HIROSHIMA MATH. J.
4(1974), 133-209

The Fourier Transform of the Schwartz Space on a Semisimple Lie Group

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(Received September 17, 1973)

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1. Introduction

Let G be a semisimple Lie group and $L^2(G)$ denote the space of square-integrable functions on G with respect to the Haar measure. The Fourier transform $\mathscr{F}$ can be regarded as an isometry of $L^2(G)$ onto the Hilbert space $L^2(\hat{G})$ which is defined by irreducible unitary representations of G.

In his paper [6(m)], Harish-Chandra introduces the Schwartz space $\mathscr{C}(G)$ consisting of functions on G. It is analogous to the Schwartz space $\mathscr{S}(\boldsymbol{R}^n)$ of rapidly decreasing functions on a eulidean space $\boldsymbol{R}^n$ and is contained densely in $L^2(G)$. It is of much interest to ask about the image of $\mathscr{C}(G)$ in $L^2(\hat{G})$ under $\mathscr{F}$. This is a Paley-Wiener type question for $\mathscr{C}(G)$. There are some results for this problem. It is solved by J.G. Arthur[1] in the real rank one case. Moreover, the problems for the Schwartz space on Riemannian globally symmetric spaces and for a certain subspace are studied by Eguchi-Okamoto[4] and Harish-

Chandra [6(h, i)] respectively.

The purpose of this paper is to give a characterization of the Fourier image of the Schwartz space for non-compact real semisimple Lie groups G with only one conjugacy class of Cartan subgroups and without any complex structure (we shall discuss, in another paper, the case that G has a complex structure). Now we assume that G has the above mentioned properties. Then it is known that only one continuous series connects with the Plancherel formula. We can obtain the Plancherel measure in a concrete form via the method of integration by parts from the character formulae of the irreducible unitary representations of the principal series. As an immediate consequence of the main theorem we obtain the theory of Fourier analysis for tempered distributions on G.

The most difficult part of this theorem is to prove the surjectivity and the continuity of the inverse Fourier transform. We shall prove this by means of induction with respect to the real rank of G. For this, we have to study in detail the Eisenstein integrals, in particular, not only their constant terms but the asymptotic behaviour at infinity along the walls of Weyl chambers, and we use some of Harish-Chandra's estimates for differential equations with respect to the center of the universal enveloping algebra of the Lie algebra of G. Arthur's methods in [1] are very effective in our discussions.

The paper is arranged as follows. We fix an Iwasawa decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ of the Lie algebra of G and a nonzero element H in $\mathfrak{a}$. In Section 2 we prove that the derived algebra $\overline{\mathfrak{m}}_1$ of the centralizer $\mathfrak{m}_1$ of H in $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras. In Section 3, we prove the character formula for the representations $\pi_{\sigma,\lambda}$ of the principal series, and in Section 4 we obtain the Plancherel measure in explicit form, which is a polynomial, and prove the Plancherel formula. We state the main theorem in Section 5 and prove the injectivity of the Fourier transform $\mathscr{F}$ for $\mathscr{C}(G)$ in Section 7. We state Theorem 8.1 in Section 8, which is a sufficient condition for the map $\mathscr{F}$ to be surjective, and in Sections 10–13 we describe Harish-Chandra's work in the form suitable for our purpose and show that his estimates are uniform in a sense. In Section 14, making use of his estimates, we obtain the functions θ_j on the analytic subgroup M'_1 of G corresponding to $\overline{\mathfrak{m}}_1$, and we prove that we can apply the induction hypothesis to $D\theta_j$, where D is the ratio of the Plancherel measure corresponding to G to that corresponding to M'_1. In Section 17, we define the Fourier transform of tempered distributions on G and characterize their images by this map.

It seems that, when we consider the analogue of Theorem 5.1 for arbitrary semisimple Lie group G, the part corresponding to continous series in the proof of the surjectivity of the Fourier transform map can be proved by induction on the real rank of G, similarly to our proof.

The author is pleased to express his gratitude to Professor K. Okamoto and Dr. M. Wakimoto for many stimulating conversations.

2. Notation and preliminaries

We shall use the standard notation $\boldsymbol{R}$ and $\boldsymbol{C}$ for the field of real numbers and the field of complex numbers respectively. We shall use i as well as $\sqrt{-1}$ to denote a square root of -1. If S is a set, T a subset of S and f a function on S, the restriction of f to T is denoted by $f|T$. If S is a finite set $[S]$ denotes the number of elements in S. If S is a topological space $cl(T)$ denotes the closure of T in S. Composition of functions and operators will often be denoted by $\circ$.

If M is a manifold (satisfying the second countability axiom), the space of infinitely differentiable functions on M and the set of those of compact support are denoted by $C^\infty(M)$ and $C_c^\infty(M)$ respectively.

If V is a vector space over $\boldsymbol{R}$, V^c denotes the complexification $V\otimes_{\boldsymbol{R}}\boldsymbol{C}$ of V. V can be seen as a differentiable manifold in the usual manner. Let $\boldsymbol{D}(V)$ denote the algebra of differential operators on V of constant coefficients. Following Schwartz [13] we denote the space of rapidly decreasing functions on V with usual topology by $\mathscr{S}(V)$. If $f\in C^\infty(V)$ and $D\in\boldsymbol{D}(V)$ the value of Df at a point v will usually be denoted by $f(v;D)$ but sometimes by $D_v(f(v))$.

Lie groups will be denoted by Latin capital letters and their Lie algebras by corresponding lower case Germann letters. If G is a Lie group and $\mathfrak{g}$ its Lie algebra the adjoint representation of G is denoted by Ad and the adjoint representation of $\mathfrak{g}$ by ad.

Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let α be a linear function on the comlex vector space $\mathfrak{h}^c$ and $\mathfrak{g}^\alpha$ denote the linear subspace of $\mathfrak{g}^c$ given by

$$\mathfrak{g}^\alpha=\{X\in\mathfrak{g}^c:\quad [H, X]=\alpha(H)X \qquad \textit{for all } H\in\mathfrak{h}^c\}.$$

The linear function α is called a root of the pair $(\mathfrak{g}^c, \mathfrak{h}^c)$ if $\mathfrak{g}^\alpha\neq\{0\}$. In this case $\mathfrak{g}^\alpha$ is called a root subspace.

Let L be a connected reductive Lie group over $\boldsymbol{R}$ with Lie algebra $\mathfrak{l}$,

$$j\colon \mathfrak{l}\subset\mathfrak{l}^c$$

be the inclusion and L^c be a complex analytic group with Lie algebra $\mathfrak{l}^c$. L^c is called a complexification of L if j extends to a homomorphism of L into L^c. Reductive Lie algebra $\mathfrak{l}$ can be written as $\mathfrak{l}=\mathfrak{l}_1+\mathfrak{c}$, where $\mathfrak{l}_1$ is a semisimple ideal of $\mathfrak{l}$ and $\mathfrak{c}$ is the center of $\mathfrak{l}$. Let L_1, $C(L_1^c, C^c)$ be the analytic subgroups of $L(L^c)$ corresponding to $\mathfrak{l}_1$, $\mathfrak{c}(\mathfrak{l}_1^c, \mathfrak{c}^c)$ respectively. L^c is said to be *quasisimply connected* (q.s.c.) if $L_1^c\cap C^c=\{1\}$, where 1 is the unit element of L^c, and if L_1^c is simply connected. L is called q.s.c. if it has a q. s. c. complexification.

Fix a complexification $j\colon L\to L^c$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$. Let A and A^c be the Cartan subgroups of L and L^c associated with $\mathfrak{h}$ and $\mathfrak{h}^c$, that is, the centralizers of $\mathfrak{h}$ and $\mathfrak{h}^c$ in L and L^c respectively. Clearly $j(A)\subset A^c$. It is known that A^c is connected ([6(k), Corollary to Lemma 27]). If λ is a linear function on $\mathfrak{h}^c$, there exists at most one complex analytic homomorphism

$$\xi_\lambda : A^c \to \boldsymbol{C}$$

such that for every $H \in \mathfrak{h}^c$

$$\xi_\lambda(\exp H) = e^{\lambda(H)}.$$

We also write ξ_λ for the homomorphism

$$\xi_\lambda \circ j : A \to \boldsymbol{C}.$$

ξ_λ can be seen to be independent of the complexification A^c used, provided that ξ_λ is defined on that complexification.

Cleary ξ_α exists for any root α of $(\mathfrak{l}^c, \mathfrak{h}^c)$. If $P_{\mathfrak{h}}$ is the set of positive roots of $(\mathfrak{l}^c, \mathfrak{h}^c)$ relative to some ordering, let

$$\rho = \frac{1}{2} \sum_{\alpha \in P_{\mathfrak{h}}} \alpha.$$

It is easy to see that the question of the existence of ξ_ρ is independent of the ordering of the roots $(\mathfrak{l}^c, \mathfrak{h}^c)$ and of the choice of Cartan subalgebra $\mathfrak{h}$. If ξ_ρ exists L^c is said to be *acceptable*. L is said to be acceptable if it has an acceptable complexification.

If L^c is q.s.c., it is known that it is acceptable ([6(k), Lemma 29]). If $L_1 \cap C$ is finite, it is clear that L has a finite, and hence acceptable cover.

Suppose L is a compact, connected acceptable Lie group with Lie algebra $\mathfrak{l}$. Let $\mathfrak{h}$, $P_{\mathfrak{h}}$, A and ρ be defined as above. For each α define an element H_α in $\mathfrak{h}^c$ by

$$B(H_\alpha, H) = \alpha(H) \qquad \text{for all } H \in \mathfrak{h}^c,$$

where B denotes the Killing form of $\mathfrak{l}^c$ restricted to $\mathfrak{h}^c$. Put

$$\varpi = \prod_{\alpha \in P_{\mathfrak{h}}} H_\alpha.$$

Then ϖ is in S, the symmetric algebra on $\mathfrak{h}^c$, and can be regarded as a polynomial function on $\mathfrak{h}^c$. Let Π be the lattice of linear functions

$$\lambda : \sqrt{-1}\mathfrak{h} \to \boldsymbol{R}$$

for which ξ_λ exists. Let $\Pi' = \{\lambda \in \Pi : \varpi(\lambda) \neq 0\}$. If $W_{\mathfrak{l}}$ is the Weyl group of $(\mathfrak{l}^c, \mathfrak{h}^c)$, $W_{\mathfrak{l}}$ acts on $\sqrt{-1}\mathfrak{h}$. Then $W_{\mathfrak{l}}$ acts on Π as follows

$$(s\mu)(H) = \mu(s^{-1}H), \qquad \mu \in \Pi,\ s \in W_{\mathfrak{l}} \text{ and } H \in \sqrt{-1}\mathfrak{h}.$$

For $s \in W_{\mathfrak{l}}$, put $\varepsilon(s) = (-1)^{n(s)}$, where $n(s)$ is the number of positive roots that are mapped by s into negative roots. For a regular element $h\in A$, put

$$\varDelta(h)=\xi_\varrho(h)\prod_{\alpha\in P_{\mathfrak{h}}}(1-\xi_\alpha(h^{-1})).$$

If σ is an irreducible unitary representation of L on a Hilbert space V_σ and $l\in L$, let $\mathrm{tr}\sigma(l)$ denote the trace of $\sigma(l)$ and let $\dim\sigma$ denote the dimension of V_σ.

