HIROSHIMA MATH. J.
3 (1973), 333-360

Oscillation and Asymptotic Behavior of Solutions of Retarded Differential Equations of Arbitrary Order

Hiroshi ONOSE

(Received April 28, 1973)

1. Introduction

We are here concerned with the oscillatory behavior of solutions of higher-order retarded differential equations of the form

(A) $$y^{(n)}(x)+y(g(x))F([y(g(x))]^2, x)=0, \qquad n\geqq 2,$$

where the following conditions are always assumed to hold:

(a) $g(x)$ is continuous for $x>0$, $g(x)\leqq x$ and $\lim_{x\to\infty} g(x)=\infty$;

(b) $yF(y^2, x)$ is continuous for $x>0$ and $|y|<\infty$, and $F(t, x)$ is nonnegative for $t\geqq 0$ and $x>0$.

Equation (A) is classified according to the nonlinearity of $F(t, x)$ with respect to t, namely (A) is called *superlinear* if F satisfies

(1.1) $$F(t_1, x)\leqq F(t_2, x), \quad t_1<t_2, \quad x\in(0, \infty),$$

and *sublinear* if F satisfies

(1.2) $$F(t_1, x)\geqq F(t_2, x), \quad t_1<t_2, \quad x\in(0, \infty).$$

Moreover, (A) is called *strongly superlinear* if there is an $\varepsilon>0$ such that

(1.3) $$t_1^{-\varepsilon}F(t_1, x)\leqq t_2^{-\varepsilon}F(t_2, x), \quad t_1<t_2, \quad x\in(0, \infty),$$

and *strongly sublinear* if there is an $\varepsilon>0$ such that

(1.4) $$t_1^{\varepsilon}F(t_1, x)\geqq t_2^{\varepsilon}F(t_2, x), \quad t_1<t_2, \quad x\in(0, \infty).$$

(See e.g. Nehari [29], Coffman and Wong [8].) The prototype of equation (A) is

(B) $$y^{(n)}(x)+p(x)|y(g(x))|^{\alpha}\operatorname{sgn} y(g(x))=0,$$

where $p(x)\geqq 0$ for $x>0$ and $\alpha>0$, which may be considered as a generalization of the Emden-Fowler equation. Equation (B) is superlinear, strongly superlinear, sublinear or strongly sublinear according as $\alpha\geqq 1$, $\alpha>1$, $\alpha\leqq 1$ or $\alpha<1$.

It will be tacitly assumed that under the initial condition

$$y(x)=\phi(x), \quad x \leqq x_0, \quad \text{and } y^{(j)}(x_0)=y_j^0, \qquad j=1,\dots, n-1,$$

equation (A) has a solution which can be extended to the interval $[x_0, \infty)$. A nontrivial solution $y(x)$ of (A) is said to be *oscillatory* if there exists a sequence $\{x_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_k=\infty$ and $y(x_k)=0$ for all k. Otherwise, a solution is said to be *nonoscillatory*; thus a nonoscillatory solution has to be of constant sign for all large x. A nonoscillatory solution is called strongly monotone if it tends monotonically to zero as $x\to\infty$ together with its first $n-1$ derivatives.

The problem of oscillation and nonoscillation of solutions of retarded differential equations is of great importance both in theory and in applications, and has drawn increasing attention in the last few years. Among numerous papers dealing with this problem (and accessible to the present author), we refer in particular to [1–5, 9, 17, 23, 32, 36, 41–43, 45, 48, 49] concerning second order oscillations, and to [10, 18–22, 24, 26–28, 30, 33–35, 37–40] concerning oscillations of higher order.

The purpose of this paper is to present a number of results concerning the oscillation and asymptotic behavior of solutions of equation (A). In Section 2 we prove two theorems on the asymptotic behavior of solutions of (A). In Section 3 we prove oscillation theorems for (A) which give conditions that all solutions of (A) be oscillatory in the case n is even, and be either oscillatory or strongly monotone in the case n is odd. In Section 4 we study the problem of maintaining the oscillations of all solutions of (A) under the effect of a forcing term.

Our results include as special cases oscillation and nonoscillation theorems not only for the retarded differential equation (B) but also for the ordinary differential equation

$$\text{(C)} \qquad y^{(n)}(x)+y(x)F([y(x)]^2, x)=0,$$

which has been the subject of investigations initiated by Nehari [29] and continued by Kiguradze [14], Coffman and Wong [6–8] and others. It seems to the author that most of the results obtained here are new even in the reduced case of equation (C).

2. Asymptotic Behavior

In this section we shall study the asymptotic behavior for $x\to\infty$ of solutions of the retarded differential equation (A). Our basic results give necessary and sufficient conditions for equation (A) to have a nonoscillatory solution which is asymptotic to a nonzero constant as $x\to\infty$, and to have a nonoscillatory solution which is asymptotic to $bx^{n-1}(b\neq 0)$ as $x\to\infty$. The special case $n=2$ and $g(x)\equiv x$ was discussed by Nehari [29] and Coffman and Wong [8].

THEOREM 2.1 *Let equation* (A) *be either superlinear or sublinear. Then, a necessary and sufficient condition in order that*:

(i) *for n even, there exists a bounded nonoscillatory solution of* (A),

(ii) *for n odd, there exists a bounded nonoscillatory solution of* (A) *with* $\lim_{x\to\infty} y(x)=a\neq 0$,

is that

$$\int^{\infty} x^{n-1}F(c^2, x)dx<\infty \qquad \textit{for some } c>0. \tag{2.1}$$

PROOF. (*Necessity*). Our proof has basic elements in common with those of Kartsatos [11] and Ladas [22].

Let $y(x)$ be a bounded nonoscillatory solution of (A). Assume that $y(x)>0$ for $x\geqq x_0$. The case $y(x)<0$ can be treated similarly. Since $\lim_{x\to\infty} g(x)=\infty$, there exists an $x_1\geqq x_0$ such that $g(x)\geqq x_0$ and so $y(g(x))>0$ for $x\geqq x_1$. Therefore, by (A), we have $y^{(n)}(x)\leqq 0$ for $x\geqq x_1$. Since $y(x)$ is bounded and positive, $y^{(k)}(x)y^{(k+1)}(x)<0$ for $x\geqq x_1$ and for $k=1, 2,\dots, n-1$. It follows that

$$(-1)^{k+1}y^{(n-k)}(x)>0, \quad x\geqq x_1, \qquad k=0, 1,\dots, n-1. \tag{2.2}$$

Since $y'(x)$ is of constant sign for $x\geqq x_1$, it follows that the limit $\lim_{x\to\infty} y(x)=y(\infty)$ exists and is finite. If n is even, then $y'(x)>0$ by (2.2), so that $y(\infty)>0$. If n is odd, then $y'(x)<0$ by (2.2), so that either $y(\infty)>0$ or $y(\infty)=0$; the latter possibility is excluded by (ii). So we assume that $y(\infty)>0$ for n even or odd. Since $\lim_{x\to\infty} y(g(x))=y(\infty)$ and $y(\infty)$ is finite, for some positive constant $c>0$, there exists an $x_2\geqq x_1$ such that

$$\frac{c}{2}\leqq y(g(x))\leqq c \text{ for all } x\geqq x_2. \tag{2.3}$$

Let (A) be sublinear. Then, from (1.2), (2.3) and (A), we have

$$y^{(n)}(x)+\frac{c}{2}F(c^2, x)\leqq 0. \tag{2.4}$$

Multiplying both sides of (2.4) by x^{n-1} and integrating from x_2 to x, we obtain

$$\int_{x_2}^{x} s^{n-1}y^{(n)}(s)ds+\frac{c}{2}\int_{x_2}^{x} s^{n-1}F(c^2, s)ds\leqq 0. \tag{2.5}$$

Successive integration by parts of the first integral in (2.5) gives

$$\int_{x_2}^{x} s^{n-1}y^{(n)}(s)ds = P(x)-P(x_2)+(-1)^{n+1}n![y(x)-y(x_2)], \tag{2.6}$$

where

$$P(x)=\sum_{k=1}^{n-1}(-1)^{k+1}(n-1)(n-2)\dots(n-k+1)x^{n-k}y^{(n-k)}(x)$$

which is positive because of (2.2). Since $y(x)$ is a bounded function, from (2.5) and (2.6), we conclude that

$$\frac{c}{2}\int_{x_2}^{\infty} x^{n-1}F(c^2, x)dx<\infty,$$

which implies (2.1).

Let (A) be superlinear. Then, from (1.1), (2.3) and (A), we have

$$y^{(n)}(x)+\frac{c}{2}F\left(\frac{c^2}{4}, x\right)\leqq 0,$$

from which, exactly as in the sublinear case, we conclude that

$$\frac{c}{2}\int_{x_2}^{\infty} x^{n-1}F\left(\frac{c^2}{4}, x\right)dx<\infty,$$

which again implies (2.1). This proves the necessity part of the theorem.

(*Sufficiency*). The proof is based on the arguments developed by Staikos and Sficas [38]. The main tool is the following fixed point theorem which is a special case of Tychonoff's theorem [46].

Fixed Point Theorem. *Let Z be a Fréchet space and Y be a convex and closed subset of Z. If S is a continouus mapping of Y into itself and the closure $\overline{SY}$ is a compact subset of Y, then there exists at least one fixed point $y\in Y$ of S, i.e. $y=Sy$.*

Now, suppose that condition (2.1) holds and we construct a solution $y(x)$ of (A) which satisfies $\lim_{x\to\infty} y(x)=a$, where a is some nonzero constant. To this end, it suffices to prove the existence of a continuous solution of the integral equation

$$y(x)=a+\frac{(-1)^{n-1}}{(n-1)!}\int_{x}^{\infty}(s-x)^{n-1}y(g(s))F([y(g(s))]^2, s)ds. \tag{2.7}$$

Sublinear Case: Suppose that (A) is sublinear. Let $a>c$ be arbitrary but fixed and put $\delta=a-c$. We choose T so large that

$$(2a-c)\int_{T}^{\infty}(s-T)^{m-1-n}F(c^2, s)ds\leqq\delta, \tag{2.8}$$

for every $m=0, 1,\dots, n-1$, and put $\tau=\min_{x\geqq T} g(x)$. Let Z be the vector space of all continuous real-valued functions on $[\tau, \infty)$ which are constants on $[\tau, T]$ and $n-1$ times continuously differentiable on $[T, \infty)$. The space Z endowed with the topology $\mathscr{T}$ given by the family of seminorms $\{p_\alpha:\ \alpha\in(T, \infty)\}$:

$$p_\alpha(z)=\max_{x\in[T,\alpha]}|z^{(n-1)}(x)|+\sum_{k=0}^{n-2}|z^{(k)}(T)|$$

becomes a Fréchet space.

