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On the Green Function of a Self-adjoint Harmonic Space

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Introduction

In [4] and [5], the author considered a self-adjoint harmonic space and defined a notion of energy for functions on such a space. By definition, a self-adjoint harmonic space is a harmonic space which possesses a symmetric Green function $G(x, y)$. We showed in [5] that, in order that energies of functions be always non-negative, it is necessary and sufficient that $G(x, y)$ be of positive type. There we assumed this property as an additional axiom (Axiom 7). The purpose of this paper is to show that this property does follow from our assumptions on the harmonic space, namely Axioms $1 \sim 6$ in [4] and the assumption that the space is self-adjoint, so that it is not necessary to introduce Axiom 7 in [5].

§ 1. Preliminaries.

Let $(\Omega, \mathfrak{H})$ be the harmonic space considered in [4], i.e., Ω is a connected, locally connected, locally compact, non-compact Hausdorff space with a countable base and $\mathfrak{H}$ is a structure of harmonic space on Ω satisfying Axioms 1, 2, 3 of M. Brelot [2] and the following additional three axioms:

Axiom 4. The constant function 1 is superharmonic.

Axiom 5. There exists a positive potential on Ω.

Axiom 6. Two positive potentials with the same point (harmonic) support are proportional.

Furthermore we assume that $(\Omega, \mathfrak{H})$ is self-adjoint, that is, there is a function $G(x, y)\colon \Omega \times \Omega \to (0, +\infty]$ such that $G(x, y) = G(y, x)$ for all $x, y \in \Omega$ and, for each $y \in \Omega$, $x \to G(x, y)$ is a potential on Ω and is harmonic on $\Omega - \{y\}$. We call such a function $G(x, y)$, which is uniquely determined up to a multiplicative constant, a *Green function* for $(\Omega, \mathfrak{H})$ and fix it throughout.

For a non-negative measure (=a Radon measure, or a regular Borel measure) μ on Ω, we write

$$U^{\mu}(x) = \int_{\Omega} G(x, y)d\mu(y) \quad \text{and} \quad I(\mu) = \int U^{\mu} d\mu.$$

Let

$$\mathbf{M}_E = \{\mu;\ \text{non-negative measure on } \Omega \text{ such that } I(\mu) < +\infty\}.$$

The Green function $G(x, y)$ is of positive type if and only if

$$(1) \qquad 2\int U^{\mu}d\nu \leqq I(\mu)+I(\nu)$$

for all μ, $\nu \in \mathbf{M}_E$.

We know ([5; Lemmas 4.1 and 4.3]):

LEMMA 1. *Given $\mu \in \mathbf{M}_E$, there is a sequence $\{\mu_n\}$ in $\mathbf{M}_E$ such that each μ_n has compact support in Ω, each U^{μ_n} is bounded continuous and $U^{\mu_n} \uparrow U^{\mu}$ as $n \to \infty$.*

LEMMA 2. *If μ_n, ν_n, μ, $\nu \in \mathbf{M}_E$ $(n=1, 2,\dots)$, $U^{\mu_n} \uparrow U^{\mu}$ and $U^{\nu_n} \uparrow U^{\nu}$, then $\int U^{\mu_n}d\nu_n \uparrow \int U^{\mu}d\nu$, $I(\mu_n) \uparrow I(\mu)$ and $I(\nu_n) \uparrow I(\nu)$.*

We shall denote by $\mathbf{B}$ the space of all bounded Borel measurable functions on Ω and by $\mathbf{C}$ the space of all bounded continuous functions on Ω. The space $\mathbf{C}$ is a Banach space with respect to the sup-norm: $\|f\| = \sup_{x\in\Omega}|f(x)|$. The space of all bounded linear operators on $\mathbf{C}$ is denoted by $\mathscr{L}(\mathbf{C})$. It is also a Banach space with respect to the norm: $\|T\| = \sup_{\|f\|\leqq 1}\|Tf\|$.

§ 2. A potential operator and its square root.

In this section, let μ be a non-negative measure on Ω such that $U^{\mu} \in \mathbf{C}$. It is easy to see that, for any $f \in \mathbf{B}$, the function $G_{\mu}f$ defined by

$$(G_{\mu}f)(x) = \int_{\Omega} G(x, y)f(y)d\mu(y)$$

belongs to $\mathbf{C}$. Furthermore, $G_{\mu} \in \mathscr{L}(\mathbf{C})$; in fact $\|G_{\mu}\| \leqq \|U^{\mu}\|$.

