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Nonlinear Operators of Monotone Type in Reflexive Banach Spaces and Nonlinear Perturbations

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Introduction

In this paper we are concerned with nonlinear operators of monotone type from a reflexive Banach space X into the dual space X^*. Such operators have been considered to make a general treatment of boundary value problems for nonlinear elliptic partial differential equations and initial-boundary value problems for nonlinear parabolic partial differential equations. Studies of nonlinear operators of monotone type have been made by many authors (e.g., [1]–[3], [5]–[7], [9]–[12], [15]–[18], [20], [22], [25]).

In [2] Brezis introduced two classes of nonlinear singlevalued operators, called of type M and pseudo-monotone respectively, from X into X^* and then established existence theorems for nonlinear functional equations of the forms

$$(a) \quad Ax=f \qquad \text{for given } f\in X^*$$

and

$$(b) \quad Ax+Tx=f \qquad \text{for given } f\in X^*,$$

where A is an operator of type M or a pseudo-monotone operator from X into X^* and T is a nonlinear monotone operator from X into X^*. Recently, the concept of pseudo-monotone operators was generalized by Browder and Hess [10] to the multivalued case. Many results in [2] on the solvability of (a) and (b) were extended to the multivalued case where the equations have the forms:

$$(a)' \quad Ax\ni f \qquad \text{for given } f\in X^*$$

and

$$(b)' \quad Ax+Tx\ni f \qquad \text{for given } f\in X^*.$$

In this paper we shall first give a natural generalization of the notion of operators of type M to the multivalued case, and investigate basic properties of such operators. Next, we shall solve nonlinear equations of types $(a)'$ and $(b)'$ for multivalued operators of type M and multivalued pseudo-monotone operators under somewhat different assumptions from those in [2], [10], [11] and [22].

In the final section, as an application we shall show the existence of a solution of a variational inequality

$$\begin{cases} (x, x^*) \in G(A) \text{ with } x \in C; \\ \langle x^* - f, x - y \rangle \leq \phi(y) - \phi(x) \qquad \text{for all } y \in C, \end{cases}$$

where A is a multivalued pseudo-monotone operator from X into X^*, $f \in X^*$ and ϕ is lower semicontinuous on a closed convex subset C of X. Furthermore, we shall study dependence of the solutions on A, C, ϕ and f by making use of results in Mosco [21].

§0. Preliminaries

Let V and W be two topological vector spaces. For a multivalued operator A from V into W (i.e., to each $x \in V$ a subset Ax of W is assigned), we define

$$D(A) = \{x \in V;\ Ax \neq \emptyset\},$$

$$R(A) = \bigcup_{x \in D(A)} Ax$$

and

$$G(A) = \{(x, x^*) \in V \times W;\ x \in D(A),\ x^* \in Ax\}.$$

In what follows an operator means a multivalued operator unless otherwise stated.

For an operator A from V into W and a real number λ, λA is an operator from V into W defined by

$$G(\lambda A) = \{(x, \lambda x^*) \in V \times W;\ (x, x^*) \in G(A)\}.$$

Let A_1 and A_2 be two operators from V into W. Then the sum $A_1 + A_2$ is an operator given by

$$G(A_1 + A_2) = \{(x, x_1^* + x_2^*) \in V \times W;\ x \in D(A_1) \cap D(A_2),\ x_1^* \in A_1 x,\ x_2^* \in A_2 x\}.$$

For an operator A from V into W, we denote by A^{-1} the inverse of A, i.e., A^{-1} is an operator from W into V given by

$$G(A^{-1}) = \{(x^*, x) \in W \times V;\ (x, x^*) \in G(A)\}.$$

Let A be an operator with $D(A) = V$ into W such that Ax is a closed subset of W for each $x \in V$. Then it is called *upper semicontinuous* (resp. *sequentially upper semicontinuous*), if for any $x \in V$ and any neighborhood U^* of Ax (resp. any sequence $\{x_n\} \subset V$ converging to $x \in V$ and any neighborhood U^* of Ax), there is a neighborhood U of x (resp. an integer n_0) such that $U^* \supset Ay$ for all $y \in U$ (resp. $U^* \supset Ax_n$ for all $n \geq n_0$). In particular, if A is single-valued (i.e., Ax consists of a single element of W for each $x \in V$), then the upper semicontinuity (resp. sequential upper semicontinuity) of A coincides with the continuity (resp. sequential continuity).

Let V and W be two real Banach spaces, and let A be an operator from V into W such that Ax is non-empty and weakly closed in W for each $x \in V$. Then A is called *weakly upper semicontinuous* (resp. *sequentially weakly upper semicontinuous*) if it is upper semicontinuous (resp. sequentially upper semicontinuous) with respect to the weak topologies of V and W. For a single-valued operator A, we say that it is *demicontinuous*, if it maps any strongly convergent sequence in V to a weakly convergent sequence in W.

Next, let V and W be real reflexive Banach spaces and let A be an operator from V into W with $D(A) = V$ such that Ax is weakly compact for each $x \in V$. We note that if A is sequentially weakly upper semicontinuous, then A is bounded (i.e., it maps bounded sets in V to bounded sets in W) and $G(A)$ is sequentially weakly closed in $V \times W$. In particular, if V is finite dimensional, then A is weakly upper semicontinuous if and only if it is bounded and $G(A)$ is sequentially weakly closed in $V \times W$.

We use symbols "$\xrightarrow{s}$" and "$\xrightarrow{w}$" to denote convergence in the strong and weak topology of a Banach space, respectively.

Throughout this paper, let X be a real reflexive Banach space, X^* be the dual space of X and $<x^*, x>$ denotes the duality pairing between $x^* \in X^*$ and $x \in X$ and $\|x\|$ (resp. $\|x^*\|$) the norm of $x \in X$ (resp. $x^* \in X^*$). We denote by J the duality mapping of X into X^*, i.e., it is defined by

$$Jx = \{x^* \in X^*;\ <x^*, x> = \|x\|^2 = \|x^*\|^2\} \quad \text{for each} \quad x \in X.$$

We know that $D(J) = X$ and $R(J) = X^*$, and that if X^* is strictly convex, then J is single-valued and demicontinuous. The inverse J^{-1} is, as easily seen from the definition of J, the duality mapping of X^* into X $(=X^{**})$. We remark that if X and X^* are strictly convex, then J is demicontinuous, one to one and onto.

Let A be an operator from X into X^*. If for any $(x_i, x_i^*) \in G(A)$, $i = 1, 2$,

$$<x_1^* - x_2^*, x_1 - x_2> \geq 0,$$

then A is called *monotone*. A monotone operator A from X into X^* is called *maximal monotone*, if there is no monotone operator $\tilde{A}$ such that $G(A)$ is a proper subset of $G(\tilde{A})$. It is well known that the duality mapping J is maximal monotone.

§1. Operators of monotone type

1.1. Definitions

We first of all recall the original definitions of operators of type M and pseudo-monotone operators as given by Brezis [2].

Let A be a single-valued operator from X into X^*. Then A is called of *type* M, if $D(A)=X$ and it has the following two properties:

(M_1) *If $\{x_\alpha\}$ is a bounded net in X, and if $x_\alpha \xrightarrow{w} x$ in X, $Ax_\alpha \xrightarrow{w} x^*$ in X^* and*

$$\limsup_{\alpha} <Ax_\alpha, x_\alpha> \leq <x^*, x>,$$

then $Ax=x^*$.

(M_2) *The restriction of A to any finite dimensional subspace of X is continuous with respect to the weak topology of X^*.*

A single-valued operator A from X into X^* is called *pseudo-monotone*, if $D(A)=X$ and two conditions below are satisfied:

(PM_1) *If $\{x_\alpha\}$ is a bounded net in X such that $x_\alpha \xrightarrow{w} x$ in X and*

$$\limsup_{\alpha} <Ax_\alpha, x_\alpha - x> \leq 0,$$

then for all $y \in X$

$$\liminf_{\alpha} <Ax_\alpha, x_\alpha - y> \geq <Ax, x-y>.$$

(PM_2) *For any fixed $y \in X$, the function $x \to <Ax, x-y>$ is bounded below on each bounded subset of X.*

Recently, the above notions were extended to the multivalued case (see Browder-Hess [10] and the author [12]). In the definition of multivalued pseudo-monotone operators by Browder and Hess, only sequences are considered instead of nets. In this direction, we give a generalization of the notion of single-valued operators of type M as follows:

DEFINITION 1.1. (cf. [12]) *Let A be an operator from X into X^*. Then A is called of type M, if it satisfies the following conditions:*

(m_1) *For each $x \in X$, Ax is a non-empty, bounded, convex and closed subset of X^*.*

(m_2) *If $\{(x_n, x_n^*)\} \subset G(A)$ is a sequence, and if $x_n \xrightarrow{w} x$ in X, $x_n^* \xrightarrow{w} x^*$ in X^* and*

$$\limsup_{n \to \infty} <x_n^*, x_n> \leq <x^*, x>,$$

then $(x, x^*) \in G(A)$.

(m_3) *The restriction of* A *to any finite dimensional subspace* F *of* X *is weakly upper semicontinuous as an operator from* F *into* X^*.

The following definition of multivalued pseudo-monotone operators is due to Browder and Hess.

DEFINITION 1.2. (*Browder-Hess* [10]) *Let* A *be an operator from* X *into* X^*. *Then* A *is called pseudo-monotone, if it has the following properties*:

(pm_1) *For each* $x \in X$, Ax *is non-empty, convex and closed in* X^*.

(pm_2) *If* $\{(x_n, x_n^*)\} \subset G(A)$ *is a sequence such that* $x_n \xrightarrow{w} x$ *in* X *and*

$$\limsup_{n\to\infty} < x_n^*, x_n - x > \leq 0,$$

then to each $y \in X$ *there exists* $x^*(y) \in Ax$ *with the property that*

$$\liminf_{n\to\infty} < x_n^*, x_n - y > \geq < x^*(y), x - y > .$$

(pm_3) *The restriction of* A *to any finite dimensional subspace* F *of* X *is weakly upper semicontinuous as an operator from* F *into* X^*.

Note that (m_1) or (pm_1) implies that $D(A) = X$.

1.2. Basic properties

We begin with the following:

PROPOSITION 1.1. (*Browder-Hess* [10; *Proposition* 7]) *Let* A *be an operator from* X *into* X^* *with* $D(A) = X$ *satisfying condition* (pm_2). *If* $\{(x_n, x_n^*)\} \subset G(A)$ *is a sequence such that* $x_n \xrightarrow{w} x$ *in* X *and*

$$\limsup_{n\to\infty} < x_n^*, x_n - x > \leq 0,$$

then $\{x_n^*\}$ *is bounded in* X^*. *If, in addition,* A *satisfies condition* (pm_1), *then every sequential weak cluster point of* $\{x_n^*\}$ *is contained in* Ax.

The class of operators of type M includes not only sequentially weakly upper semicontinuous operators with property (m_1), but also pseudo-monotone operators; in fact, we have

PROPOSITION 1.2. *Let* A *be an operator from* X *into* X^*. *If* A *is pseudo-monotone, then it is of type* M.

PROOF. It suffices to show (m_1) and (m_2). Let x be any point in X and $\{x_n^*\}$ be any sequence in Ax. Then, since $< x_n^*, x - x > = 0$ for all n, Proposition 1.1 implies that $\{x_n^*\}$ is bounded, and so Ax is bounded in X. Thus (m_1) is verified. Next, let $\{(y_n, y_n^*)\} \subset G(A)$ be any sequence such that $y_n \xrightarrow{w} y$, $y_n^* \xrightarrow{w} y^*$ and

$$\limsup_{n\to\infty} < y_n^*, y_n > \leq < y^*, y >. \tag{1.1}$$

Then, since (1.1) implies that

$$\limsup_{n\to\infty} < y_n^*, y_n - y > \leq 0,$$

it follows directly from Proposition 1.1 that $y^* \in Ay$. Thus we have (m_2).

q.e.d.