Let us now consider the set Y of all $y\in Z$ with

(I) $|y(x)-a|\leqq\delta$ for all $x\geqq\tau$,

(II) $|y^{(m)}(x)|\leqq\delta$ for all $x\geqq T$ $(m=1, 2,\dots, n-1)$.

Clearly, Y is a nonvoid convex and closed subset of the space Z.

We define a mapping S: $Y\to Z$ by the formula

$$z(x)=(Sy)(x)=\begin{cases} a+\dfrac{(-1)^{n-1}}{(n-1)!}\displaystyle\int_x^\infty (s-x)^{n-1}\hat{y}(s)F([\hat{y}(s)]^2, s)ds & \text{if } x\geqq T\\[2ex] a+\dfrac{(-1)^{n-1}}{(n-1)!}\displaystyle\int_T^\infty (s-T)^{n-1}\hat{y}(s)F([\hat{y}(s)]^2, s)ds & \text{if } \tau\leqq x\leqq T,\end{cases}$$

where we have used the notation

$$\hat{y}(x)=\begin{cases} y(g(x)) & \text{if } g(x)\geqq T\\ y(T) & \text{if } g(x)\leqq T.\end{cases}$$

Since (A) is sublinear, we have for any $y\in Y$

$$|\hat{y}(x)F([\hat{y}(x)]^2, x)|\leqq(a+\delta)F((a-\delta)^2, x)\leqq(2a-c)F(c^2, x),\quad x\geqq T, \tag{2.9}$$

and thus, the mapping S is well defined.

We shall show that S has the properties as required in the fixed point theorem.

(a) *S maps Y into Y.* In fact, by (2.8) and (2.9), we have for $z=Sy$ and $x\geqq T$

$$|z(x)-a|\leqq\frac{2a-c}{(n-1)!}\int_T^\infty(s-T)^{n-1}F(c^2, s)ds\leqq\delta$$

and

$$|z^{(m)}(x)|=\frac{(n-1)(n-2)\dots(n-m)}{(n-1)!}\left|\int_x^\infty(s-x)^{n-1-m}\hat{y}(s)F([\hat{y}(s)]^2, s)ds\right|$$

$$\leqq\frac{(n-1)(n-2)\dots(n-m)}{(n-1)!}(2a-c)\int_T^\infty(s-T)^{n-1-m}F(c^2, s)ds\leqq\delta.$$

(b) *$\overline{SY}$ is a compact subset of Y.* It follows from the definition of S that

$$|z^{(n-1)}(x_1)-z^{(n-1)}(x_2)|\leqq\left|\int_{x_1}^{x_2}\hat{y}(s)F([\hat{y}(s)]^2, s)ds\right|\leqq(2a-c)\left|\int_{x_1}^{x_2}F(c^2, s)ds\right|$$

for any $y \in Y$, which shows that the $(n-1)$th derivatives of the functions $z \in SY$ are equicontinuous at each point of the interval of $[T, \infty)$. Hence, by the Arzelà-Ascoli theorem, any sequence $\{z_k\}$ in $\overline{SY}$ contains a subsequence $\{w_k\}$ such that the sequence $\{w_k^{(n-1)}\}$ converges uniformly on every compact subinterval of $[T, \infty)$. Since, by (I) and (II), the sequences $\{w_k^{(m)}(T)\}$, $m=0, 1,\dots, n-2$ are all bounded, there exists a subsequence $\{v_k\}$ of $\{w_k\}$ for which every sequence $\{v_k^{(m)}(T)\}$, $m=0, 1,\dots, n-2$, is convergent. Thus, we conclude that

$$\mathscr{T}-\lim_{k\to\infty} v_k = v \in Y.$$

(c) *S is continuous.* Let $\{y_k\}$ be a convergent sequence in Y, i.e. $\mathscr{T}-\lim_{k\to\infty} y_k = y$, $y \in Y$. Then, for every $m=1, 2,\dots, n-1$ and $x \geqq T$, we have

$$z_k^{(m)}(x) = (-1)^{n-1-m} \frac{(n-1)(n-2)\dots(n-m)}{(n-1)!} \cdot \int_x^\infty (s-x)^{n-1-m} \hat{y}_k(s) F([\hat{y}_k(s)]^2, s) ds$$

where $z_k = Sy_k$. It is easy to see that

$$\lim_{k\to\infty} \hat{y}_k(s) = \hat{y}(s) \qquad \text{for all } s \geqq T,$$

and consequently, by the continuity of $yF(y^2, x)$, it holds that

$$\lim_{k\to\infty} \hat{y}_k(s) F([\hat{y}_k(s)]^2, s) = \hat{y}(s) F([\hat{y}(s)]^2, s), \qquad s \geqq T.$$

On the other hand, we have by (2.9)

$$|(s-x)^{n-1-m} \hat{y}_k(s) F([\hat{y}_k(s)]^2, s)| \leqq (2a-c)(s-x)^{n-1-m} F(c^2, s),$$

so that we can apply the Lebesgue dominaged convergence theorem to obtain

$$\lim_{k\to\infty} z_k^{(m)}(x) = z^{(m)}(x) \tag{2.10}$$

for every $m=0, 1,\dots, n-1$ and $x \geqq T$, where $z=Sy$. It is easily verified that for $m=n-1$ in (2.10) the convergence is uniform on every compact subinterval of $[T, \infty)$ and hence for all $m=0, 1,\dots, n-1$. Thus, we conclude that

$$\mathscr{T}-\lim_{k\to\infty} z_k = z.$$

We can now apply the fixed point theorem to conclude that the mapping S has a fixed point $y \in Y$, which is obviously a solution of the integral equation (2.7) for all sufficiently large x, and hence asymptotic to a as $x \to \infty$. Thus the required solution of (A) is obtained for the sublinear case.

Superlinear Case: Let (A) be superlinear. In this case, we take a so that $0<a<c$, put $\delta \equiv c-a$, choose T so that

$$c\int_T^\infty (s-T)^{n-1-m}F(c^2, s)ds \leqq \delta$$

for $m=0, 1, \ldots, n-1$, and then proceed exactly as in the sublinear case, except that we use instead of (2.9) the following inequality

$$|\hat{y}(x)F([\hat{y}(x)]^2, x)| \leqq cF(c^2, x), \qquad x \geqq T,$$

which holds for all $y \in Y$. Then we can obtain a solution of (A) with the desired property. This completes the proof of the sufficiency part of the theorem.

REMARK. From the above proof we see that the statements (i) and (ii) of Theorem 2.1 together are equivalent to saying that equation (A) has a nonoscillatory solution which is asymptotic to a nonzero constant as $x \to \infty$.

COROLLARY 2.1. *Let $\alpha > 0$. Assume that $p(x) \geqq 0$. Then, equation* (B) *has a nonoscillatory solution which is asymptotic to a nonzero constant as $x \to \infty$ if and only if*

$$\int^\infty x^{n-1}p(x)dx < \infty.$$

COROLLARY 2.2. *Equation* (C), *superlinear or sublinear, has a nonoscillatory solution which is asymptotic to a nonzero constant as $x \to \infty$ if and only if* (2.1) *holds.*

REMARK. In the case $n=2$, Corollary 2.1 was proved by Wong [49, Theorem (2.1)] and Corollary 2.2 by Nehari [29, Theorem I] and Coffman and Wong [8, Theorem 1]. For other related results the reader is referred to Kiguradze [15], Burkowski [3], Staikos and Sficas [38] and others.

THEOREM 2.2. *Assume that*: *Either*

(i) *equation* (A) *is superlinear*; *or*

(ii) *equation* (A) *is sublinear and $g'(x) \geqq 0$.*

Then, a necessary and sufficient condition for (A) *to have an unbounded solution with the asymptotic property*

$$\lim_{x\to\infty} \frac{y(x)}{x^{n-1}} = b \neq 0 \tag{2.11}$$

is that

$$\int^\infty [g(x)]^{n-1}F(c^2[g(x)]^{2(n-1)}, x)dx < \infty \qquad \textit{for some } c>0. \tag{2.12}$$

PROOF. Our proof was suggested by Coffman and Wong [8].

(*Necessity*). Suppose that there exists an unbounded solution $y(x)$ of (A) satisfying (2.11). We may assume that $b>0$, because a parallel argument holds if $b<0$.

