinear transformations. Crandall chose $L^1(R^n)$ as the Banach space associated with the Cauchy problem for $(*)$ and succeeded in constructing a semigroup of contractions in $L^1(R^n)$, which provides generalized solutions of the Cauchy problem in the sense of Kružkov [6] if the initial conditions lie in $L^1(R^n)\cap L^\infty(R^n)$.

In this paper we intend to treat the Cauchy problem (hereafter called (CP)) for the Hamilton-Jacobi equation

$$\text{(DE)}\qquad u_t+f(u_x)=0,\qquad -\infty<x<\infty,\quad t>0,$$

from the same point of view. In our (CP), however, we shall, suggested by Kružkov [6], choose $L^\infty(R)$ as the Banach space which may be associated with it. As we shall see, the semigroup approach enables us to treat (CP) under the assumption that $f: R\to R$ is merely continuous. Moreover, as an intermediate step in the development, the existence and uniqueness of certain bounded (possibly generalized) solutions are established for the equation

$$(1)\qquad u+f(u_x)=h,\qquad -\infty<x<\infty,$$

for given h.

When $n=1$, there is clearly an intimate relationship between generalized solutions (cf. [6]) of the Cauchy problem for the quasilinear equation $(*)$ and the Hamilton-Jacobi equation (DE): If u is a generalized solution of the latter equation, then $v=u_x$ is a generalized solution of the former, and the converse is true. In this connection it is easy to see that when applied to (CP), Crandall's result can afford a semigroup of contractions on the subspace of $L^\infty(R)$ consisting of all continuous functions u such that both $\lim_{x\to\infty}u$ and $\lim_{x\to-\infty}u$ exist and are finite. The main object of the present paper is to construct a semigroup associated with (CP) on the subspace consisting of all bounded and uniformly continuous functions on R.

We start, in Section 2, with the definition of an operator A in $L^\infty(R)$ which may be associated with (CP). Section 3 concerns the existence and uniqueness of certain bounded solutions of (1). Here the solutions are obtained as limits of solutions of the regularized equation

$$u+f(u_x)-\varepsilon u_{xx}=h, \qquad -\infty<x<\infty, \tag{2}$$

as $\varepsilon\downarrow 0$. Various results concerning (2) are obtained as needed. Section 4 is devoted to the construction of a semigroup of contractions generated by A through the generation theorem of Crandall and Liggett [3] and to the study of its properties relating to (CP).

2. Definition of the operator A.

Throughout the present paper we shall work in the Banach space $L^\infty(R)$ of all (real-valued) bounded measurable functions on $R=(-\infty, \infty)$ with the norm denoted by $\|\cdot\|_\infty$.

$W_k^\infty(R)$ denotes the subspace of $L^\infty(R)$ consisting of all measurable functions whose distribution derivatives of order at most k lie in $L^\infty(R)$. Thus, in particular, $W_1^\infty(R)$ is the subspace of all bounded and Lipschitz continuous functions on R.

Following Crandall [2] we set:

$$\operatorname{sign}_0 r=\begin{cases} 1 & \text{if } r>0 \\ 0 & \text{if } r=0 \\ -1 & \text{if } r<0. \end{cases}$$

Our first task is to define the operator A_0 associated with (CP) in $L^\infty(R)$.

Definition 2.1. *Let $f\in C$. A_0 is the operator in $L^\infty(R)$ defined by: $v\in D(A_0)$, $w=A_0v$ if v, $w\in W_1^\infty(R)$, $w(x)=f(v_x(x))$ and*

$$\int_R \operatorname{sign}_0(v_x(x)-k)\{[f(v_x(x))-f(k)]\varphi_x(x)+w_x(x)\varphi(x)\}dx\geq 0 \tag{2.1}$$

for every $\varphi\in C_0^\infty(R)$ such that $\varphi\geq 0$ and every $k\in R$.

The next lemma will clarify our definition of the operator A_0.

Lemma 2.1. *Let $f\in C^1$ and A_0 be given by Definition 2.1. If $v\in W_2^\infty(R)$, then $v\in D(A_0)$ and $A_0v=f(v_x(x))$.*

PROOF. Let $f \in C^1$. If $v \in W_2^\infty(R)$, then v, $w = f(v_x(x)) \in W_1^\infty(R)$. For $\Phi \in C^1(R)$ and $\varphi \in C_0^\infty(R)$, integration by parts shows that

$$\int_R (\Phi'(v_x) f(v_x)_x) \varphi(x) dx = \int_R \left(\int_k^{v_x(x)} \Phi'(s) f'(s) ds \right)_x \varphi(x) dx$$

$$= -\int_R \left(\int_k^{v_x(x)} \Phi'(s) f'(s) ds \right) \varphi_x(x) dx.$$

Thus, choosing $\Phi(s) = \Phi_l(s-k)$, where

$$\Phi_l(s) = \begin{cases} -s & \text{if } s \leq -1/l \\ (l/2)s^2 + 1/2l & \text{if } |s| \leq 1/l \\ s & \text{if } s \geq 1/l \end{cases} \tag{2.2}$$

and letting $l \to \infty$, we obtain

$$\int_R \operatorname{sign}_0(v_x(x) - k)\{[f(v_x(x)) - f(k)]\varphi_x(x) + f(v_x(x))_x \varphi(x)\} dx = 0,$$

which shows that $v \in D(A_0)$ and $A_0 v = f(v_x(x))$. The proof is complete.

REMARK 2.1. Let $f: R \to R$ be continuous and strictly monotone. Then $v \in D(A_0)$ implies that v_x is bounded and uniformly continuous on R, since $w = f(v_x(x))$ is Lipschitz continuous by definition.

We are now in a position to define an operator A in $L^\infty(R)$ which may be multi-valued for general f.

DEFINITION 2.2. *A is the closure of A_0, i.e., $v \in D(A)$ and $w \in Av$ if there is a sequence $\{v^k\} \subset D(A_0)$ such that $v^k \to v$, $A_0 v^k \to w$ in $L^\infty(R)$.*

PROPOSITION 2.1. *Let $f: R \to R$ be continuous and strictly monotone. If $f(R) = R$, then:*

(i) *$D(A)$ coincides with the subspace of $L^\infty(R)$ consisting of all once continuously differentiable functions u such that both u and its derivative u_x are bounded and uniformly continuous on R.*

(ii) *A is one-valued and $Av = f(v_x)$, $v \in D(A)$.*

PROOF. This follows immediately from the proof of Proposition 4.2 and the strict monotonicity of f.

3. The equation $u + f(u_x) = h$.

Our object in this section is to establish the existence and uniqueness of certain bounded generalized solutions of the equation

$$u + f(u_x) = h, \qquad -\infty < x < \infty, \tag{3.1}$$

for given h, under the assumption that $f: R \to R$ is merely continuous. For the sake of simplicity, the normalization

$$f(0)=0 \tag{3.2}$$

will be assumed throughout this section, for this can always be achieved by introducing the new unknown $\bar{u}=u+f(0)$.