LEMMA 2.1. *There is a map $\mu\to\sigma(\mu)$ from Π' onto the set of unitary equivalence classes of irreducible representations of L. $\sigma(\mu_1)=\sigma(\mu_2)$ if and only if $\mu_1=s\mu_2$ for some $s\in W_{\mathfrak{l}}$. Furthermore, if h is a regular element of A,*

$$\mathrm{tr}\sigma(\mu)(h)=(\mathrm{sign}\varpi(\mu))\cdot\varDelta(h)^{-1}\Big(\sum_{s\in W_{\mathfrak{l}}}\varepsilon(s)\xi_{s\mu}(h)\Big).$$

Also there exists a constant c_L, independent of μ, such that

$$\dim\sigma(\mu)=c_L|\varpi(\mu)|.$$

Finally, if $\mu\in\Pi'$ and $B(\mu,\alpha)>0$ for each $\alpha\in P_{\mathfrak{h}}$, then $\mu-\rho$ is the highest weight of the representation of the Lie algebra $\mathfrak{l}^c$ corresponding to $\sigma(\mu)$.

For a proof see [1, Lemma 1].

If σ is an irreducible unitary representation of L and $\sigma=\sigma(\mu)$ for a linear function $\mu\in\Pi'$ then μ is said to be associated with σ.

Let G be a connected semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a fixed Cartan decomposition with Cartan involution θ. Let $\mathfrak{a}_{\mathfrak{p}}$ be a fixed maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}$ be a fixed maximal abelian subalgebra of $\mathfrak{g}$, which is a θ-stable Cartan subalgebra, that is, $\theta\mathfrak{a}=\mathfrak{a}$. We put $\mathfrak{a}_{\mathfrak{k}}=\mathfrak{a}\cap\mathfrak{k}$. Let K be the analytic subgroup of G corresponding to $\mathfrak{k}$. We assume that G has finite center. This implies that K is compact. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, we denote the set of non-zero roots of $(\mathfrak{g}^c, \mathfrak{h}^c)$ by $\varDelta(\mathfrak{h})$, sometimes, we write simply $\varDelta$ instead of $\varDelta(\mathfrak{a})$.

Let $\mathfrak{m}$ and M be the centralizers of $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{k}$ and K respectively, and let M' be the normalizer of $\mathfrak{a}_{\mathfrak{p}}$ in K. Then $\mathfrak{a}_{\mathfrak{k}}$ is a Cartan subalgebra of reductive Lie algebra $\mathfrak{m}$. We write the finite factor group M'/M as W, which is called the little Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{a}_{\mathfrak{p}}$.

Fix compatible orders on the real dual spaces of $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}_{\mathfrak{p}}+\sqrt{-1}\mathfrak{a}_{\mathfrak{k}}=\mathfrak{a}_R$. Let P and P_+ be the set of positive roots of $(\mathfrak{g}^c, \mathfrak{a}^c)$ relative to this order and the set of roots $\alpha\in P$ which do not vanish on $\mathfrak{a}_{\mathfrak{p}}$ respectively. We also denote the complement of P_+ in P by P_M. For every non-zero root $\alpha\in\varDelta$, $H_\alpha\in\mathfrak{a}_R$ is defined by

$$B(H, H_\alpha)=\alpha(H)\qquad\text{for every } H\in\mathfrak{a},$$

where B denotes the Killing form of $\mathfrak{g}^c$. For simplicity, a root $\alpha\in\varDelta$ is often identified with H_α. Since $\mathfrak{a}_\mathfrak{k}$ is a Cartan subalgebra of m we can regard P_M as the set of positive roots of $(\mathfrak{m}^c, \mathfrak{a}_\mathfrak{k}^c)$. We put

$$\mathfrak{a}_\mathfrak{p}^+ = \{H \in \mathfrak{a}_\mathfrak{p} : \alpha(H) > 0 \quad \textit{for every} \quad \alpha \in P_+\},$$

which is called the positive Weyl chamber of $\mathfrak{a}_\mathfrak{p}$. For each root $\alpha \in \Delta$ the linear function α^θ is defined by

$$\alpha^\theta(H) = \alpha(\theta H), \qquad H \in \mathfrak{a}.$$

A root α is said to be *real* (*imaginary*) if $\alpha(H)$ is real (pure imaginary) for every $H \in \mathfrak{a}$; if α is neither real nor imaginary, then α is said to be *complex*.

Two Cartan subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$ of $\mathfrak{g}$ (two Cartan subgroups A_1, A_2 of G) are said to be conjugate under the adjoint group $\mathrm{Int}(\mathfrak{g})$ (the group of inner automorphisms of G) if there exists $\psi \in \mathrm{Int}(\mathfrak{g})$ (an inner automorphism ψ of G) such that $\psi(\mathfrak{h}_1) = \mathfrak{h}_2$ $(\psi(A_1) = A_2)$. It is known that the number $N(\mathfrak{g})$ of the conjugacy classes of Cartan sublgebras of $\mathfrak{g}$ is finite ([14, Corollary to Theorem 5]). $N(\mathfrak{g})$ is clearly equal to the number $N(G)$ of the conjugacy classes of Cartan subgroups of G.

For every Cartan sublgebra $\mathfrak{h}$ of $\mathfrak{g}$ the following two subalgebras

$$\mathfrak{h}_I = \{H \in \mathfrak{h} : \textit{ all eigenvalues of adH are pure imaginary}\},$$

$$\mathfrak{h}_R = \{H \in \mathfrak{h} : \textit{ all eigenvalues of adH are real}\}$$

are called the *toroidal part* and *the vector part* of $\mathfrak{h}$ respectively.

A θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be standard with respect to the triple $(\mathfrak{k}, \mathfrak{p}, \mathfrak{a})$ (simply, standard) if

$$\mathfrak{a}_\mathfrak{k} \subset \mathfrak{h}_\mathfrak{k} = \mathfrak{h} \cap \mathfrak{k} \quad \text{and} \quad \mathfrak{h} \cap \mathfrak{p} = \mathfrak{h}_\mathfrak{p} \subset \mathfrak{a}_\mathfrak{p}.$$

It is known that there exists a finite number of Cartan subalgebras which are standard and that any Cartan sublgabre $\mathfrak{h}$ of $\mathfrak{g}$ is conjugate under the adjoint group $\mathrm{Int}(\mathfrak{g})$ to one of them ([14, Theorem 2]).

Let $\omega : X \to \mathrm{tr}(ad(X))^2$ $(X \in \mathfrak{g}^c)$ denote the Casimir polymonial of $\mathfrak{g}^c$. For any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ set $\mathfrak{l}_-(\mathfrak{h}) = \sup(\dim(\mathfrak{h}_-))$, where $\mathfrak{h}_-$ runs over all subspaces of $\mathfrak{h}$ on which ω is negative definite; put $\mathfrak{l}_- = \sup_\mathfrak{h} \mathfrak{l}_-(\mathfrak{h})$ ($\mathfrak{h}$ running through the Cartan subalgebras of $\mathfrak{g}$). A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be fundamental if $\mathfrak{l}_- = \mathfrak{l}_-(\mathfrak{h})$. Then the rank of $\mathfrak{k}$ is equal to $\mathfrak{l}_-$ and, $\mathfrak{h}$ is fundamental if and only if the pair $(\mathfrak{g}^c, \mathfrak{h}^c)$ admits no real roots.

The following lemma permits us to make use of induction to prove the main theorem (see Section 16).

LEMMA 2.2. *Let H be a non-zero element in $cl(\mathfrak{a}_\mathfrak{p}^+)$, $\mathfrak{m}_1$ the centralizer of H in $\mathfrak{g}$ and $\overline{\mathfrak{m}}_1 = [\mathfrak{m}_1, \mathfrak{m}_1]$. If the number of conjugacy classes of Cartan* *subalgebras of* $\mathfrak{g}$ *is one, then the number of conjugacy classes of Cartan subalgebras of* $\overline{\mathfrak{m}}_1$ *is also one.*

PROOF. We have clearly,

$$\overline{\mathfrak{m}}_1^c = \sum_{\alpha\in P'} CH_\alpha + \sum_{\alpha\in P_M} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}) + \sum_{\alpha\in P'\cap P_+} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}) \tag{2.1}$$

and

$$\overline{\mathfrak{m}}_1 = \overline{\mathfrak{m}}_1^c \cap \mathfrak{g},$$

where P' denotes the set of $\alpha\in P$ such that $\alpha(H_0)=0$. Put

$$\overline{\mathfrak{m}}_{1\mathfrak{k}} = \overline{\mathfrak{m}}_1 \cap \mathfrak{k} \quad \text{and} \quad \overline{\mathfrak{m}}_{1\mathfrak{p}} = \overline{\mathfrak{m}}_1 \cap \mathfrak{p}.$$

Since $\alpha^\theta(H_0) = \alpha(\theta H_0) = -\alpha(H_0)$ the first term in the formula (2.1) is θ-stable. As is easily seen, if $\alpha\in P_+$ then $-\alpha^\theta \in P_+$ and if $\beta\in P_M$ then $\beta^\theta = \beta$. So the last two terms in (2.1) are both θ-stable. Thus $\overline{\mathfrak{m}}_1$ is θ-stable. The sum $\overline{\mathfrak{m}}_1 = \overline{\mathfrak{m}}_{1\mathfrak{k}} + \overline{\mathfrak{m}}_{1\mathfrak{p}}$ is a Cartan decomposition of $\overline{\mathfrak{m}}_1$ and $\theta|\overline{\mathfrak{m}}_1$ is the corresponding Cartan involution. In fact, if we put

$$\mathfrak{v} = \overline{\mathfrak{m}}_{1\mathfrak{k}} + \sqrt{-1}\cdot\overline{\mathfrak{m}}_{1\mathfrak{p}}$$

and denote the conjugation of $\overline{\mathfrak{m}}_1^c$ with respect to $\overline{\mathfrak{m}}_1$ by η, then

$$\eta\mathfrak{v}\subset\mathfrak{v},\ \overline{\mathfrak{m}}_{1\mathfrak{k}} = \overline{\mathfrak{m}}_1\cap\mathfrak{v},\ \overline{\mathfrak{m}}_{1\mathfrak{p}} = \mathfrak{m}_1\cap(\sqrt{-1}\mathfrak{v}) \quad \text{and} \quad \mathfrak{v}\subset\mathfrak{u} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}.$$

Since $\mathfrak{u}$ is compact and $\mathfrak{v}$ is semisimple, $\mathfrak{v}$ is also compact (cf. [10, pg. 615]). Hence $\overline{\mathfrak{m}}_1 = \overline{\mathfrak{m}}_{1\mathfrak{k}} + \overline{\mathfrak{m}}_{1\mathfrak{p}}$ is a Cartan decomposition.