From (2.11) it follows that there exists an $x_0>0$ such that for $x\geqq x_0$

$$\frac{b}{2}<\frac{y(x)}{x^{n-1}}<2b. \tag{2.13}$$

Since $\lim_{x\to\infty} g(x)=\infty$, there exists an $x_1\geqq x_0$ such that $g(x)\geqq x_0$ for $x\geqq x_1$. In view of (A), we have $y^{(n)}(x)\leqq 0$ for $x\geqq x_1$, so $y^{(n-1)}(x)$ is nonicnreasing for $x\geqq x_1$. Hence, if $y^{(n-1)}(x_2)<0$ for some $x_2\geqq x_1$, then we have $y^{(n-1)}(x)\leqq y^{(n-1)}(x_2)<0$ for $x\geqq x_2$, which implies that $\lim_{x\to\infty} y(x)=-\infty$, a contradiction to (2.13). Thus, we conclude that $y^{(n-1)}(x)\geqq 0$ for $x\geqq x_1$. Integrating (A) from x_1 to x gives

$$y^{(n-1)}(x)=y^{(n-1)}(x_1)-\int_{x_1}^{x} y(g(s))F([y(g(s))]^2, s)ds. \tag{2.14}$$

Now, letting $x\to\infty$ in (2.14) and using $y^{(n-1)}(x)\geqq 0$, we obtain

$$\int_{x_1}^{\infty} y(g(x))F([y(g(x))]^2, x)dx<\infty. \tag{2.15}$$

If (A) is superlinear, then, using (1.1) and (2.13) in (2.15), we have

$$\frac{b}{2}\int_{x_1}^{\infty}[g(x)]^{n-1}F\Big(\frac{b^2}{4}[g(x)]^{2(n-1)}, x\Big)dx$$
$$\leqq\int_{x_1}^{\infty} y(g(x))F([y(g(x))]^2, x)dx<\infty,$$

proving (2.12).

If (A) is sublinear, then, using (1.2) and (2.13) in (2.15), we have

$$\frac{b}{2}\int_{x_1}^{\infty}[g(x)]^{n-1}F(4b^2[g(x)]^{2(n-1)}, x)dx$$
$$\leqq\int_{x_1}^{\infty} y(g(x))F([y(g(x))]^2, x)dx<\infty,$$

again proving (2.12).

(*Sufficiency*). We assume that (2.12) holds and show that there exists a solution of (A) with the property (2.11). Choose x_0 so large that

$$\int_{x_0}^{\infty}[g(x)]^{n-1}F(c^2[g(x)]^{2(n-1)}, x)dx<\frac{1}{4}. \tag{2.16}$$

Superlinear Case: Let equation (A) be superlinear. We construct a solution $y(x)$ of (A) satisfying the initial conditions

$$\begin{aligned} &y(x)\equiv 0, \quad x\leqq x_0 \\ (2.17)\qquad &y^{(k)}(x_0)=0, \quad k=1, 2,\dots, n-2, \\ &y^{(n-1)}(x_0)=c. \end{aligned}$$

By hypothesis this solution can be continued to $[x_0, \infty)$. We claim that $y^{(n-1)}(x) > c/2$ for $x\geqq x_0$. Otherwise, let x_1 be the point in (x_0, ∞) such that $y^{(n-1)}(x) > c/2$ for $x_0\leqq x<x_1$ and $y^{(n-1)}(x_1)=c/2$. Integrating the inequality $y^{(n-1)}(x) \geqq c/2$ $n-1$ times from x_0 to x and using (2.17), we have

$$y(x)\geqq \frac{c}{2(n-1)!}(x-x_0)^{n-1}, \quad x_0\leqq x\leqq x_1.$$

Therefore,

$$y(g(x))\geqq \frac{c}{2(n-1)!}[g(x)-x_0]^{n-1}, \quad x_0\leqq g(x)\leqq x_1,$$

$$y(g(x))=0, \quad g(x)\leqq x_0,$$

in particular, $y(g(x))\geqq 0$ for $x_0\leqq x\leqq x_1$.

Now, integrating (A) once from x_0 to x and using (2.17) and the fact that $y(g(x))\geqq 0$, we find

$$(2.18)\qquad y^{(n-1)}(x)=c-\int_{x_0}^{x} y(g(s))F([y(g(s))]^2, s)ds\leqq c, \qquad \text{for } x_0\leqq x\leqq x_1.$$

Integration of (2.18) $n-1$ times then gives

$$y(x)\leqq \frac{c}{(n-1)!}(x-x_0)^{n-1}, \quad x_0\leqq x\leqq x_1,$$

from which there follows:

$$y(g(x))\leqq \frac{c}{(n-1)!}[g(x)-x_0]^{n-1}, \quad x_0\leqq g(x)\leqq x_1,$$

$$y(g(x))=0, \quad g(x)\leqq x_0,$$

in particular,

$$(2.19)\qquad y(g(x))\leqq c[g(x)]^{n-1}, \qquad x\leqq x_1.$$

Using (1.1) and (2.19) in (2.18), and taking (2.16) into account, we obtain for $x_0 \leqq x\leqq x_1$

$$y^{(n-1)}(x)\geqq c\left\{1-\int_{x_0}^{x} [g(s)]^{n-1}F(c^2[g(s)]^{2(n-1)}, s)ds\right\} > \frac{3}{4}c,$$

which gives a contradiction to $y^{(n-1)}(x_1)=c/2$. Therefore, we must have $y^{(n-1)}(x) \geqq c/2$, $x \geqq x_0$. This inequality together with (2.17) shows that $y(x)$ is nonnegative for all $x \geqq x_0$. Observing that the integral in (2.18) is nonnegative, we see that the limit $\lim_{x\to\infty} y^{(n-1)}(x)=b$ exists and is a finite number: $c/2 \leqq b \leqq c$. Thus, we have shown that the solution $y(x)$ under consideration has the desired asymptotic property (2.11).

Sublinear Case: Now let equation (A) be sublinear. Take a constant c' such that $c'/(n-1)!>c$, put $c'(n)=c'/(n-1)!$ for simplicity, and define x'_0 by

$$x'_0=\frac{c'(n)^{1/(n-1)}x_0}{c'(n)^{1/(n-1)}-c^{1/(n-1)}}.$$

We remark that $x'_0 \to x_0$ as $c' \to \infty$. Let x_1 and x'_1 be such that $g(x_1)=x_0$ and $g(x'_1)=x'_0$. Since $g'(x) \geqq 0$ and $\lim_{x\to\infty} g(x)=\infty$, we may assume without loss of generality that $g'(x)>0$ in a neighborhood J of x_1. Choose c' sufficiently large that $x'_1 \in J$. Observe that

$$c[g(x)]^{n-1} \leqq c'(n)[g(x)-x_0]^{n-1}, \tag{2.20}$$

if $x \geqq x'_1$, i.e. $g(x) \geqq x'_0$.

We show that the constant c' can be chosen such that

$$\int_{x_1}^{\infty} \frac{[g(x)-x_0]^{n-1}}{(n-1)!} F(c'(n)^2[g(x)-x_0]^{2(n-1)},\ x)dx \leqq \frac{1}{4}. \tag{2.21}$$

In fact, using (1.2) and (2.20), we obtain

$$\begin{aligned}
&\int_{x_1}^{x} \frac{[g(s)-x_0]^{n-1}}{(n-1)!} F(c'(n)^2[g(s)-x_0]^{2(n-1)},\ s)ds \\
&=\int_{x_1}^{x'_1} \frac{[g(s)-x_0]^{n-1}}{(n-1)!} F(c'(n)^2[g(s)-x_0]^{2(n-1)},\ s)ds \\
&\quad+\int_{x'_1}^{x} \frac{[g(s)-x_0]^{n-1}}{(n-1)!} F(c'(n)^2[g(s)-x_0]^{2(n-1)},\ s)ds \\
&\leqq \int_{x_1}^{x'_1} \frac{[g(s)-x_0]^{n-1}}{(n-1)!} F(c^2[g(x)-x_0]^{2(n-1)},\ s)ds \\
&\quad+\int_{x'_1}^{x} \frac{[g(x)]^{n-1}}{(n-1)!} F(c^2[g(x)]^{2(n-1)},\ s)ds.
\end{aligned}$$

Letting $x \to \infty$, the inequality (2.21) then follows from the above, (2.16) and the fact that $x'_1 \to x_1$ as $c' \to \infty$.

We fix such values of x_0 and c', and construct a solution of (A) satisfying the initial conditions

$$(2.22)\qquad \begin{aligned} &y(x)\equiv 0, && x\leqq x_0,\\ &y^{(k)}(x_0)=0, && k=1, 2,\dots, n-2,\\ &y^{(n-1)}(x_0)=2c'. \end{aligned}$$

We claim that $y^{(n-1)}(x)>c'$ for all $x\geqq x_0$. Suppose that there exists an $x_2\in(x_0, \infty)$ such that $y^{(n-1)}(x)>c'$ for $x_0\leqq x<x_2$ and $y^{(n-1)}(x_2)=c'$. Integrating (A) and using (2.22), we have

$$(2.23)\qquad y^{(n-1)}(x)=2c'-\int_{x_0}^{x} y(g(s))F([y(g(s))]^2, s)ds \qquad \text{for } x_0\leqq x\leqq x_2.$$

An argument similar to that we used for the superlinear case shows that $c'\leqq y^{(n-1)}(x)\leqq 2c'$ for $x_0\leqq x\leqq x_2$. This can be integrated as follows:

$$c'(n)(x-x_0)^{n-1}\leqq y(x)\leqq 2c'(n)(x-x_0)^{n-1}, \qquad x_0\leqq x\leqq x_2.$$

Therefore, we have

$$(2.24)\qquad \begin{aligned} &c'(n)[g(x)-x_0]^{n-1}\leqq y(g(x))\leqq 2c'(n)[g(x)-x_0]^{n-1}\\ &\qquad\qquad\qquad\qquad \text{for } x_0\leqq g(x)\leqq x_2,\\ &y(g(x))=0 \qquad \text{for } g(x)\leqq x_0. \end{aligned}$$

Using (2.24) in (2.23) and in view of (1.2), we obtain

$$y^{(n-1)}(x)\geqq 2c', \quad x_0\leqq x\leqq x_2, \qquad \text{if } x_1\geqq x_2,$$

$$(2.25)\qquad y^{(n-1)}(x)\geqq 2c'\left\{1-\int_{x_1}^{x} \frac{[g(s)-x_0]^{n-1}}{(n-1)!} F(c'(n)^2[g(s)-x_0]^{2(n-1)}, s)ds,\right\}$$

$$x_0\leqq x\leqq x_2, \qquad \text{if } x_1\leqq x_2.$$

In view of (2.25), (2.21), we find $y^{(n-1)}(x_2)\geqq 3c'/2$. This contradicts $y^{(n-1)}(x_2)=c'$. Therefore, it follows that $y^{(n-1)}(x)>c'$ for all $x\geqq x_0$. Now proceeding as in the superlinear case, we conclude that $y(x)$ has the property: $\lim_{x\to\infty} y(x)/x^{n-1}=b$, $c'\leqq b\leqq 2c'$.

