

Isomorphisms between Interval Sublattices of an Orthomodular Lattice

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1. Introduction.

This paper deals with the following question: given orthogonal projective elements a, b of an orthomodular lattice \mathcal{L} , under what general circumstances are the interval sublattices $\mathcal{L}(0, a)$, $\mathcal{L}(0, b)$ orthoisomorphic?

The answer that we offer provides a description of a class of lattices, called uniform, in which not only do the indicated orthoisomorphisms exist, but they are explicitly displayed as simple lattice polynomials. This desirable state of affairs is achieved through the use of a strong postulate that requires the existence of certain special kinds of elements of \mathcal{L} .

The postulate is framed in terms of a new relation "UA" between pairs of non-zero elements p, q of an orthomodular lattice \mathcal{L} . We write $p \# q$ when

$$x \leq q \Rightarrow (p \wedge (p^\perp \vee x)) \perp (q \wedge x^\perp).$$

This resembles the condition that p, q form a modular pair, and is in fact stronger (see (4) of 2.4 and remarks following the proof of 4.1). The relation UA (p, q) is then defined as the symmetrization of $\#$, subject to a side condition to rule out trivial complications. The exact definition is this: $\text{UA}(p, q) \Leftrightarrow$ both $p \# q$, $q \# p$ and $p \wedge q = p \wedge q^\perp = p^\perp \wedge q = 0$. The letters UA are intended to suggest "uniform angle", and the relationship $\text{UA}(p, q)$ may be read as " p and q have a uniform angle between them". This terminology is derived from a geometric interpretation available when \mathcal{L} is the lattice of projections of Hilbert space — see 4.4.

A uniform orthomodular lattice is defined by the following property: given any pair of non-zero orthogonal projective elements a, b , there is an element $h \leq a \oplus b$ that makes a uniform angle with both a and b . We call such an element h "splitting" for the pair a, b . The desired orthoisomorphism between the interval sublattices $\mathcal{L}(0, a)$, $\mathcal{L}(0, b)$ is constructed through the use of the special properties of the splitting element h .

This definition has the advantage of being easily verified in a large class of examples, namely the projection lattices of von Neumann algebras, and does lead swiftly to a simple, explicit formula for the desired orthoisomorphisms (Theorem 3.1). Another possible advantage is that the explicit nature of the definition may promote the building of a reasonably detailed theory of these lattices.

Clearly their usefulness must ultimately be measured by the success of such a program. There are some hopeful signs.

In his work on the lattice of left annihilators of Baer-type rings [7], Janowitz has dealt with the problem of finding lattice-induced orthoisomorphisms between interval sublattices of the projection lattice of a Rickart \ast -ring. The closing comment of his paper connects directly with uniform orthomodular lattices. It suggests that "a suitable vehicle for lattice dimension theory ought to be a complete orthomodular lattice such that $e \perp f$, $e \approx f \Rightarrow e$ is modularly perspective to f ". (The notation $e \approx f$ means that e and f are projective. To say that e is modularly perspective to f means the existence of a g satisfying $e \vee g = f \vee g = e \vee f$, $e \wedge g = f \wedge g = 0$ and such that all pairs (e, g) , (g, e) , (f, g) , (g, f) are both modular and dual modular.) A uniform orthomodular lattice comes very close to having Janowitz's property (2.9), and we may hope accordingly that these lattices will prove useful in lattice dimension theory.

As another hopeful indication of the possible future role to be played by such lattices, we mention the fact, cited by F. Maeda and S. Maeda [8; 36.14], that one may derive the 0-symmetry of an orthomodular lattice from the existence of sufficiently many orthoisomorphisms.

This paper is organized as follows: §2 contains basic material about the relations $\#$ and UA, §3 contains the statement and proof of the main theorem, and §4 is devoted to proving that the projection lattice of any von Neumann algebra is uniform, and to deriving various analytic forms of $\#$ and UA in projection lattices of von Neumann algebras.

The paper assumes some familiarity with the elementary theory of orthomodular lattices as presented for example in Birkhoff's book [1; Ch. I, §14], the Maedas' book [8; §§29, 36] or my survey article [5], and some knowledge of the basic properties of "Sasaki projections" which are summarized by Foulis in [3; Lemmas 1 and 2], and developed in more detail in his lecture notes [4]. Inasmuch as the Sasaki projections are somewhat technical, I shall for the reader's convenience repeat the definition here.

Given the element a in the orthomodular lattice \mathcal{L} , the Sasaki projection ϕ_a belonging to a is the order preserving map of \mathcal{L} onto its interval sublattice $\mathcal{L}(0, a)$ defined by $\phi_a(x) = a - (a \wedge x^\perp)$ (where I have used the convention of writing $a-b$ for $a \wedge b^\perp$ when $b \leq a$). We shall denote the composition of these maps in the usual way: $\phi_b \phi_a(x) = \phi_b(\phi_a(x))$.

2. The definition of uniform orthomodular lattices; preliminary material

2.1. *As conditions on the pair a, b of elements of an orthomodular lattice, the following are equivalent:*

$$(1) \quad x \leq b \Rightarrow \phi_a(x) \perp (b-x)$$

- (2) $x \leq b \Rightarrow \phi_a(x) = a \wedge (b-x)^\perp - (a \wedge b^\perp)$
- (3) $x \leq b \Rightarrow [(x \vee a) \wedge (x \vee a^\perp) \wedge x^\perp] \perp b$
- (4) $x \leq b \Rightarrow \phi_b \phi_a(x) = \phi_x(a)$.

PROOF. First, we shall prove the equivalence of the first three conditions, and then separately verify the equivalence of (1) and (4).

(1) \Rightarrow (2). Since $\phi_a(x) \leq a$, we can conclude, owing to (1), that $\phi_a(x) \leq a \wedge (b-x)^\perp$. Then

$$\begin{aligned} a \wedge (b-x)^\perp - \phi_a(x) &= a \wedge (b-x)^\perp - (a - a \wedge x^\perp) \\ &= a \wedge (b-x)^\perp \wedge (a^\perp \vee (a \wedge x^\perp)). \end{aligned}$$

Since a commutes with both a^\perp and $a \wedge x^\perp$, we can distribute to get

$$\begin{aligned} &= (b-x)^\perp \wedge a \wedge x^\perp \\ &= ((b-x) \oplus x)^\perp \wedge a = b^\perp \wedge a. \end{aligned}$$

Rearranged, this gives (2).