Now put

$$\mathfrak{b} = (\sum_{\alpha\in P'} CH_\alpha)\cap\mathfrak{g}.$$

Then $\mathfrak{b}$ is a Cartan subalgebra of $\overline{\mathfrak{m}}_1$ and $\theta\mathfrak{b}=\mathfrak{b}$ since $\theta(H_0)=H_0$. To prove the lemma, it is sufficient to see that $\mathfrak{b}$ has a maximal and minimal vector part. By the previous remark, to prove that $\mathfrak{b}$ has a minimal vector part we shall show that $\mathfrak{b}$ is a fundamental Cartan subalgebra. We put

$$\mathfrak{b}_\pm = \{H\in\mathfrak{b}:\ \theta H = \pm H\}.$$

Then, clearly, $\mathfrak{b} = \mathfrak{b}_+ + \mathfrak{b}_-$ is a direct sum, $\mathfrak{b}_+\subset\mathfrak{a}_\mathfrak{k}$ and $\mathfrak{b}_-\subset\mathfrak{a}_\mathfrak{p}$. We denote the center of $\mathfrak{m}_1$ by $\mathfrak{z}_1$, the centralizer of $\mathfrak{m}_1$ in $\mathfrak{a}_\mathfrak{p}$ by $\mathfrak{l}$ and put

$$\mathfrak{z}_{1\pm} = \{Z\in\mathfrak{z}_1 : \theta(Z) = \pm Z\}.$$

Then $\mathfrak{l}\subset\mathfrak{z}_1$, $\mathfrak{z}_{1+}\subset\mathfrak{a}_\mathfrak{k}$ and $\mathfrak{z}_{1-}\subset\mathfrak{a}_\mathfrak{p}$. So $\mathfrak{z}_{1-}\subset\mathfrak{l}$. Thereby $\mathfrak{z}_{1-}=\mathfrak{l}$. Let Δ_0 denote the set of non-zero roots of $(\overline{\mathfrak{m}}_1^c, \mathfrak{b}^c)$. Then each $\alpha\in\Delta_0$ can be regarded as a root of $(\mathfrak{g}^c, \mathfrak{a}^c)$ and $\alpha|\mathfrak{z}_{1+}=0$. Since $N(\mathfrak{g})=1$ by the assumption, $(\mathfrak{g}^c, \mathfrak{a}^c)$ has no real roots. Therefore $(\overline{\mathfrak{m}}_1^c, \mathfrak{b}^c)$ has no real roots. Hence $\mathfrak{b}$ is fundamental and has a minimal vector part.

The following lemma assures that $\mathfrak{b}$ has a maximal vector part, and hence it completes the proof. Q.E.D.

LEMMA 2.3. *For a θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the following conditions* 1) *and* 2) *are equivalent*:

1) $\mathfrak{h}$ *has a maximal vector part*,
2) $\Sigma_{\mathfrak{p}}=\{\alpha\in\Delta(\mathfrak{h})\colon \mathfrak{g}^\alpha\subset\mathfrak{p}^c\}$ *is empty.*

For a proof see [11, Lemma 4.3].

Now in the following, we put on G the following assumption:

$$N(G)=1,$$

that is, the number of conjugacy classes of Cartan subgroups of G is equal to one.

By going to a finite cover we can assume that G is q.s.c. and hence acceptable. Thus, if $j\colon \mathfrak{g}\subset\mathfrak{g}^c$ and G^c is a simply connected analytic group with Lie algebra $\mathfrak{g}^c$ then j extends to a homomorphism

$$j\colon G\longrightarrow G^c.$$

Since K is reductive, by going to a further finite cover of G, we may also assume that K is acceptable.

If we understand the harmonic analysis of a finite cover $\tilde{G}$ of G then we understand the theory for G. We throw out those unitary representations of G which are non-trivial on the kernel of the covering projection. Therefore, the above two assumptions can be made with no loss of generality.

Now let A be the Cartan subgroup of G associated with Cartan subalgebra $\mathfrak{a}$, that is, the centralizer of $\mathfrak{a}$ in G. We put

$$A_I=A\cap K, \qquad A_{\mathfrak{p}}=\exp\mathfrak{a}_{\mathfrak{p}}.$$

Then

$$A=A_I\cdot A_{\mathfrak{p}}.$$

We denote the inverse of the map $\exp\colon \mathfrak{a}_{\mathfrak{p}}\to A_{\mathfrak{p}}$ by log. Since $\mathfrak{a}$ is fundamental in our case, A_I is connected (cf. [16(a), Proposition 1.4.1.4 and its proof]). Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{k}$. M and M' are the centralizer and normalizer of $\mathfrak{a}_{\mathfrak{p}}$ in K respectively.

Let M^0 be the connected component of M. Let $W_{\mathfrak{g}}$ and $W_{\mathfrak{m}}$ be the Weyl groups of $(\mathfrak{g}, \mathfrak{a})$ and $(\mathfrak{m}, \mathfrak{a}_{\mathfrak{k}})$ respectively. Fix any $m\in M$. Then $\mathfrak{a}_{\mathfrak{k}}$ and

$\mathrm{Ad}(m^{-1})\mathfrak{a}_{\mathfrak{k}}$ are maximal abelian subalgebras of $\mathfrak{m}$. So, there exists an element $m_0 \in M^0$ such that $\mathrm{Ad}(m_0)\mathfrak{a}_{\mathfrak{k}} = \mathrm{Ad}(m^{-1})\mathfrak{a}_{\mathfrak{k}}$ (cf. [7(a), Theorem 6.4, pg. 212]). If we put $\gamma = mm_0$ then

$$\mathrm{Ad}(\gamma)\mathfrak{a}_{\mathfrak{k}} = \mathfrak{a}_{\mathfrak{k}}.$$

On the other hand, $\mathrm{Ad}(\gamma)$ fixes $\mathfrak{a}_{\mathfrak{p}}$ pointwise. Hence, $\mathrm{Ad}(\gamma)|\mathfrak{a}_{\mathfrak{k}}$ can be regarded as an element of the subgroup which consists of those elements of $W_{\mathfrak{g}}$ generated by roots vanishing on $\mathfrak{a}_{\mathfrak{p}}$, and as an element of $W_{\mathfrak{m}}$. This being so, there exists an element $m_1 \in M^0$ such that

$$\mathrm{Ad}(\gamma)|\mathfrak{a}_{\mathfrak{k}} = \mathrm{Ad}(m_1).$$

Hence, $\mathrm{Ad}(\gamma m_1^{-1})$ fixes $\mathfrak{a}_{\mathfrak{k}}$ pointwise. $\gamma_1 = \gamma m_1^{-1}$ is obviously in the coset $mM^0 \cap A_I$. This implies that A_I has the same number of connected components as M. Therefore M is also connected.

Let $\mathscr{E}_M(\mathscr{E}_K)$ be the set of unitary equivalence classes of irreducible representations of M(K resp.). For $\sigma \in \mathscr{E}_M$, we define the norm $|\sigma|$ of σ by

$$|\sigma|^2 = B(\mu_\sigma, \mu_\sigma),$$

where μ_σ is any real linear function on $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}$ associated with σ. Since the Killing form B of $\mathfrak{g}^c$ can be regarded as a positive definite form on either $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}$ or its real dual space and $W_{\mathfrak{m}}$ acts on $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}$ as a group of isometries under B, $|\sigma|$ is well defined. Since K is acceptable by assumption, the representation in $\mathscr{E}_K$ can be indexed by certain real linear functions on $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}$ as in Lemma 2.1. If $\tau \in \mathscr{E}_K$ and $\tau = \tau(\nu)$ for some real linear function ν on $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}$, then we write

$$|\tau|^2 = B(\nu, \nu).$$

$|\tau|$ is also well defined.

Define ρ by $\rho = \frac{1}{2} \sum_{\alpha \in P_+} \alpha$ on $\mathfrak{a}_{\mathfrak{p}}$ and $\rho = 0$ on $\mathfrak{a}_{\mathfrak{k}}$.