REMARK 1.1. In infinite dimensional Banach spaces, the converse of Proposition 1.2 is false. For example, the operator $-I$ in the space l^2 is of type M because of the weak continuity, but not pseudo-monotone, where I is the identity mapping in l^2.

Let Λ be the family of all finite dimensional subspaces of X. For each $F \in \Lambda$, we denote by j_F the natural injection from F into X and j_F^* the adjoint of j_F. We know that each j_F^* is linear, weakly continuous and surjective as an operator from X^* into F^*, and hence it is open.

The following lemma will be helpful in the later discussion.

LEMMA 1.1. *Let A be an operator from X into X^* satisfying condition (m_1). Then, setting $A_F = j_F^* A j_F$ for $F \in \Lambda$,* we have

(1) *$A_F x$ is a non-empty, bounded, convex and closed subset of F^* for each $F \in \Lambda$ and each $x \in F$.*

Furthermore, condition (m_3) $(=(pm_3))$ is satisfied if and only if the following condition holds;

(2) *for each $F \in \Lambda$, A_F is an upper semicontinuous operator from F into F^*.*

PROOF. The property (1) is easily derived from (m_1). Since, under (m_1), condition (m_3) clearly implies (2), we show only the "if" part of the second assertion of the lemma. Thus, assume (2). Let F_0 be any element of Λ, x_0 be any point in F_0 and U_0^* be any weak neighborhood of Ax_0 in X^*. By (m_1), Ax_0 is weakly compact in X^*. Therefore, there are finite sets $E^* = \{y_1^*, y_2^*, \dots, y_N^*\} \subset Ax_0$ and $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N\}$ of positive numbers such that

$$U_0^* \supset \bigcup_{k=1}^{N} U_k^* \supset Ax_0, \tag{1.2}$$

where

$$U_k^* = \{x^* \in X^*; |< y_k^* - x^*, y >| < \varepsilon_k \text{ for all } y \in E_k\}$$

with a finite subset E_k of X, $k = 1, 2, \dots, N$. Denote by F the finite dimensional subspace of X spanned by F_0 and $\bigcup_{k=1}^{N} E_k$. Then, we observe

$$\bigcup_{k=1}^{N} j_F^*(U_k^*) \supset A_F x_0$$

and

(1.3) $$j_F^{*-1}(j_F^*(U_k^*)) = U_k^*, \qquad k=1, 2,\dots N.$$

Since $\bigcup_{k=1}^{N} j_F^*(U_k^*)$ is a neighborhood of $A_F x_0$ in F^*, (2) for this subspace F implies that there is a neighborhood U of x_0 in F such that

$$\bigcup_{k=1}^{N} j_F^*(U_k^*) \supset A_F(U) = j_F^*(A(U)).$$

Hence

$$j_F^{*-1}(\bigcup_{k=1}^{N} j_F^*(U_k^*)) \supset j_F^{*-1}(A_F(U)) \supset A(U).$$

These relations together with (1.3) imply that

$$\bigcup_{k=1}^{N} U_k^* \supset A(U).$$

Therefore, setting $U_0 = U \cap F_0$, we see from (1.2) that for this neighborhood U_0 of x_0 in F_0 the relation

$$U_0^* \supset A(U_0)$$

holds. Thus A satisfies condition (m_3). q.e.d.

Next, we give results on the sum of two operators of monotone type.

PROPOSITION 1.3. *If A is an operator of type M from X into X^* and T is a sequentially weakly upper semicontinuous monotone operator from X into X^* such that Tx is non-empty, bounded, convex and closed in X^* for each $x \in X$. Then $T+A$ is of type M.*

PROOF. Since the verification of (m_1) and (m_3) for $T+A$ is easy, we verify only condition (m_2). Let $\{(x_n, z_n^*)\} \subset G(A+T)$ be a sequence with $z_n^* = x_n^* + y_n^*$, $x_n^* \in Ax_n$ and $y_n^* \in Tx_n$ such that $x_n \xrightarrow{w} x_0$ in X, $z_n^* \xrightarrow{w} z_0^*$ in X^* as $n \to \infty$ and

$$\limsup_{n\to\infty} < z_n^*, x_n > \leq < z_0^*, x_0 >.$$

Then

(1.4) $$\limsup_{n\to\infty} < x_n^*, x_n > = \limsup_{n\to\infty} < z_n^* - y_n^*, x_n >$$

$$\leq < z_0^*, x_0 > - \liminf_{n\to\infty} < y_n^*, x_n >.$$

We can choose a subsequence $\{y_{n_k}^*\}$ of $\{y_n^*\}$ weakly convergent to some $y_0^* \in X^*$ such that $\liminf_{n\to\infty} < y_n^*, x_n > = \lim_{k\to\infty} < y_{n_k}^*, x_{n_k} >$. Since $y_0^* \in Tx_0$ by the sequential weak upper semicontinuity of T, we have by the monotonicity of T

$$< y_{n_k}^*, x_{n_k} - x_0 > \geq < y_0^*, x_{n_k} - x_0 >,$$

and hence

$$\liminf_{k\to\infty} < y_{n_k}^*, x_{n_k} > \geq < y_0^*, x_0 >.$$

From (1.4) and the above inequality it follows that

$$\limsup_{k\to\infty} < x_{n_k}^*, x_{n_k} > \leq < z_0^* - y_0^*, x_0 >.$$

Therefore, by condition (m_2) for A we have $z_0^* - y_0^* \in Ax_0$. Thus $z_0^* \in Ax_0 + Tx_0$. q.e.d.

Proposition 1.4. (*Browder-Hess* [10; *Proposition* 9])
Let A_1 and A_2 be two pseudo-monotone operators from X into X^*. Then $A_1 + A_2$ is also pseudo-monotone.

The following Proposition 1.5 gives a characterization of operators of type M and pseudo-monotone operators in finite dimensional Banach spaces.

Proposition 1.5. *Suppose that X is finite dimensional. Let A be an operator from X into X^*. Then the following three statements are equivalent to each other:*

(*a*) *A is of type M.*
(*b*) *A is pseudo-monotone.*
(*c*) *Ax is non-empty, bounded, convex and closed in X^* for each $x \in X$, and A is upper semicontinuous.*

Proof. Since assertions "$(b)\to(a)$" and "$(a)\to(c)$" are easily seen from Proposition 1.2 and the definition of operators of type M, we have only to show "$(c)\to(b)$". Therefore, assume (c). Let $x_n \to x$ in X and $x_n^* \in Ax_n$ for all n. By the boundedness of A and the closedness of $G(A)$, there is a subsequence $\{(x_{n_k}, x_{n_k}^*)\} \subset G(A)$ for each $y \in X$ such that

$$\liminf_{n\to\infty} < x_n^*, x_n - y > = \lim_{k\to\infty} < x_{n_k}^*, x_{n_k} - y >$$

and for some $x^*(y) \in Ax$

$$x_{n_k}^* \to x^*(y) \qquad \text{in } X^* \text{ as } k \to \infty.$$

For this $\{(x_{n_k}, x_{n_k}^*)\}$, we have

$$< x^*(y), x - y > = \liminf_{n\to\infty} < x_n^*, x_n - y >.$$

Thus (pm_2) is verified and A is pseudo-monotone.

1.3. A generalization of Brezis' condition (PM_2)

We now give a natural generalization of condition (PM_2) to the multivalued case.

Let A be an operator with $D(A)=X$ into X^*. We consider the following condition:

(pm_4) *For each* $x_0 \in X$ *and each bounded subset* B *of* X, *there exists a constant* $N(B, x_0)$ *such that*

$$<x^*, x-x_0> \geq N(B, x_0) \qquad \text{for all } (x, x^*) \in G(A) \text{ with } x \in B.$$

Condition (pm_4) is fulfilled by monotone operators A with $D(A)=X$ as well as by bounded operators.

LEMMA 1.2. *Let A be an operator from X into X^* satisfying* (pm_1), (pm_2) *and* (pm_4). *Then A satisfies also* (pm_3).

PROOF. Let F be any element of Λ and $A_F = j_F^* A j_F$. Then we see from Proposition 1.1 that Ax is non-empty bounded, convex and closed in X^* for each $x \in X$. Hence, in view of Lemma 1.1, it is enough to show that A_F is upper semicontinuous. If A_F is not so, then there are $x_0 \in F$, a neighborhood U_F^* of $A_F x_0$, sequences $\{x_n\} \subset F$ and $\{\bar{x}_n^* = j_F^* x_n^*\}$ with $x_n^* \in Ax_n$ such that $x_n \to x_0$ in F as $n \to \infty$ and $\bar{x}_n^* \notin U_F^*$ for all n. First we show that $\{\bar{x}_n^*\}$ is bounded in F^*. In fact, if otherwise, then there is a subsequence $\{\bar{x}_{n_k}^*\}$ of $\{\bar{x}_n^*\}$ such that $\|\bar{x}_{n_k}^*\| \to \infty$ and $\bar{y}_k^* = \bar{x}_{n_k}^* / \|\bar{x}_{n_k}^*\| \to \bar{y}_0^*$ as $k \to \infty$ for some $\bar{y}_0^* \in F^*$ with $\|\bar{y}_0^*\| = 1$. Here, using condition (pm_4), we find a constant $N(x)$ for each $x \in F$ such that

$$<\bar{y}_k^*, x_{n_k} - x> \geq \frac{N(x)}{\|\bar{x}_{n_k}^*\|} \qquad \text{for all } k.$$

Letting $k \to \infty$, we have

$$<\bar{y}_0^*, x_0 - x> \geq 0 \qquad \text{for every } x \in F.$$

This implies that $\bar{y}_0^* = 0$. This is a contradiction. Thus $\{\bar{x}_n^*\}$ is bounded, and hence

$$\limsup_{n\to\infty} <x_n^*, x_n - x_0> = \lim_{n\to\infty} <\bar{x}_n^*, x_n - x_0> = 0.$$

From Proposition 1.1 we infer that $\{x_n^*\}$ is bounded in X^* and sequential weak cluster points of $\{x_n^*\}$ belong to Ax_0. Therefore, cluster points of $\{\bar{x}_n^*\}$ in F^* also belong to $A_F x_0$. This contradicts the hypothesis that $\bar{x}_n^* \notin U_F^*$ for all n.

q.e.d.

This lemma implies that an operator A satisfying conditions (pm_1), (pm_2)

and (pm_4) is pseudo-monotone.

As a consequence of the above lemma and a result on local boundedness of monotone operators (see Browder [5; Lemma 1] and Rockafellar [24; Theorem 1]), we have

PROPOSITION 1.6. (*Browder-Hess* [10; *Proposition* 8]). *A maximal monotone operator* A *from* X *into* X^* *with* $D(A)=X$ *is pseudo-monotone.*

§2. Nonlinear functional equations for operators of monotone type

2.1. Functional equations for operators of type M

We now give a result on the solvability of the equation $Ax \ni f$ for an operator of type M.

THEOREM 2.1. *Let* A *be an operator of type* M *from* X *into* X^*, *and let* C *be a bounded convex closed subset of* X *with the origin in its interior* $\mathring{C}$. *Suppose that one of the following two conditions is satisfied*:

(α) *If* $\{(x_n, x_n^*)\} \subset G(A)$ *is a sequence such that* $x_n \xrightarrow{w} x$ *in* X *and*

$$\limsup_{n\to\infty} \langle x_n^*, x_n - x\rangle \leq 0,$$

then $\{x_n^*\}$ *is bounded in* X^*.

