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Character Groups of Toral Lie Algebras

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Introduction

It is well known that the duality and categorical equivalence hold between algebraic tori and character groups (e.g., [1], ch. III). In this paper we develop an analogy for Lie algebras. General properties of toral Lie algebras are stated in [3] and [5]. Their characters are introduced by Seligman in [3] and applied to algebraic Lie algebras in [3] and [4].

Let T be a toral Lie algebra and let $X(T)$ be the character group of T. Then it is proved that the (contravariant) functor X: $T \mapsto X(T)$ is actually an equivalence of categories (Theorem 1) and in this relation every subalgebra (resp. quotient algebra) of T corresponds to a quotient group (resp. subgroup) of $X(T)$ (Proposition 3).

As an application we generalize some of the results in [3]. Namely, if T satisfies a certain condition which generalizes that the base field is finite then the properties (a) and (b) of Theorem 7 in [3] are equivalent (Theorem 2) and the direct sum decomposition of T as in Theorem 8 in [3] holds (Theorem 3).

The main tools of the paper are the rationality property for vector spaces in terms of Galois groups which is described in [1] and the direct sum decomposition, stated in [2], of a vector space on which a nilpotent Lie algebra acts.

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1. Preliminaries and notations

Let k be any field of characteristic $p>0$. Let L be a Lie p-algebra over k of finite dimension with a p-map $x \mapsto x^p$. An element $x \in L$ is said to be separable if x is represented as a linear combination of x^p, $x^{p^2}, \dots$. If $T$is an abelian Lie p-algebra over k and every element of $T$is separable then $T$is called a torus or a toral Lie algebra over k. Some criteria for tori are found in [5]. Cleary every (p-)subalgebra of a torus is itself a torus. In this paper homomorphisms of Lie p-algebras always mean Lie algebra homomorphisms which are compatible with p-maps.

Let $\bar{k}$ be the algebraic closure of k and k_s be the separable closure of k in $\bar{k}$. Then $\bar{k}$ and k_s are regarded as Lie p-algebras over k with natural p-th power.

A homomorphism of a torus Tinto $\bar{k}$ is called a character of T. If ξ is a character of $T$and $x\in T$then $\xi(x)$ is a separable algebraic element over k, so that the image $\xi(T)$ is contained in k_s. The set of all characters is denoted by $X(T)$. $X(T)$ is an elementary p-group which is regarded as a vector space over P, the prime field of k.

Let $\Gamma=\mathrm{Gal}\,(k_s/k)$ be the Galois group of k_s over k. Γ has a usual topology with which it turns out a topological group (see e.g. N. Jacobson, Lectures in Abstract Algebra, vol. II, p. 149). When Γ acts on a set S as a group of transformations, the action is said to be continuous if the stability group of each $s\in S$ is an open subgroup in Γ. In this sense Γ acts on k_s continuously.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of linear transformations in a finite-dimensional vector space V. Then V has a decomposition $V=V_0(\mathfrak{g})\oplus V_1(\mathfrak{g})$ which is called the Fitting decomposition of V relative to $\mathfrak{g}$([2], Th. 2.4, p. 39). The subspaces $V_0(\mathfrak{g})$ and $V_1(\mathfrak{g})$ are $\mathfrak{g}$-stable, and $V_0(\mathfrak{g})$ is the maximal $\mathfrak{g}$-stable subspace of V on which the elements of $\mathfrak{g}$ are all nilpotent. In particular, $V_0(\mathfrak{g})$ has a composition series with $\mathfrak{g}$-trivial factors.

When $V$and $W$are vector spaces over k we denote by $\mathfrak{L}_k(V,W)$ the set of all k-linear maps of Vinto W.

2. The duality of tori and character groups

Let Tbe an n-dimensional torus over k. Then $\mathfrak{L}_k(T,\bar{k})$ forms a vector space over $\bar{k}$ of dimension n, and it contains $X(T)$.