(2) \Rightarrow (3). A straightforward calculation verifies that

$$(x \vee a) \wedge (x \vee a^\perp) \wedge x^\perp = x \vee \phi_a(x) - x = \phi_{x^\perp} \phi_a(x).$$

Owing to (2), we have $\phi_a(x) \leq (b-x)^\perp$, whence $\phi_{x^\perp}(\phi_a(x)) \leq \phi_{x^\perp}((b-x)^\perp) = b^\perp$, a Sasaki projection being order preserving. This is the desired conclusion.

(3) \Rightarrow (1). Statement (3) says that

$$x \vee \phi_a(x) - x \leq b^\perp, \text{ so } \phi_a(x) \leq x \vee \phi_a(x) \leq b^\perp \vee x = (b-x)^\perp.$$

(1) \Rightarrow (4). Using (1) and remembering $x \leq b$, we obtain $\phi_b \phi_a(x) \leq \phi_b((b-x)^\perp) = x$. Then

$$\begin{aligned} x - \phi_b \phi_a(x) &= x \wedge (b^\perp \vee (b \wedge \phi_a(x)^\perp)) \\ &= x \wedge \phi_a(x)^\perp = x \wedge a^\perp, \end{aligned}$$

and rearranging this formula we get the result.

(4) \Rightarrow (1). Assuming (4) we have $\phi_b \phi_a(x) \leq x$ so $b^\perp \vee (\phi_b \phi_a(x)) \leq b^\perp \vee x = (b-x)^\perp$. But

$$\begin{aligned} b^\perp \vee (\phi_b \phi_a(x)) &= b^\perp \vee (b \wedge (b^\perp \vee \phi_a(x))) \\ &= b^\perp \vee \phi_a(x) \end{aligned}$$

so $\phi_a(x) \leq b^\perp \vee \phi_a(x) \leq (b-x)^\perp$ which is (1). That completes all parts of our proof.

We shall use the notation $a \# b$ to signify that the pair (a, b) satisfies the con-

dition whose equivalent formulations are listed in 2.1. It forms the basis for the following series of definitions.

2.2. DEFINITION. *The non-zero elements a, b of an orthomodular lattice are said to have a uniform angle between them, symbolized $UA(a, b)$, provided that $a \wedge b = a \wedge b^\perp = a^\perp \wedge b = 0$ and both $a \# b, b \# a$.*

The key definition then is this:

2.3. DEFINITION. *An orthomodular lattice is UNIFORM if for each pair of non-zero orthogonal projective elements a, b there is an $h \leq a \oplus b$ such that $UA(a, h)$ and $UA(h, b)$.*

An element h that satisfies the conditions of Definition 2.3 we call *splitting* for the pair a, b . Using this term we can restate the definition as follows: *An orthomodular lattice is uniform provided that every pair of orthogonal projective elements has a splitting element.*

We remind the reader that p and q are called *perspective*, written $p \sim q$, if they have a common complement, i.e. there is an x such that $p \vee x = q \vee x = 1$, $p \wedge x = q \wedge x = 0$, and are called *strongly perspective*, written $p \simeq q$, if they have a common complement in their own join, i.e. there is an x such that $p \vee x = q \vee x = p \vee q$, $p \wedge x = q \wedge x = 0$. Strongly perspective elements are perspective but not vice versa, unless the lattice is modular [6]. Finally, we say that p and q are *projective* if there are r_1, r_2, \dots, r_n such that $p \sim r_1 \sim r_2 \sim \dots \sim r_n \sim q$.

We use the rest of this section to derive some basic facts about the relations “ $\#$ ” and “UA”.

2.4. *The following assertions about the relation $\#$ are valid in any orthomodular lattice \mathcal{L} .*

- (1) *If a commutes with every $x \leq b$, then $a \# b$.*
- (2) *If b is an atom, then $a \# b$ for every a .*
- (3) *If $a \# b$, then $a^\perp \# b$.*
- (4) *If $a \# b$, then both (a, b) and (a^\perp, b) are modular pairs.*
- (5) *If $a \# b$, then $a \# y$ for every $y \leq b$.*

PROOF. If a commutes with every $x \leq b$, then $\phi_a(x) = a \wedge x$ which is always orthogonal to $b - x$; hence, according to (1) of 2.1, we have $a \# b$. That establishes (1). As a consequence of (1) we can deduce $a \# b$ in any one of the following situations: $a \geq b$, or $a \in \text{center}(\mathcal{L})$, or $a \perp b$.

Statement (2) is an immediate consequence of the fact that $x \leq b \Rightarrow x = 0$ or $x = b$ when b is an atom.

Statement (3) is easily deduced from the fact that (3) of 2.1 is unchanged when we substitute a^\perp for a , and statement (5) follows directly from (1) of 2.1.

This leaves (4), the only really non-trivial assertion in the list. Owing to (3) it is enough to prove that $a \# b \Rightarrow (a^\perp, b)M$. Now $(a^\perp, b)M$ (which is symbolic for the assertion that (a^\perp, b) is a modular pair) means that $x \leq b \Rightarrow (x \vee a^\perp) \wedge b = x \vee (a^\perp \wedge b)$. Now always $x \vee (a^\perp \wedge b) \leq (x \vee a^\perp) \wedge b$, so it is enough to prove that

$$z = (x \vee a^\perp) \wedge b - (x \vee (a^\perp \wedge b)) = 0.$$

Now

$$\begin{aligned} z &= (x \vee a^\perp) \wedge b \wedge x^\perp \wedge (a \vee b^\perp) \\ &= (x \vee a^\perp) \wedge (a \vee b^\perp) \wedge (b - x) \end{aligned}$$

and

$$\begin{aligned} (a \vee b^\perp) \wedge (x \vee a^\perp) &= a \vee b^\perp - a \wedge x^\perp \\ &= (a \vee b^\perp - a) \oplus (a - a \wedge x^\perp) \\ &= [a^\perp \wedge (a \vee b^\perp)] \oplus \phi_a(x). \end{aligned}$$

Then

$$z = \{[a^\perp \wedge (a \vee b^\perp)] \oplus \phi_a(x)\} \wedge (b - x).$$

The element $\phi_a(x)$ being $\leq a$ is orthogonal to $a^\perp \wedge (a \vee b^\perp)$; and, owing to the assumption $a \# b$ in the form (1) of 2.1, $\phi_a(x)$ is also orthogonal to $b - x$. Hence we can distribute to get

$$\begin{aligned} z &= a^\perp \wedge (a \vee b^\perp) \wedge (b - x) \leq (a \vee b^\perp) \wedge a^\perp \wedge b \\ &= (a \vee b^\perp) \wedge (a \vee b^\perp)^\perp = 0 \end{aligned}$$

which was to be proved.