(α') *A is a quasi-bounded*, i.e., *for each* $M>0$ *there is a constant* $K(M)>0$ *such that if* $\|x\| \leq M$, $\langle x^*, x\rangle \leq M\|x\|$ *and* $(x, x^*) \in G(A)$, *then* $\|x^*\| \leq K(M)$.

Suppose furthermore, given $f \in X^*$,

(β) *for any* $x \in \partial C$ *and any* $x^* \in Ax$,

$$\langle x^* - f, x\rangle \geq 0.$$

Then $S_f = \{x \in C; f \in Ax\}$ *is non-empty and weakly compact.*

REMARK 2.1. In the case of single-valued operators, the above theorem was shown by Petryshyn and Fitzpatrick [22; Proposition 1.2] under (α'). In the multivalued case, we know [12; Theorem 1] that if A is of type M in the sense of [12], that is, it satisfies conditions (m_1), (m_3) and $(m_2)'$ given by replacing sequences by nets in (m_2), then Theorem 2.1 is valid without the assumption (α) or (α').

The proof of Theorem 2.1 is based on the following lemma due to Browder [8; Theorem 11].

LEMMA 2.1. *Suppose that* X *is finite dimensional. Let* A *be an operator from* X *into* X^* *such that*

(1) *Ax is a non-empty, bounded, convex and closed subset of X^* for each $x \in X$,*

(2) *A is upper semicontinuous,*

and let T be a monotone operator from X into X^*. Then for any given bounded convex closed subset C $(\neq \emptyset)$ of $D(T)$ and $f \in X^*$, there exist $x_0 \in C$ and $x_0^* \in Ax_0$ such that

$$<x_0^* - f + x^*, x_0 - x> \leq 0 \qquad \text{for all } (x, x^*) \in G(T) \text{ with } x \in C.$$

Before proving Theorem 2.1, we recall the following remarkable result by Browder and Hess [10; Proposition 11] that allows us to dispense with nets and to consider only sequences in our arguments.

PROPOSITION A. *Let X_0 be a linear subspace of X, and let Λ_0 be the family of all finite dimensional subspaces of X_0 and B the closed ball of radius R about the origin in X. Suppose that we are given a mapping $\psi: \Lambda_0 \to 2^B$, with $\psi(F)$ a non-empty subset of $F \cap B$ for each $F \in \Lambda_0$. For F_0 in Λ_0, set*

$$V_{F_0} = \bigcup_{\substack{F \supset F_0 \\ F \in \Lambda_0}} \psi(F)$$

and let

$$x_0 \in \bigcap_{F \in \Lambda_0} \tilde{V}_F,$$

where $\tilde{V}_F$ is the weak closure of V_F. Then for each $F' \in \Lambda_0$, there exists an increasing sequence $\{F_k\}_{k=1}^{\infty} \subset \Lambda_0$ with $F' \subset F_1$, and exists for each k an element $x_k \in \psi(F_k)$ such that x_k converges weakly to x_0 as $k \to \infty$.

PROOF of THEOREM 2.1. Let Λ, j_F, j_F^* and A_F be as in paragraph 1.2. For each $F \in \Lambda$, we set

$$S_F = \{x \in C \cap F;\ \text{there is } x^* \in Ax \text{ such that } <x^* - f, y> = 0 \text{ for all } y \in F\}.$$

We first show

(2.1) $$S_F \neq \emptyset \qquad \text{for every } F \in \Lambda.$$

In fact, as we have seen in Lemma 1.1, $A_F x$ is non-empty, convex and closed in F^* for each $x \in F$ and A_F is upper semicontinuous. Therefore, applying Lemma 2.1 for A_F and T given by $G(T) = \{(x, 0) \in F \times F^*;\ x \in F\}$, we obtain $x_F \in C \cap F$ and $x_F^* \in Ax_F$ such that

(2.2) $$<x_F^* - f, x_F - x> \leq 0 \qquad \text{for all } x \in C \cap F.$$

In the case where $x_F \in \mathring{C}$, since x_F is in the interior of $C \cap F$ in F, we have

(2.3) $$<x_F^*-f, x> = 0 \qquad \text{for all } x \in F.$$

In the case where $x_F \in \partial C$, noting that $<x_F^*-f, x_F> \geq 0$ by (β) and $<x_F^*-f, x_F> \leq 0$ by taking $x=0$ in (2.2), we see that

$$<x_F^*-f, x_F> = 0.$$

Thus, also in this case, (2.3) holds. Hence we have (2.1).

Now, we set for each $F \in \Lambda$

$$V_F = \bigcup_{\substack{F' \supset F \\ F' \in \Lambda}} S_{F'}.$$

Then, clearly, $V_F \subset C$ for every $F \in \Lambda$ and the family $\{V_F; F \in \Lambda\}$ has the finite intersection property. Since C is weakly compact, it follows that

$$\bigcap_{F \in \Lambda} \tilde{V}_F \neq \emptyset,$$

where $\tilde{V}_F$ is the weak closure of V_F in X. We take an x_0 in the intersection of all $\tilde{V}_F$ and fix it.

Next, let z be any point in X and take $F_0 \in \Lambda$ with $z, x_0 \in F_0$. Applying the above proposition, we find an increasing sequence $\{F_k\}_{k=1}^{\infty}$ with $F_0 \subset F_1$ and a sequence $\{(x_k, x_k^*)\} \subset G(A)$ with $x_k \in S_{F_k}$ and $x_k^* \in Ax_k$ such that $x_k \xrightarrow{w} x_0$ in X as $k \to \infty$ and

(2.4) $$<x_k^*-f, x> = 0 \qquad \text{for all } x \in F_k,\ k=1, 2, \ldots.$$

This implies that

(2.5) $$<x_k^*-f, x_k - x_0> = 0,\ <x_k^*, x_k> \leq \|f\| \cdot \|x_k\| \qquad \text{for all } k.$$

Hence, by hypothesis (α) or (α'), $\{x_k^*\}$ is bounded in X^*. Choose a subsequence $\{x_{k'}^*\}$ of $\{x_k^*\}$ weakly convergent to some $x_0^* \in X^*$. Then, by (2.5),

$$\limsup_{k' \to \infty} <x_{k'}^*, x_{k'}> = <x_0^*, x_0>.$$

Therefore, by condition (m_2), $x_0^* \in Ax_0$. Moreover, it follows from (2.4) that

$$0 = \lim_{k' \to \infty} <x_{k'}^* - f, z> = <x_0^* - f, z>.$$

We have seen that for each $z \in X$ there is $x^*(z) \in Ax_0$ such that

$$<x^*(z) - f, z> = 0.$$

Since Ax_0 is convex and closed by (m_1), Hahn-Banach theorem implies that $f \in Ax_0$. Thus $S_f \neq \emptyset$. Finally, the weak compactness of S_f immediately follows from condition (m_2). q.e.d.

COROLLARY 1. *Let A be an operator of type M from X into X^*. Suppose that A satisfies condition* (α) *or* (α') *in Theorem* 2.1 *and that A is coercive, i.e.,*

$$\inf_{x^* \in Ax} \frac{<x^*, x>}{\|x\|} \to \infty \qquad as\ \|x\| \to \infty.$$

Then $R(A) = X^*$.

COROLLARY 2. *Let A be a pseudo-monotone operator from X into X^*. If it is coercive, then* $R(A) = X^*$.

COROLLARY 3. *Let A be an operator from X into X^* such that Ax is non-empty, bounded, convex and closed in X^* for each $x \in X$. If A is sequentially weakly upper semicontinuous and is coercive, then* $R(A) = X^*$.

PROPOSITION 2.1. *Let A be an operator of type M from X into X^*, and let C be a bounded and weakly closed subset of X. Then the image $A(C)$ is closed in X^*.*

PROOF. Let $\{x_n^*\}$ be a sequence in $A(C)$ converging strongly to some $x_0^* \in X^*$. For each n, there is $x_n \in C$ such that $x_n^* \in Ax_n$. Since C is weakly compact by the reflexivity of X, there is a weakly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and the weak limit x_0 is contained in C. Besides, as easily seen,

$$\lim_{k \to \infty} <x_{n_k}^*, x_k> = <x_0^*, x_0>.$$

Hence, by condition (m_2), $x_0^* \in Ax_0$. Thus $x_0^* \in A(C)$. q.e.d.

An analogous result for pseudo-monotone operators was proved by Browder and Hess [10; Lemma 1].

PROPOSITION 2.2. *Suppose that there is a coercive monotone and sequentially weakly upper semicontinuous operator T from X into X^* with $D(T) = X$ such that $(0, 0) \in G(T)$, Tx is bounded, convex and closed in X^* for each $x \in X$ and there is $\delta > 0$ with*

$$<x^*, x> \geq \delta \|x\| \cdot \|x^*\| \qquad for\ all\ (x, x^*) \in G(T). \tag{2.6}$$

Let A be an operator of type M from X into X^* satisfying condition (α) *or* (α') *in Theorem* 2.1 *and assume that there is $N > 0$ such that*

$$<x^*, x> \geq -N\|x\| - N \qquad for\ all\ (x, x^*) \in G(A). \tag{2.7}$$

Suppose further that A^{-1} is bounded. Then $R(A) = X^*$.

PROOF. We first observe from Proposition 1.3, assumptions on T and (2.7) that for each $\lambda>0$, the operator $\lambda T+A$ is of type M and coercive. Furthermore $\lambda T+A$ satisfies condition (α) or (α') according as A. Therefore, by Corollary 1 to Theorem 2.1 we have $R(\lambda T+A)=X^*$ for $\lambda>0$. Let f be any element of X^*. Then we find $(x_\lambda, x_\lambda^*)\in G(A)$ and $(x_\lambda, y_\lambda^*)\in G(T)$ for each $\lambda>0$ such that

$$\lambda y_\lambda^* + x_\lambda^* = f. \tag{2.8}$$

By considering $<f, x_\lambda>$ and using (2.6) and (2.7) we obtain

$$\delta\lambda\|y_\lambda^*\|\cdot\|x_\lambda\| - N\|x_\lambda\| - N \leq \|f\|\cdot\|x_\lambda\|.$$

This implies that $\{x_\lambda\}_{\lambda>0}$ is bounded in X. For, if otherwise, there would exist a sequence $\{\lambda_n\}$ such that $\|x_{\lambda_n}\|\uparrow\infty$, and hence $\{\lambda_n\|y_{\lambda_n}^*\|\}$ is bounded, so that $\{\|x_{\lambda_n}^*\|\}$ is bounded by (2.8). This contradicts the boundedness of A^{-1}. Thus for a suitable $M>0$ we have

$$\|x_\lambda\|\leq M \qquad \text{for every } \lambda>0.$$

Since T is bounded, $\{y_\lambda^*\}$ is also bounded in X^*. Denote by B_M the closed ball of radius M about the origin in X. Then from Proposition 2.1 and (2.8), it follows that $f\in A(B_M)$, because $\|x_\lambda^*-f\|=\lambda\|y_\lambda^*\|\to 0$ as $\lambda\downarrow 0$. q.e.d.

This proposition is an analogue of Theorem 2 in [10].

2.2. Functional equations for pseudo-monotone operators

In this paragraph, we discuss the solvability of the equations

$$f\in Ax, \qquad x\in C \tag{2.9}$$

for a closed convex subset C of X and an $f\in X^*$ under a boundary condition. In case A is a pseudo-monotone operator from X into X^*, $C=B_r=\{x\in X;\|x\|\leq r\}$ with $r>0$ and $f=0$, the solvability of (2.9) was discussed by Browder and Hess [10; Theorem 11] and DeFigueiredo [11; Theorem 1] under the following boundary condition:

$$Ax+\lambda Jx\not\ni 0 \qquad \text{for all } \lambda>0 \text{ and all } x \text{ with } \|x\|=r. \tag{2.10}$$

We shall establish an existence theorem for (2.9) under a more general boundary condition (2.13) below.