Let $l = \dim \mathfrak{a}_{\mathfrak{p}}$. The Killing form induces euclidean measures on $A_{\mathfrak{p}}$, $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}_{\mathfrak{p}}^*$; multiplying these by the factor $(2\pi)^{-(1/2)l}$ we obtain invariant measures da, dH and $d\lambda$ so that the following Fourier transform holds without any multiplicative constant:

$$f^*(\lambda) = \int_{A_{\mathfrak{p}}} f(a)\, e^{-i\lambda(\log a)} da,$$

$$f(a) \int_{\mathfrak{a}_{\mathfrak{p}}^*} f^*(\lambda) e^{i\lambda(\log a)} d\lambda, \ \lambda \in \mathfrak{a}_{\mathfrak{p}}^*, \qquad f \in \mathscr{S}(A).$$

We put $\mathfrak{n}^c = \sum_{\alpha\in P_+} \mathfrak{g}^\alpha$, $\mathfrak{n} = \mathfrak{n}^c \cap \mathfrak{g}$ and $\bar{\mathfrak{n}} = \theta\mathfrak{n}$. Let N and $\bar{N}$ be the analytic subgroups of G corresponding to $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ respectively. Let $G = KA_\mathfrak{p}N$ be the Iwasawa decomposition of G (cf. [7(a), pg. 373]). Each $g \in G$ can be uniquely written as

$$g = \kappa(g)\cdot\exp H(g)\cdot n(g), \qquad \kappa(g)\in K,\ H(g)\in\mathfrak{a}_\mathfrak{p},\ n(g)\in N.$$

We normalize the Haar measures dk, dm and du on the compact groups K, M and A_I respectively, so that the total measures are 1. The Haar measures of the nilpotent groups N and $\bar{N}$ are normalized so that

$$\theta(dn) = d\bar{n}, \qquad \int_{\bar{N}} e^{-2\varrho(H(\bar{n}))} d\bar{n} = 1.$$

The Haar measure dg on G can be normalized so that

$$\int_G f(g)dg = \int_{K\times A_\mathfrak{p}\times N} f(kan)e^{2\varrho(\log a)} dk da dn, \qquad f\in C_c^\infty(G).$$

Now $P = MA_\mathfrak{p}N$ is clearly a subgroup of G. If $\sigma\in\mathscr{E}_M$ acts on the finite dimensional Hilbert space V_σ and if $\lambda\in\mathfrak{a}_\mathfrak{p}^*$, then the map $\sigma\lambda$ from P into $\mathrm{End}(V_\sigma)$ (the algebra of linear endomorphisms of V_σ) given by

$$(\sigma\lambda)(m\cdot\exp H\cdot n) = \sigma(m)e^{-i\lambda(H)}, \qquad m\in M,\ n\in N,\ \lambda\in\mathfrak{a}_\mathfrak{p}^*,$$

is an irreducible unitary representation of P. Let $\pi_{\sigma,\lambda}$ be the unitary representation of G on the Hilbert space $\mathscr{H}_{\sigma,\lambda}$ obtained by inducing $\sigma\lambda$ from P to G; here $\mathscr{H}_{\sigma,\lambda}$ is the set of functions Φ from G into V_σ such that

(i) $\Phi(x\xi^{-1}) = (\sigma\lambda)(\xi) \quad \Phi(x)$, $x\in G$, $\xi\in P$,

(ii) $\Phi(k)$ is a Borel function on K,

(iii) $\int_K |\Phi(k)|^2 dk < +\infty$,

and the inner product on $\mathscr{H}_{\sigma,\lambda}$ is given by

$$(\Phi, \Psi) = \int_K (\Phi(k), \Psi(k))_{V_\sigma} dk, \qquad \Phi, \Psi\in\mathscr{H}_{\sigma,\lambda},$$

where $(\ ,\ )_{V_\sigma}$ is the inner product in V_σ. If $\Phi\in\mathscr{H}_{\sigma,\lambda}$, $\pi_{\sigma,\lambda}(y)\Phi$ is given by

$$(\pi_{\sigma,\lambda}(y)\Phi)(x) = \Phi(y^{-1}x)e^{-\varrho(H(y^{-1}x))+\varrho(H(x))}, \qquad x, y\in G.$$

For any $\lambda\in\mathfrak{a}_\mathfrak{p}^*$ and any $\Phi\in\mathscr{H}_{\sigma,\lambda}$ we can define a function $\tilde{\Phi}$ from K to V_σ by restricting Φ to K. This identifies $\mathscr{H}_{\sigma,\lambda}$ with a Hilbert space $\mathscr{H}_\sigma$ of square-integrable functions from K into V_σ. $\mathscr{H}_\sigma$ is the Hilbert space on which π_σ acts. The above equivalence between $\mathscr{H}_\sigma$ and $\mathscr{H}_{\sigma,\lambda}$ gives an intertwining operator between π_σ and $\pi_{\sigma,\lambda}|K$, the restriction of $\pi_{\sigma,\lambda}$ to K.

Any element s in W (the little Weyl group) acts on $\mathfrak{a}_\mathfrak{p}$ by reflection and so on its dual space $\mathfrak{a}_\mathfrak{p}^*$. s also induces an automorphism of M, modulo the group of inner automorphisms. Therefore s defines a bijection:

$$s: \sigma \longrightarrow s\sigma$$

of $\mathscr{E}_M$ onto itself. If we let s act on P, we can transform the representation $\sigma\lambda$ into the representation $(s\sigma)(s\lambda)$. Now, if $\lambda \in \mathfrak{a}_\mathfrak{p}^*$ and $\sigma \in \mathscr{E}_M$, it is known that $\pi_{\sigma,\lambda}$ is equivalent to $\pi_{s\sigma,s\lambda}$. Furthermore, the representations $\{\pi_{\sigma,\lambda}\}_{\sigma\in\mathscr{E}_M,\lambda\in\mathfrak{a}_\mathfrak{p}^{*+}}$, where $\mathfrak{a}_\mathfrak{p}^{*+}$ is the positive Weyl chamber in $\mathfrak{a}_\mathfrak{p}^*$, are all irreducible and inequivalent ([2, Theorem 7; 2]).

For each $\sigma \in E_M$ and regular $\lambda \in \mathfrak{a}_\mathfrak{p}^*$, that is, $\varpi(\lambda) \neq 0$ for $\varpi = \sum_{\alpha\in P_+} H_\alpha$, let $N_\sigma^s(\lambda)$ be a fixed unitary intertwining operator between $\pi_{\sigma,\lambda}$ and $\pi_{s\sigma,s\lambda}$. Then

$$N_\sigma^s(\lambda)\pi_{\sigma,\lambda}(x)N_\sigma^s(\lambda)^{-1} = \pi_{s\sigma,s\lambda}(x), \qquad x \in G.$$

3. The character of $\pi_{\sigma,\lambda}$

In order to obtain the Plancherel measure and the Plancherel formula we shall study the character of the unitary representation $\pi_{\sigma,\lambda}$ of G. To do this, we use notations by Harish-Chandra.

Put

$$\Delta_M(ah) = \xi_\varrho(a) \prod_{\alpha\in P_M} (1 - \xi_\alpha(a^{-1})), \qquad a \in A_I,\ h \in A_\mathfrak{p}.$$

For $g \in C_c^\infty(MA_\mathfrak{p})$, write

$$F_g^M(ah) = \Delta_M(a) \int_{M/A_I} g(m^*ahm^{*-1})dm^*, \qquad a \in A_I,\ h \in A_\mathfrak{p}$$

and ah is a regular element in A and dm^* is the invariant measure on the homogeneous space M/A_I. Hereafter we denote the set of regular elements in A by A'. It is known that there exists a constant $c_1 > 0$ such that for any $g \in C_c^\infty(MA_\mathfrak{p})$

$$\text{(3.1)} \qquad \int_{M\times A_\mathfrak{p}} g(mh)dmdh = c_1 \int_{A_I\times A_\mathfrak{p}} \overline{\Delta_M(a)} F_g^M(ah)dadh,$$

(see [6(k), Lemma 41]).

For $f \in C_c^\infty(G)$, write

$$\text{(3.2)} \qquad F_f(a) = \Delta(a) \int_{G^*} f(a^{x^*})dx^*, \qquad a \in A',$$

where $G^*=G/A$, $a^{x^*}=xax^{-1}(a\in A,\ x\in G)$ and dx^* denotes the invariant measure on G^*. Here we remark that ε_R in the definition of F_f in [6(k)] is 1 in our case bacause $\mathfrak{a}$ is fundamental. Since $\mathfrak{a}$ is fundamental we can apply the Harish-Chandra's limit formula to F_f. Namely, there is a positive constant c such that for any $f\in C_c^\infty(G)$

$$cf(1)=(-1)^q F_f(1;\varpi), \tag{3.3}$$

where $q=\frac{1}{2}[P_+]$ and $\varpi=\prod_{\alpha\in P} H_\alpha$. We write $\varpi^{\mathrm{m}}=\prod_{\alpha\in P_M} H_\alpha$. Let $\tilde{A}$ be the normalizer of A in G. Then $W_A=\tilde{A}/A$ is a finite group (see [6(k), pg. 488]). If $y\in\tilde{A}$, $s\in W_A$ and $yA=s$ then W_A acts on A and hence on $\mathfrak{a}$ by

$$h^s=yhy^{-1},\qquad h\in A.$$

We put

$$(\varpi^{\mathrm{m}})^s=\varepsilon_0(s)\varpi^{\mathrm{m}},$$

where $\varepsilon_0(s)=1$ or -1. It is clear that for any $f\in C_c^\infty(G)$

$$F_f(h^s)=\varepsilon_0(s)F_f(h),\qquad h\in A',\ s\in W_A. \tag{3.4}$$

LEMMA 3.1. *Let U be a compact real semisimple Lie group with Lie algebra $\mathfrak{u}$ and $\mathfrak{g}$ the complexification of $\mathfrak{u}$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{u}$, $\mathfrak{h}$ its complexification and let $W_{\mathfrak{g},\mathfrak{h}}$ denote the Weyl group of $(\mathfrak{g},\mathfrak{h})$. If $s\in W_{\mathfrak{g},\mathfrak{h}}$ then there exists an element $u\in U$ such that $sH=Adu(H)$ $(H\in\mathfrak{h})$.*

PROOF. Let Δ denote the set of non-zero roots of $(\mathfrak{g},\mathfrak{h})$ and $\mathfrak{g}^\alpha$ the root subspace of $\alpha\in\Delta$. Let $E_{\pm\alpha}\in\mathfrak{g}^{\pm\alpha}$ such that $[E_\alpha, E_{-\alpha}]=-H_\alpha$ and put $X=\pi(2(\alpha,\alpha))^{-1/2}(E_\alpha+E_{-\alpha})$, where the number π is the ratio of the circumference of the circle to the diameter. We put $\sigma=\exp\operatorname{ad}(X)$. Then σ is an automorphism of $\mathfrak{g}$. We shall prove that $\mathfrak{h}$ is invariant under σ and that, if we denote the dual of σ which acts on the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ by σ^*, then $(\sigma^*)^{-1}=\hat{\sigma}$ can be regarded as the reflection s_α associated with α. By induction on p, we can prove easily that

$$(\operatorname{ad}X)^{2p+1}\cdot H=(-1)^{p+1}\pi^{2p+1}\alpha(H)(2(\alpha,\alpha))^{-1/2}(E_\alpha-E_{-\alpha}),$$

$$(\operatorname{ad}X)^{2p+2}\cdot H=(-1)^{p+1}\pi^{2p+2}\alpha(H)(\alpha,\alpha)^{-1}H_\alpha,\qquad H\in\mathfrak{h}.$$

Hence,

$$\sigma(H)=H+\sum_{p=0}^\infty\frac{1}{(2p+1)!}(\operatorname{ad}X)^{2p+1}H+\sum_{p=0}^\infty\frac{1}{(2p+2)!}(\operatorname{ad}X)^{2p+2}H$$

$$=H+\alpha(H)(2(\alpha,\alpha))^{-1/2}\sum_{p=0}^{\infty}(-1)^{p+1}\frac{\pi^{2p+1}}{(2p+1)!}(E_\alpha-E_{-\alpha})$$

$$+\alpha(H)(\alpha,\alpha)^{-1}\sum_{p=0}^{\infty}(-1)^{p+1}\frac{\pi^{2p+2}}{(2p+2)!}H_\alpha$$

$$=H-\alpha(H)(2(\alpha,\alpha))^{-1/2}\sin\pi(E_\alpha-E_{-\alpha})+\alpha(H)(\alpha,\alpha)^{-1}(\cos\pi-1)H_\alpha$$

$$=H-2\alpha(H)(\alpha,\alpha)^{-1}H_\alpha.$$

Therefore we obtain the above assertion.