This completes the proof of Theorem 2.2.

COROLLARY 2.3. *Let $\alpha>0$. Assume that $p(x)\geqq 0$. Assume, moreover, that $g'(x)\geqq 0$. Then, equation* (B) *has an unbounded solution which is asymptotic to bx^{n-1} $(b\neq 0)$ as $x\to\infty$ if and only if*

$$\int^{\infty}[g(x)]^{\alpha(n-1)}p(x)dx<\infty.$$

COROLLARY 2.4. *Equation* (C), *superlinear or sublinear, has an unbound-*

ed solution which is asymptotic to bx^{n-1} $(b\neq 0)$ *as* $x\to\infty$ *if and only if*

$$\int^{\infty} x^{n-1}F(c^2x^{2(n-1)}, x)dx<\infty \qquad \textit{for some } c>0. \tag{2.26}$$

REMARK. Corollary 2.3 was given for the case $n=2$ by Wong [49, Theorem (2.2)] and for the case of arbitrary $n\geqq 2$ by Kusano and Onose [20, Theorem 3] without the assumption $g'(x)\geqq 0$. Corollary 2.4 extends results of Nehari [29, Theorem II] and Coffman and Wong [8, Theorem 2] for the second order equation (C). For related results see Burkowski [4], Kiguradze [15, 16], Ladas [22], Marušiak [27, 28], Waltman [47] and others.

3. Oscillation Theorems

In this section, we prove several oscillation theorems for the retarded differential equation (A). As witness the recent results of Kusano and Onose [18–20], Ševelo and Odarič [32], Ševelo and Vareh [33, 34] and Wong [49], in order to obtain sharp oscillation criteria it is natural to exclude linear equations from our considerations. Here, we focus our attention on the equation (A) which is in the strongly superlinear case or in the strongly sublinear case, and establish sufficient conditions (Theorems 3.1, 3.2 and 3.3) and necessary and sufficient conditions (Theorems 3.4 and 3.5) in order that all solutions of (A) be oscillatory when n is even, and be either oscillatory or strongly monotone when n is odd.

Before stating the theorems, we give the following lemmas, which can be found in [31].

LEMMA 3.1. *Suppose that* $y(x)\in C^m[a, \infty)$, $y(x)\geqq 0$ *and* $y^{(m)}(x)$ *is monotone on* $[a, \infty)$. *Then, exactly one of the following is true*:

(i) $\lim_{x\to\infty} y^{(m)}(x)=0$;

(ii) $\lim_{x\to\infty} y^{(m)}(x)>0$ *and* $y'(x),\dots, y^{(m-1)}(x)$ *tend to* ∞ *as* $x\to\infty$.

LEMMA 3.2. *Suppose that* $y(x)\in C^n[a, \infty)$, $y(x)\geqq 0$ *and* $y^{(n)}(x)\leqq 0$ *on* $[a, \infty)$. *Then, exactly one of the following is true*:

(I) $y'(x),\dots, y^{(n-1)}(x)$ *tend monotonically to zero as* $x\to\infty$;

(II) *there exists an odd integer* k, $1\leqq k\leqq n-1$, *such that* $\lim_{x\to\infty} y^{(n-j)}(x)=0$ *for* $1\leqq j\leqq k-1$, $\lim_{x\to\infty} y^{(n-k)}(x)\geqq 0$, $\lim_{x\to\infty} y^{(n-k-1)}(x)>0$, *and* $y(x)$, $y'(x),\dots, y^{(n-k-2)}(x)$ *tend to* ∞ *as* $x\to\infty$.

First, we consider the strongly superlinear case.

THEOREM 3.1. *Let equation* (A) *be strongly superlinear. Assume that* $g'(x)\geqq 0$ *for* $x>0$. *A sufficient condition in order that*;

(i) *for n even, every solution of* (A) *be oscillatory,*

(ii) *for n odd, every solution of* (A) *be either oscillatory or strongly monotone,*

is that

$$\int^{\infty}[g(x)]^{n-1}F(c^2, x)dx=\infty \qquad \textit{for any } c>0, \tag{3.1}$$

PROOF. Our proof is based on the use of techniques introduced by Kusano and Onose in [18].

Let $y(x)$ be a nonoscillatory solution of (A). Without loss of generality, we may suppose that $y(x)>0$ for $x \geqq x_0$. Since $\lim_{x\to\infty} g(x)=\infty$, there is an $x_1 \geqq x_0$ such that $y(g(x))>0$ for $x \geqq x_1$. From (A) we have

$$y^{(n)}(x)=-y(g(x))F([y(g(x))]^2, x) \leqq 0 \tag{3.2}$$

for $x \geqq x_1$, so that, by Lemma 3.1, $y^{(n-1)}(x)$ decreases to a nonnegative limit as $x\to\infty$. Integration of (3.2) from x to infinity yields

$$y^{(n-1)}(x) \geqq \int_x^{\infty} y(g(s))F([y(g(s))]^2, s)ds, \qquad x \geqq x_1,$$

from which, using the nonincreasing character of $y^{(n-1)}(x)$, we obtain

$$y^{(n-1)}(g(x)) \geqq \int_x^{\infty} y(g(s))F([y(g(s))]^2, s)ds, \qquad x \geqq x_1. \tag{3.3}$$

Suppose that Case (I) of Lemma 3.2 holds. Multiplying both sides of (3.3) by $g'(x)$ and integrating from x to u, $x_1<x<u$, we have

$$\begin{aligned} &y^{(n-2)}(g(u))-y^{(n-2)}(g(x)) \geqq \\ &\qquad \int_x^u [g(s)-g(x)]y(g(s))F([y(g(s))]^2, s)ds. \end{aligned} \tag{3.4}$$

Letting $u\to\infty$ in (3.4) we have

$$-y^{(n-2)}(g(x)) \geqq \int_x^{\infty}[g(s)-g(x)]y(g(s))F([y(g(s))]^2, s)ds.$$

Proceeding in this way, we arrive at

$$(-1)^n y'(g(x)) \geqq \int_x^{\infty} \frac{[g(s)-g(x)]^{n-2}}{(n-2)!} y(g(s))F([g(s))]^2, s)ds. \tag{3.5}$$

Let n be even. We integrate (3.5) multiplied by $g'(x)$ from T to x, $x_1<T<x$, to obtain

$$(3.6)\qquad y(g(x))\geqq\int_T^x\frac{[g(s)-g(T)]^{n-1}}{(n-1)!}y(g(s))F([y(g(s))]^2,\ s)ds.$$

Since $g'(x)\geqq 0$ by hypothesis and $y'(x)\geqq 0$ for $x\geqq g(T)$ by (3.5), we get

$$(3.7)\qquad [y(g(x))]^2\geqq[y(g(T))]^2 \qquad \text{for } x\geqq T.$$

Using (3.7) and the strong superlinearity of (A), i.e.,

$$t_1^{-\varepsilon}F(t_1,\ x)\leqq t_2^{-\varepsilon}F(t_2,\ x),\quad t_1<t_2,\quad \varepsilon>0,$$

we obtain

$$(3.8)\qquad \begin{aligned} &y(g(x))F([y(g(x))]^2,\ x)\\ &=[y(g(x))]^{1+2\varepsilon}\cdot[y(g(x))]^{-2\varepsilon}F([y(g(x))]^2,\ x)\\ &\geqq[y(g(x))]^{1+2\varepsilon}\cdot[y(g(T))]^{-2\varepsilon}F([y(g(T))]^2,\ x)\\ &=c^{1-\alpha}[y(g(x))]^{\alpha}F(c^2,\ x), \end{aligned}$$

where $c=y(g(T))$ and $\alpha=1+2\varepsilon$. It follows from (3.6) and (3.8) that

$$(3.9)\quad [y(g(x))]^{-\alpha}\leqq\left\{c^{1-\alpha}\int_T^x\frac{[g(s)-g(T)]^{n-1}}{(n-1)!}[y(g(s))]^{\alpha}F(c^2,\ s)ds\right\}^{-\alpha}.$$

Multiplying both sides of (3.9) by $\frac{[g(x)-g(T)]^{n-1}}{(n-1)!}[y(g(x))]^{\alpha}F(c^2,\ x)$ and integrating from x_1 to u, $T<T_1<u$, we find

$$(3.10)\qquad \begin{aligned} &\int_{T_1}^u\frac{[g(x)-g(T)]^{n-1}}{(n-1)!}F(c^2,\ x)dx\leqq\\ &\frac{c^{\alpha(\alpha-1)}}{1-\alpha}\left\{\int_T^x\frac{[g(s)-g(T)]^{n-1}}{(n-1)!}[y(g(s))]^{\alpha}F(c^2,\ s)ds\right\}^{1-\alpha}\Bigg|_{T_1}^u. \end{aligned}$$

Since $\alpha>1$, the right side remains finite as $u\to\infty$; consequently, the integral on the left converges as $u\to\infty$:

$$\int_{T_1}^{\infty}\frac{[g(x)-g(T)]^{n-1}}{(n-1)!}F(c^2,\ x)dx<\infty,$$

which contradicts (3.1).