DEFINITION 3.1. *Let $f \in C$ and A_0 be given by Definition 2.1. Let $h \in W_1^\infty(R)$. Then $u \in W_1^\infty(R)$ is a generalized solution of (3.1) provided $u \in D(A_0)$ and $u+A_0u=h$.*

Our main results concerning (3.1) are:

THEOREM 3.1 (*Existence*). *Let $f: R \to R$ be continuous and the normalization (3.2) be assumed. Then $R(I+A_0)=W_1^\infty(R)$, i.e., for each $h \in W_1^\infty(R)$, there is a generalized solution u of (3.1) such that*

$$\|u\|_\infty \leq \|h\|_\infty, \qquad \|u_x\|_\infty \leq \|h_x\|_\infty, \tag{3.3}$$

and

$$\begin{aligned} &\|u_x(x+y)-u_x(x)\|_{L^1(I)} \\ &\leq \|h_x(x+y)-h_x(x)\|_{L^1(I)}+4\|h_x\|_\infty|y| \end{aligned} \tag{3.4}$$

for every compact interval I and every $y \in R$.

REMARK 3.1. *If $f: R \to R$ is continuous and strictly monotone, then for $h \in W_1^\infty(R)$ there is a solution of (3.1) which lies in $C^1(R) \cap W_1^\infty(R)$.*

PROOF. This follows from Theorem 3.1 and Remark 2.1.

THEOREM 3.2. *Let $f: R \to R$ be continuous and $u, v \in D(A_0)$ satisfy*

$$\begin{aligned} u+f(u_x)&=h, \\ v+f(v_x)&=g. \end{aligned} \tag{3.5}$$

Then:

(i) *A_0 is accretive in $L^\infty(R)$, i.e., we have for (3.5)*

$$\|u-v\|_\infty \leq \|h-g\|_\infty.$$

(ii) *If $g \geq h$, then $v \geq u$.*

(iii) *If $h_x-g_x \in L^1(R)$ (Note that we do not assume $h_x, g_x \in L^1(R)$), then $u_x-v_x \in L^1(R)$ and*

$$\|u_x-v_x\|_1 \leq \|h_x-g_x\|_1.$$

The following corollaries are direct consequences of Theorem 3.2.

COROLLARY 3.1 (*Uniqueness*). *Under the assumption of Theorem* 3.2, *the generalized solution* $u \in W_1^\infty(R)$ *of* (3.1) *is unique for* $h \in W_1^\infty(R)$.

COROLLARY 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied and* $u \in W_1^\infty(R)$ *be the generalized solution of* (3.1). *If* $h_x \in L^1(R)$, *then* $u_x \in L^1(R)$ *and* $\|u_x\|_1 \leq \|h_x\|_1$.

As was stated in the introduction, the generalized solution of (3.1) will be obtained as a limit of solutions of the regularized equation

$$u+f(u_x)-\varepsilon u_{xx}=h, \qquad -\infty<x<\infty, \tag{3.6}$$

as $\varepsilon \downarrow 0$. Consequently, in order to prove Theorems 3.1 and 3.2, it will suffice to prove the corresponding results for solutions of (3.6). To this end, we shall borrow a technique from the work [1] of M. M. Belova, in which the existence and uniqueness of bounded solutions are established for second order equations of the general form $y''=F(x, y, y')$ under various sets of assumptions on F, but always under the assumption that F is once continuously differentiable with respect to the arguments y and y' (cf. in particular, Theorem, p. 467 and Theorem 8, pp. 474–475).

We begin with a lemma due to Belova [1] (see also Kusano [8]) which is a variant of the maximum principle. We shall give a proof here for the sake of completeness.

LEMMA 3.1. *Let* $a \in L^\infty(R)$ *and* $\varepsilon>0$. *If* $v \in C^2(R)$ *is bounded from above and satisfies*

$$Lv \equiv v+a(x)v_x-\varepsilon v_{xx} \leq 0, \qquad -\infty<x<\infty,$$

then $v \leq 0$ *on* R.

PROOF. To prove the lemma by a contradiction, suppose there is a point x^0 such that $v(x^0)>0$. Set

$$w=(v(x^0)-\eta)\cosh k(x-x^0),$$

where $\eta(0<\eta<v(x^0))$ and k are positive constants. A simple calculation shows that we can choose a sufficiently small k in such a way that $Lw>0$ on R. Then $L(v-w)<0$ and hence $v-w$ can not have a positive maximum at finite points of R. But this contradicts the fact that $v-w>0$ at x^0 and $v-w \to -\infty$ as $|x| \to \infty$. The proof is complete.

PROPOSITION 3.1. *Let* $f: R \to R$ *be continuous and* $u, v \in C^2(R) \cap W_1^\infty(R)$ *satisfy*

$$
(3.7) \qquad \begin{aligned} u+f(u_x)-\varepsilon u_{xx}&=h, \\ v+f(v_x)-\varepsilon v_{xx}&=g, \end{aligned}
$$

where $\varepsilon>0$. *If* $h, g\in L^\infty(R)$, *then*:

(i) $\|u-v\|_\infty \le \|h-g\|_\infty$.

(ii) *If* $g\ge h$, *then* $v\ge u$.

PROOF. We shall only give a proof of the first part (i), since the second part (ii) can be proved quite similarly.

First step. Let $f\in C^1$. Then $w=u-v$ satisfies

$$
\begin{aligned} Lw &\equiv w+f(u_x)-f(v_x)-\varepsilon w_{xx} \\ &= w+f'(v_x+\theta(u_x-v_x))w_x-\varepsilon w_{xx} \\ &= w+a(x)w_x-\varepsilon w_{xx}=h-g, \end{aligned}
$$

where $0<\theta=\theta(x)<1$ and $a(x)=f'(v_x+\theta(u_x-v_x))\in L^\infty(R)$, since u_x, $v_x\in L^\infty(R)$ by assumption. Hence an application of Lemma 3.1 yields

$$
\|w\|_\infty=\|u-v\|_\infty\le\|h-g\|_\infty,
$$

since

$$
L(\pm w-\|h-g\|_\infty)=\pm(h-g)-\|h-g\|_\infty\le 0.
$$

Second step. Let $f\in C$ and K be a constant such that both $K\ge\|u_x\|_\infty$ and $K\ge\|v_x\|_\infty$ hold. As is well known, we can find a sequence $\{f_l\}$ of C^1 functions satisfying

$$
|f_l(p)-f(p)|\le 1/l \qquad (l=1, 2....)
$$

for every p such that $|p|\le K$. Then, since

$$
\begin{aligned} u+f_l(u_x)-\varepsilon u_{xx}&=h+f_l(u_x)-f(u_x), \\ v+f_l(v_x)-\varepsilon v_{xx}&=g+f_l(v_x)-f(v_x), \end{aligned}
$$

and

$$
\|h-g+f_l(u_x)-f(u_x)-(f_l(v_x)-f(v_x))\|_\infty\le\|h-g\|_\infty+2/l,
$$

we have by the result of the first step

$$
\|u-v\|_\infty\le\|h-g\|_\infty+2/l
$$

and hence, letting $l\to\infty$, $\|u-v\|_\infty\le\|h-g\|_\infty$. Thus the proof is complete.

Immediate consequences of Proposition 3.1 are:

COROLLARY 3.3. *Let $f: R \to R$ be continuous. Then for each $h \in L^\infty(R)$ there is at most one solution $u \in C^2(R) \cap W_1^\infty(R)$ of* (3.6).