Now let η be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}$. Then $\mathfrak{h}$ is invariant under η and there exists a vector $X_\alpha\in\mathfrak{g}^\alpha$ such that for all $\alpha\in\Delta$

$$\sqrt{-1}(X_\alpha+X_{-\alpha})\in\mathfrak{u},\qquad [X_\alpha, X_{-\alpha}]=2(\alpha,\alpha)^{-1}H_\alpha,$$

(cf. [7(a), pg. 219, Lemma 3.1]). If we put

$$E_{\pm\alpha}=\sqrt{-1}\{(\alpha,\alpha)/2\}^{1/2}X_{\pm\alpha},$$

then they satisfy $[E_\alpha, E_{-\alpha}]=-H_\alpha$ and $X=\pi(2(\alpha,\alpha))^{-1/2}(E_\alpha+E_{-\alpha})\in\mathfrak{u}$. This proves Lemma 3.1. Q.E.D.

By the above lemma it is clear that, for any $s\in W_{\mathfrak{m}}$, there exists an element $m\in M$ such that $sH=\mathrm{Ad}(m)H$ $(H\in\mathfrak{a}_{\mathfrak{k}}^c)$. Since m fixes $\mathfrak{a}_{\mathfrak{p}}$ pointwise, we have $m\in\tilde{A}$. Hence, from the formulae (3.2) and (3.4) we obtain the following: For any $f\in C_c^\infty(G)$,

$$F_f(a^sh)=\varepsilon_0(s)F_f(h),\qquad a\in A_I,\ h\in A_{\mathfrak{p}},\ ah\in A',\ s\in W_{\mathfrak{m}}. \tag{3.5}$$

For $f\in C_c^\infty(G)$ define a function g_f in $C_c^\infty(MA_{\mathfrak{p}})$ by

$$g_f(mh)=e^{\varrho(\log h)}\int_N\int_K f(kmhnk^{-1})dkdn,\qquad m\in M,\ h\in A_{\mathfrak{p}}.$$

By Lemma 52 in [6(k)] we know, in our case, that there exists a positive constant c_2 scuh that

$$F_f(ah)=c_2F_{g_f}^M(ah),\qquad a\in A_I,\ ah\in G'. \tag{3.6}$$

The map $K\times\mathfrak{a}_{\mathfrak{p}}^+\times K\to G$ given by

$$(k_1, H, k_2)\longrightarrow k_1\cdot\exp H\cdot k_2$$

is a diffeomorphism onto G'. Furthermore, there exists a constant $c>0$ such that for any $f\in C_c^\infty(G)$

$$\int_G f(x)dx = c\int_{\mathfrak{a}_\mathfrak{p}}\int_{K\times K} f(k_1 \exp H\cdot k_2)|D(H)|dk_1 dk_2 dH,$$

where

$$D(H) = \prod_{\alpha\in P_+} \sinh \alpha(H).$$

Let π be an irreducible unitary representation of G on a Hilbert space $\mathscr{H}$. Let $f\in C_c^\infty(G)$. Then it is known that the operator

$$\pi(f) = \int_G f(x)\pi(x)dx$$

is of trace class and that the map

$$f \longrightarrow \operatorname{tr}\pi(f)$$

is a distribution on $C_c^\infty(G)$ (see [6(c), § 5]). This distribution is called the character of π.

Let $\sigma\in\mathscr{E}_M$, $\lambda\in\mathfrak{a}_\mathfrak{p}^{*\prime}$ and $\Theta_{\sigma,\lambda}$ be the character of the representation $\pi_{\sigma,\lambda}$. Put $t=\frac{1}{2}(\dim\mathfrak{g}-\operatorname{rank}\mathfrak{g})$ and choose $\mu\in\Pi'$ such that $\sigma=\sigma(\mu)$ as in Lemma 2.1.

THEOREM 3.1. *There exists a constant $c_0>0$ such that for every $f\in C_c^\infty(G)$,*

$$\Theta_{\sigma,\lambda}(f) = c_0(-1)^t(\operatorname{sign}\varpi^{\mathfrak{m}}(\mu))\int_{A_I\times A_\mathfrak{p}} F_f(ah)\xi_\mu(a)e^{-i\lambda(\log h)}dadh.$$

PROOF. Let A be the operator on $\mathscr{H}_{\sigma,\lambda}$ defined by

$$A = \int_G f(x)\pi_{\sigma,\lambda}(x)dx, \qquad f\in C_c^\infty(G).$$

We want to compute the trace of A. $\operatorname{tr}A$ is equal to $\operatorname{tr}\overline{A^*}$, where A^* is the adjoint operator of A and the bar denotes complex conjugation. For $\Phi\in\mathscr{H}_{\sigma,\lambda}$ and $k_1\in K$,

$$\begin{aligned}(A^*\Phi)(k_1) &= \Big(\int_G \overline{f(x)}\pi_{\sigma,\lambda}(x^{-1})dx\cdot\Phi\Big)(k_1)\\ &= \int_G \overline{f(x)}\Phi(xk_1)e^{-\rho(H(xk_1))}dx\\ &= \int_G \overline{f(xk_1^{-1})}\Phi(x)e^{-\rho(H(x))}dx\\ &= \int_{K\times A_\mathfrak{p}\times N} \overline{f(khnk_1^{-1})}\Phi(k)e^{(i\lambda+\rho)(\log h)}dkdhdn.\end{aligned}$$

In the last integral, substitute km for k and integrate with respect to M. Then

$$(A^*\Phi)(k_1)=\int_{K\times M\times A_{\mathfrak{p}}\times N}\overline{f(kmhnk_1^{-1})}\sigma(m^{-1})e^{(i\lambda+\varrho)(\log h)}\Phi(k)\,dkdmdhdn.$$

Now to deal with this expression further we consider the principal fiber bundle

$$M\longrightarrow K\longrightarrow K/M.$$

The map $m\to\sigma(m^{-1})$ defines a complex vector bundle E_σ over K/M with fiber V_σ, the space on which σ acts. Let $F(k_1, k)$ be the operator-valued function defined by

$$F(k_1, k)=\int_{M\times A_{\mathfrak{p}}\times N}\overline{f(kmhnk_1^{-1})}\sigma(m^{-1})e^{(i\lambda+\varrho)(\log h)}\,dmdhdn.$$

Now it is easy to see that M normalizes N and that for fixed $m\in M$ the measures dn and $d(mnm^{-1})$ on N are equal. Then for $\overline{m}_1, \overline{m}\in M$

$$F(k_1\overline{m}_1, k\overline{m})=(\overline{m}_1^{-1})F(k_1, k)\sigma(\overline{m}).$$

Therefore $F(k_1, k)$ can be regarded as a section of $E_\sigma\boxtimes E_\sigma^*$, where E_σ^* denotes the adjoint bundle of E_σ and $E_\sigma\boxtimes E_\sigma^*$ denotes the exterior tensor product of E_σ and E_σ^*, a bundle with base space $K/M\times K/M$ and fiber $V_\sigma\otimes V_\sigma^*$.

In §2 we note that there exists a natural equivalence between $\mathscr{H}_{\sigma,\lambda}$ and $\mathscr{H}_\sigma$. However, $\mathscr{H}_\sigma$ is the space of square-integrable sections of E_σ with respect to a K-invariant measure on K/M. $F(k_1, k)$ can be regarded as the kernel of the linear operator A^* on this space. Then for any Φ on $\mathscr{H}_\sigma$

$$(A^*\Phi)(k_1)=\int_K F(k_1, k)\Phi(k)dk.$$

To evaluate the trace of A^* we need the following lemma.

LEMMA 3.2. *Let X be a compact infinitely differentiable manifold of dimension n. Let dx be a positive nowhere vanishing differentiable n-form on X. If $E\to X$ is a differentiable Hilbert bundle of fiber dimension s, let $L^2(E)$ be the Hilbert space of square-integrable sections of E. If $F(x_1, x)$ is a continuous section of $E\boxtimes E^*$, $F(x_1, x)$ defines a bounded linear operator F on $L^2(E)$ in the obvious manner. Then if $F(x_1, z)$ is differentiable in both variables F is of trace class. Furthermore*

$$\operatorname{tr} F=\int_X(\operatorname{tr} F(x, x)dx.$$

For a proof see [1, Lemma 4].