If $b \wedge a^\perp = 0$, then we can express the relation $a \# b$ in a particularly nice form using Sasaki projections.

2.5. *Both relations $a \# b$ and $b \wedge a^\perp = 0$ hold for elements a, b of an orthomodular lattice \mathcal{L} when and only when $\phi_b \phi_a \phi_b = \phi_b$.*

This latter condition resembles von Neumann's "regularity" axiom in ring theory.

We know from (4) of 2.1 that $a \# b$ is equivalent to $x \leq b \Rightarrow \phi_b \phi_a(x) = \phi_x(a) = x - x \wedge a^\perp$. If $b \wedge a^\perp = 0$, then $x \wedge a^\perp = 0$ so $\phi_b \phi_a(x) = x$ whenever $x \leq b$. For any $y \in \mathcal{L}$, $\phi_b(y) \leq b$ so $\phi_b \phi_a \phi_b(y) = \phi_b(y)$, which is to say $\phi_b \phi_a \phi_b = \phi_b$. Conversely, if this equation is valid, then

$$b \wedge a^\perp = \phi_b(b \wedge a^\perp) = \phi_b \phi_a(b \wedge a^\perp) = \phi_b(0) = 0,$$

and then $x \leq b \Rightarrow \phi_b \phi_a(x) = \phi_b \phi_a \phi_b(x) = \phi_b(x) = x = \phi_x(a)$ which is (4) of 2.1, an equivalent form of $a \# b$.

Using 2.5, we can express the relation UA in a very convenient form:

2.6. For non-zero elements a, b , we have $UA(a, b) \Leftrightarrow a \wedge b = 0$, $\phi_b \phi_a \phi_b = \phi_b$, and $\phi_a \phi_b \phi_a = \phi_a$.

The proof of the main theorem is based on 2.1 through 2.6. We close this section with two results, 2.8 and 2.9, that provide additional information about the relation UA. The first result asserts a kind of transitivity of UA, and requires an elementary preliminary fact about Sasaki projections.

2.7. If $a \perp (b \vee c)$, then $\phi_{a \oplus b} \phi_c = \phi_b \phi_c$.

Owing to the fact that for any $x \in \mathcal{L}$, a is orthogonal to both b and $\phi_c(x)$, we have

$$\begin{aligned} \phi_{a \oplus b} \phi_c(x) &= (a \vee b) \wedge ((a^\perp \wedge b^\perp) \vee \phi_c(x)) \\ &= (a \vee b) \wedge (a^\perp \vee \phi_c(x)) \wedge (b^\perp \vee \phi_c(x)) \\ &= (a \vee b) \wedge a^\perp \wedge (b^\perp \vee \phi_c(x)) \\ &= b \wedge (b^\perp \vee \phi_c(x)) = \phi_b \phi_c(x). \end{aligned}$$

2.8. If $UA(a, b)$, $UA(b, c)$ and $(a \vee b - b) \perp (c \vee b - b)$, then both $\phi_a \phi_c \phi_a = \phi_a$ and $\phi_c \phi_a \phi_c = \phi_c$. Hence we can conclude $UA(a, c)$ provided that $a \wedge c = 0$.

Introduce the auxiliary notations $m = a \vee b - b$, $n = b \vee c - b$. We have $m \oplus b = a \vee b$, $n \oplus b = b \vee c$. Owing to the fact that $c \leq b \vee c = n \oplus b$, we have $\phi_c = \phi_c \phi_{n \oplus b}$, so $\phi_a \phi_c \phi_a = \phi_a \phi_c \phi_{n \oplus b} \phi_a$. Part of the hypothesis of 2.8 asserts that $n \perp m$, and since evidently $n \perp b$, we have $n \perp (b \oplus m) = (a \vee b)$. According to 2.7, $\phi_{n \oplus b} \phi_a = \phi_b \phi_a$, so $\phi_a \phi_c \phi_a = \phi_a \phi_c \phi_b \phi_a$. Now using similar justifications, we argue that $\phi_a \phi_c = \phi_a \phi_{m \oplus b} \phi_c = \phi_a \phi_b \phi_c$, so $\phi_a \phi_c \phi_a = \phi_a \phi_c \phi_b \phi_a = \phi_a \phi_b \phi_c \phi_b \phi_a$. According to the hypothesis, $\phi_b \phi_c \phi_b = \phi_b$ and $\phi_a \phi_b \phi_a = \phi_a$, hence $\phi_a \phi_c \phi_a = \phi_a$ as was to be proved. The other equation is proved similarly: we write $\phi_c \phi_a \phi_c = \phi_c \phi_a \phi_{m \oplus b} \phi_c = \phi_c \phi_a \phi_b \phi_c = \phi_c \phi_{n \oplus b} \phi_a \phi_b \phi_c = \phi_c \phi_b \phi_a \phi_b \phi_c = \phi_c \phi_b \phi_c = \phi_c$. That completes the verification of 2.8.

The final result establishes a connection with Janowitz's concept of "modularly perspective" [7].

2.9. If h is a splitting element for the orthogonal pair a, b , then all the pairs (a, h) , (h, a) , (h, b) , (b, h) are modular. The pairs (h, a) (h, b) are also dual modular.