Let n be odd. Then (3.5) reduces to

$$(3.11)\qquad -y'(g(x))\geqq\int_x^{\infty}\frac{[g(s)-g(x)]^{n-2}}{(n-2)!}y(g(s))F([y(g(s))]^2,\ s)ds,$$

which implies that $y'(x)\leqq 0$ for $x\geqq g(T)$. Hence, $y(x)$ decreases to a limit $h\geqq 0$. Suppose that $h>0$. Integrating (3.11) multiplied by $g'(x)$ from K to x, $g(T)\leqq K<x$, we have

$$y(g(K))-y(g(x))\geqq \tag{3.12}$$
$$\int_K^x \frac{[g(s)-g(K)]^{n-1}}{(n-1)!}y(g(s))F([y(g(s))]^2, s)ds.$$

We observe that the strong superlinearity of (A) implies that $yF(y^2, x)$ is a nondecreasing function of $y>0$. Therefore, we obtain

$$y(g(x))F([y(g(x))]^2, x)\geqq hF(h^2, x), \qquad x\geqq T, \tag{3.13}$$

since $y(g(x))\geqq h>0$ for $x\geqq T$. It follows from (3.12) and (3.13) that

$$\begin{aligned} y(g(K)) &\geqq y(g(K))-h \\ &\geqq \int_K^\infty \frac{[g(s)-g(K)]^{n-1}}{(n-1)!}y(g(s))F([y(g(s))]^2, s)ds \\ &\geqq \frac{h}{(n-1)!}\int_K^\infty [g(s)-g(K)]^{n-1}F(h^2, s)ds, \end{aligned}$$

which is again a contradiction to (3.1). Thus we must have $h=0$, and this completes the proof of Case (I).

Suppose that Case (II) of Lemma 3.2 holds. We note that there exists an $x_2\geqq x_1$ such that $y^{(j)}(x)>0$ for $x\geqq x_2$ and $j=0, 1,\ldots, n-k-1$. Proceeding as in Case (I), we have

$$y^{(n-k)}(g(x))\geqq\int_x^\infty \frac{[g(s)-g(x)]^{k-1}}{(k-1)!}y(g(s))F([y(g(s))]^2, s)ds.$$

Multiplying the above inequality by $g'(x)$ and integrating from x_2 to x, we obtain

$$y^{(n-k-1)}(g(x))\geqq\frac{[g(x)-g(x_2)]^k}{k!}\int_x^\infty y(g(s))F([y(g(s))]^2, s)ds.$$

Repetition of this procedure yields

$$y'(g(x))\geqq\frac{[g(x)-g(x_2)]^{n-2}}{(n-2)!}\int_x^\infty y(g(s))F([y(g(s))]^2, s)ds,$$

and upon an integration of the above, we find

$$y(g(x))\geqq\int_{x_2}^x \frac{[g(s)-g(x_2)]^{n-1}}{(n-1)!}y(g(s))F([y(g(s))]^2, s)ds.$$

Now the proof proceeds exactly as in Case (I).

COROLLARY 3.1. *Let $\alpha>1$. Assume that $p(x)\geqq 0$, $g'(x)\geqq 0$ and the following condition holds*:

$$\int^{\infty}[g(x)]^{n-1}p(x)dx=\infty.$$

Then, every solution of (B) *is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd.*

COROLLARY 3.2. *Let equation* (C) *be strongly superlinear. Assume that the following condition holds*:

$$\int^{\infty}x^{n-1}F(c^2, x)dx=\infty \qquad \text{for any } c>0. \tag{3.14}$$

Then, every solution of (C) *is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd.*

REMARK. Corollary 3.1 was obtained independently by Kusano and Onose [18], and Ševelo and Vareh [34].

In the above theorem it was assumed that $g'(x)\geqq 0$. In the next theorem an attempt is made to avoid this smoothness assumption on $g(x)$ and produce an oscillation criterion which applies at least to the case where the delay $\tau(x)\equiv x-g(x)$ is bounded.

THEOREM 3.2. *Let* (A) *be strongly superlinear and let there exist a function $g_*(x)$ such that*

$$g_*'(x)\geqq 0, \quad g_*(x)\leqq g(x) \quad \text{and} \quad \lim_{x\to\infty}g_*(x)=\infty.$$

Assume that

$$\int^{\infty}[g_*(x)]^{n-1}F(c^2, x)dx=\infty \qquad \text{for any } c>0. \tag{3.15}$$

Then, every solution of (A) *is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd.*

PROOF. The proof patterns after that of Kusano and Onose [19].

Let $y(x)$ be a nonoscillatory solution of (A) which can be assumed positive, say $y(x)>0$ for $x\geqq x_0$. Since $\lim_{x\to\infty}g_*(x)=\infty$, there exists an $x_1\geqq x_0$ such that $g(x)\geqq g_*(x)\geqq x_0$ and $y(g(x))>0$ for $x\geqq x_1$. Proceeding in the same manner as the proof of Theorem 3.1, we obtain for $x\geqq x_1$

$$y^{(n-1)}(g_*(x))\geqq\int_x^{\infty}y(g(s))F([y(g(s))]^2, s)ds, \tag{3.16}$$

which corresponds to (3.3).

We consider Case (I) of Lemma 3.2. Multiply both sides of (3.16) by $g'_*(x)$, integrate from x to u and let u tend to infinity. Repeated application of this procedure then gives

$$(3.17)\qquad \begin{aligned}&(-1)^n y'(g_*(x))\\ &\geqq \int_x^\infty \frac{[g_*(s)-g_*(x)]^{n-2}}{(n-2)!} y(g(s))F([y(g(s))]^2, s)ds, \qquad x \geqq x_1,\end{aligned}$$

which corresponds to (3.5) in the preceding proof.

Let n be even. Then, by (3.17), $y'(x) \geqq 0$ for $x \geqq x_1$. Let T be fixed so that $T > x_1$. Using (1.3), $g(x) \geqq g_*(x)$ and the nondecreasing character of $y(x)$, we obtain

$$(3.18)\qquad \begin{aligned}&y(g(x))F([y(g(x))]^2, x)\\ &=[y(g(x))]^{1+2\varepsilon}\cdot[y(g(x))]^{-2\varepsilon}F([y(g(x))]^2, x)\\ &\geqq [y(g_*(x))]^{1+2\varepsilon}\cdot[y(g_*(x))]^{-2\varepsilon}F([y(g_*(x))]^2, x)\\ &\geqq c^{1-\alpha}[y(g_*(x))]^{\alpha}F(c^2, x),\end{aligned}$$

where $c = y(g_*(T))$ and $\alpha = 1+2\varepsilon$. From (3.17) and (3.18) it follows that

$$(3.19)\qquad y'(g_*(x)) \geqq c^{1-\alpha}\int_x^\infty \frac{[g_*(s)-g_*(x)]^{n-2}}{(n-2)!}[y(g_*(s))]^{\alpha}F(c^2, s)ds.$$

It is a matter of easy computation to derive from (3.19) the following inequality

$$\int^\infty [g_*(x)]^{n-1}F(c^2, x)dx < \infty,$$

which contradicts (3.15).

Let n be odd. Then, (3.17) becomes

$$(3.20)\qquad -y'(g_*(x)) \geqq \int_x^\infty \frac{[g_*(s)-g_*(x)]^{n-2}}{(n-2)!} y(g(s))F([y(g(s))]^2, s)ds,$$

and this implies that $y(x)$ is nonincreasing for all sufficiently large x, say for $x \geqq T$. Let $\lim_{x\to\infty} y(x) = h \geqq 0$. Suppose that $h > 0$. Integrating (3.20) sultiplied by $g'_*(x)$ from K to x with $g(T) \leqq K < x$, and using the inequality

$$y(g(x))F([y(g(x))]^2, x) \geqq hF(h^2, x)$$

which is implied by the strong superlinearity of (A), we find

$$(3.21)\qquad y(g_*(K)) - y(g_*(x)) \geqq \frac{h}{(n-1)!}\int_K^x [g_*(s)-g_*(K)]^{n-1}F(h^2, s)ds.$$

Letting $x \to \infty$ in (3.21), we get a contradiction to (3.15). Therefore, we must have $h=0$.

The proof for Case (II) of Lemma 3.2 also proceeds, with necessary modifications, in a manner similar to the corresponding part of the proof of Theorem 3.1. So, we omit the details.

Now, we turn to the strongly sublinear case.

THEOREM 3.3. *Assume that equation* (A) *is in the strongly sublinear case. Then, a sufficient condition in order that*:

(i) *for n even, every solution of* (A) *be oscillatory,*

(ii) *for n odd, every solution of* (A) *be either oscillatory or strongly monotone, is that*

$$\int^{\infty}[g(x)]^{n-1}F(c^2[g(x)]^{2(n-1)}, x)dx=\infty \qquad \text{for any } c>0. \tag{3.22}$$

We need the following lemma due to Kiguradze [16].

LEMMA 3.3. *If $y(x)$ is a function such that it and all its derivatives up to order $(n-1)$ inclusive, are absolutely continuous and of constant sign in the interval $[x_1, \infty)$, and $y(x)y^{(n)}(x)\leqq 0$, then there is an integer l, $0\leqq l\leqq n-1$, which is odd if n is even and even if n is odd, such that for $x\geqq x_1$ we have*

$$\begin{aligned} &y(x)y^{(j)}(x)\geqq 0, \qquad j=0,1,\dots, l,\\ &(-1)^{n+j-1}y(x)y^{(j)}(x)\geqq 0, \qquad j=l+1,\dots, n, \end{aligned} \tag{3.23}$$

and if $l>0$,

$$|y(x)|\geqq \frac{(x-x_1)^{n-1}}{(n-1)\dots(n-l)}|y^{(n-1)}(2^{n-l-1}x)|. \tag{3.24}$$

PROOF of Theorem 3.3. Let $y(x)$ be a nonoscillatory solution of (A). We may assume that $y(x)>0$ for $x\geqq x_0$. There is an $x_1\geqq x_0$ such that $y(g(x))>0$ for $x\geqq x_1$. In view of (A), $y^{(n)}(x)\leqq 0$ for $x\geqq x_1$, so we can find an integer l such that (3.23) holds.