Let ϕ be a function on X, i.e., a mapping of X into $[-\infty, \infty]$. If for a subset S of X, $\phi(x)\in(-\infty, \infty]$ for every $x\in S$ and $\phi\not\equiv\infty$ on S, then ϕ is called *proper* on S.

We now consider subdifferentials of proper convex functions. Let ϕ be a proper lower semicontinuous convex function on X. Then the *subdifferential* $\partial\phi$ is an operator from X into X^* given by

$$\partial\phi(x)=\{x^*\in X^*;\ <x^*, y-x> \leq \phi(y)-\phi(x) \text{ for all } y\in X\}$$

for $x\in X$ with $\phi(x)<\infty$ and by $\partial\phi(x)=\emptyset$ for $x\in X$ with $\phi(x)=\infty$. It is well-known that $\partial\phi$ is maximal monotone (see Rockafellar [23], [26]).

Let C be a non-empty closed convex subset of X and let us consider a function ϕ_C on X defined by

$$\phi_C(x)=\begin{cases} 0 & \text{if } x\in C, \\ \infty & \text{otherwise.} \end{cases} \tag{2.11}$$

ϕ_C is a proper lower semicontinuous convex function on X. Then, as easily seen, $x^*\in\partial\phi_C(x)$ for $x\in C$ if and only if

$$<x^*, y-x> \leq 0 \qquad \text{for all } y\in C.$$

From this we see that

$$\begin{cases} D(\partial\phi_C)=C, & 0\in\partial\phi_C(x) \quad \text{for all } x\in C \text{ and} \\ \partial\phi_C(x)=\{0\} & \text{for all } x\in \mathring{C}. \end{cases} \tag{2.12}$$

Theorem 2.2. *Let A be a pseudo-monotone operator from X into X^* and let C be a non-empty bounded, convex and closed subset of X. Given $f\in X^*$, assume that*

$$Ax+(\partial\phi_C(x)\backslash\{0\})\not\ni f^{1)} \qquad \textit{for all } x\in\partial C, \tag{2.13}$$

where ϕ_C is the function given by (2.11). *Then $S_f=\{x\in C;\ f\in Ax\}$ is non-empty and weakly compact.*

This theorem will follow from the following proposition.

Proposition 2.3. *Let A be a pseudo-monotone operator from X into X^* and let T be a monotone operator from X into X^* with bounded closed convex domain $D(T)$. Then for any given $f\in X^*$, there is $x_0\in D(T)$ with the following property: for each $x\in D(T)$ there is $x_0^*(x)\in Ax_0$ such that*

$$<x_0^*(x)-f+x^*, x_0-x> \leq 0 \qquad \textit{for all } x^*\in Tx.$$

Proof. In view of Lemma 1.2, we apply Lemma 2.1 for $A_F=j_F^*Aj_F$ and $T_F=j_F^*Tj_F$, $F\in\Lambda$, and see that the set S_F of all $x_F\in D(T)\cap F$ such that there is $x_F^*\in Ax_F$ with the property that

$$<x_F^*-f+x^*, x_F-x> \leq 0 \qquad \text{for all } (x, x^*)\in G(T) \text{ with } x\in F, \tag{2.14}$$

is non-empty. Here, set

1) For subsets S_1 and S_2, $S_1\backslash S_2=\{x;\ x\in S_1,\ x\notin S_2\}$.

$$V_F = \bigcup_{\substack{F' \supset F \\ F' \in \Lambda}} S_{F'}.$$

Then, just as in the proof of Theorem 2.1, we observe that

$$\emptyset \neq \bigcap_{F \in \Lambda} \tilde{V}_F \subset D(T),$$

where $\tilde{V}_F$ is the weak closure of V_F in X. Let x_0 be a point of the intersection.

Let x be an arbitrary point in $D(T)$ and, by using Proposition A, choose an increasing sequence $\{F_n\}_{n=1}^{\infty}$ in Λ with x_0, $x \in F_1$ and $\{x_n\}$ with $x_n \in S_{F_n}$ weakly convergent to x_0. Then, by (2.14), for each n there is $x_n^* \in Ax_n$ such that

$$(2.15) \quad \langle x_n^* - f + y^*, x_n - y \rangle \leq 0 \qquad \text{for all} \quad (y, y^*) \in G(T) \text{ with } y \in F_n.$$

Substituting some $(x_0, \tilde{x}_0^*) \in G(T)$ for (y, y^*) in (2.15), we have

$$\begin{aligned} 0 &\geq \limsup_{n\to\infty} \langle x_n^* - f + \tilde{x}_0^*, x_n - x_0 \rangle \\ &= \limsup_{n\to\infty} \langle x_n^*, x_n - x_0 \rangle . \end{aligned}$$

Therefore from condition (pm_2) it follows that for some $x_0^*(x) \in Ax_0$

$$\langle x_0^*(x), x_0 - x \rangle \leq \liminf_{n\to\infty} \langle x_n^*, x_n - x \rangle .$$

This inequality and (2.15) imply that for all $x^* \in Tx$

$$\begin{aligned} 0 &\geq \liminf_{n\to\infty} \langle x_n^* - f + x^*, x_n - x \rangle \\ &\geq \langle x_0^*(x) - f + x^*, x_0 - x \rangle . \end{aligned}$$

q.e.d.

We now state another lemma due to Browder [8; Lemma 1] which is needed in our proof of Theorem 2.2.

LEMMA 2.2. *Let C_0 be a convex subset of X and C_0' be a bounded, convex and closed subset of X^*. Suppose that for each $x \in C_0$ there is $x^*(x) \in C_0'$ such that $\langle x^*(x), x \rangle \leq 0$. Then there is $x_0^* \in C_0'$ such that $\langle x_0^*, x \rangle \leq 0$ for all $x \in C_0$.*

PROOF of THEOREM 2.2: Applying Proposition 2.3 for A and $T = \partial\phi_C$, we obtain a point $x_0 \in D(\partial\phi_C) = C$ with the property that for each $x \in C$ there is $x_0^*(x) \in Ax_0$ such that

$$\langle x_0^*(x) - f + x^*, x_0 - x \rangle \leq 0 \qquad \text{for all } x^* \in \partial\phi_C(x).$$

Taking $x^* = 0$ (cf. (2.12)),

$$<x_0^*(x)-f,\ x_0-x> \leq 0.$$

We infer from Lemma 2.2 that for some $x_0^* \in Ax_0$

$$<x_0^*-f,\ x_0-x> \leq 0 \qquad \text{for all } x \in C. \tag{2.16}$$

If $x_0 \in \overset{\circ}{C}$, then (2.16) implies that $x_0^*=f$. If $x_0 \in \partial C$, then, since $f-x_0^* \in \partial\phi_C(x_0)$ by (2.16), our boundary condition (2.13) implies that $x_0^*=f$. Thus $x_0 \in S_f$. That S_f is weakly compact is easily seen from the pseudo-monotonicity of A. q.e.d.

REMARK 2.1. We remark that the boundary condition (2.10) is a special case of (2.13) with $f=0$ and $C=B_r$. In fact,

$$\partial\phi_{B_r}=N_r(x) \equiv \begin{cases} \{0\} & \text{if } \|x\|<r, \\ \{\lambda x^*;\ \lambda \geq 0,\ x^* \in Jx\} & \text{if } \|x\|=r, \\ \emptyset & \text{if } \|x\|>r. \end{cases}$$

To prove this, first, let $x_0 \in \partial B_r$ and $x_0^* \in \partial\phi_{B_r}(x_0)$. Then by the definition of $\partial\phi_{B_r}$

$$<x_0^*,\ x_0-x> \geq 0 \qquad \text{for all } x \in B_r.$$

Putting $\rho=<x_0^*,\ x_0>$, we have

$$\rho=\sup_{x \in B_r} <x_0^*,\ x> = r\|x_0^*\|.$$

If $\rho=0$, then $x_0^*=0 \in N_r(x_0)$. In case $\rho>0$, we see that

$$\frac{r^2}{\rho}\|x_0^*\|=r=\|x_0\|$$

and

$$<\frac{r^2}{\rho}x_0^*,\ x_0> = r^2=\|x_0\|^2.$$

Therefore, by the definition of the duality mapping J,

$$\frac{r^2}{\rho}x_0^* \in Jx_0.$$

Hence $x_0^* \in N_r(x_0)$, i.e., $\partial\phi_{B_r}(x_0) \subset N_r(x_0)$. Thus we have proved

$$N_r(x) \supset \partial\phi_{B_r}(x) \qquad \text{for all } x \in \partial B_r.$$

In view of (2.12), this inclusion holds for all $x \in X$. Since N_r is monotone and $\partial\phi_{B_r}$ is maximal monotone, the above relation implies that $N_r=\partial\phi_{B_r}$.

REMARK 2.2. If C is a bounded, convex and closed subset of X with the origin in its interior, then (2.13) is more general than a boundary condition of the following type (cf. (β) in Theorem 2.1):

$$<x^*-f, x> \geq 0 \text{ for all } (x, x^*) \in G(A) \text{ with } x \in \partial C. \tag{2.17}$$

Indeed, assume (2.17) and let $x_0 \in \partial C$, $x_0^* \in Ax_0$ and $f-x_0^* \in \partial\phi_C(x_0)$. Then, since

$$<f-x_0^*, x-x_0> \leq 0 \qquad \text{for all } x \in C,$$

we have by (2.17)

$$<x_0^*-f, x> \geq 0 \qquad \text{for all } x \in C.$$

Hence $x_0^*=f$, because $0 \in \overset{\circ}{C}$. Thus (2.13) holds.

§3. Perturbation of maximal monotone operators

3.1. Perturbation of linear maximal monotone operators

In this papagraph, we treat the range of operators of the form $L+A$ with L linear maximal monotone and A of type M.

THEOREM 3.1. *Suppose that X is separable. Let A be an operator of type M from X into X^* and L be a maximal monotone operator from X into X^* with linear graph $G(L)$ in $X \times X^*$. Suppose further that A is coercive, i.e.,*

$$\inf_{x^* \in Ax} \frac{<x^*, x>}{\|x\|} \to \infty \qquad as\ \|x\| \to \infty$$

and that A is quasi-bounded, i.e., for each $N>0$ there is $K(N)>0$ such that if $(x, x^*) \in G(A)$, $\|x\| \leq N$ and $<x^*, x> \leq N\|x\|$, then $\|x^*\| \leq K(N)$. Then $R(L+A)=X^*$.

This theorem is a consequence of the following proposition.

PROPOSITION 3.1. *Let X, A and L be as in the above theorem, and let C be a bounded, convex and closed subset of X with the origin in its interior. Suppose that A is quasi-bounded and that*

$$<x^*, x> \geq 0 \qquad \textit{for any } (x, x^*) \in G(A) \textit{ with } x \in \partial C. \tag{3.1}$$

Then the set $S=\{x \in C;\ 0 \in Lx+Ax\}$ is non-empty and weakly compact.

REMARK 3.1. In case A is bounded and of type M in the sense of [12], the above proposition was shown in [12; Theorem 2] without the separability of X, so that, under the separability of X, the present result is a slight generalization of that in [12].