By the lemma,

$$\operatorname{tr} A^* = \int_{K\times M\times A_{\mathfrak{p}}\times N} \overline{f(kmhnk^{-1})}\cdot \operatorname{tr}\sigma(m^{-1}) e^{(i\lambda+\varrho)(\log h)} dk dm dh dn.$$

Therefore,

$$\begin{aligned}\operatorname{tr} A &= \overline{\operatorname{tr} A^*}\\ &= \int_{K\times M\times A_{\mathfrak{p}}\times N} f(kmhnk^{-1})\, \overline{\operatorname{tr}\sigma(m^{-1})} e^{(-i\lambda+\varrho)(\log h)} dk dm dh dn\\ &= \int_{M\times A_{\mathfrak{p}}} g_f(mh)\cdot \operatorname{tr}\sigma(m)\, e^{-i\lambda(\log h)} dm dh,\end{aligned}$$

since

$$\overline{\operatorname{tr}\sigma(m^{-1})} = \overline{\operatorname{tr}\sigma(m)^*} = \operatorname{tr}\sigma(m).$$

Recall that

$$q = \frac{1}{2}[P_+], \qquad [P] = t.$$

Then

$$[P_M] = [P] - [P_+] = t - 2q.$$

If $a \in A_I$ then

$$\overline{\Delta_M(a)} = \Delta_M(a)(-1)^{[P_M]} = \Delta_M(a)(-1)^t.$$

Now for any $m \in M$

$$\operatorname{tr}\sigma(mam^{-1}) = \operatorname{tr}\sigma(a), \qquad (a \in A_I).$$

Therefore, from (3.1) we see that

$$\operatorname{tr} A = c_1 \int_{A_I\times A_{\mathfrak{p}}} F^M_{g_f}(ah) \overline{\Delta_M(a)} \operatorname{tr}\sigma(a) e^{-i\lambda(\log h)} da dh.$$

By Lemma 2.1, this equals

$$(\operatorname{singn} \varpi^{\mathfrak{m}}(\mu))(-1)^t c_1 \int_{A_I\times A_{\mathfrak{p}}} F^M_{g_f}(ah) e^{-i\lambda(\log h)} (\sum_{s\in W_{\mathfrak{m}}} \varepsilon_0(s)\, \xi_{s\mu}(a)) da dh.$$

By formula (3.5) this expression equals

$$(\operatorname{singn} \varpi^{\mathfrak{m}}(\mu))(-1)^t (c_1/c_2)\cdot \int_{A_I\times A_{\mathfrak{p}}} F_f(ah) e^{-i\lambda(\log h)} (\sum_{s\in W_{\mathfrak{m}}} \varepsilon_0(s) \xi_{s\mu}(a)) da dh.$$

Now if $s \in W_{\mathfrak{m}}$, substitute sa for a in the above expression. From (3.5) we obtain the formula tr A

$$=(\operatorname{sign}\varpi^{\mathfrak{m}}(\mu))(-1)^{t}(c_1/c_2)[W_{\mathfrak{m}}]\int_{A_I\times A_{\mathfrak{p}}}F_f(ah)e^{-i\lambda(\log h)}\xi_\mu(a)\,da\,dh.$$

This implies the theorem if we put $c_0=(c_1/c_2)[W_{\mathfrak{m}}]$. Q.E.D.

4. The Plancherel measure and the Plancherel formula

By means of the character formula for $\pi_{\sigma,\lambda}$, we shall find the explicit Plancherel measure.

For any real linear function $\mu\in\Pi'$ on $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}$ and any $\lambda\in\mathfrak{a}_{\mathfrak{p}}^*$ we extend them to real linear functions on $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}+\mathfrak{a}_{\mathfrak{p}}$ defining $\mu=0$ on $\mathfrak{a}_{\mathfrak{p}}$ and $\lambda=0$ on $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}$. We write

$$\varpi(\mu:\lambda)=\varpi(\mu+i\lambda),$$

which is clearly equal to

$$\varpi^{\mathfrak{m}}(\mu)\prod_{\alpha\in P_+}<\mu+i\lambda,\ H_\alpha>.$$

Since the Cartan subalgebra $\mathfrak{a}$ is fundamental in our case, all elements in P_+ are positive complex roots. If we denote the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$ by η, then for each $\alpha\in P_+$ $\alpha^\eta\in P_+$, so the complex roots occur in pairs. We have the formula

$$<\mu+i\lambda,\ H_\alpha>\cdot<\mu+i\lambda,\ H_{\alpha^\eta}>=-(\mu(H_\alpha)^2+\lambda(H_\alpha)^2). \tag{4.1}$$

Therefore,

$$\operatorname{sign}\{\prod_{\alpha\in P_+}<\mu+i\lambda,\ H_\alpha>\}=(-1)^q,$$

where $q=\frac{1}{2}[P_+]$.

It is clear that

$$\varpi(\mu:\lambda)|\varpi(\mu:\lambda)|^{-1}=(-1)^q\cdot\operatorname{sign}\varpi^{\mathfrak{m}}(\mu) \tag{4.2}$$

and

$$\varpi(\mu:-\lambda)=\varpi(\mu:\lambda). \tag{4.3}$$

Let $\sigma\in\mathscr{E}_M$, $\mu\in\Pi'$ and $\sigma=\sigma(\mu)$. In §2 we define $s\sigma\in\mathscr{E}_M$ (which we write also as σ^s) for each $s\in W$. Then we can choose $\mu_s\in\Pi'$ such that $\sigma^s=\sigma^s(\mu_s)$. For a given μ, μ_s is not uniquely defined. However the expression

$$\operatorname{sign} \varpi^{\mathfrak{m}}(\mu_s)\varpi(\mu_s : \lambda)$$

is well defined for any $\lambda \in \mathfrak{a}_{\mathfrak{p}}^*$. Furthermore

$$\operatorname{sign} \varpi^{\mathfrak{m}}(\mu_s)\varpi(\mu_s : \lambda) = \operatorname{sign} \varpi(\mu)\varpi(\mu : \lambda). \tag{4.4}$$

Now let $f \in C_c^\infty(G)$, then from the formula (3.3)

$$f(1) = (1/c)(-1)^q F_f(1; \varpi). \tag{3.3'}$$

By the Fourier inversion formula on the connected abelian group $A_I \times A_{\mathfrak{p}}$,

$$f(1) = (1/c)(-1)^q \sum_{\mu \in \Pi} \int_{\mathfrak{a}_{\mathfrak{p}}^*} \left[\int_{A_I \times A_{\mathfrak{p}}} F_f(ah; \varpi)\xi_\mu(a) e^{-i\lambda(\log h)} da dh \right] d\lambda.$$

Since imaginary roots are all compact in our case, by [6(k), Lemma 40] and [6(g), Theorem 2] we obtain $F_f \in C_c^\infty(A)$. So we can apply the integration by parts to the above and we see that

$$f(1) = (1/c)\cdot(-1)^{q+t} \sum_{\mu \in \Pi} \int_{\mathfrak{a}_{\mathfrak{p}}^*} \varpi(\mu : \lambda) \left[\int_{A_I \times A_{\mathfrak{p}}} F_f(ah)\xi_\mu(a) e^{-i\lambda(\log h)} da dh \right] d\lambda,$$

where $t = [P]$. By Theorem 3.1 this expression equals

$$(1/c_0 c)\cdot(-1)^q \sum_{\mu \in \Pi} \int_{\mathfrak{a}_{\mathfrak{p}}^*} \varpi(\mu : -\lambda) \operatorname{sign} \varpi^{\mathfrak{m}}(\mu) \Theta_{\sigma(\mu),\lambda}(f) d\lambda.$$

Now we define

$$\beta(\mu : \lambda) = (\omega/c_0 c)(-1)^q \varpi(\mu; -\lambda) \operatorname{sign} \varpi^{\mathfrak{m}}(\mu),$$

where $\omega = [W]$. Then we see from (4.2) that $\beta(\mu : \lambda)$ is nonnegative. Also from (4.3) we see that

$$\beta(\mu : -\lambda) = \beta(\mu : \lambda). \tag{4.5}$$

For $s \in W$, it is clear that the expression $\beta(\mu_s : \lambda)$ is well defined. (4.4) implies the formula

$$\beta(\mu_s : \lambda) = \beta(\mu : \lambda) = \beta(\mu_s : s\lambda).$$

Since, for any $\lambda \in \mathfrak{a}_{\mathfrak{p}}^{*\prime}$ and $\sigma \in \mathcal{E}_M$, $\pi_{s\sigma,\lambda}$ and $\pi_{\sigma,s^{-1}\lambda}$ are equivalent,

$$\Theta_{s\sigma,\lambda} = \Theta_{\sigma,s^{-1}\lambda}.$$

Therefore we obtain the formula

$$f(1) = \sum_{\mu \in \Pi} \int_{\mathfrak{a}_{\mathfrak{p}}^{*+}} \beta(\mu : \lambda) \Theta_{\sigma(\mu),\lambda}(f) d\lambda.$$

It is clear that

$$\beta(s\mu:\lambda)=\beta(\mu:\lambda), \qquad (s\in W_{\mathfrak{m}}).$$

Then if $\sigma\in\mathscr{E}_M$, choose any $\mu\in\Pi'$ such that $\sigma=\sigma(\mu)$. Define

$$\beta(\sigma:\lambda)=[W_{\mathfrak{m}}]\beta(\mu:\lambda).$$

Then $\beta(\sigma:\lambda)$ is well defined and $\beta(\sigma:\lambda)$ satisfies the formula

$$f(1)=\sum_{\sigma\in\mathscr{E}_M}\int_{\mathfrak{a}_{\mathfrak{p}}^{*+}}\beta(\sigma:\lambda)\Theta_{\sigma,\lambda}(f)d\lambda.$$

If $\sigma\in\mathscr{E}_M$ and $\lambda\in\mathfrak{a}_{\mathfrak{p}}^{*\prime}$, then it is clear that $\beta(\sigma:\lambda)\neq 0$ from the definition of $\beta(\sigma:\lambda)$. Furthermore, since $\beta(\sigma:\lambda)$ is a polynomial in μ and λ, for every $d\in\boldsymbol{D}(\mathfrak{a}_{\mathfrak{p}}^*)$, there exist polynomials p_1, p_2 such that for $\sigma\in\mathscr{E}_M$, $\lambda\in\mathfrak{a}_{\mathfrak{p}}^*$

$$|\beta(\sigma:\lambda;d)|\leqq p_1(|\sigma|)p_2(|\lambda|).$$

Thus we obtain the following lemma.

LEMMA 4.1. *$\beta(\sigma:\lambda)$ is a non-negative function on $\mathscr{E}_M\times\mathfrak{a}_{\mathfrak{p}}^*$ such that for any $f\in C_c^\infty(G)$*

$$f(1)=\sum_{\sigma\in\mathscr{E}_M}\int_{\mathfrak{a}_{\mathfrak{p}}^{*+}}\beta(\sigma:\lambda)\Theta_{\sigma,\lambda}(f)d\lambda. \tag{4.6}$$

Moreover $\beta(\sigma:\lambda)$ has the following properties.

(i) $\beta(\sigma:\lambda)=\beta(\sigma,-\lambda)=\beta(s\sigma,s\lambda) \qquad (s\in W)$.

(ii) *For every $d\in\boldsymbol{D}(\mathfrak{a}_{\mathfrak{p}}^*)$, there exist polynomials p_1, p_2 such that for $\sigma\in\mathscr{E}_M$, $\lambda\in\mathfrak{a}_{\mathfrak{p}}^*$*

$$|\beta(\sigma:\lambda;d)|\leqq p_1(|\sigma|)p_2(|\lambda|). \tag{4.7}$$

Now let $\mathscr{H}_2(\sigma)$ be the space of Hilbert-Schmidt operators on $\mathscr{H}_\sigma$ with the Hilbert-Schmidt norm $\|\cdot\|_2$.