Assume that $l>0$. Then, by (3.23), $y'(x)\geqq 0$ for $x\geqq x_1$. It follows from Kiguradze's formula (3.24) and the nondecreasing character of $y(x)$ that for $x\geqq x_1$

$$y(x)\geqq y(2^{l-n+1}x)\geqq \frac{2^{(l-n+1)(n-1)}(x-x_1)^{n-1}}{(n-1)\dots(n-l)}y^{(n-1)}(x).$$

Therefore,

$$y(x)\geqq Ax^{n-1}y^{(n-1)}(x), \qquad x\geqq x_2=2x_1, \tag{3.25}$$

where $A=2^{(l-n+1)(n-1)}/(n-1)\ldots(n-l)$. Since $\lim_{x\to\infty} g(x)=\infty$, there is an $x_3\geqq x_2$ such that $g(x)\geqq x_2$ for $x\geqq x_3$. From (3.25) and the fact that $y^{(n-1)}(x)$ is nonincreasing, we then have

$$y(g(x))\geqq A[g(x)]^{n-1}y^{(n-1)}(x), \qquad x\geqq x_3. \tag{3.26}$$

On the other hand, since $y^{(n)}(x)\leqq 0$ for $x\geqq x_1$, by Taylor's theorem, there exists a constant $a\geqq 1$ such that $y(x)\leqq ax^{n-1}$ for $x\geqq x_1$, which implies

$$y(g(x))\leqq a[g(x)]^{n-1}, \qquad x\geqq x_3. \tag{3.27}$$

Using (3.27) in (1.4), i.e.,

$$t_1^\varepsilon F(t_1, x)\geqq t_2^\varepsilon F(t_2, x), \quad t_1<t_2, \quad \varepsilon>0,$$

we have

$$[y(g(x))]^{2\varepsilon}F([y(g(x))]^2, x)\geqq a^{2\varepsilon}[g(x)]^{2(n-1)\varepsilon}F(a^2[g(x)]^{2(n-1)}, x). \tag{3.28}$$

Now, suggested by Coffman and Wong [8], we consider

$$\begin{aligned} -([y^{(n-1)}(x)]^{2\varepsilon})' &= -2\varepsilon[y^{(n-1)}(x)]^{2\varepsilon-1}y^{(n)}(x) \\ &= 2\varepsilon[y^{(n-1)}(x)]^{2\varepsilon-1}y(g(x))F([y(g(x))]^2, x) \\ &= 2\varepsilon[y^{(n-1)}(x)]^{2\varepsilon-1}\cdot[y(g(x))]^{1-2\varepsilon}\cdot[y(g(x))]^{2\varepsilon}F([y(g(x))]^2, x). \end{aligned} \tag{3.29}$$

Using (3.26) and (3.28) in (3.29), we obtain

$$\begin{aligned} -([y^{(n-1)}(x)]^{2\varepsilon})' &\geqq 2\varepsilon[y^{(n-1)}(x)]^{2\varepsilon-1}\cdot \\ &\quad (A[g(x)]^{n-1}y^{(n-1)}(x))^{1-2\varepsilon}\cdot a^{2\varepsilon}[g(x)]^{2(n-1)\varepsilon}F(a^2[g(x)]^{2(n-1)}, x) \\ &= k[g(x)]^{n-1}F(a^2[g(x)]^{2(n-1)}, x), \qquad x\geqq x_3, \end{aligned}$$

where $k=2\varepsilon a^{2\varepsilon}A^{1-2\varepsilon}$ and we have assumed that $\varepsilon<1/2$ without restricting generality. Integrating the above inequality from x_3 to x, we have

$$\begin{aligned} &[y^{(n-1)}(x_3)]^{2\varepsilon}-[y^{(n-1)}(x)]^{2\varepsilon}\geqq \\ &\qquad k\int_{x_3}^{x}[g(s)]^{n-1}F(a^2[g(s)]^{2(n-1)}, s)ds. \end{aligned} \tag{3.30}$$

Since $y^{(n-1)}(x)>0$ for all large x, it follows from (3.30) that

$$\int_{x_3}^{\infty}[g(x)]^{n-1}F(a^2[g(x)]^{2(n-1)}, x)dx<\infty,$$

which is a contradiction to (3.22). Thus, we conclude that $l=0$. Obviously, this is possible only when n is odd. In view of (3.23), $y(x)$ decreases to a limit

$h \geqq 0$ as $x \to \infty$. Let $h > 0$. Then, since the strong sublinearity implies the sublinearity, it follows from Theorem 2.1 that

$$\int^{\infty} x^{n-1} F(c^2, x) dx < \infty \qquad \text{for some } c > 0.$$

Noting that $g(x) \leqq x$ and $c^2 \leqq c^2[g(x)]^{2(n-1)}$ for all large x, and using the sublinearity of (A), we obtain from the above

$$\int^{\infty} [g(x)]^{n-1} F(c^2[g(x)]^{2(n-1)}, x) dx < \infty \qquad \text{for some } c > 0,$$

contradicting (3.22). Therefore, we must have $h = 0$, that is, $y(x)$ must tend monotonically to zero together with its first $n-1$ derivatives. This completes the proof.

Corollary 3.3. *Let $\alpha < 1$. Then, a sufficient condition that every solution of* (B) *be oscillatory if n is even, and be either oscillatory or strongly monotone if n is odd is that*

$$\int^{\infty} [g(x)]^{\alpha(n-1)} p(x) dx = \infty. \tag{3.31}$$

Corollary 3.4. *Let equation* (C) *be strongly sublinear. Then, a sufficient condition that every solution of* (C) *be oscillatory if n is even, and be either oscillatory or strongly monotone if n is odd is that*

$$\int^{\infty} x^{n-1} F(c^2 x^{2(n-1)}, x) dx = \infty \qquad \textit{for any } c > 0. \tag{3.32}$$

Remark. Corollary 3.3 was proved independently by Kusano and Onose [18–20] and Ševelo and Vareh [34].

Combining Theorems 2.2 and 3.3 we obtain the following

Theorem 3.4. *Let equation* (A) *be strongly sublinear. Assume that $g'(x) \geqq 0$ for $x > 0$. Then, a necessary and sufficient condition in order that every solution of* (A) *be oscillatory in the case n is even, and be either oscillatory or strongly monotone in the case n is odd is that* (3.22) *be valid.*

Proof. The sufficiency part follows from Theorem 3.3. The necessity part is contained in Theorem 2.2, because the strong superlinearity of (A) implies the superlinearity of (A).

A question naturally arises as to whether (3.1) is a necessary and sufficient condition for oscillation of all solutions of (A) which is in the strongly superlinear case. A partial answer to this question is given in the following theorem.

Theorem 3.5. *Let equation* (A) *be strongly superlinear. Assume that*

$g(x)$ satisfies

$$\liminf_{x\to\infty}\frac{g(x)}{x}=\gamma>0. \tag{3.33}$$

Then, a necessary and sufficient condition that every solution of (A) *be oscillatory if n is even, and be either oscillatory or strongly monotone if n is odd is that* (3.14) *be valid.*

PROOF. Since the strong superlinearity of (A) implies the superlinearity of (A), the necessity part is contained in Theorem 2.1. To prove the sufficiency, we observe from (3.33) that $\gamma/2\leqq g(x)/x$, i.e. $\gamma x/2\leqq g(x)$ for all sufficiently large x. Put $g_*(x)=\gamma x/2$. Then, condition (3.14) can also be written as

$$\int^{\infty}[g_*(x)]^{n-1}F(c^2, x)dx=\infty \qquad \text{for any } c>0.$$

Applying Theorem 3.2, we conclude that every solution of (A) is oscillatory if n is even, and is either oscillatory or strongly monotone if n is odd.

REMARK. The condition (3.33) was proposed by Wong [49]. It contains as special cases the following:

(i) $g(x)=x-\tau(x),\quad 0\leqq\tau(x)\leqq M$;

(ii) $g'(x)\geqq 0$ and $\lim_{x\to\infty}g'(x)=\gamma>0$.

The following corollaries are immediate consequences of Theorems 3.4 and 3.5.

COROLLARY 3.5. *With regard to equation* (B) *assume that either* (i) $\alpha<1$ *and* $g'(x)\geqq 0$, *or* (ii) $\alpha>1$ *and* $g(x)$ *satisfies* (3.33).

Then, a necessary and sufficient condition that every solution of (B) *be oscillatory if n is even, and be oscillatory or strongly monotone if n is odd is that* (3.31) *be valid in the case* (i), *and that*

$$\int^{\infty}x^{n-1}p(x)dx=\infty$$

in the case (ii).

COROLLARY 3.6. *A necessary and sufficient condition that every solution of* (C) *be oscillatory if n is even, and be oscillatory or strongly monotone if n is odd is that* (3.14) *be valid in the strongly superlinear case, and* (3.32) *be valid in the strongly sublinear case.*

REMARK. Corollary 3.5 (i) was given in a stronger form by Wong [49, Theorem (3.1)] and Kusano and Onose [20, Theorem 3] where the assumption $g'(x)\geqq 0$ is not required. Corollary 3.5 (ii) is an extension of a result of Wong [49, Theorem

(3.2)]. Gollwitzer's result [9] is contained in Corollary 3.5. Corollary 3.6 generalizes results of Ličko and Švec [25] and Kiguradze [15, Theorems 3,4]. Related results can be found in Kiguradze [14, 16], Ryder and Wend [31], Staikos and Sficas [37–39] and others.

We close this section by stating two propositions concerning the ordinary differential equation (C), which are duals to one another in the sense specified in Coffman and Wong [8].

COROLLARY 3.7. *Let equation* (C) *be strongly superlinear. Then, the following statements are equivalent*:

(i) *Equation* (C) *has a bounded solution which is asymptotic to a nonzero constant* as $x \to \infty$;

(ii) *Equation* (C) *has a nonoscillatory solution which is not strongly monotone*;

(iii) *For some* $c>0$, (2.1) *holds.*

COROLLARY 3.8. *Let equation* (C) *be strongly sublinear. Then, the following statements are equivalent*:

(i) *Equation* (C) *has an unbounded solution which is asymptotic to* bx^{n-1} $(b \neq 0)$ *as* $x \to \infty$;

(ii) *Equation* (C) *has a nonoscillatory solution which is not strongly monotone*;

(iii) *For some* $c>0$, (2.26) *holds.*

REMARK. Corollary 3.7 follows from Corollaries 2.2 and 3.6 and extends a result of Coffman and Wong [7] (see also [8, Theorem E]). Corollary 3.8 is derived by combining Corollaries 2.4 and 3.6; it contains a theorem due to Coffman and Wong [8, Theorem 2].