We prepare lammas to prove Proposition 3.1. The first is as follows:

LEMMA 3.1. (*Browder* [7; *Theorem* 1]) *Let T_1 be a maximal monotone operator from X into X^* with the origin in $D(T_1)$ and T_2 be a single-valued, bounded, coercive, demicontinuous and monotone operator from X into X^*. Then T_1+T_2 is maximal monotone and $R(T_1+T_2)=X^*$.*

LEMMA 3.2. *Suppose that X and X^* are strictly convex, and let T be a maximal monotone operator from X into X^*. Then*

(i) *The graph $G(T)$ is sequentially closed in the strong-weak topology of $X \times X^*$.*

(ii) *For each $\varepsilon>0$, the operator $T_\varepsilon=(T^{-1}+\varepsilon J^{-1})^{-1}: X \to X^*$ is a single-valued, bounded, demicontinuous and maximal monotone operator with $D(T_\varepsilon)=X$. Moreover, if $(0, 0) \in G(T)$, then $(0, 0) \in G(T_\varepsilon)$.*

PROOF. Let $\{(x_n, x_n^*)\} \subset G(T)$ be a sequence such that $x_n \xrightarrow{s} x$ in X and $x_n^* \xrightarrow{w} x^*$ in X^*. Then, from the monotonicity of T it follows that

$$<x_n^*-y^*, x_n-y> \geq 0 \qquad \text{for any } (y, y^*) \in G(T).$$

Letting $n \to \infty$, we have

$$<x^*-y^*, x-y> \geq 0 \qquad \text{for any } (y, y^*) \in G(T).$$

The maximal monotonicity of T implies that $(x, x^*) \in G(T)$, and thus (i) is proved.

Since X and X^* are strictly convex, we note that $J^{-1}: X^* \to X$ is one to one, demicontinuous, bounded, coercive and monotone. Now, we show (ii). Let $a^* \in Ta$. It is easy to see that $x^* \to T^{-1}(x^*+a^*)$ is a maximal monotone operator with the origin in its domain and $x^* \to \varepsilon J^{-1}(x^*+a^*)$ is single-valued, bounded, coercive, demicontinuous and monotone. Therefore, by Lemma 3.1, $R(T^{-1}+\varepsilon J^{-1})=X$, i.e., $D(T_\varepsilon)=X$, and simultaneously we see that T_ε is maximal monotone and bounded (the boundedness of T_ε follows from the coerciveness of the operator $x^* \to (T^{-1}+\varepsilon J^{-1})(x^*+a^*)$).

Let x^* and y^* be contained in $T_\varepsilon x$. Then $T^{-1}x^*+\varepsilon J^{-1}x^* \ni x$ and $T^{-1}y^* +\varepsilon J^{-1}y^* \ni x$. Therefore, for some suitable $x' \in T^{-1}x^*$ and $y' \in T^{-1}y^*$, we have $x=x'+\varepsilon J^{-1}x^*=y'+\varepsilon J^{-1}y^*$. Moreover, we observe

$$\begin{aligned} 0 &= <x^*-y^*, x'+\varepsilon J^{-1}x^*-y'-\varepsilon J^{-1}y^*> \\ &= <x^*-y^*, x'-y'> + \varepsilon <x^*-y^*, J^{-1}x^*-J^{-1}y^*> \\ &\geq \varepsilon\{\|x^*\|^2 - <x^*, J^{-1}y^*> - <y^*, J^{-1}x^*> + \|y^*\|^2\} \\ &\geq \varepsilon(\|x^*\| - \|y^*\|)^2. \end{aligned}$$

Hence $\|x^*\|=\|y^*\|$ and $<y^*, J^{-1}x^*> =\|y^*\|^2$. This implies that $x^*=y^*$. Thus T_ε is single-valued.

Let $x_n^*=T_\varepsilon x_n$ and $x_n \xrightarrow{s} x$. By the boundedness of T_ε, $\{x_n^*\}$ is bounded in X^*. Now, let x^* be any weak cluster point of $\{x_n^*\}$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k}^* \xrightarrow{w} x^*$. Since T_ε is also maximal monotone, from (i) we infer that $x^*=T_\varepsilon x$. Hence $x_n^* \xrightarrow{w} T_\varepsilon x$. Thus T_ε is demicontinuous.

Finally, if $(0, 0) \in G(T)$, then we have

$$0 \in T^{-1}0=T^{-1}0+\varepsilon J^{-1}0=(T^{-1}+\varepsilon J^{-1})0$$

and hence, $0=T_\varepsilon 0$. q.e.d.

LEMMA 3.3. *Let A, L and C be as in Proposition* 3.1; (3.1) *is assumed as well. For each $\varepsilon>0$ and $F\in \Lambda$, we set*

$$A_{\varepsilon,F}=j_F^*(L_\varepsilon+A)j_F$$

where $L_\varepsilon=(L^{-1}+\varepsilon J^{-1})^{-1}$ and Λ, j_F and j_F^* are as in paragraph 1.2. *Then each $A_{\varepsilon,F}$ has the following properties*:

(1) *$A_{\varepsilon,F}X$ is bounded, convex and closed in F^* for every $x\in F$.*

(2) *$A_{\varepsilon,F}$ is an upper semicontinuous operator from F into F^*.*

(3) *For any boundary point x of $C\cap F$ in F and any $x^*\in A_{\varepsilon,F}x$,*

$$<x^*, x> \geq 0.$$

PROOF. By Lemma 1.1, for each $F\in \Lambda$, $A_F=j_F^*Aj_F$ is an upper semicontinuous operator from F into F^* such that A_Fx is bounded, convex and closed for every $x\in F$. Since L_ε, $\varepsilon>0$, is demicontinuous from X into X^*, $L_{\varepsilon,F}=j_F^*L_\varepsilon j_F$ is a continuous operator from F into F^*. Hence $A_{\varepsilon,F}=L_{\varepsilon,F}+A_F$ has the properties (1) and (2). Condition (3) is easily obtained from (3.1), the monotonicity of L_ε and the fact that $(0, 0)\in G(L_\varepsilon)$ (cf. Lemma 3.2).

PROOF of PROPOSITION 3.1: Since X is reflexive, there exists a norm on X equivalent to the initial norm with respect to which X and X^* are strictly convex (see [4]). Thus, we may assume from the beginning that X and X^* are strictly convex.

First step. For each ε, $1>\varepsilon>0$, and each finite dimensional subspace F of X, we denote by $S_{\varepsilon,F}$ the set of all $y\in C\cap F$ such that there is $y^*\in Ay$ with the property that

$$<L_\varepsilon y+y^*, x> =0 \qquad \text{for all } x\in F. \tag{3.2}$$

Just as in the proof of Theorem 2.1, we see that each $S_{\varepsilon,F}\neq\emptyset$. We now show that there is a constant $M>0$ independent of ε and F such that for any (y, y^*) $\in G(A)$ with $y\in S_{\varepsilon,F}$ satisfying (3.2),

$$\|y^*\| \leq M. \tag{3.3}$$

In fact we have by the definition of L_ε

$$L_\varepsilon y \in L(y - \varepsilon J^{-1}(L_\varepsilon y)) \tag{3.4}$$

and hence, using (3.2) and the monotonicity of L,

$$\varepsilon \|L_\varepsilon y\|^2 = <L_\varepsilon y, \varepsilon J^{-1}(L_\varepsilon y)> \leq <L_\varepsilon y, y> = -<y^*, y>. \tag{3.5}$$

Thus $<y^*, y> \leq 0$. This and the quasi-boundedness of A imply that (3.3) holds for some $M > 0$ independent of ε and F.

Second step. Fix ε with $1 > \varepsilon > 0$. By the separability of X there is an increasing sequence $\{F_n\}$ of finite dimensional subspaces of X such that $\bigcup_{n=1}^{\infty} F_n$ is dense in X. For simplicity we write $S_{\varepsilon,n}$ for S_{ε,F_n}. We take sequences $\{x_{\varepsilon,n}\}$ with $x_{\varepsilon,n} \in S_{\varepsilon,n}$ for all n and $\{x^*_{\varepsilon,n}\}$ with $x^*_{\varepsilon,n} \in Ax_{\varepsilon,n}$ satisfying

$$<L_\varepsilon x_{\varepsilon,n} + x^*_{\varepsilon,n}, x> = 0 \qquad \text{for all } x \in F_n. \tag{3.6}$$

Set $K = \sup_{x \in C} \|x\|$. Then, since $\|x_{\varepsilon,n}\| \leq K$ and $\|x^*_{\varepsilon,n}\| \leq M$ by (3.3), we derive from (3.5) that

$$\varepsilon \|J^{-1}(L_\varepsilon x_{\varepsilon,n})\|^2 = \varepsilon \|L_\varepsilon x_{\varepsilon,n}\|^2 \leq -<x^*_{\varepsilon,n}, x_{\varepsilon,n}> \leq MK. \tag{3.7}$$

Therefore, there is a subsequence $\{n_k\}$ such that as $k \to \infty$,

$$x_{\varepsilon,n_k} \xrightarrow{w} x_\varepsilon \quad \text{in } X,$$

$$x^*_{\varepsilon,n_k} \xrightarrow{w} x^*_\varepsilon \quad \text{in } X^*,$$

$$L_\varepsilon x_{\varepsilon,n_k} \xrightarrow{w} X^*_\varepsilon \quad \text{in } X^*,$$

$$\sqrt{\varepsilon}\, J^{-1}(L_\varepsilon x_{\varepsilon,n}) \xrightarrow{w} \rho_\varepsilon \quad \text{in } X$$

for some $x_\varepsilon \in X$, $x^*_\varepsilon \in X^*$, $X^*_\varepsilon \in X^*$ and $\rho_\varepsilon \in X$. For these limit points we see that $x_\varepsilon \in C$ by the weak compactness of C, that

$$\|\rho_\varepsilon\| \leq \sqrt{M \cdot K}, \qquad \|x^*_\varepsilon\| \leq M \tag{3.8}$$

by (3.7), and that by (3.6)

$$<X^*_\varepsilon + x^*_\varepsilon, x> = 0 \qquad \text{for all } x \in \bigcup_{k=1}^{\infty} F_{n_k},$$

so that

$$X^*_\varepsilon + x^*_\varepsilon = 0 \qquad \text{in } X^*,$$

because $\bigcup_{k=1}^{\infty} F_{n_k}$ is dense in X. Moreover, since $G(L)$ is sequentially strong-weak-closed ((i) of Lemma 3.2) and linear in $X \times X^*$, it is sequentially weak-weak-closed. Hence, by (3.4),

$$-x_\varepsilon^* = X_\varepsilon^* \in L(x_\varepsilon - \sqrt{\varepsilon}\,\rho_\varepsilon). \tag{3.9}$$

Next, we show $x_\varepsilon^* \in Ax_\varepsilon$. In fact, given $\delta>0$, we take $x_\delta \in \bigcup_{k=1}^{\infty} F_{n_k}$ so that $\|x_\delta - x_\varepsilon\| \leq \delta$. Using the monotonicity of L_ε and noting that $<L_\varepsilon x_{\varepsilon,n_k} + x_{\varepsilon,n_k}^*, x_{\varepsilon,n_k} - x_\varepsilon> = 0$ for large k by (3.6), we have

$$\begin{aligned}
&\limsup_{k\to\infty} <x_{\varepsilon,n_k}^*, x_{\varepsilon,n_k} - x_\varepsilon> \\
&= \limsup_{k\to\infty} <L_\varepsilon x_\varepsilon + x_{\varepsilon,n_k}^*, x_{\varepsilon,n_k} - x_\varepsilon> \\
&\leq \limsup_{k\to\infty} <L_\varepsilon x_{\varepsilon,n_k} + x_{\varepsilon,n_k}^*, x_{\varepsilon,n_k} - x_\varepsilon> \\
&\leq \limsup_{k\to\infty} \{<L_\varepsilon x_{\varepsilon,n_k} + x_{\varepsilon,n_k}^*, x_{\varepsilon,n_k} - x_\delta> + \delta \|L_\varepsilon x_{\varepsilon,n_k} + x_{\varepsilon,n_k}^*\|\} \\
&= \delta \limsup_{k\to\infty} \|L_\varepsilon x_{\varepsilon,n_k} + x_{\varepsilon,n_k}^*\|.
\end{aligned}$$

Since $\{L_\varepsilon x_{\varepsilon,n_k} + x_{\varepsilon,n_k}^*\}$ is bounded in X^* for fixed ε, and δ is arbitrary, the above inequalities imply that

$$\limsup_{k\to\infty} <x_{\varepsilon,n_k}^*, x_{\varepsilon,n_k}> \leq <x_\varepsilon^*, x_\varepsilon>.$$

Thus $x_\varepsilon^* \in Ax_\varepsilon$ by condition (m_2).