LEMMA 1. *Let $\xi_1,\dots,\xi_m$ be (P-)linearly independent characters of T. Then they are linearly independent over $\bar{k}$.*

PROOF. Assume not. Let $\xi_1,\dots,\xi_r$ be linearly independent over $\bar{k}$ and let $\xi_{r+1}=\sum_{i=1}^{r}a_i\xi_i$, $a_i\in\bar{k}$, be a non-trivial linear relation. Then $\xi_{r+1}(z^p)=\Sigma a_i\xi_i(z^p)$. On the other hand $\xi_{r+1}(z^p)=(\xi_{r+1}(z))^p=\Sigma a_i^p\xi_i(z)^p=\Sigma a_i^p\xi_i(z^p)$. Since $\{z^p|z\in T\}$ spans T we have $\Sigma a_i\xi_i=\Sigma a_i^p\xi_i$. Therefore $a_i^p=a_i$, $i=1,\dots,r$, which implies $a_i\in P$ for all i. This contradicts the fact that $\xi_1,\dots,\xi_{r+1}$ are linearly independent over P.

Now we have $\mathfrak{L}_k(T,\bar{k})\simeq\mathfrak{L}_{\bar{k}}(\bar{k}\otimes_k T,\bar{k})$ and $\bar{k}\otimes T$ is a torus over $\bar{k}$ ([5], Cor. 2.6). On the other hand it is obvious that the p-map of a torus is 1: 1. Thus the torus $\bar{k}\otimes T$ is isomorphic to the direct sum of n copies of $\bar{k}$ ([2], Th. 5.13, p. 192), whose canonical projections are also characters. And then their restrictions to $T$are characters of T. Consequently we have seen that Thas at least n characters which are linearly independent over P. Therefore we have

COROLLARY 1. $\bar{k}\otimes_P X(T)\simeq\mathfrak{L}_k(T,\bar{k})$.

Taking the dual of the diagram in Corollary 1 we obtain the following

COROLLARY 2. $\mathfrak{L}_{\bar{k}}(\bar{k}\otimes_P X(T),\ \bar{k})\simeq \bar{k}\otimes_k T.$

REMARK. In Corollaries above we may replace $\bar{k}$ by k_s since every character is k_s-valued.

Next we define the action of Γ on $X(T)$ by the rule:

$$\xi^\sigma(x)=(\xi(x))^\sigma, \qquad \xi\in X(T), \quad \sigma\in\Gamma, \quad x\in T.$$

Since T is of finite dimension this action of Γ on $X(T)$ is continuous, i.e., for every $\xi\in X(T)$ $\Gamma_\xi=\{\sigma\in\Gamma | \xi^\sigma=\xi\}$ is an open subgroup (see § 1).

Let $f\colon T\to T'$ be a homomorphism of tori. Then f induces a Γ-homomorphism $X(f)$ of $X(T')$ into $X(T)$: $X(f)(\xi')=\xi'\circ f$ for $\xi'\in X(T')$. Then as easily seen X is a (contravariant) functor from a category of tori over k and homomorphisms to a category of elementary p-groups of finite rank on which Γ acts continuously and Γ-homomorphisms. If dim $T=n$ then clearly the order of the group $X(T)$ is p^n. From this fact we have

PROPOSITION 1. *X is an exact functor.*

To prove that the functor X is fully faithful we need some general notions of Galois criteria for rationality on vector spaces described in [1](§ 14.1, p. 52). Let V be a vector space over k. Then Γ acts on $k_s\otimes_k V$ in the following manner:

$$(a\otimes v)^\sigma=a^\sigma\otimes v, \qquad a\in k_s, \quad v\in V, \quad \sigma\in\Gamma.$$

Then $1\otimes V$ is the set of Γ-fixed elements. If W is a vector space over k_s on which Γ acts semi-linearly, i.e.,

$$(aw)^\sigma=a^\sigma w^\sigma, \qquad a\in k_s, \quad w\in W, \quad \sigma\in\Gamma,$$

then the dual space $\mathfrak{L}_{k_s}(W,k_s)$ of W permits the action of Γ by the rule:

$$u^\sigma(w)=(u(w^{\sigma^{-1}}))^\sigma, \qquad u\in\mathfrak{L}_{k_s}(W,k_s), \quad w\in W, \quad \sigma\in\Gamma.$$

Now we have by the Remark to Corollary 2 $\mathfrak{L}_{k_s}(k_s\otimes_P X(T))\simeq k_s\otimes_k T$. Since Γ acts semi-linearly on $k_s\otimes X(T)$ we have from the above discussion two actions of Γ on $k_s\otimes T$. However we have

LEMMA 2. *The two actions of Γ on $k_s\otimes T$ coincide.*

PROOF. Note that the above isomorphism is as follows:

$$(a\otimes x)(b\otimes\xi)=ab\xi(x), \qquad a,\ b\in k_s, \quad x\in T, \quad \xi\in X(T).$$