4. Forced Oscillations

Of mathematical and physical importance is the problem of maintaining the oscillation of all solutions of equation (A) under the effect of a forcing term. In the case of differential equations without delay, that is, when $g(x) \equiv x$, this problem was studied by Kartsatos [12, 13] and Teufel [44]. An attempt was made by Kusano and Onose [21] to extend part of Kartsatos' results to retarded differential equations.

In this section, we present theorems to the effect that all solutions of the equation

$$\text{(D)} \qquad y^{(n)}(x) + y(g(x))F([y(g(x))]^2, x) = q(x)$$

are oscillatory if the forcing term $q(x)$ is sufficiently small or periodic (in the sense specified below).

First, we consider the maintenance of the oscillations of the strongly superlinear equation (A).

THEOREM 4.1. *Let $F(t, x)$ satisfy* (1.3). *Assume that $g'(x) \geqq 0$ and*

$$\int^{\infty} [g(x)]^{n-1} F(c^2, x) dx = \infty \qquad \text{for any } c > 0. \tag{4.1}$$

Let there exist a function $Q \in C^n[(0, \infty), R]$, $R = (-\infty, \infty)$, such that $Q^{(n)}(x) = q(x)$ for $x > 0$ and either

(I) $\lim_{x \to \infty} Q(x) = 0$; *or*

(II) *there exist constants q_1, q_2 and sequences $\{x'_m\}$, $\{x''_m\}$ with the following property*: $\lim_{m \to \infty} x'_m = \lim_{m \to \infty} x''_m = \infty$, $Q(x'_m) = q_1$, $Q(x'_m) = q_2$, $q_1 \leqq Q(x) \leqq q_2$ *for $x > 0$.*

Let (I) *hold. Then, every solution $y(x)$ of* (D) *is oscillatory or such that $\lim_{x \to \infty} y(x) = 0$.*

Let (II) *hold. Then, if n is even, every solution $y(x)$ of* (D) *is oscillatory, while if n is odd, every solution is either oscillatory or such that $\lim_{x \to \infty} [y(x) - Q(x)] = -q_1$ or $-q_2$.*

PROOF. We follow closely the arguments developed by Kartsatos [12, 13] and Kusano and Onose [21].

Case (I). Let $y(x)$ be a nonoscillatory solution of (D) such that $y(x) > 0$ for $x \geqq x_0$. Choose an $x_1 \geqq x_0$ such that $y(g(x)) > 0$ for $x \geqq x_1$. If we put $Y(x) \equiv y(x) - Q(x)$, then $Y(x)$ is a solution of the equation

$$Y^{(n)}(x) + [Y(g(x)) + Q(g(x))] F([Y(g(x)) + Q(g(x))]^2, x) = 0 \tag{4.2}$$

with the property: $Y(g(x)) + Q(g(x)) > 0$ for $x \geqq x_1$. From (4.2) it follows that

$$Y^{(n)}(x) \leqq 0 \qquad \text{for } x \geqq x_1. \tag{4.3}$$

Suppose that $y(x)$ is unbounded for all large x; then, so is $Y(x)$ and, in view of (4.3), it is easily verified that $Y(x)$ and all its first $n-1$ derivatives are eventually of fixed sign. In particular, we have $Y'(x) \geqq 0$, $x \geqq x_1$ and $\lim_{x \to \infty} Y(x) = \infty$. Therefore, we can choose an $x_2 \geqq x_1$ and an $\varepsilon > 0$ such that

$$Y(g(x)) + Q(g(x)) > Y(g(x)) - \varepsilon, \qquad x \geqq x_2. \tag{4.4}$$

Put $Z(x) \equiv Y(x) - \varepsilon$. Then, we see that $\lim_{x \to \infty} Z(x) = \infty$. On the other hand, $Z(x)$ satisfies the retarded differential equation

$$Z^{(n)}(x) + Z(g(x)) F_1([Z(g(x))]^2, x) = 0, \tag{4.5}$$

where

$$(4.6)\quad F_1(Z^2, x)=\frac{[Y(g(x))+Q(g(x))]F([Y(g(x))+Q(g(x))]^2, x)}{[Y(g(x))-\varepsilon]F([Y(g(x))-\varepsilon]^2, x)}F(Z^2, x).$$

Equation (4.5) is again strongly superlinear. From (4.4) and (4.6) we find $F_1(Z^2, x)\geqq F(Z^2, x)$, $Z>0$, which in view of (4.1) gives

$$\int^{\infty}[g(x)]^{n-1}F_1(c^2, x)dx=\infty \qquad \text{for } c>0.$$

Thus it follows from Theorem 3.1 that $Z(x)$ has to be oscillatory or tends to zero as $x\to\infty$. But this contradicts the fact that $\lim_{x\to\infty} Z(x)=\infty$.

Next suppose that $y(x)$ remains bounded as $x\to\infty$. Then, $Y(x)$ is also bounded and from (4.3) we conclude that

$$(4.7)\qquad (-1)^{n+j}Y^{(j)}(x)\leqq 0, \quad x\geqq x_1, \qquad j=1,\dots, n.$$

Let n be even. Then, (4.7) implies $Y'(x)\geqq 0$ for $x\geqq x_1$. If $Y(x)>0$ eventually, then the limit $\lim_{x\to\infty} Y(x)=\eta$ exists and is a positive number. Introducing the function $Z(x)\equiv Y(x)-\varepsilon$, $0<\varepsilon<\eta/2$, and arguing as in the case $y(x)$ is unbounded, we arrive at a contradiction. Consequently, we must have $Y(x)\leqq 0$, i.e., $y(x)\leqq Q(x)$ for all large x. Of course, this is possible only when $Q(x)>0$ eventually, and in this case we have $\lim_{x\to\infty} y(x)=0$. Let n be odd. Then, $Y'(x)\leqq 0$, $x\geqq x_1$, by (4.7), so that $Y(x)$ decreases to a limit η as $x\to\infty$. Again it cannot happen that $\eta>0$. Since $Y(x)+Q(x)$ is eventually positive, we conclude that $\eta=0$, which implies $\lim_{x\to\infty} y(x)=0$.

A parallel argument holds if we assume that $y(x)$ is eventually negative. This completes the proof of Case (I).

Case (II). Let $y(x)$ be a nonoscillatory solution of (D) such that $y(g(x))>0$ for $x\geqq x_1$.

Assume that $y(x)$ is unbounded; then, as in the corresponding part of the proof of Case (I), we can show that $Y(x)\equiv y(x)-Q(x)$ has the property: $Y'(x)\geqq 0$, $x\geqq x_1$, $\lim_{x\to\infty} Y(x)=\infty$, and there exists an $x_2\geqq x_1$ such that

$$(4.8)\qquad Y(g(x))+Q(g(x))\geqq Y(g(x))+q_1>0, \qquad x\geqq x_2.$$

The function $W(x)\equiv Y(x)+q_1$ satisfies the retarded differential equation

$$(4.9)\qquad W^{(n)}(x)+W(g(x))F_2([W(g(x))]^2, x)=0,$$

where

$$F_2(W^2, x)=\frac{[Y(g(x))+Q(g(x))]F([Y(g(x))+Q(g(x))]^2, x)}{[Y(g(x))+q_1]F([Y(g(x))+q_1]^2, x)}F(W^2, x).$$

In view of (4.8) we have $F_2(W^2, x)\geqq F(W^2, x)$. This implies that

$$\int^{\infty}[g(x)]^{n-1}F_2(c^2, x)dx=\infty \qquad \text{for any } c>0,$$

so, by Theorem 3.1 applied to (4.9), $W(x)$ must be either oscillatory or tending to zero as $x\to\infty$. This contradicts the fact $\lim_{x\to\infty} W(x)=\infty$.

Assume now that $y(x)$ is bounded. If n is even, then $Y(x)\equiv y(x)-Q(x)$ also satisfies $Y'(x)\geqq 0$ for $x\geqq x_1$, and (4.8) holds for all large x. Thus, we arrive again at equation (4.9) which implies a contradiction to $\lim_{x\to\infty}[Y(x)+q_1]>0$. If n is odd, then $Y(x)\equiv y(x)-Q(x)$ satisfies $Y'(x)\leqq 0$ for $x\geqq x_1$. If we suppose that $Y(x_2)+q_1\leqq 0$ for some $x_2\geqq x_1$, then $Y(x)+q_1\leqq 0$ for all $x\geqq x_2$, contradicting the eventual positivity of $Y(x)+Q(x)$. Hence, we have $Y(x)+q_1>0$ for all $x\geqq x_1$ and, applying Theorem 3.1 to equation (4.9), we conclude that $\lim_{x\to\infty}[Y(x)+q_1]=0$, that is, $\lim_{x\to\infty}[y(x)-Q(x)]=-q_1$.

The case where $y(x)$ is eventually negative can be treated quite analogously. This completes the proof of Case (II).

The maintenance of the oscillations of the strongly sublinear equation (A) is contained in the following

THEOREM 4.2. *Let $F(t, x)$ satisfy* (1.4). *Assume that*

$$y_1F(y_1^2, x)\leqq y_2F(y_2^2, x),\quad 0\leqq y_1<y_2,\quad x\in(0, \infty), \tag{4.10}$$

and

$$\int^{\infty}[g(x)]^{n-1}F(c^2[g(x)]^{2(n-1)}, x)dx=\infty \qquad \textit{for any } c>0.$$

Let $q(x)$ be as in Theorem 4.1. *Then, the conclusion of* Theorem 4.1 *holds.*

The proof of this theorem may be omitted.