Third step. Since $\{x_\varepsilon;\ 0<\varepsilon<1\} \subset C$ and $\{x_\varepsilon^*;\ 0<\varepsilon<1\}$ are bounded as we have seen in the Second step, there is a sequence $\{\varepsilon_k\}$ tending to 0 such that as $k\to\infty$,

$$\begin{aligned}
x_{\varepsilon_k} &\xrightarrow{w} x_0 \quad \text{in } X, \\
x_{\varepsilon_k}^* &\xrightarrow{w} x_0^* \quad \text{in } X^*
\end{aligned}$$

for some $x_0 \in C$ and $x_0^* \in X^*$. Write simply x_k and x_k^* for x_{ε_k} and $x_{\varepsilon_k}^*$, respectively. Then $\sqrt{\varepsilon_k}\rho_k \xrightarrow{s} 0$ in X by (3.8), and hence $x_k - \sqrt{\varepsilon_k}\rho_k \xrightarrow{w} x_0$ in X as $k\to\infty$, where $\rho_k = \rho_{\varepsilon_k}$. Furthermore, (3.9) and the weak-weak closedness of $G(L)$ imply $-x_0^* \in Lx_0$. Finally, we prove that $x_0^* \in Ax_0$. From the monotonicity of L it follows that

$$<-x_k^* + x_0^*, x_k - \sqrt{\varepsilon_k}\rho_k - x_0> \geq 0,$$

i.e.,

$$<x_k^*, x_k> \leq <x_k^*, x_0> + <x_0^*, x_k - x_0> + \sqrt{\varepsilon_k} < -x_k^* + x_0^*, \rho_k>$$

and hence

$$\limsup_{k\to\infty} <x_k^*, x_k> \leq <x_0^*, x_0>.$$

Condition (m_2) implies that $x_0^* \in Ax_0$. Thus $0 = -x_0^* + x_0^* \in Lx_0 + Ax_0$, that is, $S \neq \emptyset$.

Finally we show the weak compactness of S. Let $\{x_n\}$ be any sequence in S weakly convergent to $x \in C$ and $\{x_n^*\}$ be a sequence in X^* such that $x_n^* \in Ax_n$ and $-x_n^* \in Lx_n$ for all n. Then, $<x_n^*, x_n> = - < -x_n^*, x_n> \leq 0$ by the monotonicity of L and (iii) of Lemma 3.2. From this and the quasi-boundedness of A it follows that $\{x_n^*\}$ is bounded in X^*. Now, let $\{x_{n_k}^*\}$ be a subsequence of $\{x_n^*\}$ weakly convergent to some $x^* \in X^*$. We have $-x^* \in Lx$ because of the sequential weak-weak closedness of $G(L)$. Hence, by the monotonicity of L again, we obtain

$$\limsup_{k\to\infty} <x_{n_k}^*, x_{n_k}> \leq <x^*, x>,$$

so that $x^* \in Ax$ by condition (m_2). Thus $x \in S$. q.e.d.

PROOF of THEOREM 3.1: For any $f \in X^*$, we consider an operator A_f given by $A_f x = Ax - f$. By the coerciveness of A, there is $r > 0$ such that

$$<x^*, x> \geq 0 \qquad \text{for all } (x, x^*) \in G(A_f) \text{ with } \|x\| = r.$$

Therefore, applying Proposition 3.1, we obtain x with $\|x\| \leq r$ such that $f \in Ax + Lx$. Thus $R(A+L) = X^*$. q.e.d.

3.2. Perturbation of nonlinear maximal monotone operators

The purpose of this paragraph is to prove

THEOREM 3.2. *Let A be an operator from X into X^* satisfying (pm_1), (pm_2) and (pm_4), and let T be a maximal monotone operator from X into X^*. Suppose that for some $a \in D(T)$*

$$\inf_{x^* \in Ax} \frac{<x^*, x-a>}{\|x\|} \to \infty, \qquad \text{as } \|x\| \to \infty. \tag{3.10}$$

Then $R(A+T) = X^*$.

PROOF. Without loss of generality we may assume that X and X^* are strictly convex.

Now, define for each positive number ε

$$T_\varepsilon = (T^{-1} + \varepsilon J^{-1})^{-1}.$$

Then, by Lemma 3.2, $D(T_\varepsilon)=X$ and T_ε is a single-valued maximal monotone operator from X into X^*. The operator $T_\varepsilon+A$ is pseudo-monotone by Propositions 1.4 and 1.6. The mapping $x\to T_\varepsilon(x+a)+A(x+a)$ is coercive by (3.10). Hence we have $R(T_\varepsilon+A)=X^*$ by Corollary 2 to Theorem 2.1, that is, for each $f\in X^*$ there are elements $(x_\varepsilon, x_\varepsilon^*)\in G(A)$ and $(x_\varepsilon, y_\varepsilon^*)\in G(T_\varepsilon)$ such that $x_\varepsilon^*+y_\varepsilon^*=f$. Let $(a, a^*)\in G(T)$. Since

$$y_\varepsilon^*\in T(x_\varepsilon-\varepsilon J^{-1}y_\varepsilon^*),$$

we have by the monotonicity of T

$$\begin{aligned}< y_\varepsilon^*, x_\varepsilon-a> &=< y_\varepsilon^*, x_\varepsilon-\varepsilon J^{-1}y_\varepsilon^*-a> +\varepsilon\| y_\varepsilon^*\|^2\\ &\geq <a^*, x_\varepsilon-\varepsilon J^{-1}y_\varepsilon^*-a> +\varepsilon\| y_\varepsilon^*\|^2\\ &\geq <a^*, x_\varepsilon> - <a^*, a> +\varepsilon\| y_\varepsilon^*\|(\| y_\varepsilon^*\|-\|a^*\|)\\ &\geq <a^*, x_\varepsilon> - <a^*, a> -\frac{\varepsilon}{4}\|a^*\|^2\end{aligned}$$

and hence

$$\begin{aligned}<f, x_\varepsilon-a> &=<x_\varepsilon^*, x_\varepsilon-a> + < y_\varepsilon^*, x_\varepsilon-a>\\ &\geq <x_\varepsilon^*, x_\varepsilon-a> + <a^*, x_\varepsilon> - <a^*, a> -\frac{\varepsilon}{4}\|a^*\|^2.\end{aligned}$$

From (3.10) we see that $\{x_\varepsilon;\ \varepsilon_0>\varepsilon>0\}$ is bounded in X and so $\{<x_\varepsilon^*, x_\varepsilon-a>;\ \varepsilon_0>\varepsilon>0\}$ is bounded above for some $\varepsilon_0>0$. Moreover, by condition (pm_4),

$$\begin{aligned}< y_\varepsilon^*, x> &=<f, x> - <x_\varepsilon^*, x>\\ &=<f, x> + <x_\varepsilon^*, x_\varepsilon-x-a> - <x_\varepsilon^*, x_\varepsilon-a>\end{aligned}$$

is bounded below for each $x\in X$. By considering $-x$ instead of x, we see that $< y_\varepsilon^*, x>$ is also bounded above. Thus we have seen

$$\sup_{0<\varepsilon<\varepsilon_0} |< y_\varepsilon^*, x>|<\infty \qquad \text{for all } x\in X.$$

It follows from the uniform boundedness theorem that $\{ y_\varepsilon^*;\ \varepsilon_0>\varepsilon>0\}$ is bounded in X^*, so that $\{x_\varepsilon^*;\ \varepsilon_0>\varepsilon>0\}$ is bounded in X^*. Therefore we can choose a sequence $\{\varepsilon_k\}$ tending to 0 such that for some $x_0\in X$, $x_0^*\in X^*$ and $y_0^*\in X^*$

$$\begin{aligned}&x_{\varepsilon_k}\xrightarrow{w} x_0 && \text{in } X,\\ &x_{\varepsilon_k}^*\xrightarrow{w} x_0^* && \text{in } X^*,\\ &y_{\varepsilon_k}^*\xrightarrow{w} y_0^* && \text{in } X^*,\end{aligned}$$

as $k\to\infty$, and

$$\chi_1=\lim_{k\to\infty}<x^*_{\varepsilon_k}, x_{\varepsilon_k}-x_0>$$

exists. For simplicity, write x_k, x^*_k and y^*_k for x_{ε_k}, $x^*_{\varepsilon_k}$ and $y^*_{\varepsilon_k}$, respectively. Clearly $x^*_0+y^*_0=f$. Let $\chi_2=\lim_{k\to\infty}<y^*_k, x_k-x_0>$. Noting that

$$<x^*_k, x_k-\varepsilon_k J^{-1}y^*_k-x_0>+<y^*_k, x_k-\varepsilon_k J^{-1}y^*_k-x_0>$$
$$=<f, x_k-\varepsilon_k J^{-1}y^*_k-x_0>\to 0\ (k\to\infty),$$
$$\chi_1=\lim_{k\to\infty}<x^*_k, x_k-x_0>=\lim_{k\to\infty}<x^*_k, x_k-\varepsilon_k J^{-1}y^*_k-x_0>$$

and

$$\chi_2=\lim_{k\to\infty}<y^*_k, x_k-x_0>=\lim_{k\to\infty}<y^*_k, x_k-\varepsilon_k J^{-1}y^*_k-x_0>,$$

we have either $\chi_1\leq 0$ or $\chi_2\leq 0$. We first consider the case where $\chi_2\leq 0$. In this case, we see that

(3.11) $$\lim_{k\to\infty}<y^*_k, x_k>\leq<y^*_0, x_0>.$$

By the monotonicity of T, we have

(3.12) $$<y^*-y^*_k, y-(x_k-\varepsilon_k J^{-1}y^*_k)>\geq 0 \qquad \text{for all } (y, y^*)\in G(T).$$

By letting $k\to\infty$ in this inequality and using (3.11), we obtain

$$<y^*-y^*_0, y-x_0>\geq 0 \qquad \text{for all } (y, y^*)\in G(T).$$

This implies that $(x_0, y^*_0)\in G(T)$, since T is maximal monotone. Then taking $y=x_0$ and $y^*=y^*_0$ in (3.12), we have

$$<y^*_k-y^*_0, x_k-\varepsilon_k J^{-1}y^*_k-x_0>\geq 0.$$

It follows that $\lim_{k\to\infty}<y^*_k, x_k-x_0>\geq 0$, and hence the equality holds in (3.11). Therefore $\chi_2=0$. Consequently, $\chi_1=0$ because $\chi_1+\chi_2=0$. Hence from Proposition 1.1 we infer that $x^*_0\in Ax_0$. Thus $f=x^*_0+y^*_0\in Ax_0+Tx_0$. In case $\chi_1\leq 0$, we first use Proposition 1.1 and obtain $x^*_0\in Ax_0$. Then, by the pseudo-monotonicity of A, we see that $\chi_1=0$, so that $\chi_2=0$. As above we see that $y^*_0\in Tx_0$.

§4. Variational inequalities

4.1. Existence theorem for variational inequalities

As an application of our preceding results, we give an existence theorem for a variational inequality.