COROLLARY 3.4. *Under the assumption of Corollary* 3.3, *let $u \in C^2(R) \cap W_1^\infty(R)$ satisfy* (3.6). *If $h \in W_1^\infty(R)$, then $\|u_x\|_\infty \le \|h_x\|_\infty$. Moreover, if the normalization* (3.2) *is assumed, then $\|u\|_\infty \le \|h\|_\infty$.*

Next we shall prove the

PROPOSITION 3.2 (*Existence*). *Let $f: R \to R$ be continuous and the normalization* (3.2) *be assumed. Then for each $h \in W_1^\infty(R)$ there is a solution $u \in C^2(R) \cap W_1^\infty(R)$ of* (3.6) *such that*

$$\|u\|_\infty \le \|h\|_\infty, \qquad \|u_x\|_\infty \le \|h_x\|_\infty. \tag{3.8}$$

In proving the proposition, we may assume without loss of generality that $\varepsilon = 1$ in (3.6). Our proof below will be based on the following

LEMMA 3.2. *Let $h \in C(R) \cap L^\infty(R)$. Then a unique solution $u \in C^2(R) \cap L^\infty(R)$ of the equation*

$$u - u_{xx} = h, \qquad -\infty < x < \infty,$$

is expressed by

$$u(x) = C_1 e^x + C_2 e^{-x} - \frac{1}{2} e^x \int_0^x e^{-s} h(s) ds + \frac{1}{2} e^{-x} \int_0^x e^s h(s) ds,$$

where

$$C_1 = \frac{1}{2} \int_0^\infty e^{-s} h(s) ds, \quad C_2 = -\frac{1}{2} \int_0^{-\infty} e^s h(s) ds.$$

Moreover, if $h \in W_1^\infty(R)$, then $\|u\|_\infty \le \|h\|_\infty$ and $\|u_x\|_\infty \le \|h_x\|_\infty$.

PROOF OF PROPOSITION 3.2. *First step.* Let $f \in C^1(R) \cap W_1^\infty(R)$ and $\mathscr{F}$ be defined by

$$\mathscr{F} = \{u \in C^1(R);\ \|u\|_\infty \le K_1,\ \|u_x\|_\infty \le K_2 \text{ and for } y \in R,$$
$$\|u_x(x+y) - u_x(x)\|_\infty \le K_3 |y|\},$$

where

$$K_1 = \|h\|_\infty + \|f\|_\infty,$$

$$K_2 = \|h_x\|_\infty + 2\|f'\|_\infty (\|h\|_\infty + \|f\|_\infty),$$

$$K_3 = 2(\|h\|_\infty + \|f\|_\infty).$$

Then $\mathscr{F}$ is a compact and convex subset of the locally convex, linear topological space $\mathscr{E}^1(R)$ whose topology is given by the family of seminorms

$$p_m(u) = \sup_{|x|\le m}|u(x)| + \sup_{|x|\le m}|u_x(x)|, \qquad u\in C^1(R), \qquad (m=1, 2,\dots).$$

Let T be the operator in $\mathscr{E}^1(R)$ defined by: $u\in D(T)$ and $v=Tu$ if $u\in\mathscr{F}$ and v is a solution in $C^2(R)\cap W_1^\infty(R)$ of the equation

$$v - v_{xx} = h - f(u_x), \qquad -\infty < x < \infty,$$

whose existence is guaranteed by Lemma 3.2. Then, in view of Lemma 3.2, it is not difficult to show that $T\mathscr{F}\subset\mathscr{F}$ and T is continuous. Hence the well known fixed point theorem of Schauder-Tychonoff applies: there is a fixed point $u=Tu$, which provides a solution $u\in C^2(R)\cap W_1^\infty(R)$ of (3.6).

The estimate (3.8) follows from Corollary 3.4.

Second step. Let $f\in C$ and K be a constant such that $K\ge\|h_x\|_\infty$. Then we can find a sequence $\{f_l\}$ of functions in $C^1(R)\cap W_1^\infty(R)$ such that $f_l(0)=0$ and

$$|f_l(p) - f(p)| \le 1/l \qquad (l=1, 2,\dots)$$

for every p with $|p|\le K$. By virtue of the result of the first step, there is, for each l, a solution $u^l\in C^2(R)\cap W_1^\infty(R)$ of the equation

$$u + f_l(u_x) - u_{xx} = h, \qquad -\infty < x < \infty,$$

such that

$$\|u^l\|_\infty \le \|h\|_\infty \quad\text{and}\quad \|u_x^l\|_\infty \le \|h_x\|_\infty,$$

from which

$$\|u_{xx}^l\|_\infty \le 2\|h\|_\infty + \sup_{|p|\le K}|f_l(p)| \le 2\|h\|_\infty + \sup_{|p|\le K}|f(p)| + 1/l.$$

Hence the sequence $\{u^l\}$ is precompact in $\mathscr{E}^1(R)$ and there is a subsequence $\{u^{l(i)}\}$ of $\{u^l\}$ which converges, in $\mathscr{E}^1(R)$, to a limit $u\in C^1(R)$. But this implies that the sequence $\{u_{xx}^{l(i)}\}$ also converges to u_{xx} uniformly on every compact set. Obviously, the limit u is a solution in $C^2(R)\cap W_1^\infty(R)$ of (3.6) which satisfies (3.8). Hereby the proof of Proposition 3.2 has been completed.

The next proposition is a refined version of a result of Crandall [2] (Corollary 2.1, p. 121) and is a core in our proof of Theorem 3.1.

PROPOSITION 3.3. *Let $f: R\to R$ be continuous and u, $v\in C^2(R)\cap W_1^\infty(R)$ satisfy* (3.7), *where $\varepsilon>0$. If h, $g\in W_1^\infty(R)$, then*

$$\|u_x - v_x\|_{L^1(I)} \le \|h_x - g_x\|_{L^1(I)} + 4\|h-g\|_\infty \tag{3.9}$$

for every compact interval $I=[a, b]$.

PROOF. *First step.* Let $f\in C^1(R)\cap W_1^\infty(R)$. Define α_l by $\alpha_l=\Phi_l'$, where Φ_l is given by (2.2). Let $\varphi_m\in C(R)$ be defined by

$$\varphi_m(x)=\begin{cases} 1 & \text{inside } I \\ 0 & \text{outside } I_m \\ \text{linear} & \text{elsewhere,} \end{cases}$$

where we have set $I_m=[a-1/m, b+1/m]$ $(m=1, 2, ..)$. Then $w=u_x-v_x$ satisfies

$$w+(f(u_x)-f(v_x))_x-\varepsilon w_{xx}=h_x-g_x.$$

Multiplying the above by $\alpha_l(w)\varphi_m$ and integrating we have

$$\int_R\{w\alpha_l(w)\varphi_m+(f(u_x)-f(v_x))_x\alpha_l(w)\varphi_m-\varepsilon w_{xx}\alpha_l(w)\varphi_m\}dx \leq \|h_x-g_x\|_{L^1(I_m)}, \tag{3.10}$$

since $|\alpha_l(w)|\leq 1$ and $0\leq\varphi_m\leq 1$.