The eight relations $(a, h)M$, $(a^\perp, h)M$, $(h, a)M$, $(h^\perp, a)M$, $(h, b)M$, $(h^\perp, b)M$, $(b, h)M$, $(b^\perp, h)M$ are all immediate consequences of (4) of 2.4, which establishes the modularity of all pairs in question. To prove the dual modularity, we set $c = a^\perp \wedge b^\perp$, and note that c commutes with h^\perp , because $h \leq a \oplus b = c^\perp$, and c also commutes with b . Since $(h^\perp, b)M$, we may conclude by Schreiner's lemma, 36.11 of [8], that $(h^\perp \vee c, b \vee c)M$. But $h^\perp \geq c$ so $h^\perp \vee c = h^\perp$. And $b \vee c = b \vee$

$(a^\perp \wedge b^\perp) = (b \vee a^\perp) \wedge (b \vee b^\perp) = a^\perp$. Thus $(h^\perp, a^\perp)M$, so $(h, a)M^*$ by 29.6 of [8]. We prove $(h, b)M^*$ by an analogous argument, starting with $(h^\perp, a)M$ and using the same c .

It may be that $(a, h)M^*$ and $(b, h)M^*$ as well, in which case a would be modularly perspective to b in the sense of Janowitz.

Our principal result is this:

3. The theorem

3.1. **THEOREM.** *If h is a splitting element for the orthogonal pair (a, b) , then the function*

$$\Psi_h(x) = (x \vee h) \wedge (x \vee h^\perp) \wedge x^\perp$$

- (1) *maps $\mathcal{L}(0, a)$ orthoisomorphically onto $\mathcal{L}(0, b)$ so that $x \approx \Psi_h(x)$ for every $x \in \mathcal{L}(0, a)$,*
- (2) *maps $\mathcal{L}(0, b)$ orthoisomorphically onto $\mathcal{L}(0, a)$ so that $x \approx \Psi_h(x)$ for every $x \in \mathcal{L}(0, b)$,*
- (3) *is its own inverse in the sense that the maps in (1) and (2) are mutually inverse, and*
- (4) *is expressible explicitly in terms of Sasaki projections by the formulas*

$$\Psi_h(x) = \phi_b \phi_h(x) \quad \text{When } x \in \mathcal{L}(0, a),$$

$$\Psi_h(x) = \phi_a \phi_h(x) \quad \text{When } x \in \mathcal{L}(0, b).$$

We shall devote this section to the proof of this result. We first rephrase the result so as to answer explicitly the question raised in the introduction:

3.2. **COROLLARY.** *Let \mathcal{L} be a uniform orthomodular lattice. If a, b are orthogonal projective elements in \mathcal{L} , then the sublattices $\mathcal{L}(0, a)$, $\mathcal{L}(0, b)$ are orthoisomorphic.*

Of course, considerably more information is conveyed by the theorem itself.

As the last statement in the theorem asserts, we can view the orthoisomorphism as the composition of two Sasaki projections, one from $\mathcal{L}(0, a)$ onto $\mathcal{L}(0, h)$, and the other from $\mathcal{L}(0, h)$ onto $\mathcal{L}(0, b)$. We begin our proof of the theorem by analyzing these individual maps.

3.3. *Suppose that $UA(a, h)$. Then:*

- (1) *ϕ_a is an orthoisomorphism of $\mathcal{L}(0, h)$ onto $\mathcal{L}(0, a)$, ϕ_h is an orthoisomorphism of $\mathcal{L}(0, a)$ onto $\mathcal{L}(0, h)$, and ϕ_a, ϕ_h are mutually inverse.*
- (2) *$x \approx \phi_h(x)$ When $x \in \mathcal{L}(0, a)$, and $x \approx \phi_a(x)$ When $x \in \mathcal{L}(0, h)$.*

PROOF. The proof is based on 2.6 which asserts that under the above

hypotheses we have $a, h \neq 0, a \wedge h = 0, \phi_h \phi_a \phi_h = \phi_h$, and $\phi_a \phi_h \phi_a = \phi_a$. If $x \in \mathcal{L}(0, h)$, then $x = \phi_h(x) = \phi_h \phi_a \phi_h(x) = \phi_h \phi_a(x)$, from which we conclude that $\phi_h \phi_a = \text{identity}$ on $\mathcal{L}(0, h)$. Similarly, $\phi_a \phi_h = \text{identity}$ on $\mathcal{L}(0, a)$. Now given $y \in \mathcal{L}(0, h)$, we have $y = \phi_h \phi_a(y) = \phi_h(x)$ where $x = \phi_a(y) \in \mathcal{L}(0, a)$, which shows that ϕ_h maps $\mathcal{L}(0, a)$ onto $\mathcal{L}(0, h)$. Similarly, ϕ_a maps $\mathcal{L}(0, h)$ onto $\mathcal{L}(0, a)$. The one-to-one character of both maps clearly follows from the fact that their product in either order is the identity, and, since each is order-preserving, each is a lattice isomorphism.

The next step of the proof is the verification that ϕ_a and ϕ_h are orthoisomorphisms, which (for ϕ_h) is the assertion $\phi_h(a-x) = h - \phi_h(x)$. According to (2) of 2.1,

$$y \leq a \Rightarrow \phi_h(y) = h \wedge (a-y)^\perp - h \wedge a^\perp.$$

Under the present hypotheses we have $h \wedge a^\perp = 0$. Setting $y = a-x$ we get $\phi_h(a-x) = h \wedge x^\perp = h - (h - h \wedge x^\perp) = h - \phi_h(x)$ as was to be proved. A parallel argument establishes that ϕ_a is an orthoisomorphism.

The strong perspectivities cited in (2) are a consequence of the general "parallelogram law": $x \vee y - y \leq x - x \wedge y$ [6]. We have $x \in \mathcal{L}(0, a) \Rightarrow x = x - x \wedge h^\perp \leq x \vee h^\perp - h^\perp = h - h \wedge x^\perp = \phi_h(x)$, and similarly, $x \in \mathcal{L}(0, h) \Rightarrow x \leq \phi_a(x)$. That completes the proof of 3.3.

At this point we could deduce the existence of the orthoisomorphisms of Corollary 3.2 directly from the above result by composition of maps. But we prefer to examine the situation in more detail, concentrating attention specifically on the map Ψ_h , in order to get the more precise information stated in the main theorem.