Let A be an operator from X into X^*, C be a convex closed subset of X and ϕ be a proper lower semicontinuous convex function on C. Then for given $f \in X^*$ we consider the variational problem $V[A, C, \phi, f]$: *find* $x_0 \in C$ *such that there is an* $x_0^* \in Ax_0$ *satisfying*

$$(4.1) \qquad <x_0^* - f, x_0 - x> \leq \phi(x) - \phi(x_0) \qquad \textit{for all } x \in C.$$

Theorem 4.1. *Let A be an operator from X into X^* satisfying* (pm_1), (pm_2) *and* (pm_4), *C be a convex closed subset of X and ϕ be a proper lower semicontinuous convex function on X. Suppose that for some $a \in C$ with $\phi(a) < \infty$*

$$(4.2) \qquad \inf_{x^* \in Ax} \frac{<x^*, x-a> + \phi(x)}{\|x\|} \to \infty \qquad \textit{as } \|x\| \to \infty,\ x \in C.$$

Then for any given $f \in X^*$, there is $(x_0, x_0^*) \in G(A)$ with $x_0 \in C$ satisfying (4.1), *that is, $V[A, C, \phi, f]$ has a solution.*

Remark 4.1. If the assumption (4.2) is replaced by the following condition:

$$(4.3) \qquad \begin{cases} \textit{there is an } a \in D(\partial\phi) \textit{ such that} \\ \inf_{x^* \in Ax} \dfrac{<x^*, x-a>}{\|x\|} \to \infty \qquad \textit{as } \|x\| \to \infty, \end{cases}$$

then Theorem 4.1 is a direct consequence of Theorem 3.2. Obviously (4.3) implies (4.2). But, in general, (4.2) does not imply (4.3).

Remark 4.2. For proper convex functions, lower semicontinuity in the strong topology is the same as sequential lower semicontinuity in the weak topology.

To obtain Theorem 4.1 we apply the following proposition.

Proposition 4.1. *Let A, C and ϕ be as in Theorem* 4.1. *Suppose, in addition, that C is bounded. Then for any given $f \in X^*$, the problem $V[A, C, \phi, f]$ has a solution.*

The main tool for the proof of this proposition is the following:

Lemma 4.1. *Let C and ϕ be as in Proposition* 4.1 *and λ be a positive number. Define*

$$\phi_\lambda(x) = \inf_{y \in C} \left(\frac{1}{\lambda} \|x - y\| + \phi(y) \right) \qquad \textit{for } x \in X.$$

Then ϕ_λ is finite in X and

(i) $\inf_{y \in C} \phi(y) \leq \phi_\lambda(x)$ *for* $x \in X$ *and* $\phi_\lambda(x) \leq \phi(x)$ *for* $x \in C$,

(ii) *for each* $x \in X$ *there is a point* $\tilde{x} \in C$ *such that*

$$\phi_\lambda(x) = \frac{1}{\lambda}\|x - \tilde{x}\| + \phi(\tilde{x}),$$

(iii) ϕ_λ *is convex and continuous in* X,

(iv) $D(\partial\phi_\lambda) = X$, *where* $\partial\phi_\lambda$ *is the subdifferential of* ϕ_λ.

PROOF. Since C is bounded closed and $\phi(a) < \infty$, we see that ϕ_λ is finite in X. From the definition of ϕ_λ we immediately obtain (i). For each $x \in X$ there is a sequence $\{x_n\}$ in C such that

$$\frac{1}{\lambda}\|x - x_n\| + \phi(x_n) \downarrow \phi_\lambda(x) \qquad \text{as } n \to \infty.$$

Since C is weakly compact, we may assume that $x_n \xrightarrow{w} \tilde{x}$ as $n \to \infty$ for some $\tilde{x} \in C$. Now, since

$$\phi(\tilde{x}) \le \liminf_{n\to\infty} \phi(x_n)$$

and

$$\|x - \tilde{x}\| \le \liminf_{n\to\infty} \|x - x_n\|,$$

we have

$$\frac{1}{\lambda}\|x - \tilde{x}\| + \phi(\tilde{x}) \le \phi_\lambda(x).$$

This implies that

$$\phi_\lambda(x) = \frac{1}{\lambda}\|x - \tilde{x}\| + \phi(\tilde{x}).$$

Thus (ii) is proved. Next, let x_1, $x_2 \in X$ and $0 < t < 1$. Then, by (ii) we have

$$\phi_\lambda(x_i) = \frac{1}{\lambda}\|x_i - \tilde{x}_i\| + \phi(\tilde{x}_i) \qquad \text{for some } \tilde{x}_i \in C,\ i = 1, 2.$$

Hence

$$\begin{aligned} & t\phi_\lambda(x_1) + (1-t)\phi_\lambda(x_2) \\ &= \frac{1}{\lambda}(t\|x_1 - \tilde{x}_1\| + (1-t)\|x_2 - \tilde{x}_2\|) + t\phi(\tilde{x}_1) + (1-t)\phi(\tilde{x}_2) \\ &\ge \frac{1}{\lambda}\|tx_1 + (1-t)x_2 - t\tilde{x}_1 - (1-t)\tilde{x}_2\| + \phi(t\tilde{x}_1 + (1-t)\tilde{x}_2) \\ &\ge \phi_\lambda(tx_1 + (1-t)x_2). \end{aligned}$$

Thus ϕ_λ is convex. Moreover, we see easily that ϕ_λ is locally bounded at every point in X. Therefore ϕ_λ is continuous in X. This implies the subdifferentiability of ϕ_λ at every point in X, that is, $D(\partial\phi_\lambda)=X$ (see Moreau [20]).

q.e.d.

PROOF of PROPOSITION 4.1: Let $\lambda>0$. Since $\partial\phi_\lambda$ is maximal monotone and $D(\partial\phi_\lambda)=X$ by (iv) of the above lemma, the operator $A+\partial\phi_\lambda$ is pseudo-monotone by Propositions 1.4 and 1.6. First, apply Proposition 2.3 for $A+\partial\phi_\lambda$ and T given by $G(T)=\{(x, 0);\ x\in C\}$ to obtain a point $x_\lambda\in C$ with the following property: for each $x\in C$ there is $z_\lambda^*(x)\in Ax_\lambda+\partial\phi_\lambda(x_\lambda)$ such that

$$<z_\lambda^*(x)-f,\ x_\lambda-x>\leq 0.$$

Furthermore, apply Lemma 2.2 for $C_0=x_\lambda-C$ and $C_0'=Ax_\lambda+\partial\phi_\lambda(x_\lambda)-f$. Then we find $x_\lambda^*\in Ax_\lambda$ and $y_\lambda^*\in\partial\phi_\lambda(x_\lambda)$ such that

$$<x_\lambda^*+y_\lambda^*-f,\ x_\lambda-x>\leq 0 \qquad \text{for all } x\in C.$$

Therefore we have by the definition of $\partial\phi_\lambda$ and (i) of Lemma 4.1

$$\begin{aligned} <x_\lambda^*-f,\ x_\lambda-x> &\leq <y_\lambda^*,\ x-x_\lambda> \\ &\leq \phi_\lambda(x)-\phi_\lambda(x_\lambda) \\ &\leq \phi(x)-\phi_\lambda(x_\lambda) \qquad \text{for all } x\in C. \end{aligned} \tag{4.4}$$

By condition (pm_4), the left side of (4.4) is bounded below for fixed $x\in C$, and hence $\{\phi_\lambda(x_\lambda);\ \lambda>0\}$ is bounded above. Therefore, by this and (i) of Lemma 4.1, we see that $\{\phi_\lambda(x_\lambda);\ \lambda>0\}$ is bounded. Next, let $\tilde{x}_\lambda$ be a point in C given by (ii) of Lemma 4.1 for x_λ and take a sequence $\{\lambda_n\}$ tending to 0 such that

$$x_{\lambda_n}\xrightarrow{w}x_0 \qquad \text{in } X \qquad \text{for some } x_0\in C,$$

$$\tilde{x}_{\lambda_n}\xrightarrow{w}\tilde{x}_0 \qquad \text{in } X \qquad \text{for some } \tilde{x}_0\in C.$$

Write x_n and $\tilde{x}_n$ for x_{λ_n} and $\tilde{x}_{\lambda_n}$, respectively, for simplicity. Since, as is proved above, $\{\phi_{\lambda_n}(x_n)\}$ is bounded,

$$\|x_n-\tilde{x}_n\|=\lambda_n\{\phi_{\lambda_n}(x_n)-\phi(\tilde{x}_n)\}\to 0,$$

and hence $x_0=\tilde{x}_0$ and

$$\begin{aligned} &\liminf_{n\to\infty}\phi_{\lambda_n}(x_n) \\ &=\liminf_{n\to\infty}\left\{\frac{1}{\lambda_n}\|x_n-\tilde{x}_n\|+\phi(\tilde{x}_n)\right\}\geq\phi(x_0). \end{aligned} \tag{4.5}$$

Taking $x=x_0$ in (4.4) with $\lambda=\lambda_n$ and letting $n\to\infty$, we obtain by (4.5) and (i) of Lemma 4.1

$$\begin{aligned}&\limsup_{n\to\infty} < x_n^*, x_n - x_0 > \\ &= \limsup_{n\to\infty} < x_n^* - f, x_n - x_0 > \\ &\leq \limsup_{n\to\infty} \{\phi_{\lambda_n}(x_0) - \phi_{\lambda_n}(x_n)\} \\ &\leq \phi(x_0) - \liminf_{n\to\infty} \phi_{\lambda_n}(x_n) \leq 0.\end{aligned}$$

Hence, by Proposition 1.1, $\{x_n^*\}$ is bounded in X^* and, if $\{x_{n'}^*\}$ is a subsequence of $\{x_n^*\}$ weakly convergent to some $x_0^* \in X^*$, then $x_0^* \in Ax_0$ and by (pm_2) we have

(4.6) $$\lim_{n'\to\infty} < x_{n'}^*, x_{n'} > = < x_0^*, x_0 >.$$

Letting $n'\to\infty$ in the inequality

$$< x_{n'}^* - f, x_{n'} - x > \leq \phi_{\lambda_{n'}}(x) - \phi_{\lambda_{n'}}(x_{n'}) \qquad \text{for all } x \in C,$$

we have by using (4.5) and (4.6)

$$< x_0^* - f, x_0 - x > \leq \phi(x) - \phi(x_0) \qquad \text{for all } x \in C.$$

Proof of Theorem 4.1: For $r>0$, we set $C_r = C \cap \{x; \|x\| \leq r\}$. Apply Proposition 4.1 with C_r in place of C. Then there are $x_r \in C_r$ and $x_r^* \in Ax_r$ for each $r>0$ such that

(4.7) $$< x_r^* - f, x_r - x > \leq \phi(x) - \phi(x_r) \quad \text{for all } x \in C_r.$$

Since $\{x_r;\ r>r_0\}$ is bounded in X for some r_0 because of (4.2), we can choose a sequence $\{r_n\}$ tending to ∞ such that

$$x_{r_n} \xrightarrow{w} x_0 \in C \qquad \text{as } n\to\infty.$$

Now, taking $x=x_0$ in (4.7) with $r=r_n$ and letting $n\to\infty$ in the inequality

$$< x_{r_n}^* - f, x_{r_n} - x_0 > \leq \phi(x_0) - \phi(x_{r_n}),$$

we obtain by the lower semicontinuity of ϕ

$$\limsup_{n\to\infty} < x_{r_n}^*, x_{r_n} - x_0 > \leq 0.$$

Hence, just as in the proof of Proposition 4.1, we see that $\{x_{r_n}^*\}$ is bounded in X^*, a weak cluster point x_0^* of $\{x_{r_n}^*\}$ belongs to Ax_0 and (x_0, x_0^*) satisfies (4.1).

q.e.d.