Each term on the left-hand side of (3.10) can be estimated from below in the following manner:

$$\int_R w\alpha_l(w)\varphi_m dx\geq\int_I w\alpha_l(w)dx\to\int_I|w|dx=\|w\|_{L^1(I)} \qquad \text{as } l\to\infty \tag{3.11}$$

by using the oddness of α_l. Integration by parts yields

$$\int_R(f(u_x)-f(v_x))_x\alpha_l(w)\varphi_m dx$$
$$=-\int_{\Omega_l}(f(u_x)-f(v_x))\alpha_l'(w)w_x\varphi_m dx-\int_R(f(u_x)-f(v_x))\alpha_l(w)\varphi_m' dx$$

where $\Omega_l=\{x\in I_1;\ |u_x(x)-v_x(x)|\leq 1/l\}$ and we have used the fact that $\alpha_l'(w)=0$ outside Ω_l. For the first term we note that

$$|f(u_x)-f(v_x)|\alpha_l'(w)\leq l\|f'\|_\infty|u_x-v_x|\leq\|f'\|_\infty, \qquad x\in\Omega_l.$$

Hence, by the bounded convergence theorem,

$$\limsup_{l\to\infty}\left|\int_{\Omega_l}(f(u_x)-f(v_x))\alpha_l'(w)w_x\varphi_m dx\right|\leq\|f'\|_\infty\int_\Omega|w_x|dx, \tag{3.12}$$

where $\Omega=\cap_l\Omega_l$. But $u_x=v_x$ a.e. on Ω implies $w_x=(u_x-v_x)_x=0$ a.e. on Ω and the integral on the right is zero. Next, by using (3.7) and Proposition 3.1 (i), we have

$$(3.13)\qquad -\int_R (f(u_x)-f(v_x))\alpha_l(w)\varphi_m' dx \geq -4\|h-g\|_\infty - \varepsilon\int_R w_x\alpha_l(w)\varphi_m' dx,$$

since $|\alpha_l(w)|\leq 1$ and $\int_R |\varphi_m'|dx=2$. Integration by parts again yields

$$(3.14)\qquad -\varepsilon\int_R w_{xx}\alpha_l(w)\varphi_m dx$$

$$=\varepsilon\int_R w_x^2\alpha_l'(w)\varphi_m dx+\varepsilon\int_R w_x\alpha_l(w)\varphi_m' dx \geq \varepsilon\int_R w_x\alpha_l(w)\varphi_m' dx,$$

since $\alpha_l'\geq 0$.

Using (3.11)–(3.14) in (3.10) and letting l, $m\to\infty$ we obtain (3.9).

Second step. Let $f\in C$ and $\{f_l\}$ be a sequence of C^1 functions given in the second step of the proof of Proposition 3.1. Then, by virtue of Corollary 3.3 and what was shown in the second step of the proof of Proposition 3.2, it is easily seen that the sequences $\{u^l\}$, $\{v^l\}$ of solutions in $C^2(R)\cap W_1^\infty(R)$ of the equations

$$u+f_l(u_x)-\varepsilon u_{xx}=h,$$

$$v+f_l(v_x)-\varepsilon v_{xx}=g$$

converge, in $\mathscr{E}^1(R)$, to u, v of solutions in $C^2(R)\cap W_1^\infty(R)$ of (3.7) respectively. Now, by the result of the first step,

$$\|u_x^l-v_x^l\|_{L^1(I)}\leq \|h_x-g_x\|_{L^1(I)}+4\|h-g\|_\infty$$

and hence, letting $l\to\infty$, we obtain (3.9). The proof is complete.

An immediate consequence of Proposition 3.3 is:

COROLLARY 3.5. *Under the assumption of Proposition* 3.3, *let* $u\in C^2(R)\cap W_1^\infty(R)$ *satisfy* (3.6), *where* $\varepsilon>0$. *If* $h\in W_1^\infty(R)$, then

$$(3.15)\qquad \|u_x(x+y)-u_x(x)\|_{L^1(I)}$$

$$\leq \|h_x(x+y)-h_x(x)\|_{L^1(I)}+4\|h_x\|_\infty|y|$$

for every compact interval I and every $y\in R$.

PROOF OF THEOREM 3.1. Choose a sequence $\{f_l\}_{l=1}^\infty$ of C^1 functions such that $f_l(0)=0$ and $\{f_l\}$ converges to f uniformly on compact sets. Given $h\in W_1^\infty(R)$, let $u^l\in C^2(R)\cap W_1^\infty(R)$ be the unique solution of the equation

$$u+f_l(u_x)-(1/l)u_{xx}=h,\qquad -\infty<x<\infty,$$

guaranteed by Proposition 3.2 and Corollary 3.3. Then the estimates (3.8) and

(3.15) imply that $\{u^l\}$ is precompact in $\mathscr{E}^0(R)$ and $\{u_x^l\}$ precompact in $L^1_{\mathrm{loc}}(R)$. Hence we can find a subsequence $\{u^{l(i)}\}$ of $\{u^l\}$ and a $u \in W_1^\infty(R)$ such that $u^{l(i)} \to u$ in $\mathscr{E}^0(R)$ and $\{u_x^{l(i)}\}$ converges a.e. and in $L^1_{\mathrm{loc}}(R)$ to u_x. We denote this convergence in $W_1^\infty(R)$ by $\twoheadrightarrow$, $u^{l(i)} \twoheadrightarrow u$. It is obvious that the limit u enjoys the properties (3.3) and (3.4).

We shall show that the limit u satisfies (3.1) a.e.. To see this, let $\varphi \in C_0^\infty(R)$. Multiplying the equation satisfied by u^l by φ and integrating we have

$$\int_R \{(u^l + f_l(u_x^l))\varphi + (1/l)u_x^l \varphi_x\} dx = \int_R h\varphi dx.$$

Letting l tend to ∞ through the subsequence $\{l(i)\}$ and using the convergences $u^{l(i)} \twoheadrightarrow u$ and $f_{l(i)} \to f$ uniformly on compact sets, we obtain

$$\int_R (u + f(u_x))\varphi dx = \int_R h\varphi dx,$$

since $\int_R u_x^l \varphi_x dx$ is bounded in l by (3.8). But this implies $u + f(u_x) = h$ a.e., since $\varphi \in C_0^\infty(R)$ is arbitrary.

It remains to show that $u \in D(A_0)$ and $u + A_0 u = h$. In doing this, we shall proceed exactly in the same way as in the proof of [2, Corollary 2.2]. First we note that both u and $f(u_x)$ lie in $W_1^\infty(R)$, since $f(u_x) = h - u \in W_1^\infty(R)$ by (3.8). To prove (2.1), let $\varphi \in C_0^\infty(R)$ and Φ: $R \to R$ have a piecewise continuous second derivative. Multiply the equation

$$u_x^l + f_l'(u_x^l)u_{xx}^l - (1/l)u_{xxx}^l = h_x, \quad -\infty < x < \infty,$$

satisfied a.e. by u_x^l by $\Phi'(u_x^l)\varphi$ and integrate over R. After some integration by parts we find

$$\begin{aligned}\int_R \Big\{ & u_x^l \Phi'(u_x^l)\varphi - (\Phi'(u_x^l)f_l(u_x^l) - \Phi'(k)f_l(k))\varphi_x \\ & + \Big(\int_k^{u_x^l} \Phi''(s)f_l(s)ds\Big)\varphi_x + (1/l)(\Phi''(u_x^l)|u_{xx}^l|^2\varphi \\ & - \Phi(u_x^l)\varphi_{xx})\Big\} dx = \int_R h_x \Phi'(u_x^l)\varphi dx\end{aligned}$$