3.4. For a fixed element h in the orthomodular lattice \mathcal{L} the mapping Ψ_h defined by

$$\Psi_h(x) = (x \vee h) \wedge (x \vee h^\perp) \wedge x^\perp$$

has the following properties (each property holding for all $x \in \mathcal{L}$):

- (1) $\Psi_h(x) = \Psi_{h^\perp}(x)$
- (2) $\Psi_h(x) \perp x$
- (3) $\Psi_h(x) = x \vee \phi_h(x) - x = \phi_{x^\perp} \phi_h(x)$
- (4) $\Psi_h(x) = 0 \Leftrightarrow xCh$
- (5) $h \wedge \Psi_h(x) = h^\perp \wedge \Psi_h(x) = 0$
 $h \vee \Psi_h(x) = (h \vee x) \wedge (h \vee x^\perp)$

$$h^\perp \vee \Psi_h(x) = (h^\perp \vee x) \wedge (h^\perp \vee x^\perp)$$

(6) $\Psi_h(x) \lesssim \Psi_x(h)$

(7) $x \vee \phi_h(x) = x \oplus \Psi_h(x)$

$$\Psi_h(x) \vee \phi_h(x) = x \oplus \Psi_h(x) - x \wedge h^\perp$$

$$x \wedge \phi_h(x) = x \wedge h \quad \Psi_h(x) \wedge \phi_h(x) = 0$$

(8) $\Psi_h(x) \lesssim \phi_h(x) - h \wedge x$

(9) $\Psi_h^2(x) = \Psi_h(\Psi_h(x)) = x - [(x \wedge h) \oplus (x \wedge h^\perp)].$

Thus $\Psi^2(x) = x \Leftrightarrow x \wedge h = x \wedge h^\perp = 0.$

Statements (1) and (2) are obvious. The validity of assertion (3) is established by the following direct calculation: $x \vee \phi_h(x) - x = [x \vee (h \wedge (h^\perp \vee x))] \wedge x^\perp = (x \vee h) \wedge (x \vee h^\perp) \wedge x^\perp.$ Number (4) follows from the known calculus for Sasaki projections:

$$\Psi_h(x) = \phi_{x^\perp} \phi_h(x) = 0 \Leftrightarrow \phi_h(x) \perp x^\perp \Leftrightarrow \phi_h(x) \leq x \Leftrightarrow xCh.$$

Assertion (5) can be verified easily as follows:

$$h \wedge \Psi_h(x) = h \wedge (x \vee h) \wedge (x \vee h^\perp) \wedge x^\perp$$

$$= (x \vee h^\perp)^\perp \wedge (x \vee h^\perp) \wedge (x \vee h) = 0, \quad h^\perp \wedge \Psi_h(x)$$

$$= h^\perp \wedge x^\perp \wedge (x \vee h) \wedge (x \vee h^\perp) = (x \vee h)^\perp \wedge (x \vee h)$$

$$\wedge (x \vee h^\perp) = 0, \quad h \vee \Psi_h(x) = h \vee [(x \vee h) \wedge (x \vee h^\perp) \wedge x^\perp]$$

$$= \{h \vee [(x \vee h) \wedge (x \vee h^\perp)]\} \wedge (h \vee x^\perp) = (h \vee x) \wedge (h \vee x^\perp),$$

$$h^\perp \vee \Psi_h(x) = h^\perp \vee [(x \vee h) \wedge (x \vee h^\perp) \wedge x^\perp]$$

$$= \{h^\perp \vee [(x \vee h) \wedge (x \vee h^\perp)]\} \wedge (h^\perp \vee x^\perp)$$

$$= (h^\perp \vee x) \wedge (h^\perp \vee x^\perp).$$

Number (6) is a direct consequence of the parallelogram law

$$\Psi_h(x) = \Psi_h(x) - h \wedge \Psi_h(x) \lesssim h \vee \Psi_h(x) - h$$

$= (h \vee x) \wedge (h \vee x^\perp) \wedge h^\perp = \Psi_x(h).$ The computations verifying (7) are also routine:

$$x \oplus \Psi_h(x) = x \oplus (x \vee \phi_h(x) - x) = x \vee \phi_h(x) \quad (\text{using (3)}),$$

$$\Psi_h(x) \vee \phi_h(x) = (x \vee \phi_h(x) - x) \vee \phi_h(x)$$

$$= ((x \vee \phi_h(x)) \wedge x^\perp) \vee \phi_h(x)$$

$$\begin{aligned}
&= (x \vee \phi_h(x)) \wedge (x^\perp \vee \phi_h(x)) \\
&= (x \vee \phi_h(x)) \wedge (x^\perp \vee h) \\
&= x \vee \phi_h(x) - x \wedge h^\perp, \\
x \wedge \phi_h(x) &= x \wedge (h \wedge (h^\perp \vee x)) = x \wedge h, \\
\Psi_h(x) \wedge \phi_h(x) &\leq \Psi_h(x) \wedge h = 0 \quad (\text{using (5)}).
\end{aligned}$$

Number (8) follows from the parallelogram law, (7), and (3):

$$\begin{aligned}
\phi_h(x) - h \wedge x &= \phi_h(x) - \phi_h(x) \wedge x \stackrel{s}{=} \phi_h(x) \vee x - x \\
&= \Psi_h(x) \quad \text{by (3)}.
\end{aligned}$$

Item (9) is a matter of direct calculation using (5):

$$\begin{aligned}
\Psi_h(\Psi_h(x)) &= (\Psi_h(x) \vee h) \wedge (\Psi_h(x) \vee h^\perp) \wedge \Psi_h(x)^\perp \\
&= (h \vee x) \wedge (h \vee x^\perp) \wedge (h^\perp \vee x) \wedge (h^\perp \vee x^\perp) \wedge \Psi_h(x)^\perp \\
&= [(h \vee x) \wedge (h^\perp \vee x) \wedge \Psi_h(x)^\perp] \wedge [(h \vee x^\perp) \wedge (h^\perp \vee x^\perp) \wedge \Psi_h(x)^\perp] \\
&= \{(h \vee x) \wedge (h^\perp \vee x) \wedge [(h \vee x) \wedge (h^\perp \vee x)]^\perp \vee x\} \\
&\quad \wedge \{(h \vee x^\perp) \wedge (h^\perp \vee x^\perp) \wedge \Psi_h(x)^\perp\} \\
&= x \wedge (h \vee x^\perp) \wedge (h^\perp \vee x^\perp) \wedge \Psi_h(x)^\perp \\
&= x \wedge (h \vee x^\perp) \wedge (h^\perp \vee x^\perp) \quad \text{since } x \leq \Psi_h(x)^\perp \\
&= x - ((h^\perp \wedge x) \oplus (h \wedge x)).
\end{aligned}$$

That completes the proof of 3.4.