LEMMA 3. *There exists a family $\{R_{\mu,p}\}_{p\geqq 0}$ of operators in $\mathscr{L}(\mathbf{C})$ which has the following properties:*

(a) $R_{\mu,0} = G_{\mu}$;

(b) *For any p, $q \geqq 0$, the resolvent equation*

$$R_{\mu,p} - R_{\mu,q} = (q-p)R_{\mu,p}R_{\mu,q}$$

holds;

(c) $\|R_{\mu,p}\| \leqq \|G_{\mu}\|$ *for all* $p \geqq 0$;

(d) $p\|R_{\mu,p}\| \leqq 1$ *for all* $p \geqq 0$.

The proof of this lemma may be carried out as in [6; § 3]. We remark that the proof there is based on the following "weak principle of positive maximum": If $f \in \mathbf{C}$ and $\omega = \{x \in \Omega;\ f(x) > 0\}$ is non-empty, then

$$\max(0, \sup_{x\in\Omega} G_\mu f(x)) = \max(0, \sup_{x\in\omega} G_\mu f(x)).$$

In [6], it is assumed that constant functions are harmonic; its proof is still valid under our Axiom 4.

On account of the above lemma,

$$Q_\mu = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{p}} R_{\mu,p}\, dp$$

is defined as an element of $\mathscr{L}(\mathbf{C})$, the integral being norm-convergent. This operator Q_μ is in fact the fractional power of G_μ of order 1/2 as defined in [1; § 1] (also, cf. [3; § 5]). Thus, by considering fractional powers of G_μ of order α $(0<\alpha<1)$, we can prove the following proposition as in [1; Lemmas 2.3 and 2.5]:

PROPOSITION. $Q_\mu^2 = G_\mu$.

Here, we give a direct proof of this proposition without the aid of fractional powers of general order.

PROOF of PROPOSITION. First we observe

$$\int_0^\infty \frac{1}{t(1+t)} \log(\sqrt{t}+\sqrt{1+t})\,dt = \frac{\pi^2}{4}, \tag{2}$$

which can be obtained from an elementary calculus.

For simplicity, let $G=G_\mu$, $Q=Q_\mu$ and $R_p=R_{\mu,p}$. By using (b) of Lemma 3, we have

$$\begin{aligned} Q^2 &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \frac{1}{\sqrt{pq}} R_p R_q\, dp\, dq \\ &= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty \frac{R_p - R_q}{\sqrt{pq}(q-p)}\, dp\, dq \\ &= \frac{2}{\pi^2}\lim_{\varepsilon\downarrow 0}\int_{D\varepsilon} \frac{R_p}{\sqrt{pq}(q-p)}\, dp\, dq, \end{aligned}$$

where $D_\varepsilon = \{(p, q);\ |p-q| \geqq \varepsilon\}$ for $\varepsilon>0$. By integrating with respect to q and then by making the change of variable $p=\varepsilon t$, we deduce

$$\begin{aligned} Q^2 &= \frac{4}{\pi^2}\lim_{\varepsilon\downarrow 0}\left\{\int_0^\infty \frac{R_{\varepsilon t}}{t}\log(\sqrt{t}+\sqrt{1+t})\,dt - \int_1^\infty \frac{R_{\varepsilon t}}{t}\log(\sqrt{t}+\sqrt{t-1})\,dt\right\} \\ &= \frac{4}{\pi^2}\lim_{\varepsilon\downarrow 0}\int_0^\infty \left\{\frac{R_{\varepsilon t}}{t} - \frac{R_{\varepsilon(1+t)}}{1+t}\right\}\log(\sqrt{t}+\sqrt{1+t})\,dt. \end{aligned}$$

Using (b) of Lemma 3 again and remarking that $\|R_{\varepsilon t}R_{\varepsilon(1+t)}\| \leq \|G\|^2$ and $\leqq \|G\|/\varepsilon t$ for all $t>0$ (Lemma 3, (c) and (d)), we obtain

$$Q^2 = \frac{4}{\pi^2} \lim_{\varepsilon \downarrow 0} \int_0^\infty \frac{R_{\varepsilon t}}{t(1+t)} \log(\sqrt{t} + \sqrt{1+t}) dt.$$

Thus, by (2) and Lemma 3,

$$\begin{aligned} G - Q^2 &= \frac{4}{\pi^2} \lim_{\varepsilon \downarrow 0} \int_0^\infty \frac{R_0 - R_{\varepsilon t}}{t(1+t)} \log(\sqrt{t} + \sqrt{1+t}) dt \\ &= \frac{4}{\pi^2} \lim_{\varepsilon \downarrow 0} \varepsilon \int_0^\infty \frac{t R_0 R_{\varepsilon t}}{t(1+t)} \log(\sqrt{t} + \sqrt{1+t}) dt = 0. \end{aligned}$$

§ 3. Theorems.