Let $L^2(\hat{G})$ be the set of functions

$$a:\mathscr{H}_M\times\mathfrak{a}_{\mathfrak{p}}^*\longrightarrow\bigoplus_{\sigma\in\mathscr{E}_M}\mathscr{H}_2(\sigma)$$

which satisfy the following conditions:

(i) *$a(\sigma:\lambda)\in\mathscr{H}_2(\sigma)$ for each $\sigma\in\mathscr{E}_M$ and $\lambda\in\mathfrak{a}_{\mathfrak{p}}^*$.*

(ii) *$a(s\sigma:s\lambda)=N_\sigma^s(\lambda)a(\sigma:\lambda)N_\sigma^s(\lambda)^{-1}$, $\sigma\in\mathscr{E}_M$, $\lambda\in\mathfrak{a}_{\mathfrak{p}}^{*\prime}$ and $s\in W$.*

(iii) *For any $\sigma\in\mathscr{E}_M$, $a(\sigma:\lambda)$ is a Borel function of λ.*

(iv) $\|a\|^2 = \omega^{-1} \sum_{\sigma \in \mathscr{E}_M} \int_{\mathfrak{a}_\mathfrak{p}^*} \|a(\sigma : \lambda)\|_2^2 \beta(\sigma : \lambda) d\lambda < +\infty.$

(In (ii) we can regard the operators $N_\sigma^s(\lambda)$ as maps from $\mathscr{H}_\sigma$ to $\mathscr{H}_{s\sigma}$ if we recall the canonical isomorphisms $\mathscr{H}_{\sigma,\lambda} \leftrightarrow \mathscr{H}_\sigma$, $\mathscr{H}_{s\sigma, s\lambda} \leftrightarrow \mathscr{H}_{s\sigma}$).

Since every $N_\sigma^s(\lambda)$ is unitary, condition (ii) implies that

$$\|a(s\sigma : s\lambda)\|_2^2 = \|a(\sigma : \lambda)\|_2^2.$$

Hence

$$\|a\|^2 = \sum_{\sigma \in \mathscr{E}_M} \int_{\mathfrak{a}_\mathfrak{p}^{*+}} \|a(\sigma : \lambda)\|_2^2 \beta(\sigma : \lambda) d\lambda. \tag{4.8}$$

It is easy to see that $L^2(\hat{G})$ is a Hilbert space. For $f \in C_c^\infty(G)$, define $\hat{f} \in L^2(\hat{G})$ by

$$\hat{f}(\sigma, \lambda) = \int_G f(x) \pi_{\sigma,\lambda}(x) dx, \qquad \sigma \in \mathscr{E}_M,\ \lambda \in \mathfrak{a}_\mathfrak{p}^*.$$

Then $\hat{f}(\sigma, \lambda)$ can be regarded as an operator on $\mathscr{H}_\sigma$. We call the map

$$f \longrightarrow \hat{f}, \qquad f \in C_c^\infty(G),$$

the Fourier transform.

The Fourier transform $f \to \hat{f}$ ($f \in C_c^\infty(G)$) clearly extends uniquely to a map from $L^2(G)$ into $L^2(\hat{G})$. Now we shall prove the following Plancherel formula by the same method as in [1].

THEOREM 4.1. (*Plancherel formula*). *The Fourier transform*

$$f \longrightarrow \hat{f}, \qquad f \in C_c^\infty(G)$$

extends uniquely to an isometry from $L^2(G)$ onto $L^2(\hat{G})$.

PROOF. Fix $f \in C_c^\infty(G)$. Define

$$g(x) = \int_G f(y) \overline{f(x^{-1}y)} dy, \qquad x \in G,$$

then clearly $g \in C_c^\infty(G)$ and $g(1) = \|f\|_2^2$. If π is an irreducible unitary representation of G,

$$\begin{aligned} \pi(g) &= \int_{G \times G} f(y) \overline{f(x^{-1}y)} dy \pi(x) dx \\ &= \int_{G \times G} f(y) \overline{f(x^{-1})} \pi(yx) dy dx \end{aligned}$$

$$=\left(\int_G f(y)\pi(y)dy\right)\left(\int_G f(x)\pi(x)dx\right)^*$$
$$=\pi(f)\cdot\pi(f)^*,$$

where $\pi(f)^*$ is the adjoint of $\pi(f)$. Therefore

$$\operatorname{tr}\pi(g)=\|\pi(f)\|_2^2=\|\hat{f}(\pi)\|_2^2.$$

Therefore, applying Lemma 4.1 to $g(x)$ we see that

$$\|f\|_2^2=\|\hat{f}\|^2.$$

Thus, the map $f\to\hat{f}$ is an isometry.

We have to show that the map is surjective.

Let $\hat{\rho}$ be the representation of $G\times G$ on $L^2(\hat{G})$ given by

$$(\hat{\rho}(x, y)a)(\sigma, \lambda)=\pi_{\sigma,\lambda}(x)a(\sigma, \lambda)\pi_{\sigma,\lambda}(y^{-1})$$

for $\sigma\in\mathscr{E}_M$, $\lambda\in\mathfrak{a}_\mathfrak{p}^*$, and $(x, y)\in G\times G$. Then we can see that $\hat{\rho}$ is multiplicity free, and hence the algebra $R(\hat{\rho}, \hat{\rho})$ of intertwining operators of $\hat{\rho}$ is commutative (see the proof of Theorem 2 in [1]).

Let ρ be the two-sided regular representation of $G\times G$ on $L^2(G)$. Then the map

$$f\longrightarrow\hat{f},\qquad f\in L^2(G)$$

is an intertwining operator between ρ and $\hat{\rho}$. Thus if L is the closure of the set $\{\hat{f}\colon f\in L^2(G)\}$, and P is the orthogonal projection of $L^2(\hat{G})$ onto L, then P is contained in $R(\hat{\rho}, \hat{\rho})$. But since $R(\hat{\rho}, \hat{\rho})$ is commutative, it is well known that P is of the form P_E, where E is a Borel subset of $S=\mathscr{E}_M\times\mathfrak{a}_\mathfrak{p}^{*+}$ and

$$P_E=\{a\in L^2(\hat{G})\colon\ a \textit{ vanishes outside } E\}.$$

In order to complete the proof of the surjectivity of the map $f\to\hat{f}$, we prove that the complement of E in S is a null set with respect the measure class C on S defined by the discrete measure on $\mathscr{E}_M$ and the Lebesgue measure on $\mathfrak{a}_\mathfrak{p}^{*+}$.

Let us assume the contrary. Then there is a $\sigma\in\mathscr{E}_M$ and a subset R_1 of $\mathfrak{a}_\mathfrak{p}^{*\prime}$ of positive Lebesque measure such that for any $f\in C_c^\infty(G)$,

$$\hat{f}(\sigma, \lambda)=0\qquad\text{for almost all } \lambda\in R_1.$$

Choose a $\tau\in\mathscr{E}_K$ for which there is a non-zero intertwining operator T between the restriction of τ to M and σ. Choose a unit vector ξ in the space on which τ acts such that $T\xi\neq 0$. Define

$$\Phi(k)=T(\tau(k^{-1})\xi),\qquad k\in K.$$

Then $\Phi\in\mathscr{H}_\sigma$. For any $f\in C_c^\infty(G)$,

$$\begin{aligned}(\hat{f}(\sigma, \lambda)\Phi)(1) &= \left(\int_G f(x)\pi_{\sigma,\lambda}(x)dx\cdot\Phi\right)(1) \\ &= \int_G f(x^{-1})\Phi(x)e^{-\varrho(H(x))}dx.\end{aligned}$$

Then

$$\begin{aligned}&(\hat{f}(\sigma, \lambda)\Phi)(1) \\ &= \int_{K\times\mathfrak{a}_\mathfrak{p}^*\times N} f(n^{-1}\cdot\exp(-H)\cdot k^{-1})\, e^{(i\lambda+\varrho)(H)}\,\Phi(k)\, dkdHdn.\end{aligned}$$

Let $f(n^{-1}\cdot\exp(-H)\cdot k^{-1})=\chi(k)\cdot\alpha(H)\cdot\nu(n)$, where $\chi(k)=(\tau(k)\xi, \xi)$ and ν is any function in $C_c^\infty(N)$ such that $\int_N \nu(n)dn=1$, and α is some function in $C_c^\infty(\mathfrak{a}_\mathfrak{p}^*)$ such that $\int_{\mathfrak{a}_\mathfrak{p}^*}\alpha(H)e^{(i\lambda+\varrho)(H)}dH$ is not equal to zero for any λ belonging to a subset R_2 of R_1 of positive measure. Clearly such an α exists.

For a fixed $\lambda_0\in R_2$,

$$(\hat{f}(\sigma, \lambda_0)\Phi)(1)=T(\xi)\int_{\mathfrak{a}_\mathfrak{p}^*}\alpha(H)e^{(i\lambda+\varrho)(H)}dH.$$

This is a non-zero vector in the space on which σ acts. However, $(\hat{f}(\sigma, \lambda)\Phi)(k)$ is a continuous function of k, so $(\hat{f}(\sigma, \lambda_0)\Phi)(k)$ is nonzero on a subset of K of positive measure. Therefore $\hat{f}(\sigma, \lambda_0)\Phi$ is a nonzero vector in $\mathscr{H}_\sigma$. This means that the operators $\hat{f}(\sigma, \lambda)$ do not vanish for any $\lambda\in R_2$. We have a contradiction. The proof of Theorem 4.1 is now complete. Q.E.D.

5. Statement of the main theorem

We shall define the Schwartz space for G according to [6(m)] and state the main theorem.

For $x\in G$, define

$$\Xi(x)=\int_K e^{-\varrho(H(xk))}dk.$$

As usual we define a norm on $\mathfrak{g}$ by

$$|X|^2=-B(X, \theta X), \qquad x\in\mathfrak{g},$$

where B is the Killing form on $\mathfrak{g}$ and θ is the Cartan involution of $\mathfrak{g}$. Since $G=KA_\mathfrak{p}K$ there exists a unique function σ on G such that

(i) $\sigma(k_1xk_2)=\sigma(x)$, k_1, $k_2\in K$ and $x\in G$,

(ii) $\sigma(\exp H)=|H|$, $\quad H\in\mathfrak{a}_{\mathfrak{p}}$.