When restricted to an appropriate subset of \mathcal{L} the properties of Ψ_h are simple and easily described. We single out the relevant subset in 3.5 and list the properties in 3.6.

3.5. Define $\text{NC}(h) = \{y \in \mathcal{L}; y \wedge h = y \wedge h^\perp = 0\}$. An equivalent form is $y \in \text{NC}(h) \Leftrightarrow$ every subelement of y save 0 fails to commute with h .

3.6. The image (range) of Ψ_h is $\text{NC}(h)$. Restricted to $\text{NC}(h)$, Ψ_h is a one-to-one map of $\text{NC}(h)$ onto itself, and is there its own inverse. The orthogonal elements x and $\Psi_h(x)$ are strongly perspective for each $x \in \text{NC}(h)$.

The proof of 3.6 follows directly by specializing 3.4. According to (5) of 3.4, we have $h \wedge \Psi_h(x) = h^\perp \wedge \Psi_h(x) = 0$ which says that $\Psi_h(x) \in \text{NC}(h)$ for all $x \in \mathcal{L}$. And (9) of 3.4 asserts that $\Psi^2(x) = x$ for all $x \in \text{NC}(h)$, which shows that Ψ is one-to-one and onto there. And item (7) of 3.4 says that $\phi_h(x)$ is an axis of

strong perspectivity between x and $\Psi_h(x)$ for each $x \in \text{NC}(h)$.

There is one property missing from the attractive list in 3.6, namely that Ψ_h be order-preserving. Indeed, since Ψ_h is the meet of two maps, an order-preserving one, $x \rightarrow (x \vee h) \wedge (x \vee h^\perp)$, and an order-inverting one, $x \rightarrow x^\perp$, the possible order-preserving character of Ψ_h is not at all clear. In fact, if Ψ_h is order-preserving on $\mathcal{L}(0, a)$, then we must have $x \leq a \Rightarrow \Psi_h(x) \leq \Psi_h(a) \leq a^\perp$ which already requires $h \# a$ by (3) of 2.1. We complete the proof of the main theorem by proving a converse:

3.7. *Suppose $a \in \text{NC}(h)$ and $h \# a$. Let $b = \Psi_h(a)$. Then $\Psi_h(x) = \phi_b \phi_h(x)$ for $x \in \mathcal{L}(0, a)$.*

Before beginning the proof of 3.7, we assemble here the results of 2.1 under the substitution $a \rightarrow h, b \rightarrow a$ for our present use. We have $h \# a \Leftrightarrow \{x \leq a \Rightarrow \phi_h(x) \perp (a - x)\} \Leftrightarrow \{x \leq a \Rightarrow \phi_h(x) = h \wedge (a - x)^\perp - h \wedge a^\perp\} \Leftrightarrow \{x \leq a \Rightarrow (x \vee h) \wedge (x \vee h^\perp) \wedge x^\perp = \Psi_h(x) \perp a\} \Leftrightarrow \{x \leq a \Rightarrow \phi_a \phi_h(x) = \phi_x(h)\}$. According to 2.4, $h \# a \Rightarrow h^\perp \# a$ and $h \# a \Rightarrow h \# y$ for every $y \leq a$. According to 2.5, $h \# a \ \& \ a \wedge h^\perp = 0 \Leftrightarrow \phi_a \phi_h \phi_a = \phi_a$.

Let x represent an element $\leq a$. We first prove that $\phi_b \phi_h(x) = a \vee \phi_h(x) - a$ by the following series of computations.

$$\begin{aligned} \phi_b \phi_h(x) &= b \wedge (b^\perp \vee \phi_h(x)) \\ &= a^\perp \wedge [(a \vee h) \wedge (a \vee h^\perp)] \\ &\quad \wedge ([(a \vee h) \wedge (a \vee h^\perp)]^\perp \vee (a \vee \phi_h(x))) \end{aligned}$$

where we have used $b = \Psi_h(a)$. Since we are assuming $h \# a$, we can use the following formula written directly above

$$\begin{aligned} \phi_h(x) &= h \wedge (a - x)^\perp \wedge (h^\perp \vee a) \\ &= h \wedge (a^\perp \vee x) \wedge (h^\perp \vee a) \leq a \vee h^\perp. \end{aligned}$$

Since also $a \leq a \vee h^\perp$, we have $a \vee \phi_h(x) \leq a \vee h^\perp$. Also $\phi_h(x) \leq h$, so $a \vee \phi_h(x) \leq a \vee h$. Hence $a \vee \phi_h(x) \leq (a \vee h) \wedge (a \vee h^\perp)$, so $a \vee \phi_h(x)$ commutes with $[(a \vee h) \wedge (a \vee h^\perp)]$ and its orthocomplement. Using this information, we can distribute the expression $[\] \wedge ([\]^\perp \vee (a \vee \phi_h(x)))$:

$$\phi_b \phi_h(x) = a^\perp \wedge (a \vee \phi_h(x)) = a \vee \phi_h(x) - a.$$

Next, we prove that $\phi_b \phi_h(x) = \Psi_h(x)$.

$$\begin{aligned} \phi_b \phi_h(x) &= (a \vee \phi_h(x)) \wedge a^\perp \\ &= ((a - x) \vee (x \vee \phi_h(x))) \wedge a^\perp, \text{ using } a = (a - x) \vee x. \end{aligned}$$

Since $h \# a$, $\phi_h(x) \perp (a-x)$, and, evidently $x \perp (a-x)$, so $(x \vee \phi_h(x)) \perp (a-x)$. Clearly a^\perp commutes with $a-x$, so we can distribute

$$=(x \vee \phi_h(x)) \wedge a^\perp = (x \vee \phi_h(x)) \wedge (a-x)^\perp \wedge x^\perp,$$

$$\text{using } a^\perp = (a-x)^\perp \wedge x^\perp,$$

$$=[(x \vee \phi_h(x)) \wedge x^\perp] \wedge (a-x)^\perp = \Psi_h(x) \wedge (a-x)^\perp.$$

Now $h \# a$ implies that $\Psi_h(x) \perp a$, so $\Psi_h(x) \leq a^\perp \leq (a-x)^\perp$. Therefore $\phi_b \phi_h(x) = \Psi_h(x) \wedge (a-x)^\perp = \Psi_h(x)$, which completes the proof.