Let $\sigma\in\Gamma$. We start to calculate $(a\otimes x)^\sigma(b\otimes\xi)$ along the action on the dual space.

$$\begin{aligned}(a\otimes x)^\sigma(b\otimes\xi)&=((a\otimes x)((b\otimes\xi)^{\sigma^{-1}}))^\sigma\\ &=((a\otimes x)(b^{\sigma^{-1}}\otimes\xi^{\sigma^{-1}}))^\sigma\end{aligned}$$

$$=(ab^{\sigma^{-1}}\xi^{\sigma^{-1}}(x))^{\sigma}$$

$$=a^{\sigma}b\xi(x)$$

$$=(a^{\sigma}\otimes x)(b\otimes\xi).$$

This shows that the action equals that on the tensor product.

PROPOSITION 2. *The functor X is fully faithful, that is,* X: $\mathrm{Hom}(T,T')\to \mathrm{Hom}_{\Gamma}(X(T'), X(T))$ *is bijective. In particular, if* $X(T')\simeq X(T)$ *then* $T\simeq T'$.

PROOF. Injectivity. Assume $X(f)=X(g)$. This implies that $\xi'(f(x)-g(x))=0$ for all $\xi'\in X(T')$ and all $x\in T$. By Corollary 1 we have $f(x)=g(x)$ so that $f=g$.

Surjectivity. Let ψ: $X(T')\to X(T)$ be a Γ-homomorphism. Then $1\otimes\psi$: $k_s\otimes_P X(T')\to k_s\otimes_P X(T)$ is a Γ-homomorphism, where Γ acts on these vector spaces semi-linearly. Taking the dual of this diagram we have ${}^t(1\otimes\psi)$: $\mathfrak{L}_{k_s}(k_s\otimes X(T), k_s)\to\mathfrak{L}_{k_s}(k_s\otimes X(T'), k_s)$ and this is a k_s-linear map. By the Remark to corollary we have $\mathfrak{L}_{k_s}(k_s\otimes X(T), k_s)\simeq k_s\otimes_k T$ and a similar isomorphism for T'. These are provided with the action of Γ and ${}^t(1\otimes\psi)$ is a Γ-homomorphism. Moreover it is a homomorphism of k_s-tori since $1\otimes\psi(\xi')(a^p\otimes x^p)=a^p\otimes\psi(\xi')(x^p)=a^p\otimes(\psi(\xi')(x))^p=(a\otimes\psi(\xi')(x))^p=(1\otimes\psi(\xi')(a\otimes x))^p$ for $a\in k_s$ and $x\in T$. By Lemma 2 and the previous discussion the set of Γ-fixed elements of $k_s\otimes T$(resp. $k_s\otimes T'$) is $1\otimes T$ (resp. $1\otimes T'$) and ${}^t(1\otimes\psi)$ maps $1\otimes T$ into $1\otimes T'$. Let f be the restriction of ${}^t(1\otimes\psi)$ to $1\otimes T$. Identify $1\otimes T$ (resp. $1\otimes T'$) with T (resp. T'). Since ${}^t(1\otimes\psi)$ is a homomorphism of k_s-tori, we see that f is a homomorphism of k-tori, and we have $\psi=X(f)$ as directly checked.

THEOREM 1. *The functor X is an equivalence of categories.*

PROOF. Since the functor X is fully faithful by Proposition 2, it remains to prove that for an elementary p-group X of finite rank on which Γ acts continuously there exists a torus T over k such that $X\simeq X(T)$. Let n be the rank (the dimension over P) of X. And let $V=\mathrm{Hom}(X, k_s)$. This is an n-dimensional vector space over k_s. Moreover it is a torus over k_s with the following p-map:

$$z^p(\xi)=z(\xi)^p, \qquad z\in\mathrm{Hom}(X, k_s), \quad \xi\in X.$$

In fact, let $x_1,\dots, x_n$ be a basis of V. Then it suffices to see that $x_1^p,\dots, x_n^p$ are linearly independent ([4], Prop. 2.5, (2)). Let $\xi_1,\dots, \xi_n$ be a basis of X (as a vector space over P). Then we have $\det(x_i(\xi_j))\neq 0$. It follows that $\det(x_i(\xi_j))^p=\det(x_i(\xi_j)^p)=\det(x_i^p(\xi_j))\neq 0$, which shows linear independence of x_i^p's. On the other hand we have $V\simeq\mathfrak{L}_{k_s}(k_s\otimes_P X, k_s)$ on which Γ acts continuously. In fact, let $x\in V$ and let Γ_x be the stability group of x. By the definition of the action we have

$$\Gamma_x = \{\sigma \in \Gamma \mid x(\xi)^\sigma = x(\xi^\sigma) \qquad \text{for all } \xi \in X\}.$$