REMARK 4.3. Theorem 4.1 is an extension of Theorem 24 in [2] to the multi-valued case.

4.2. Convergence of sets and of functions

Let $\{C_n\}$ be a sequence of subsets of X. Then we define

$$\text{s-}\operatorname*{Liminf}_{n\to\infty} C_n = \{x\in X;\ \text{there is a sequence } \{x_n\} \text{ with } x_n\in C_n \text{ for all } n \text{ such that } x_n \xrightarrow{s} x \text{ in } X\}$$

and

$$\text{w-}\operatorname*{Limsup}_{n\to\infty} C_n = \{x\in X;\ \text{there is } \{x_k\} \text{ with } x_k\in C_{n_k} \text{ for a subsequence } \{C_{n_k}\} \text{ of } \{C_n\} \text{ such that } x_k \xrightarrow{w} x \text{ in } X\}.$$

DEFINITION 4.1. (*Mosco* [21]) *A sequence $\{C_n\}$ of subsets of X converges to a subset C of X in X, if*

$$C = \text{s-}\operatorname*{Liminf}_{n\to\infty} C_n = \text{w-}\operatorname*{Limsup}_{n\to\infty} C_n.$$

We then write

$$C = \operatorname*{Lim}_{n\to\infty} C_n \qquad \textit{in } X.$$

Let ϕ be a function on X, that is, it is a mapping of X into $[-\infty, \infty]$. Then the set

$$\{(x, r)\in X\times R;\ \phi(x)\le r\}$$

is called the epigraph of ϕ and denoted by epi (ϕ).

DEFINITION 4.2. (*Mosco* [21]) *A sequence $\{\phi_n\}$ of functions on X converges to a function ϕ on X, if*

$$\text{epi}\,(\phi) = \operatorname*{Lim}_{n\to\infty} \text{epi}\,(\phi_n) \qquad \textit{in } X\times R$$

in the sense of Definition 4.1. *We then write*

$$\phi = \operatorname*{Lim}_{n\to\infty} \phi_n \qquad \textit{in } X.$$

The following lemma is also due to Mosco [21; Lemma 1.10].

LEMMA 4.2. *Let $\{\phi_n\}$ be a sequence of functions on X. Then $\phi = \operatorname*{Lim}_{n\to\infty} \phi_n$ in X if and only if* (1) *and* (2) *below hold*:

(1) *For each $x\in X$, there is a sequence $\{x_n\}$ in X such that $x_n \xrightarrow{s} x$ in X and*

$$\limsup_{n\to\infty} \phi_n(x_n) \le \phi(x).$$

(2) *If* $\{\phi_{n_k}\}$ *is a subsequence of* $\{\phi_n\}$ *and* $\{x_k\}$ *is a sequence in* X *weakly convergent to* x, *then*

$$\liminf_{k\to\infty} \phi_{n_k}(x_k) \ge \phi(x).$$

Here, limsup, liminf and lim are taken in the wide sense, that is, those may take the values ∞ or $-\infty$. Moreover, we note that if $\phi = \operatorname*{Lim}_{n\to\infty} \phi_n$ in X, then (1) and (2) of Lemma 4.2 imply that for each $x \in X$ there is a sequence $\{x_n\}$ strongly convergent to x such that

$$\lim_{n\to\infty} \phi_n(x_n) = \phi(x).$$

4.3. Convergence of solutions of variational inequalities

Let A be an operator from X into X^* with $D(A) = X$, C be a closed convex subset of X, ϕ be a proper lower semicontinuous convex function on X and f be an element of X^*. We suppose that

(i) $\{A_n\}$ is a sequence of bounded pseudo-monotone operators from X into X^* with the following properties:

(a_1) $\{A_n\}$ is uniformly bounded, i.e., for each bounded subset B of X, the union $\bigcup_{n=1}^{\infty} A_n(B)$ is bounded in X^*.

(a_2) Given a subsequence $\{A_{n_k}\}$ of $\{A_n\}$, let $\{(x_k, x_k^*)\}$ be any sequence such that $(x_k, x_k^*) \in G(A_{n_k})$, $x_k \xrightarrow{w} x$ in X, $x_k^* \xrightarrow{w} x^*$ and

$$\limsup_{k\to\infty} < x_k^*, x_k > \le < x^*, x >.$$

Then $(x, x^*) \in G(A)$ and

$$\lim_{k\to\infty} < x_k^*, x_k > = < x^*, x >.$$

(ii) $\{C_n\}$ is a sequence of closed convex subsets of X such that

$$C = \operatorname*{Lim}_{n\to\infty} C_n.$$

(iii) $\{\phi_n\}$ is a sequence of functions on X such that each ϕ_n is proper lower semicontinuous and convex on X, $\{x \in X;\ \phi_n(x) < \infty\} \subset C_n$ and

$$\phi = \operatorname*{Lim}_{n\to\infty} \phi_n \qquad \text{in } X.$$

(iv) $\{f_n\}$ is a sequence in X^* such that $f_n \xrightarrow{s} f$ in X^* as $n \to \infty$.

Now, we consider the variational inequalities $V[A, C, \phi, f]$ and $V[A_n,$

C_n, ϕ_n, $f_n]$, $n=1, 2, \ldots$. We denote by S the set of all solutions of $V[A, C, \phi, f]$ and by S_n the set of all solutions of $V[A_n, C_n, \phi_n, f_n]$.

THEOREM 4.2. *In addition to the above hypotheses, assume that there is a bounded sequence* $\{a_n\}$ *with* $a_n \in C_n$ *and* $\phi_n(a_n) < \infty$ *for all n such that*

(4.8) $$\limsup_{n\to\infty} \phi_n(a_n) < \infty,$$

(4.9) *for any sequence* $\{(x_n, x_n^*)\}$ *with* $(x_n, x_n^*) \in G(A_n)$, $\|x_n\| \to \infty$ *implies that*

$$\frac{<x_n^*, x_n - a_n> + \phi_n(x_n)}{\|x_n\|} \to \infty,$$

(4.10) *for each n,*

$$\inf_{x^* \in A_n x} \frac{<x^*, x - a_n - a_n> + \phi_n(x)}{\|x\|} \to \infty \qquad \text{as } \|x\| \to \infty,\ x \in C_n.$$

Then we have:

$$\text{w-}\limsup_{n\to\infty} S_n \neq \emptyset \qquad \text{and} \qquad \text{w-}\limsup_{n\to\infty} S_n \subset S.$$

PROOF.. First, applying Theorem 4.1 for each n, we have $S_n \neq \emptyset$. Let $\{x_n\}$ be any sequence with $x_n \in S_n$ for all n. Then for each n there is $x_n^* \in A_n x_n$ such that

(4.11) $$<x_n^* - f_n, x_n - x> \leq \phi_n(x) - \phi_n(x_n) \qquad \text{for all } x \in C_n.$$

In particular, taking a_n for x in (4.11), we obtain

(4.12) $$<x_n^* - f_n, x_n - a_n> + \phi_n(x_n) \leq \phi_n(a_n) \qquad \text{for every } n.$$

Hence our assumptions (4.8) and (4.9) imply that $\{x_n\}$ has a bounded subsequence, so that $\{x_n\}$ has a sequential weak cluster point. This proves $\text{w-}\limsup_{n\to\infty} S_n \neq \emptyset$. Next, in order to show $\text{w-}\limsup_{n\to\infty} S_n \subset S$, we must prove that every sequential weak cluster point x_0 of $\{x_n\}$ belongs to S. By assumption (a_1) on the uniform boundedness of $\{A_n\}$, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ weakly convergent to x_0 such that the corresponding subsequence $\{x_{n_k}^*\}$ of $\{x_n^*\}$ converges weakly to some $x_0^* \in X^*$. We see that $x_0 \in C$, since $C = \lim_{n\to\infty} C_n$ in X. For simplicity we write x_k, x_k^*, f_k, a_k and ϕ_k for x_{n_k}, $x_{n_k}^*$, f_{n_k}, a_{n_k} and ϕ_{n_k}, respectively. Then we observe from (2) of Lemma 4.2, (4.12) and (4.8) that

(4.13) $$\begin{aligned}\phi(x_0) &\leq \liminf_{k\to\infty} \phi_k(x_k) \\ &\leq \limsup_{k\to\infty} \{\phi_k(a_k) - <x_k^* - f_k, x_k - a_k>\} \\ &< \infty.\end{aligned}$$

By Lemma 4.2 (see the remark after it), there is a sequence $\{y_k\}$ such that $y_k \xrightarrow{s} x_0$ in X and

(4.14) $$\lim_{k\to\infty} \phi_k(y_k) = \phi(x_0).$$

Here, note that $y_k \in C_{n_k}$ for all k, which follows from the assumption (iii), so that

$$<x_k^* - f_k, x_k - y_k> \le \phi_k(y_k) - \phi_k(x_k) \qquad \text{for all } k.$$

Hence, letting $k\to\infty$ in this inequality, we have by using (4.13) and (4.14)

$$\begin{aligned} &\limsup_{k\to\infty} <x_k^*, x_k - x_0> \\ &= \limsup_{k\to\infty} <x_k^* - f_k, x_k - y_k> \\ &\le \limsup_{k\to\infty} \phi_k(y_k) - \liminf_{k\to\infty} \phi_k(x_k) \\ &\le 0. \end{aligned}$$

Therefore we infer from the assumption (a_2) that $(x_0, x_0^*) \in G(A)$ and

(4.15) $$\lim_{k\to\infty} <x_k^*, x_k> = <x_0^*, x_0>.$$

We shall show that

(4.16) $$<x_0^* - f, x_0 - x> \le \phi(x) - \phi(x_0) \qquad \text{for all } x \in C.$$

Let x be any point in C. If $\phi(x) = \infty$, then (4.16) is trivial. Thus, assume $\phi(x) < \infty$. Then, by Lemma 4.2 and (iii) again, there is a sequence $\{z_k\}$ with $z_k \in C_{n_k}$ for all k strongly convergent to x such that

$$\lim_{k\to\infty} \phi_k(z_k) = \phi(x).$$

Since $f_k \xrightarrow{s} f$ in X^* as $k\to\infty$ and

$$<x_k^* - f_k, x_k - z_k> \le \phi_k(z_k) - \phi_k(x_k) \qquad \text{for all } k,$$

we obtain (4.16) by letting $k\to\infty$ and using (4.13) and (4.15). Thus $x_0 \in S$.
q.e.d.

REMARK 4.4. The following can be proved as above: Let $(x, x^*) \in X \times X^*$ and $\{(x_k, x_k^*)\}$ be a sequence in $X \times X^*$ such that $(x_k, x_k^*) \in G(A_{n_k})$ and $x_k \in S_{n_k}$ for some subsequence $\{n_k\}$, $x_k \xrightarrow{w} x$ in X and $x_k^* \xrightarrow{w} x^*$ in X^* as $k\to\infty$ (hence $x \in S$ by the above theorem) and

$$<x_k^* - f_{n_k}, x_k - y> \le \phi_{n_k}(y) - \phi_{n_k}(x_k) \qquad \text{for all } y \in C_{n_k}.$$

Then we have

$$\lim_{k\to\infty} < x_k^*, x_k > = < x^*, x >$$

and

$$\lim_{k\to\infty} \phi_{n_k}(x_k) = \phi(x).$$

REMARK 4.5. In case A and A_n, $n=1, 2,\dots$, are bounded hemicontinuous monotone operators, a sharper result than our theorem was proved by Mosco [21]. Some interesting applications of Theorem 4.2 to boundary value problems are given in [13] and [14].

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