The proof of the main theorem now follows directly from these results. The hypotheses of the main theorem are $UA(a, h)$, $UA(h, b)$, and $h \leq a \oplus b$, and we note first that if these are fulfilled, then so are the hypotheses of 3.7. First both $a \in NC(h)$ and $h \# a$ are contained in $UA(a, h)$. Secondly, $b = \Psi_h(a)$ is a consequence of $h \leq a \oplus b$, $a^\perp \wedge h = b \wedge h^\perp = 0$ as follows: $\Psi_h(a) = (a \vee h) \wedge (a \vee h^\perp) \wedge a^\perp = (a \vee h) \wedge a^\perp = (((a \oplus b) \wedge b^\perp) \vee h) \wedge a^\perp = ((a \oplus b) \vee h) \wedge (b^\perp \vee h) \wedge a^\perp = (a \oplus b) \wedge a^\perp = b$. Thus we are in the situation of 3.7 and are entitled to conclude $\Psi_h(x) = \phi_b \phi_h(x)$ for $x \in \mathcal{L}(0, a)$. Now by 3.3, from $UA(a, h)$ it follows that ϕ_h is an orthoisomorphism of $\mathcal{L}(0, a)$ onto $\mathcal{L}(0, h)$, and from $UA(h, b)$ it follows that ϕ_b is an orthoisomorphism of $\mathcal{L}(0, h)$ onto $\mathcal{L}(0, b)$. Thus, as the composition of two orthoisomorphisms, Ψ_h is itself an orthoisomorphism of $\mathcal{L}(0, a)$ on $\mathcal{L}(0, b)$. By symmetry, we conclude that $\Psi_h(x) = \phi_a \phi_h(x)$ when $x \in \mathcal{L}(0, b)$ and that Ψ_h maps $\mathcal{L}(0, b)$ orthoisomorphically on $\mathcal{L}(0, a)$. Finally, we note that both $\mathcal{L}(0, a)$ and $\mathcal{L}(0, b)$ are contained in $NC(h)$, so that the assertions of 3.1 relating to the strong perspectivities $x \preceq \Psi_h(x)$ and the fact that Ψ_h is its own inverse now follow directly from 3.6. That completes the proof of the main theorem.

4. Examples and further comments

The properties of splitting elements are so special that one might easily surmise that they exist rarely, if at all, and that correspondingly the class of uniform orthomodular lattices is a small class. Our first result shows that, to the contrary, there is a large class of orthomodular lattices easily verified to be uniform.

4.1. *The projection lattice of a von Neumann algebra is uniform.*

PROOF. Let \mathcal{A} be the von Neumann algebra in question, \mathcal{L} its projection lattice, and let A, B be orthogonal non-zero, projective elements of \mathcal{L} . Then A, B are equivalent, which is to say, there is a partial isometry W in \mathcal{A} such that

$WW^*=A, W^*W=B$. Choose λ satisfying $0 < \lambda < 1$ and set $H = \lambda A + (1 - \lambda)B + (\lambda(1 - \lambda))^{1/2}(W + W^*)$. Direct computations show that H is a projection, and that $AHA = \lambda A, HAH = \lambda H, BHB = (1 - \lambda)B, HBH = (1 - \lambda)H$. For an element X of \mathcal{L} , one verifies easily that the Sasaki projection $\phi_H(X)$ is the projection on the closure of $\text{im}(HX)$. According to the criterion of 2.1, we have $H \# A \Leftrightarrow \{X \leq A \Rightarrow \phi_H(X) \perp (A - X)\}$, which is equivalent to $\{X \leq A \Rightarrow (A - X)HX = 0\}$, or $\{X \leq A \Rightarrow AHX = XHX\}$. But from $AHA = \lambda A$ we derive immediately $AHX = \lambda X = XHX$. Hence $H \# A$. Similarly, we have also $A \# H, H \# B$, and $B \# H$. The equations $A \wedge H = A \wedge H^\perp = A^\perp \wedge H = B \wedge H = B \wedge H^\perp = B^\perp \wedge H = 0$ being easily verified, we have that H is splitting for A, B . This completes the proof of 4.1.

If h is a splitting element for the pair a, b , then all pairs $(a, h), (h, a), (h, b), (b, h)$ are modular, and the pairs $(h, a), (h, b)$ are also dual modular. However, the requirement that h be splitting is much stronger than the modularity of these various pairs: We give an example in the modular lattice of subspaces of a four dimensional real vector space V where $A \perp B, H \leq A \oplus B = 1, A \vee H = B \vee H = 1, A \wedge H = B \wedge H = 0$, (so that H effects a strong perspectivity between A and B) but not $\text{UA}(A, H)$. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of V , let A be the subspace spanned by $\{e_1, e_2\}$, B that spanned by $\{e_3, e_4\}$, and let H be the subspace spanned by $\{e_1 + e_2 + e_3, e_2 + e_4\}$. Then one checks easily that H effects the claimed strong perspectivity. Let X be the subspace spanned by $\{e_1\}$. Then $0 < X < A$, and, as computation shows, $\phi_H(X) = H - H \wedge X^\perp$ is spanned by $-2e_1 - e_2 - 2e_3 + e_4$. But if $\text{UA}(A, H)$, then by (1) of 2.1 this vector would have to be orthogonal to e_2 which is clearly not the case.