Since the action of Γ on X is continuous the stability group Γ_ξ of ξ is an open subgroup for each $\xi \in X$. Similarly the stability group $\Gamma_{x(\xi)}$ of $x(\xi) \in k_s$ is an open subgroup. Therefore the intersection $\Gamma_\xi \cap \Gamma_{x(\xi)}$ is an open subgroup of Γ. Since X is finite the intersection $\bigcap_{\xi \in X} \Gamma_\xi \cap \Gamma_{x(\xi)}$ is also an open subgroup of Γ and it is contained in Γ_x. Consequently Γ_x is also an open subgroup of Γ since Γ is a topological group. Thus V has a k-structure $T = V^\Gamma = \mathrm{Hom}_\Gamma(X, k_s)$, the set of Γ-fixed elements in V ([1], § 14 Ch. AG). It is easy to see that T is an n-dimensional torus over k and the map $\xi \mapsto (z \mapsto z(\xi))$ is a Γ-homomorphism of X onto $X(T)$.

Let S be a subtorus of T. Then

$$S^\circ = \{\xi \in X(T) \mid \xi(x) = 0 \qquad \text{for all } x \in S\}$$

is a Γ-stable subgroup of $X(T)$. Conversely if Y is a subgroup of $X(T)$ then the set

$$Y^\circ = \{x \in T \mid \xi(x) = 0 \qquad \text{for all } \xi \in Y\}$$

is a subtorus of T. Then we have

PROPOSITION 3. *The maps $S \mapsto S^\circ$ and $Y \mapsto Y^\circ$ define reciprocal bijections between the collection of subtori of T and the collection of Γ-stable subgroups of $X(T)$. Moreover we have canonical isomorphisms $S^\circ \simeq X(T/S)$ for every subtorus S of T and $Y \simeq X(T/Y^\circ)$ for every Γ-stable subgroup Y of $X(T)$.*

PROOF. Let S be a subtorus of T. The canonical isomorphism $S^\circ \simeq X(T/S)$ follows from the fact that the functor X is exact and S° is the kernel of the restriction map $X(T) \to X(S)$. It is clear that $S \subset S^{\circ\circ}$ and $S^{\circ\circ\circ} = S^\circ$. From the exact sequence $0 \to S \to T \to T/S \to 0$ we obtain an exact sequence

$$(1) \qquad 0 \leftarrow X(S) \leftarrow X(T) \leftarrow X(T/S) \leftarrow 0,$$

where $X(T/S) \simeq S^\circ$. In the same way we have

$$(2) \qquad 0 \to S^{\circ\circ} \to T \to T/S^{\circ\circ} \to 0$$

and

$$(3) \qquad 0 \leftarrow X(S^{\circ\circ}) \leftarrow X(T) \leftarrow S^{\circ\circ\circ} \leftarrow 0.$$

Calculating the dimension of $S^{\circ\circ}$, we have

$$\begin{aligned} \dim S^{\circ\circ} &= \log |X(S^{\circ\circ})| / \log p \\ &= \log(|X(T)| / |S^\circ|) / \log p \qquad \text{(by (3))} \end{aligned}$$

$$=\log|X(T)|/\log p \qquad \text{(by (1))}$$
$$=\dim S.$$

Therefore $S^{\circ\circ}=S$.

Conversely let Y be any Γ-stable subgroup of $X(T)$. Then Y is represented as a character group $X(U)$ for some torus U over k by Theorem 1. Then the bijectivity of X: $\mathrm{Hom}(T,U)\to\mathrm{Hom}_{\Gamma}(Y, X(T))$ gives a unique homomorphism $f: T\to U$ such that $X(f)$ is the inclusion map of Y into $X(T)$. It is easy to see that f is surjective. Let $S=\mathrm{Ker} f$. Then we have $Y=S^{\circ}$. Furthermore this implies $Y^{\circ}=S$ so that $Y=S^{\circ}=X(T/S)=X(T/Y^{\circ})$. This completes the proof.