for every $k \in R$. Assuming $\Phi'' \geq 0$ and $\varphi \geq 0$, we find that the term involving $\Phi''(u_x^l)|u_{xx}^l|^2\varphi$ is nonnegative. Moreover, $\int_R \Phi(u_x^l)\varphi_{xx} dx$ is bounded in l by (3.8). Thus, letting $l \to \infty$ through the subsequence $\{l(i)\}$, we obtain

$$\int_R \Big\{ u_x \Phi'(u_x)\varphi - \Phi'(u_x)f(u_x)\varphi_x + \Big(\int_k^{u_x} \Phi''(s)f(s)ds\Big)\varphi_x \Big\} dx \leq \int_R h_x \Phi'(u_x)\varphi dx$$

for $\varphi \in C_0^\infty(\mathrm{R})$, $\varphi \geq 0$ and $k \in R$. Next choose $\Phi(s) = \Phi_l(s-k)$ where Φ_l is given by (2.2) and let $l \to \infty$. Then, since

$$\int_k^{u_x} \Phi''(s) f(s) ds \to \mathrm{sign}_0(u_x - k) f(k),$$

this yields

$$\int_R \mathrm{sign}_0(u_x - k)[(f(u_x) - f(k))\varphi_x + (h_x - u_x)\varphi] dx \geq 0$$

for every $\varphi \in C_0^\infty(R)$ such that $\varphi \geq 0$ and every $k \in R$. Hence, according to Definition 2.1, $u \in D(A_0)$ and $h - u = A_0 u$. The proof of Theorem 3.1 has been completed.

Let $u: R \to R$ be measurable. We shall denote by sign u the set of all measurable $\alpha: R \to R$ such that $|\alpha(x)| \leq 1$ a.e. and $\alpha(x)u(x) = |u(x)|$ a.e.. Notice that $\mathrm{sign}_0\, u \in \mathrm{sign}\, u$.

LEMMA 3.3. *Let $\{u^k\}$ and u be measurable functions on R such that $u^k \to u$ in $L^1(S)$, where S is a measurable subset of R. If $\alpha^k \in \mathrm{sign}\, u^k$, then there is a subsequence $\{\alpha^{k(i)}\}$ and $\alpha \in \mathrm{sign}\, u$ (depending, perhaps, on S) such that $\{\alpha^{k(i)}\}$ converges to α in the weak-star topology on $L^\infty(S)$.*

PROOF. This follows easily from the fact that $(L^1(S))^* = L^\infty(S)$.

Proceeding in the same way as in the proof of [2, Proposition 2.1] and using Lemma 3.3, we can prove the following:

LEMMA 3.4. *Let $f \in C$ and $u, v \in D(A_0)$. Then, for each $\varphi \in C_0^\infty(R)$, there exists an $\alpha \in \mathrm{sign}(u_x - v_x)$ (depending, perhaps, on φ) such that*

$$\int_R \alpha\{(f(u_x) - f(v_x))\varphi_x + (w_x - z_x)\varphi\} dx \geq 0, \tag{3.16}$$

where we set $w = f(u_x)$ and $z = f(v_x)$.

PROOF OF THEOREM 3.2. We shall begin with the proof of (iii). Let $I = [a, b]$ be any compact interval and $\{\varphi_m\}$ be a sequence of piecewise C^1 functions given in the proof of Proposition 3.3. Using Lemmas 3.3 and 3.4 in the obvious way, we can find a sequence $\{\alpha^m\} \subset \mathrm{sign}(u_x - v_x)$ such that (3.16) holds for every pair of $\varphi = \varphi_m$ and $\alpha = \alpha^m$. Now let K be a constant such that both $K \geq \|u_x\|_\infty$ and $K \geq \|v_x\|_\infty$ hold. Then, since

$$\left| \int_R \alpha^m (f(u_x) - f(v_x)) \varphi'_m dx \right| \leq 4 \sup_{|p| \leq K} |f(p)|,$$

we obtain by letting $m \to \infty$ and using Lemma 3.3 again

$$\int_I \alpha(w_x - z_x)dx \geq -4 \sup_{|p|\leq K} |f(p)|$$

for some $\alpha \in \operatorname{sign}(u_x - v_x)$. But this shows that

$$\int_I |h_x - g_x| dx \geq \int_I \alpha(h_x - g_x)dx = \int_I \alpha(u_x - v_x)dx + \int_I \alpha(w_x - z_x)dx$$

$$\geq \int_I |u_x - v_x| dx - 4 \sup_{|p|\leq K} |f(p)|$$

and hence, by the arbitrariness of I, $u_x - v_x \in L^1(R)$ if $h_x - g_x \in L^1(R)$.

It remains to prove $\|u_x - v_x\|_1 \leq \|h_x - g_x\|_1$. To do this, let $\{f_k\}$ be a sequence of C^1 functions such that

$$|f_k(p) - f(p)| \leq 1/k \qquad (k = 1, 2, \ldots)$$

for every p such that $|p| \leq K$. Choose a function $\kappa \in C_0^\infty(R)$ such that $\kappa \geq 0$ and $\kappa(s) = 1$ for $|s| \leq 1$. Set $\varphi(x) = \kappa(x/l)$ in (3.16), let $l \to \infty$ and use Lemma 3.3 in the obvious way to find that there is an $\alpha \in \operatorname{sign}(u_x - v_x)$ such that

$$\int_R \alpha(w_x - z_x)dx \geq -(2/k)\|\kappa'\|_1$$

for every k, since $f_k(u_x) - f_k(v_x) \in L^1(R)$ and

$$\left| \int_R \alpha^l \{f(u_x) - f_k(u_x) - (f(v_x) - f_k(v_x))\} \varphi_x dx \right| \leq (2/k)\|\kappa'\|_1$$

($\alpha^l \in \operatorname{sign}(u_x - v_x)$ is determined by $\varphi(x) = \kappa(x/l)$). Thus, letting $k \to \infty$ yields

$$\int_R \alpha(w_x - z_x)dx \geq 0$$

for some $\alpha \in \operatorname{sign}(u_x - v_x)$. But this implies, as before, the estimate $\|u_x - v_x\|_1 \leq \|h_x - g_x\|_1$.

Now the proof of (i) and (ii) can be carried out as follows: From (iii) it follows immediately that u, $v \in D(A_0)$ are unique generalized solutions of the equations

$$u + A_0 u = h, \qquad v + A_0 v = g$$

respectively. Hence, by what was shown in the proof of Theorem 3.1, u, v can be obtained as limits of solutions u^l, v^l in $C^2(R) \cap W_1^\infty(R)$ of the equations

$$u + f(u_x) - (1/l)u_{xx} = h,$$

$$v + f(v_x) - (1/l)v_{xx} = g$$

as $l \to \infty$ through a suitable subsequence $\{l(i)\}$. Since $u^{l(i)} \twoheadrightarrow u$, $v^{l(i)} \twoheadrightarrow v$ in $W_1^\infty(R)$, Proposition 3.1 can be used to prove (i) and (ii). The proof is complete.

4. The semigroup of contractions associated with (CP).

The Cauchy problem (CP) consists of (DE) and the initial condition

(IC) $$u(x, 0)=u^0(x), \qquad -\infty<x<\infty,$$

where u^0 is a given function on R.