We next examine in some detail the meaning of $A \# B$ and $\text{UA}(A, B)$ in the projection lattice of a von Neumann algebra.

4.2. THEOREM. *Let \mathcal{L} represent the lattice of projections of a von Neumann algebra \mathcal{A} . The following conditions on the elements A, B of \mathcal{L} are equivalent:*

- (1) $A \# B$
- (2) BAB commutes with every $X \leq B, X \in \mathcal{L}$.
- (3) There exists a unique operator T such that
 (a) $T \in \text{center}(\mathcal{A})$, (b) $BAB = TB$, and (c) $T = e(B)T$ (where $e(B)$ is the central cover of B). The operator T (uniquely determined by (a), (b), and (c)) also satisfies $0 \leq T \leq I$.

PROOF. (1) \Rightarrow (2). The relation $A \# B$ means this: If $X \in \mathcal{L}$ and $X \leq B$, then $(A - (A \wedge X^\perp)) \perp (B - X)$. Now $(\ker(XA))^\perp = A - (A \wedge X^\perp)$, so the previous condition is equivalent to $X \leq B \Rightarrow (B - X) \leq \ker(XA)$, or $X \leq B \Rightarrow XA(B - X) = 0$,

which is the same as $XAB = XAX$. Taking adjoints, we get $BAX = XAX = XAB$. Since $X \leq B$, we have $BX = XB = X$, so we can recast the previous equation in the form $(BAB)X = X(BAB)$, which is (2).

(2) \Rightarrow (3). The operator BAB belongs to the $*$ -algebra $B\mathcal{A}B$, and the projections in $B\mathcal{A}B$ are precisely those $X \in \mathcal{L}$, $X \leq B$. Now $B\mathcal{A}B$ is $*$ -isomorphic to \mathcal{A}_B , the von Neumann algebra of operators on the Hilbert space B that are restrictions to B of elements of $B\mathcal{A}B$ [2; Ch. I, § 2, 1°]. Then BAB considered as an operator in \mathcal{A}_B commutes with every projection in \mathcal{A}_B and so is in the center of \mathcal{A}_B . But the center of \mathcal{A}_B is \mathcal{Z}_B where \mathcal{Z} is the center of \mathcal{A} [2; Ch. I, § 2, Cor. to Prop. 2]. It follows that $BAB = BTB$ for $T \in \text{center}(\mathcal{A})$. The operator T can be replaced by $e(B)T$, and the new T satisfies both $BAB = TB$, $= e(B)T$. That proves the existence.

To prove the uniqueness, we observe that if M is the subspace of all finite sums $T_1x_1 + T_2x_2 + \dots + T_nx_n$, $T_i \in \mathcal{A}$, $x_i \in B$, then $e(B)$ is the projection on the closure of M . If $y \in M$, then $Ty = T(T_1x_1 + \dots + T_nx_n) = T_1Tx_1 + \dots + T_nTx_n = T_1TBx_1 + \dots + T_nTBx_n = T_1BAx_1 + \dots + T_nBAx_n$ so that T is already determined on M by the conditions $TB = BAB$, $T \in \text{center}(\mathcal{A})$. By continuity, T is determined on $e(B)$. The condition $T = Te(B)$ implies that $T = 0$ on $I - e(B)$; thus T is determined everywhere. That proves uniqueness.

(3) \Rightarrow (1). If $X \leq B$, $BAB = TB$, then $XA(B - X) = XAB - XAX = XBAB - XBABX = XBT - XTBX = XT - XT = 0$, which is equivalent to $A \# B$.

We turn our attention now to the other properties claimed for T . The assertion $0 \leq T \leq I$ is equivalent to $0 \leq (Tx, x) \leq \|x\|^2$ for every $x \in H$. Owing to (c) we have $(Tx, x) = (Te(B)x, e(B)x)$, so we may assume $x \in e(B)$. Since $TB = (AB)^*(AB) \geq 0$, and $B - TB = ((I - A)B)^*((I - A)B) \geq 0$, we have $0 \leq TB \leq B$. Inasmuch as elements of the form $x = T_1x_1 + \dots + T_nx_n$, $T_i \in \mathcal{A}, x_i \in B$ are dense in $e(B)$, it is enough to verify $0 \leq (Tx, x) \leq \|x\|^2$ for such x 's. We have

$$\begin{aligned} (Tx, x) &= (T(T_1x_1 + \dots + T_nx_n), T_1x_1 + \dots + T_nx_n) \\ &= \Sigma(TT_ix_i, T_jx_j) = \Sigma(T^*TT_ix_i, x_j) \\ &= \Sigma(T^*T_iTx_i, x_j) = \Sigma(T^*T_iTBx_i, Bx_j) \\ &= \Sigma((BT^*_jT_iB)(TB)x_i, x_j) \\ &= \Sigma((BT^*_jT_iB)(TB)^{1/2}(TB)^{1/2}x_i, x_j) \\ &= \Sigma((TB)^{1/2}(BT^*_jT_iB)(TB)^{1/2}x_i, x_j) \\ &= \Sigma((BT^*_jT_iB)y_i, y_j) \quad \text{where } y_i = (TB)^{1/2}x_i \\ &= \|T_1By_1 + \dots + T_nBy_n\|^2 \geq 0 \end{aligned}$$

and similarly for $I - T$.

It is a reasonable conjecture that $\ker(T) = e(B \wedge A^\perp) \oplus e(B)^\perp$, $\ker(I - T) = e(B \wedge A)$, but I have not been able to prove this.

4.3. COROLLARY. *If \mathcal{A} is a factor (of any type), then the following conditions on A and B are equivalent:*

- (1) $A \# B$
- (2) *There exists a unique scalar λ such that*
 (i) $BAB = \lambda B$, (ii) $\lambda I = \lambda e(B)$. *The number λ necessarily satisfies $0 \leq \lambda \leq 1$.*

4.4. COROLLARY. *If \mathcal{A} is the full algebra of all operators on the Hilbert space, and A, B are non-zero projections in \mathcal{A} , then $UA(A, B) \Leftrightarrow ABA = \lambda A$ and $BAB = \lambda B$ for some λ satisfying $0 < \lambda < 1$.*

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