3. Some structure theorems of tori

Let K be a subfield of k_s containing k. K is called a splitting field of T if every character of T is K-valued ([3], Th. 6).

PROPOSITION 4. *T has a unique minimal splitting field K, which is a finite Galois extension field of k. And there exists a canonical isomorphism of the Galois group* $\mathrm{Gal}(K/k)$ *onto a subgroup of the group of all automorphisms of* $X(T)$.

PROOF. Let N be the kernel of the representation of Γ on $X(T)$: $N=\mathrm{Ker}(\Gamma\to\mathrm{Aut}(X(T)))$. Since the action of Γ on $X(T)$ is continuous N is an open normal subgroup with finite index. Then the subfield of N-invariants is a finite Galois extension of k and we have an isomorphism $\mathrm{Gal}(K/k)\simeq\Gamma/N$. It is easy to see that K is a splitting field of T. K is the minimal one. In fact, let $K'\subset k_s$ be any splitting field of T. Let N' be the subgroup of Γ of elements σ such that $a^{\sigma}=a$ for all $a\in K'$. Then N' is a closed subgroup. Since K' is a splitting field of T every element of N' acts identically on $X(T)$. It follows that $N'\subset N$. Consequently we have $K\subset K'$ which asserts the minimality of K.

Now let U be a subtorus of T and let L be the minimal splitting field of U. Then L is a subfield of K, the minimal splitting field of T. And let H be the Galois group of L/k. Then H is the quotient group of G, the Galois group of K/k, by the normal subgroup $\{\sigma\in G|a^{\sigma}=a \text{ for all } a\in L\}$. Let $\pi: G\to H$ be the natural projection and let $\phi: X(T)\to X(U)$ be the homomorphism corresponding to the inclusion map $U\to T$.

Since G and H are quotient groups of Γ and since ϕ is a Γ-homomorphism we have the following lemma concerning the action of G and H on $X(T)$ and $X(U)$ respectively.

LEMMA 3. For $\xi\in X(T)$ *and* $\sigma\in G$

$$\phi(\xi^{\sigma})=(\phi(\xi))^{\pi(\sigma)}.$$

Now by Proposition 4 we may consider G (resp. H) as a subgroup of the general linear group $\mathrm{GL}(X(T))$ (resp. $\mathrm{GL}(X(U))$). Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be a Lie algebra generated by the set $\{\sigma-1|\sigma\in G\}$ (resp. $\{\sigma-1|\sigma\in H\}$). An easy calculation shows that $\mathfrak{g}$(resp. $\mathfrak{h}$) is given in fact as a linear span of the set in $\mathfrak{gl}(X(T))$ (resp. $\mathfrak{gl}(X(U))$). Then we have

LEMMA 4. *There exists a surjective Lie algebra homomorphism* $\bar{\pi}: \mathfrak{g}\to\mathfrak{h}$ *such that*

$$\bar{\pi}(\sigma-1)=\pi(\sigma)-1, \qquad \sigma\in G.$$

PROOF. It suffices to prove that the map π' of the set $\{\sigma-1|\sigma\in G\}$ onto the set $\{\sigma-1|\sigma\in H\}$ defined by $\pi'(\sigma-1)=\pi(\sigma)-1$ can be extended to a linear map $\bar{\pi}: \mathfrak{g}\to\mathfrak{h}$. In this case the map $\bar{\pi}$ is in fact a Lie algebra homomorphism as directly checked. Now let $\sigma_1,\dots,\sigma_r\in G$. Then we have only to prove the following fact:

$$\text{If } \textstyle\sum_{i=1}^{r} a_i(\sigma_i-1)=0 \quad (a_i\in P) \text{ then } \textstyle\sum_{i=1}^{r} a_i(\pi(\sigma_i)-1)=0.$$

Let ξ' be any element of $X(U)$. Then there is a $\xi\in X(T)$ such that $\xi'=\phi(\xi)$. Then

$$\begin{aligned}\xi'\Sigma a_i(\pi(\sigma_i)-1) &= \Sigma a_i(\phi(\xi)^{\pi(\sigma_i)}-\phi(\xi))\\ &= \Sigma a_i(\phi(\xi^{\sigma_i})-\phi(\xi)) \quad \text{(by Lemma 3)}\\ &= \Sigma a_i\phi(\xi(\sigma_i-1))\\ &= \phi(\xi\Sigma a_i(\sigma_i-1))\\ &= 0.\end{aligned}$$

Therefore $\Sigma a_i(\pi(\sigma_i)-1)=0$.