It is assumed throughout the section that $f: R \to R$ is merely continuous and satisfies the normalization (3.2), for this can always be achieved by introducing the new unknown $\bar{u}=u+f(0)t$.

We shall choose $L^\infty(R)$ as the Banach space associated with (CP) and regard the function u in (DE) as a map: $[0, \infty) \ni t \mapsto u(\cdot, t) \in L^\infty(R)$. Let A be given by Definition 2.2. Then (CP) can be rewritten in the abstract form

(ACP) $$\frac{du}{dt}+Au \ni 0, \qquad u(0)=u^0$$

(Note that A may be multi-valued for general f).

In order to apply the abstract theory to (ACP), we shall state the generation theorem of Crandall and Liggett [3] in a form suitable for our later use. Let X be a Banach space and A be an operator in X (which is allowed to be multi-valued). A is said to be accretive in X if

$$\|(u+\lambda w)-(v+\lambda z)\| \geq \|u-v\|$$

for $\lambda>0$, u, $v \in D(A)$, $w \in Au$ and $z \in Av$, where $\|\cdot\|$ denotes the norm in X. For $\lambda>0$, let $D_\lambda=D(J_\lambda)=R(I+\lambda A)$, $J_\lambda=(I+\lambda A)^{-1}$ and $A_\lambda=\lambda^{-1}(I-J_\lambda)$. Set $\mathscr{D} = \bigcup_{\kappa>0}(\bigcap_{0<\lambda<\kappa} D_\lambda)$ and define, if $\mathscr{D} \supseteq D(A)$,

$$\hat{D}(A)=\{v \in \mathscr{D};\ |Av|<\infty\},$$

where we have set for $v \in \mathscr{D}$

$$|Av|=\lim_{\lambda \downarrow 0}\|A_\lambda v\|.$$

The following generation theorem is a combined version of [3, Theorem I] and [4, Proposition 2.3].

GENERATION THEOREM. *Let A be an accretive operator in a Banach space X. If $R(I+\lambda A) \supset \overline{D(A)}$ for all sufficiently small positive λ, then*

(4.1) $$\lim_{n\to\infty}\left(I+\frac{t}{n}A\right)^{-n} u^0$$

exists for $u^0 \in \overline{D(A)}$ *and* $t \geq 0$. *Moreover, if* $S(t)u^0$ *is defined as the limit in* (4.1), *then* $S(t)$ *is a semigroup of contractions on* $\overline{D(A)}$:

(i) *We have* $S(t)$: $\overline{D(A)} \to \overline{D(A)}$ *for* $t \geq 0$; $S(t)S(\tau) = S(t+\tau)$ *for* $t, \tau \geq 0$; $\|S(t)v - S(t)w\| \leq \|v - w\|$ *for* $v, w \in \overline{D(A)}$ *and* $t \geq 0$; $S(0) = I$ *and* $S(t)v$ *is continuous in* (t, v).

(ii) *If* $v \in \hat{D}(A)$, *then* $S(t)v$ *is Lipschitz continuous in* t *on every compact interval.*

(iii) *For each* $\varepsilon > 0$ *and* $u^0 \in \overline{D(A)}$, *the problem*

$$\begin{cases} \varepsilon^{-1}(u^\varepsilon(t) - u^\varepsilon(t-\varepsilon)) + Au^\varepsilon(t) \ni 0, & t \geq 0, \\ u^\varepsilon(t) = u^0, & t < 0 \end{cases} \tag{4.2}$$

has a unique solution $u^\varepsilon(t)$ *on* $[0, \infty)$ *and* $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t) = S(t)u^0$ *uniformly in* t *on compact sets.*

We have to verify the hypotheses of the Generation Theorem for the A of Definition 2.2. First we make the following:

REMARK 4.1. *If* $u \in D(A)$ *and* $w \in Au$, *then both* u *and* w *are bounded and uniformly continuous functions on* R.

PROOF. This is obvious, since A is the closure of A_0 and $v \in D(A_0)$ implies $v, A_0 v \in W_1^\infty(R)$.

From Theorem 3.2 we easily have:

PROPOSITION 4.1. *Let* $f: R \to R$ *be continuous. If* $u, v \in D(A)$, $w \in Au$ *and* $z \in Av$ *satisfy*

$$u + \lambda w = h, \qquad v + \lambda z = g,$$

where $\lambda > 0$, *then*:

(i) A *is accretive in* $L^\infty(R)$, *i.e., we have*

$$\|u - v\|_\infty \leq \|h - g\|_\infty.$$

(ii) *If* $g \geq h$, *then* $v \geq u$.

In what follows, $BU(R)$ denotes the closed linear subspace of $L^\infty(R)$ consisting of all bounded and uniformly continuous functions on R.

Now we shall give another definition of generalized solutions for (3.1).

DEFINITION 4.1. *Let* $h \in BU(R)$. *Then* $u \in BU(R)$ *is a generalized solution of* (3.1) *provided* $u \in D(A)$ *and* $h \in u + Au$.

It follows from Theorem 3.1 that $R(I + \lambda A) = BU(R)$ for $\lambda > 0$, since $R(I + \lambda A_0) = W_1^\infty(R)$ is dense in $BU(R)$ and A is the closure of A_0 (Note that $R(I + \lambda A)$ is closed for $\lambda > 0$ when A is closed and accretive). Clearly, we have $\overline{D(A)} \subset BU(R)$.

Therefore we have proved the

THEOREM 4.1. *Let f: $R\to R$ be continuous. Then the operator A of Definition 2.2 satisfies the assumptions of the Generation Theorem. In particular, $u=(I+A)^{-1}h$ is the unique generalized solution of (3.1) for $h\in BU(R)$.*

COROLLARY 4.1. *Let f: $R\to R$ be continuous and strictly monotone. If $f(R)=R$, then $u=(I+A)^{-1}h$ is the unique solution in $C^1(R)\cap W_1^\infty(R)$ of (3.1) for $h\in BU(R)$.*

With the next proposition the construction of our semigroup is complete.

PROPOSITION 4.2. *If f: $R\to R$ is continuous, then $\overline{D(A)}=BU(R)$.*

PROOF. It suffices to prove that $C^2(R)\cap W_2^\infty(R)\subset D(A)$, since $C^2(R)\cap W_2^\infty(R)$ is dense in $BU(R)$. Let $u\in C^2(R)\cap W_2^\infty(R)$ and $\{f_l\}$ be a sequence of C^1 functions which converges to f uniformly on compact sets. Set

$$u+f(u_x)=h, \qquad u+f_l(u_x)=h_l.$$

Since $h\in BU(R)$ and $h_l\in W_1^\infty$, it then follows from Theorems 4.1 and 3.1 that there are a unique $v\in D(A)$ such that $v+w=h$ for some $w\in Av$ and a unique $v^l\in D(A_0)$ such that $v^l+A_0v^l=h_l$. By Proposition 4.1 (i) we have $\|v^l-v\|_\infty\le\|h_l-h\|_\infty$ and so $v^l\to v$ in $L^\infty(R)$ as $l\to\infty$, since $h_l-h=f_l(u_x)-f(u_x)\to 0$ in $L^\infty(R)$. Now, let I be any compact interval. By using what was shown in the proof of Theorem 3.1 it is seen that for each l, there is an integer $k(l)$ such that $k(l)>l$ and the unique solution $v^{k(l)}\in C^2(R)\cap W_1^\infty(R)$ of the equation

$$v+f_l(v_x)-(1/k(l))v_{xx}=h_l$$

satisfies $\|v^{k(l)}-v^l\|_{L^\infty(I)}<1/l$. But then Proposition 3.2 (i) yields that $\|v^{k(l)}-u\|_\infty\le(1/k(l))\|u_{xx}\|_\infty$, since $u\in C^2(R)\cap W_2^\infty(R)$ satisfies the equation

$$u+f_l(u_x)-(1/k(l))u_{xx}=h_l-(1/k(l))u_{xx}.$$

Hence the sequence $\{v^{k(l)}\}_{l=0}^\infty$ converges to u as well as to v uniformly on I so that, by the arbitrariness of I, we have $u=v$ on R. Consequently, $w=f(u_x)$, which shows that $u\in D(A)$ and $f(u_x)\in Au$. The proof is complete.