Now we are ready to prove what we have aimed at:

THEOREM 1. *The Green function $G(x, y)$ is a kernel of positive type.*

PROOF. Let $\mathbf{M}_C = \{\mu \in \mathbf{M}_E$; the support of μ is compact and $U^\mu \in \mathbf{C}\}$. Since $G(x, y)$ is symmetric, we see that

$$\int (G_\mu f) g \, d\mu = \int f(G_\mu g) \, d\mu \tag{3}$$

for all $\mu \in \mathbf{M}_E$ and $f, g \in \mathbf{B}$. We shall first show that

$$\int (R_{\mu,p} f) g \, d\mu = \int f(R_{\mu,p} g) \, d\mu \tag{4}$$

for all $\mu \in \mathbf{M}_C$, $f, g \in \mathbf{C}$ and $p \geqq 0$. Fix μ, f and g for a moment and let P be the set of numbers $p \geqq 0$ for which (4) is valid. If $p_0 \in P$, then, for any $\varepsilon \in [0, 1/\|G_\mu\|)$,

$$R_{\mu, p_0 + \varepsilon} = \sum_{n=0}^{\infty} (-1)^n \varepsilon^n (R_{\mu, p_0})^{n+1},$$

where the series on the right is norm-convergent in $\mathscr{L}(\mathbf{C})$. Hence the equality (4) for $p = p_0$ implies that for $p = p_0 + \varepsilon$ whenever $\varepsilon \in [0, 1/\|G_\mu\|)$. Since $0 \in P$ by (3), it follows that $P = [0, +\infty)$, so that (4) is valid for all $p \geqq 0$. Then, by the definition of Q_μ, (4) implies that

$$\int (Q_\mu f) g \, d\mu = \int f(Q_\mu g) \, d\mu \tag{5}$$

for all $\mu \in \mathbf{M}_C$ and $f, g \in \mathbf{C}$. Therefore, by the above proposition,

$$\int (G_\mu f) f \, d\mu = \int (Q_\mu(Q_\mu f)) f \, d\mu = \int (Q_\mu f)^2 d\mu \geqq 0$$

for all $\mu \in \mathbf{M}_C$ and $f \in \mathbf{C}$. It then follows that

(6) $$\int (G_\mu f) f \, d\mu \geqq 0$$

for all $\mu \in \mathbf{M}_C$ and $f \in \mathbf{B}$. If μ, $\nu \in \mathbf{M}_C$, then $\mu + \nu \in \mathbf{M}_C$ and there is $f \in \mathbf{B}$ such that $d\mu - d\nu = f d(\mu + \nu)$. Hence, by (6)

$$\int (U^\mu - U^\nu)(d\mu - d\nu) = \int (G_{\mu+\nu} f) f \, d(\mu + \nu) \geqq 0.$$

Therefore, (1) holds for any μ, $\nu \in \mathbf{M}_C$. Now, given μ, $\nu \in \mathbf{M}_E$, Lemma 1 asserts that there are $\{\mu_n\}$ and $\{\nu_n\}$ in $\mathbf{M}_C$ such that $U^{\mu_n} \uparrow U^\mu$ and $U^{\nu_n} \uparrow U^\nu$. Since (1) holds for $\mu = \mu_n$ and $\nu = \nu_n$, it follows from Lemma 2 that (1) holds for the given μ, $\nu \in \mathbf{M}_E$. Thus the theorem is proved.

REMARK.*) This theorem may be proved without using the Proposition above. Namely, by Lemma 3, we see that $dR_{\mu,p}/dp = -R_{\mu,p}^2$ for each $p > 0$, so that $G_\mu = \int_0^\infty R_{\mu,p}^2 dp$. Then, by (4) in the above proof, we have

$$\int (G_\mu f) f d\mu = \int_0^\infty \left\{ \int (R_{\mu,p}^2 f) f d\mu \right\} dp = \int_0^\infty \left\{ \int (R_{\mu,p} f)^2 d\mu \right\} dp \geqq 0$$

for all $\mu \in \mathbf{M}_C$ and $f \in \mathbf{C}$. The rest of the proof is the same as above.

Now the following assertions are consequences of [5; Theorem 4.1]:

THEOREM 2. (i) *The Green function $G(x, y)$ satisfies the energy principle*; (ii) *The following domination principle holds for $(\Omega, \mathfrak{H})$: If p is a potential on Ω which is locally bounded on its (harmonic) support $\sigma(p)$ and if s is a non-negative superharmonic function such that $s \geqq p$ on $\sigma(p)$, then $s \geqq p$ on Ω*; (iii) *The continuity principle holds for $(\Omega, \mathfrak{H})$, i.e., if s is a superharmonic function on Ω such that $s|\sigma(s)$ is finite continuous, then s is continuous everywhere on Ω.*

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