By Lemma 3 and 4 we immediately have

LEMMA 5. *For* $A\in\mathfrak{g}$ *and* $\xi\in X(T)$

$$\phi(\xi A)=\phi(\xi)\bar{\pi}(A).$$

LEMMA 6. *Let* $\mathfrak{g}$ *and therefore* $\mathfrak{h}$ *be nilpotent. Let*

$$X(T)=X(T)_0\oplus X(T)_1 \quad (\textit{resp. } X(U)=X(U)_0\oplus X(U)_1)$$

be the Fitting decomposition of $X(T)$ (*resp.* $X(U)$) *relative to* $\mathfrak{g}$ (*resp.* $\mathfrak{h}$). *Then* $\phi(X(T)_0)\subset X(U)_0$ *and* $\phi(X(T)_1)\subset X(U)_1$.

PROOF. Let $\xi\in X(T)_0$. For any $B\in\mathfrak{h}$, we have $B=\bar{\pi}(A)$ for some $A\in\mathfrak{g}$ by Lemma 4. It follows from Lemma 5 that $\phi(\xi)B^m=\phi(\xi A^m)=0$ for a large m.

This implies that $\phi(\xi)\in X(U)_0$.

Next let $\xi\in X(T)_1$ and let $l\geqq 0$ be any integer. Then ξ is of the form $\xi=\Sigma\,\eta A_1\dots A_l(A_i\in\mathfrak{g},\ \eta\in X(T))$. Therefore by Lemma 5 $\phi(\xi)=\Sigma\phi(\eta)\bar{\pi}(A_1)\dots\bar{\pi}(A_l)\in X(U)(\mathfrak{h}^*)^l$. Hence $\phi(\xi)\in\bigcap_{l\geqq 0}X(U)(\mathfrak{h}^*)^l=X(U)_1$.

We now have a generalization of Theorem 7 in [3].

THEOREM 2. *Let T, K, G and $\mathfrak{g}$ be as above. If $\mathfrak{g}$ is a nilpotent Lie algebra then the following two conditions on T are equivalent.*

a) *The only k-valued character of T is zero,*

b) *T contains no subtorus isomorphic to k.*

PROOF. a)$\Rightarrow$b). Let $U\simeq k$ be a subtorus of T. Then $\mathfrak{h}=0$ in the previous notation. Consequently $X(U)_1=0$ so that by Lemma 6 $\phi(X(T)_1)=0$, that is, $\phi(X(T)_0)=X(U)_0=X(U)$. This implies that $X(T)_0\neq 0$. On the other hand every element of $\mathfrak{g}$ acts on $X(T)_0$ as a nilpotent linear transformation. Therefore there exists a $\xi\neq 0$ in $X(T)_0$ such that $\xi A=0$ for all $A\in\mathfrak{g}$. Hence $\xi^\sigma=\xi$ for all $\sigma\in G$. It follows that $\xi^\sigma=\xi$ for all $\sigma\in\Gamma$ which implies that ξ is k-valued.

b)$\Rightarrow$a). Let $\xi\neq 0$ be a k-valued character of T. Then ξ is a Γ-fixed and so G-fixed element in $X(T)$. Thus ξ is in $X(T)_0$, that is, $X(T)_0\neq 0$. But $X(T)_0$ has a composition series with $\mathfrak{g}$-trivial and so G-trivial factors. Hence there exists a G-stable subgroup Y of $X(T)_0$ such that $X(T)_0/Y\simeq P\simeq X(k)$. Now let ϕ: $X(T)\to X(k)$ be the natural map with the kernel $Y+X(T)_1$. Since ϕ is surjective the corresponding homomorphism $k\to T$ is injective. This implies that T has a subtorus isomorphic to k.

COROLLARY. *If G is abelian then conditions* a) *and* b) *in Theorem* 2 *are equivalent. In particular, it is the case if k is finite.*

PROOF. If G is abelian then the corresponding Lie algebra is also abelian. In particular, when k is finite then G is a cyclic group.

As in [3] a torus T is said to be anisotropic if T satisfies condition a) of Theorem 2, and semisplit if T has a composition series with factors isomorphic to k. We have the first part of Theorem 8 in [2].