According to Theorem 4.1, Proposition 4.2 and the Generation Theorem, a semigroup of contractions $S(t)$ on $BU(R)$ is determined by the operator A. Concerning the properties of this semigroup, we have first the

THEOREM 4.2. *Let f: $R\to R$ be continuous and $S(t)$ be the semigroup of contractions on $BU(R)$ obtained from A through the Generation Theorem. Let u, $v\in BU(R)$ and $t\ge 0$. Then:*

(i) *If $y\in R$, then*

$$\sup_{x\in R}|S(t)v(x+y)-S(t)v(x)|\le\sup_{x\in R}|v(x+y)-v(x)|.$$

Moreover, if $v\in W_1^\infty(R)$, *then* $S(t)v\in W_1^\infty(R)$ *and*

$$\|S(t)v\|_\infty\le\|v\|_\infty,\qquad \|(S(t)v)_x\|_\infty\le\|v_x\|_\infty$$

(Note that the normalization (3.2) is assumed).

(ii) *If* $v\ge u$, *then* $S(t)v\ge S(t)u$.

PROOF. The solution $u^\varepsilon(t)$ of (4.2) is given by $u^\varepsilon(t)=(I+\varepsilon A)^{-[t/\varepsilon]-1}u^0$, where $[t/\varepsilon]$ is the greatest integer in t/ε. Since $\lim_{\varepsilon\downarrow 0}u^\varepsilon(t)=S(t)u^0$ uniformly in t on compact sets, the proofs of (i) and (ii) follow immediately from Proposition 4.1 (i) and (ii). The proof is complete.

Kružkov [6] treats (CP) in the case where f: $R\to R$ is continuously differentiable and u^0 is a Lipschitz continuous (and not necessarily bounded) function on R, and establishes the existence and uniqueness of certain generalized solutions. A Lipschitz continuous function $u(x, t)$ on $R\times[0, \infty)$ is called a generalized solution of (CP) if: 1) u satisfies (DE) *a.e.* as well as (IC); 2) for every $\varphi(x, t)\in C_0^\infty(R\times(0, T))$ such that $\varphi\ge 0$ and every $k\in R$ and $T>0$, we have

$$\int_0^T\int_R\{|u_x-k|\varphi_t+\mathrm{sign}_0(u_x-k)[f(u_x)-f(k)]\varphi_x\}dxdt\ge 0,$$

where the derivative u_x is continuous in x in the L^1-norm on every compact interval, uniformly with respect to $t\in[0, T]$. Below we shall show that the semigroup $S(t)$ obtained above provides a generalized solution $S(t)u^0$ of (CP) in the sense of Kružkov if u^0 lies in $W_1^\infty(R)$ and its derivative u_x^0 is continuous in the L^1-norm on R. If $f\in C^1$, the hyperbolic character of (CP) and Theorem 4.3 below can be used in the usual way to deduce the existence of generalized solutions of Kružkov's type for Lipschitz continuous (and not necessarily bounded) functions u^0.

A measurable function v on R is said to be continuous in the L^1-norm on R if there is a positive number δ depending on v such that $v(x+y)-v(x)\in L^1(R)$ for every y with $|y|<\delta$ and

$$\lim_{|y|\to 0}\int_R|v(x+y)-v(x)|dx=0.$$

Notice that $v\in L^1(R)$ is continuous in the L^1-norm on R.

THEOREM 4.3. *Let* f: $R\to R$ *be continuous and* $S(t)$ *be the semigroup of contractions on* $BU(R)$ *obtained from* A *through the Generation Theorem. Let* $v\in W_1^\infty(R)$. *If the derivative* v_x *is continuous in the* L^1-*norm on* R, *then*:

(i) *If* $|y|<\delta$, *where* δ *is a positive number depending on* v, *then*

$$\int_R |(S(t)v)_x(x+y)-(S(t)v)_x(x)|dx \leq \int_R |v_x(x+y)-v_x(x)|dx$$

for every $t \geq 0$.

(ii) $S(t)v(x)$ *is Lipschitz continuous on* $R\times[0, \infty)$ *and satisfies* (DE) *a.e.*.

(iii) *We have*

$$\int_0^T\int_R \{|(S(t)v(x))_x-k|\varphi_t+\mathrm{sign}_0((S(t)v(x))_x-k)$$
$$\times[f((S(t)v(x))_x)-f(k)]\varphi_x\}dxdt \geq 0$$

for every $\varphi(x, t)\in C_0^\infty(R\times(0, T))$ *such that* $\varphi\geq 0$ *and every* $k\in R$ *and* $T>0$.

Proof. Let $v\in W_1^\infty(R)$ and $u^\varepsilon(t)$ satisfy

$$\text{(4.3)}\qquad \begin{cases}\varepsilon^{-1}(u^\varepsilon(t)-u^\varepsilon(t-\varepsilon))+A_0u^\varepsilon(t)=0, & t\geq 0,\\ u^\varepsilon(t)=v, & t<0.\end{cases}$$

Then $u^\varepsilon(t)=(I+\varepsilon A_0)^{-[t/\varepsilon]-1}v$ for $t\geq 0$ and, by Theorem 3.2 (i),

$$\text{(4.4)}\qquad \|u^\varepsilon(t)\|_\infty\leq\|v\|_\infty,\qquad \|(u^\varepsilon(t))_x\|_\infty\leq\|v_x\|_\infty$$

for $t\geq 0$. Next we note that $W_1^\infty(R)\subset\hat{D}(A)$. Indeed, if $v\in W_1^\infty(R)$, then $u=J_\lambda v$ satisfies $u+\lambda A_0u=v$ and so $A_\lambda v=\lambda^{-1}(I-J_\lambda)v=A_0u=f(u_x)$ for $\lambda>0$ (cf. Definition 2.1). Hence we have $\|A_\lambda v\|_\infty\leq\sup_{|p|\leq\|v_x\|_\infty}|f(p)|$ for $\lambda>0$, which implies $v\in\hat{D}(A)$. Therefore, according to the Generation Theorem (ii), $S(t)v$ is Lipschitz continuous in t on compact intervals.