COROLLARY 4.1. *Consider the equation*

$$y^{(n)}(x)+p(x)|y(g(x))|^{\alpha}\operatorname{sgn} y(g(x))=q(x), \tag{E}$$

where $p(x)\geqq 0$, $\alpha>0$, $\alpha\neq 1$, and $q(x)$ is as in Theorem 4.1. *If*

$$\int^{\infty}[g(x)]^{\alpha_*(n-1)}p(x)dx=\infty, \qquad \alpha_*=\min(\alpha, 1),$$

then the conclusion of Theorem 4.1 *holds for equation* (E).

COROLLARY 4.2. *Consider the equation*

$$y^{(n)}(x)+y(x)F([y(x)]^2, x)=q(x), \tag{F}$$

where $q(x)$ *is as in* Theorem 4.1. *Assume that either* (i) $F(t, x)$ *satisfies* (1.3) *and*

$$\int^{\infty} x^{n-1}F(c^2, x)dx=\infty \qquad \text{for any } c>0,$$

or (ii) $F(t, x)$ *satisfies* (1.4), (4.10) *and*

$$\int^{\infty} x^{n-1}F(c^2x^{2(n-1)}, x)dx=\infty \qquad \text{for any } c>0.$$

Then, the conclusion of Theorem 4.1 *holds for equation* (F).

REMARK. Corollary 4.2 overlaps with but is not covered by the results of Kartsatos [12, 13].

ACKNOWLEDGMENT. The author would like to express his heartfelt thanks to Professor Takaŝi Kusano for his interest in this work and for many helpful suggestions and stimulating discussions. Without his encouragement and assistance the author could not have completed this work.

References

[1] L. E. Bobisud, Oscillation of second-order differential equations with retarded argument, J. Math. Anal. Appl. **43** (1973), 201–206.

[2] J. S. Bradley, Oscillation theorems for a second-order delay equation, J. Differential Equations **8** (1970), 397–403.

[3] F. Burkowski, Oscillation theorems for a second order nonlinear functional differential equation, J. Math. Anal. Appl. **33** (1971), 258–262.

[4] F. Burkowski, Nonlinear oscillation of a second order sublinear functional differential equation, SIAM J. Appl. Math. **21** (1971), 486–490.

[5] T. Burton and R. Grimmer, Oscillatory solutions of $x''(t)+a(t)f(x(q(t)))=0$, "Delay and Functioanl Differential Equations and Their Applications" (Ed. by K. Schmitt), pp. 335–343, Academic Press, New York, 1972.

[6] C. V. Coffman and J. S. W. Wong, On a second order nonlinear oscillation problem, Trans. Amer. Math. Soc. **147** (1970), 357–366.

[7] C. V. Coffman and J. S. W. Wong, Oscillation and nonoscillation of solutions of generalized Emden-Fowler equations, Trans. Amer. Math. Soc. **167** (1972), 399–434.

[8] C. V. Coffman and J. S. W. Wong, Oscillation and nonoscillation theorems for second order ordinary differential equations, Funkc. Ekvac. **15** (1972), 119–130.

[9] H. E. Gollwitzer, On nonlinear oscillations for a second order delay equation, J. Math. Anal. Appl. **26** (1969), 385–389.

[10] G. W. Grefsrud, Existence and oscillation of solutions of certain functioanl differential equations, Ph. D. Thesis, Montana State University, June, 1971.

[11] A. G. Kartsatos, On oscillation of solutions of evenorder nonlinear differential equa-

tion, J. Differential Equations **6** (1969), 232–237.

[12] A. G. Kartsatos, On the maintenance of oscillations of nth order equations under the effect of a small forcing term, J. Differential Equations **10** (1971), 355–363.

[13] A. G. Kartsatos, Maintenance of oscillations under the effect of a periodic forcing term, Proc. Amer. Math. Soc. **33** (1972), 377–383.

[14] I. T. Kiguradze, Oscillation properties of solutions of certain ordinary differential equations, Dokl. Akad. Nauk SSSR **144** (1962), 33–36. (Russian) = Soviet Math. Dokl. **3** (1962), 649–652.

[15] I. T. Kiguradze, On the oscillation of solutions of the equation $\frac{d^m u}{dt^m}+a(t)\,|u|^n \operatorname{sign} u=0$ Mat. Sb. **65** (1964), 172–187. (Russian)

[16] I. T. Kiguradze, The problem of oscillation of solutions of nonlinear differential equations, Differencial'nye Uravnenija **1** (1965, 995–1006. (Russian) = Differential Equations **1** (1965), 773–782.

[17] R. G. Koplatadze, A note on the oscillation of solutions of second order differential equations with retarded argument, Mat. Časopis Sloven. Akad. Vied **22** (1972), 253–261. (Russian)

[18] T. Kusano and H. Onose, Oscillation of solutions of nonlinear differential delay equations of arbitrary order, Hiroshima Math. J. **2** (1972), 1–13.

[19] T. Kusano and H. Onose, Oscillation theorems for delay equations of arbitrary order, Hiroshima Math. J. **2** (1972), 263–270.

[20] T. Kusano and H. Onose, Nonlinear oscillation of sublinear delay equation of arbitrary order, Proc. Amer. Math. Soc. (to appear)

[21] T. Kusano and H. Onose, Oscillations of functional defferential equations with retarded argument, J. Differential Equations (to appear).

[22] G. Ladas, Oscillation and asymptotic behavior of solutions of differential equations with retarded argument, J. Differential Equations **10** (1971), 281–290.

[23] G. Ladas, G. Ladde and J. S. Papadakis, Oscillations of functional-differential equations generated by delays, J. Differential Equations **12** (1972), 385–395.

[24] G. Ladas, V. Lakshmikantham and J. S. Papadakis, Oscillations of higher-order retarded differential equations generated by the retarded argument, "Delay and Functional Differential Equations and Their Applications" (Ed. by K. Schmitt), pp. 219–231, Academic Press, New York, 1972.

[25] I. Ličko and M. Švec, Le caractère oscillatoire des solutions de l'équation $y^{(n)}+f(x)y^{\alpha}=0$, $n>1$, Czechoslovak Math. J. **13** (1963), 481–491.

[26] P. Marušiak, On some properties of solutions of the differential equation, Acta Fac. Rerum Natur. Univ. Comenian, 26 (1972), 39–51.

[27] P. Marušiak, Note on Ladas' paper on oscillation and asymptotic behavior of solutions of differential equations with retarded argument, J. Differential Equations **13** (1973), 150–156.

[28] P. Marušiak, A differential equation with retarded argument asymptotically equivalent to the equation $y^{(n)}=0$, Mat. Časopis Sloven. Acad. Vied **23** (1973), 45–54. (Russian)

[29] Z. Nehari, On a class of nonlinear second order differential equations, Trans. Amer. Math. Soc. **95** (1960), 101–123.

[30] H. Onose, Some oscillation criteria for n-th order nonlinear delay-defferential equations, Hiroshima Math. J. **1** (1971), 171–176.

[31] G. H. Ryder and D. V. V. Wend, Oscillation of solutions of certain ordinary differential equations of nth order, Proc. Amer. Math. Soc. **25** (1970), 463–469.

[32] V. N. Ševelo and O. N. Odarič, Some problems of the theory of oscillation (nonoscillation) of solutions of second order differential equations with retarded argument, Ukrain. Mat. Ž. **23** (1971), 508–516. (Russian)

[33] V. N. Ševelo and N. V. Vareh, On the oscillation of solutions of higher order linear differential equations with retarded argument, Ukrain. Mat. Ž. **24** (1972), 513–520. (Russian)

[34] V. N. Ševelo and N. V. Vareh, On some properties of solutions of differential equations with delay, Ukrain. Mat. Ž. **24** (1972), 807–813. (Russian)

[35] Y. G. Sficas, On oscillation and asymptotic behavior of a certain class of differential equations with retarded argument, Utilitas Mathematica **3** (1973), 239–249.

[36] V. A. Staikos and A. G. Petsoulas, Some oscillation criteria for second order nonlinear delay-differential equations, J. Math. Anal. Appl. **30** (1970), 695–701.

[37] V. A. Staikos and Y. G. Sficas, Oscillatory and asymptotic behavior of functional differential equations, J. Differential Equations **12** (1972), 426–437.

[38] V. A. Staikos and Y. G. Sficas, Criteria for asymptotic and oscillatory character of functional differential equations of arbitrary order, Boll. Un. Mat. Ital. **6** (1972), 185–192.

[39] V. A. Staikos and Y. G. Sficas, Oscillatory and asymptotic ptoperties of differential equations with retarded arguments, Applicable Anal. (to appear)

[40] R. D. Terry and P.-K. Wong, Oscillatory properties of a fourth-order delay differential equation, Funkc. Ekvac. **15** (1972), 209–220.

[41] H. Teufel, Jr., Second order nonlinear oscillation — deviating arguments, Monatsh. Math. **75** (1971), 341–345.

[42] H. Teufel, Jr., A note on second order differential inequalities and functional differential equations, Pacific J. Math. **41** (1972), 537–541.

[43] H. Teufel, Jr., Second order almost linear functional differential equations — oscillation, Proc. Amer. Math. Soc. **35** (1972), 117–119.

[44] H. Teufel, Jr., Forced second order nonlinear oscillation, J. Math. Anal. Appl. **40** (1972), 148–152.

[45] C. C. Travis, Oscillation theorems for second-order differential equations with functional arguments, Proc. Amer. Math. Soc. **31** (1972), 199–202.

[46] A. Tychonoff, Ein Fixpundtsatz, Math. Ann. **111** (1935), 767–776.

[47] P. Waltman, On the asymptotic behavior of solutions of an nth order equation, Monatsh. Math. 69 (1965), 427–430.

[48] P. Waltmen, A note on an oscillation criterion for an equation with a functional argument, Canad. Math. Bull. **11** (1968), 593–595.

[49] J. S. W. Wong, Second order oscillation with retarded arguments, "Ordinary Differential Equations" (Ed. by L. Weiss), pp. 581–596, Academic Press, New York, 1972.

Department of Mathematics,
College of General Education,
Ibaraki University, Mito
JAPAN