PROPOSITION 5. *Let T be a torus over k. Then T has a unique maximal anisotropic subtorus and a unique maximal semisplit subtorus.*

PROOF. Since if T_1 and T_2 are subtori of T then T_1+T_2 is also a subtorus it suffices to see that if they are anisotropic (resp. semisplit) so is T_1+T_2. But these are immediate consequences of definitions.

If ξ is a character of T then ξ is k-valued if and only if it is a Γ-fixed element in $X(T)$, that is, $\xi^\sigma=\xi$ for every $\sigma\in\Gamma$. Therefore we have proved the first part of the follosing

LEMMA 7. *T is anisotropic if and only if the only Γ-fixed element in $X(T)$ is zero and T is semisplit if and only if $X(T)$ has a composition series with Γ-trivial factors.*

PROOF. It remains to prove the last part. Now let T be semisplit. Then by definition there exists a chain of subtori $0=T_0\subset T_1\subset\cdots\subset T_n=T$ such that $T_i/T_{i-1}\simeq k$ for $i=1,\dots,n$. Therefore we have a chain of Γ-stable subgroups $X(T)=T_0^\circ\supset T_1^\circ\supset\cdots\supset T_n^\circ=0$. Consider an exact sequence of tori $0\to T_i/T_{i-1}\to T/T_{i-1}\to T/T_i\to 0$. From this we have an exact sequence of Γ-modules and Γ-homomorphisms $0\to X(T_i/T_{i-1})\to X(T/T_{i-1})\to X(T/T_i)\to 0$, where by Proposition 3 $X(T/T_{i-1})\simeq T_{i-1}^\circ$ and $X(T/T_i)\simeq T_i^\circ$. Therefore we have $T_{i-1}{}^\circ/T_i^\circ\simeq X(T_i/T_{i-1})\simeq X(k)$ on which Γ acts trivially. The converse is proved in a similar way.

THEOREM 3. *Let T, G and $\mathfrak{g}$ be as in Theorem 2. If $\mathfrak{g}$ is nilpotent then $T=A\oplus S$ where A is the maximal anisotropic subtorus and S the maximal semisplit subtorus.*

PROOF. Let $X(T)=X(T)_0\oplus X(T)_1$ be the Fitting decomposition of $X(T)$ relative to $\mathfrak{g}$. Note that $X(T)_i$ $(i=0,1)$ is a Γ-stable sugbroup of $X(T)$. Thus by Theorem 1 and Proposition 3 we have a decomposition of T into a derect sum of two subtori, say $T=A\oplus S$, where $A=X(T)_0^\circ$ and $S=X(T)_1^\circ$. In this case $X(A)\simeq X(T)_1$ and $X(S)\simeq X(T)_0$, so we identify these respectively.

To prove that A is anisotropic let $\xi\in X(A)$ such that $\xi^\sigma=\xi$ for all $\sigma\in\Gamma$. Then $\xi^\sigma=\xi$ for all $\sigma\in G$ since the action of G on $X(T)$ is induced by that of Γ. Therefore $\xi B=0$ for all $B\in\mathfrak{g}$ which implies $\xi\in X(T)_0$ so that $\xi=0$. By Lemma 7 A is in fact anisotropic. On the other hand since $X(T)_0$ has a composition series with $\mathfrak{g}$-trivial factors which are also G-trivial. Therefore these factors are also Γ-trivial. Hence S is semisplit by Lemma 7.

Finally we must prove the maximality of A and S. Now let A' be any anisotropic subtorus of containing A. We can apply Lemma 6 for $U=A'$. Then ϕ maps $X(S)$ onto $X(A')_0$ and $X(A)$ onto $X(A')_1$. But since A' is anisotropic by Lemma 7 and the construction of $\mathfrak{g}$ we have $X(A')_0=0$ and then $X(A')_1=X(A')$. Therefore $\phi(X(A))=X(A')$. Consequently we have $|X(A)|\geqq|X(A')|$ so that $\dim A\geqq\dim A'$. It follows that $A=A'$. By Proposition 5 this shows the maximality of A. The maximality of S can be proved similarly.

By the same reasoning as in the proof of the Corollary to Theorem 2 we obtain the following

COROLLARY. *If G is abelian then the derect sum decomposition of T holds. In particular it is the case if k is finite.*

References

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