Now, let v_x be continuous in the L^1-norm on R and $u^\varepsilon(x, t)=u^\varepsilon(t)(x)$. It then follows from Theorem 3.2 (iii) that $u_x^\varepsilon(x+y, t)-u_x^\varepsilon(x, t)\in L^1(R)$ and

$$\text{(4.5)}\qquad \int_R|u_x^\varepsilon(x+y, t)-u_x^\varepsilon(x, t)|dx\leq\int_R|v_x(x+y)-v_x(x)|dx$$

for every $t\geq 0$ and every y with $|y|<\delta$. For each fixed $t\geq 0$, the sequence $\{(u^\varepsilon(t))_x\}$ must converge in $L_{loc}^1(R)$ to $(S(t)v)_x$ as $\varepsilon\downarrow 0$, since $u^\varepsilon(t)\to S(t)v\in W_1^\infty(R)$ in $L^\infty(R)$ as $\varepsilon\downarrow 0$ and $\{(u^\varepsilon(t))_x\}$ is precompact in $L_{loc}^1(R)$ by (4.4) and (4.5). Thus, letting $\varepsilon\downarrow 0$ in (4.5), we easily obtain (i) for $t\geq 0$.

By Definition 2.1, $u^\varepsilon(t)$ satisfies the equation

$$\text{(4.6)}\qquad \varepsilon^{-1}(u^\varepsilon(t)-u^\varepsilon(t-\varepsilon))+f((u^\varepsilon(t))_x)=0$$

for $t\geq 0$. Let $T>0$. Since $S(t)v$ is Lipschitz continuous for $0\leq t\leq T$ and $S(t)v\in W_1^\infty(R)$ for every $t\geq 0$, $S(t)v(x)$ is (totally) differentiable *a.e.* on $R\times[0, T]$. Moreover, by what was proved above, $\{(u^\varepsilon(t))_x\}$ converges in $L_{loc}^1(R)$ to $(S(t)v)_x$ as $\varepsilon\downarrow 0$ for every $t\geq 0$ so that, by the bounded convergence theorem, we find that $(u^\varepsilon(t))_x\to$

$(S(t)v)_x$ in $L^1_{\mathrm{loc}}(R\times[0, T])$ as $\varepsilon\downarrow 0$. Therefore we can find a subsequence $\{\varepsilon(i)\}$ such that $\{(u^{\varepsilon(i)}(t))_x\}$ converges *a.e.* and in $L^1_{\mathrm{loc}}(R\times[0, T])$ to $(S(t)v)_x$ as $\varepsilon(i)\downarrow 0$. Multiply (4.6) by $\varphi\in C_0^\infty(R\times(0, T))$ and integrate over $R\times[0, T]$. Integrating by parts and letting $\varepsilon\downarrow 0$ through the subsequence $\{\varepsilon(i)\}$ yield

$$\int_0^T\int_R\{-(S(t)v)\varphi_t+f((S(t)v)_x)\varphi\}dxdt=0,$$

which can be rewritten as

$$\int_0^T\int_R\{(S(t)v)_t+f((S(t)v)_x)\}\varphi dxdt=0.$$

But this implies that $S(t)v(x)$ satisfies (DE) *a.e.* on $R\times(0, T)$. Since $T>0$ is arbitrary, $S(t)v(x)$ satisfies (DE) *a.e.* on $R\times(0, \infty)$, which in turn shows that $S(t)v(x)$ is Lipschitz continuous on $R\times[0, \infty)$ by Theorem 4.2 (i). The proof of (ii) is complete.

It remains to prove (iii). To do so, we shall follow the proof of [2, Theorem 1.2 (ii)]. By Definition (2.1),

$$\tag{4.7} \begin{aligned}\int_R\{&\mathrm{sign}_0(u^\varepsilon_x(x, t)-k)[f(u^\varepsilon_x(x, t))-f(k)]\varphi_x(x, t)\\ &+\varepsilon^{-1}(u^\varepsilon_x(x, t-\varepsilon)-u^\varepsilon_x(x, t))\mathrm{sign}_0(u^\varepsilon_x(x, t)-k)\varphi(x, t)\}dx\geq 0\end{aligned}$$

for every $\varphi(x, t)\in C_0^\infty(R\times(0, T))$ such that $\varphi\geq 0$ and every $k\in R$. Let $h^\varepsilon(x, t)=(u^\varepsilon_x(x, t)-k)\mathrm{sign}_0(u^\varepsilon_x(x, t)-k)$. Integrating (4.7) over $0\leq t\leq T$ we have

$$\tag{4.8} \begin{aligned}\int_0^T\int_R\{&\mathrm{sign}_0(u^\varepsilon_x(x, t)-k)[f(u^\varepsilon_x(x, t))-f(k)]\varphi_x(x, t)\\ &+\varepsilon^{-1}(h^\varepsilon(x, t-\varepsilon)-h^\varepsilon(x, t))\varphi(x, t)\}dxdt\geq 0,\end{aligned}$$

since

$$(u^\varepsilon_x(x, t-\varepsilon)-u^\varepsilon_x(x, t))\mathrm{sign}_0(u^\varepsilon_x(x, t)-k)\leq h^\varepsilon(x, t-\varepsilon)-h^\varepsilon(x, t).$$

The second integral tends, as is easily verified, to

$$\int_0^T\int_R|(S(t)v(x))_x-k|\varphi_t(x, t)dxdt$$

as $\varepsilon\downarrow 0$, since $u^\varepsilon_x(x, t)\to(S(t)v(x))_x$ in $L^1_{\mathrm{loc}}(R\times[0, T])$. Hence the inequality (iii) is obtained by letting $\varepsilon\downarrow 0$ in (4.8). The proof of Theorem 4.3 has been completed.

REMARK 4.2. If $v\in W_1^\infty(R)$, we have seen above that $S(t)v(x)$ is Lipschitz continuous on $R\times[0, T]$ for every $T>0$. Thus $S(t)v(x)$ is (totally) differentiable *a.e.* on $R\times(0, \infty)$. It is most probable that the semigroup $S(t)$ provides a generalized solution $S(t)v$ of (CP) for $v\in W_1^\infty(R)$. However, we have not as yet succeeded in proving this conjecture without assuming the continuity of v_x in the L^1-norm on R.

References

[1] M. M. Belova, On bounded solutions of nonlinear differential equations of second order, Mat. Sbornik, **56** (98) (1962), 469–503.

[2] M. G. Crandall, The semigroup approach to first order quasilinear equations in several space variables, Israel J. Math., **12** (1972), 108–132.

[3] M. G. Crandall and T. M. Liggett, Generation of semigroups of nonlinear transformations on general Banach spaces, Amer. J. Math., **93** (1971), 265–298.

[4] M. G. Crandall and A. Pazy, Nonlinear evolution equations in Banach spaces, Israel J. Math., **11** (1972), 57–94.

[5] Y. Konishi, On $u_t=u_{xx}-F(u_x)$ and the differentiability of the nonlinear semigroup associated with it, Proc. Japan Acad., **48** (1972), 281–286.

[6] S. N. Kružkov, Generalized solutions of the Cauchy problem in the large for nonlinear equations of first order, Dokl. Akad. Nauk SSSR, **187** (1969), 29–32.

[7] ——, First order quasilinear equations in several independent variables, Mat. Sbornik, **81** (123) (1970), 228–255.

[8] T. Kusano, On bounded solutions of elliptic partial differential equations of the second order, Funkcial. Ekvac., **7** (1965), 1–13.

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