It is clear that

$$\Xi(x^{-1})=\Xi(x),\qquad \sigma(x^{-1})=\sigma(x)\quad x\in G. \tag{5.1}$$

It is known that there exist positive numbers c, d such that

$$1\leqq\Xi(a)e^{\varrho(\log a)}\leqq c(1+\sigma(a))^d,\qquad a\in cl(A_{\mathfrak{p}}^+) \tag{5.2}$$

(see [6(h), Theorem 3 and Lemma 36]), and that

$$\sigma(xy)\leqq\sigma(x)+\sigma(y),\qquad x,\ y\in G \tag{5.3}$$

(see [6(m), Lemma 10]), and also that there exists an $r_0>0$ such that

$$\int_G \Xi(x)^2(1+\sigma(x))^{-r_0}dx=N(r_0)<+\infty \tag{5.4}$$

(see [6(m), Lemma 11]).

Let $\mathfrak{B}$ be the universal enveloping algebra of $\mathfrak{g}^c$. We can identify $\mathfrak{B}$ with the algebra of left invariant differential operators on G. Let ξ be the canonical anti-isomorphism of $\mathfrak{B}$ with the algebra of right invariant differential operators on G. If g_1, $g_2\in\mathfrak{B}$ and $f\in C^\infty(G)$, then the actions of $\xi(g_1)$ and g_2 on f commute. We denote the resultant of this action at any $x\in G$ by $f(g_1;x;g_2)$.

Now for $f\in C^\infty(G)$ and g_1, $g_2\in\mathfrak{B}$ and $r\in\boldsymbol{R}$ we put

$$\|f\|_{g_1,g_2,r}=\sup_{x\in G}|f(g_1;x;g_2)|\Xi(x)^{-1}(1+\sigma(x))^r.$$

Let

$$\mathscr{C}(G)=\{f\in C^\infty(G)\colon \|f\|_{g_1,g_2,r}<+\infty,\quad \textit{for any } g_1,\ g_2\in\mathfrak{B} \textit{ and } r\in\boldsymbol{R}\}.$$

These semi-norms make $\mathscr{C}(G)$ into a Fréchet space. The space $\mathscr{C}(G)$ is called the *Schwartz space* of G.

Clearly

$$C_c^\infty(G)\subset\mathscr{C}(G)$$

and the inclusion is continuous, and it is known that $C_c^\infty(G)$ is dense in $\mathscr{C}(G)$ ([6(m), Theorem 2]). Also from (5.4) we see that there is a continuous inclusion of $\mathscr{C}(G)$ into $L^2(G)$.

We wish to define a subspace of $L^2(\hat{G})$ which will ultimately turn out to be the image of $\mathscr{C}(G)$ under the Fourier transform $\mathscr{F}: f \to \hat{f}$. We shall need to fix an appropriate basis for the Hilbert space $\mathscr{H}_{\sigma,\lambda}$. As we remarked earlier, there is a canonical intertwining operator between the representations $\pi_{\sigma,\lambda}|K$ and π_σ of K. Therefore we shall choose a fixed orthonormal basis for the Hilbert space $\mathscr{H}_\sigma$.

The multiplicity of τ in $\pi_{\sigma,\lambda}|K$ equals the multiplicity of τ in π_σ. But π_σ is just the representation σ induced to K. Therefore by the Frobenius reciprocity theorem for compact groups ([9(a), Theorem 8.2]), these multiplicities are just equal to $[\tau:\sigma]$, the multiplicity of σ in $\tau|M$.

Fix $\tau \in \mathscr{E}_K$ and $\sigma \in \mathscr{E}_M$ acting on the Hilbert spaces V_τ and V_σ of dimension t and s respectively. Let $R(\tau, \sigma)$ be the set of intertwining operators from V_τ to V_σ for $\tau|M$ and σ. The Hilbert-Schmidt norm makes $R(\tau, \sigma)$ into a Hilbert space of dimension $[\tau:\sigma]$.

Now suppose $T \in R(\tau, \sigma)$. Since σ is irreducible, we can assume that there are orthonormal bases $\{\xi_1, \dots, \xi_t\}$ and $\{\eta_1, \dots, \eta_s\}$ of V_τ and V_σ respectively so that there is a constant c for which

$$T\xi_i = c\eta_i, \qquad 1 \leqq i \leqq s$$

$$T\xi_i = 0, \qquad s < i \leqq t.$$

Suppose T has been normalized such that $c = (t/s)^{1/2}$. Then

$$\begin{aligned} T\xi_i &= (t/s)^{1/2}\eta_i, \qquad 1 \leqq i \leqq s, \\ \|T\| &= t^{1/2}. \end{aligned} \tag{5.5}$$

Fix an element $\xi \in V_\tau$ of norm 1. Write $\tau^*(k)$ for $\tau(k^{-1})$, $k \in K$. Define

$$\Phi(k) = T(\tau^*(k)\xi), \qquad k \in K.$$

Then it is not difficult to see that

$$\Phi \in H_\sigma, \qquad \|\Phi\| = 1.$$

Conversely, let Φ be any unit vector in $\mathscr{H}_\sigma$ so that Φ transforms under π_σ according to τ. Then there exist a unit vector $\xi \in V_\tau$ and an intertwining operator $T \in R(\tau, \sigma)$ with $\|T\| = (\dim \tau)^{1/2}$ such that

$$\Phi(k) = T(\tau^*(k)\xi), \qquad k \in K.$$

For Φ defined as above and $s \in W$, the little Weyl group, the vector $N_\sigma^s(\lambda)\Phi$ is contained in $\mathscr{H}_{s\sigma}$. Clearly $N_\sigma^s(\lambda)\Phi$ transforms under $\pi_{s\sigma}$ according to τ. Then there exists a unique $T_s \in R(\tau, s\sigma)$ with $\|T_s\| = (\dim \tau)^{1/2}$ such that

$$(N_\sigma^s(\lambda)\Phi)(k) = T_s(\tau^*(k)\xi), \qquad k \in K.$$

The map $T \to T_s$ from $R(\tau, \sigma)$ into $R(\tau, s\sigma)$ will be denoted by $n_\sigma^s(\lambda)$. So $T_s = n_\sigma^s(\lambda)T$. $n_\sigma^s(\lambda)$ is norm preserving and hence unitary.

Fix an orthonormal base $\{T_1, \ldots, T_r\}$ of $R(\tau, \sigma)$ of elements of norm equal to $(\dim \tau)^{1/2}$. For $1 \leqq l \leqq r$, $1 \leqq j \leqq t$, and $k \in K$, define

$$\Phi_{\tau,\,(l-1)t+j}(k) = T_l(\tau^*(k)\xi_j). \tag{5.6}$$

Then $\{\Phi_{\tau,i}\colon \tau \in \mathscr{E}_K,\ 1 \leqq i \leqq [\tau:\sigma]\cdot\dim\tau\}$ is an orthonormal base for $\mathscr{H}_\sigma$.

Let $\mathscr{C}(\hat{G})$ denote the set of functions $a(\sigma, \lambda)$ of $\mathscr{E}_M \times \mathfrak{a}_\mathfrak{p}^*$ into $\mathscr{H}_2(\sigma)$ which satisfy the following conditions (i), (ii) and (iii):

(i) *For each* $\sigma \in \mathscr{E}_M$, $a(\sigma, \lambda)$ *is a matrix-valued* C^∞ *function on* $\mathfrak{a}_\mathfrak{p}^*$.

(ii) $a(s\sigma, s\lambda) = N_\sigma^s(\lambda)a(\sigma, \lambda)N_\sigma^s(\lambda)^{-1}$, $\sigma \in \mathscr{E}_M$, $\lambda \in \mathfrak{a}_\mathfrak{p}^{*\prime}$, $\quad s \in W$.

(iii) *For every quartet* (p_1, p_2, q_1, q_2) *of polynomials and each* $d \in \boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^*)$,

$$\|a\|_{(p_1,p_2,q_1,q_2,d)} \tag{5.6}$$

$$= \sup_{\lambda,\sigma,\tau_1,i_1,\tau_2,i_2} |d_\lambda(\Phi_{\tau_1,i_1}, a(\sigma,\lambda)\Phi_{\tau_2,i_2})| p_1(|\sigma|)p_2(|\lambda|)q_1(|\tau_1|)q_2(|\tau_2|) < +\infty.$$

Then the above semi-norms define a topology on $\mathscr{C}(\hat{G})$ so that $\mathscr{C}(\hat{G})$ is a Fréchet space. $\mathscr{C}(\hat{G})$ is contained densely in $L^2(\hat{G})$.

THEOREM 5.1. *The Fourier transform* $\mathscr{F}\colon f \to \hat{f}$ *is a topological isomorphism of* $\mathscr{C}(G)$ *onto* $\mathscr{C}(\hat{G})$.

We shall spend the most of the rest of this paper to prove this theorem.

6. Eisenstein integrals

In this section we shall define τ-spherical functions and study the matrix elements.

Let τ be a unitary double representation of the compact group K on a Hilbert space V_τ, that is, V_τ is a left and right unitary K-module and the K-actions from the left and right commute with each other. We denote both the left and right K-actions of K by τ.

A function Φ from G to V_τ is said to be τ-spherical if

$$\Phi(k_1xk_2) = \tau(k_1)\Phi(x)\tau(k_2), \qquad k_1, k_2 \in K,\ x \in G.$$

We write the norm of $\Phi(x)$ in V_τ as $|\Phi(x)|$.

Let $f(x)$ be a continuous complex valued function on G such that the left and right translations of f by elements in K span a finite dimensional space of functions on G. Let ϕ be the function from G into $L^2(K \times K)$ defined by

$$\phi(x)(k_1, k_2)=f(k_1^{-1}xk_2), \qquad x\in G,\ k_1,\ k_2\in K.$$

Define a double representation μ of K on $L^2(K\times K)$ by

$$[\mu(\bar{k}_1)u](k_1, k_2)=u(\bar{k}_1^{-1}k_1, k_2),$$

$$[u\mu(\bar{k}_2)](k_1, k_2)=u(k_1, k_2\bar{k}_2^{-1})$$

for $u\in L^2(K\times K)$, k_1, k_2, $\bar{k}_1$, $\bar{k}_2\in K$. Let $V_\mu=\operatorname{sp}_{x\in G}\{\phi(x)\}$, the finite dimensional subspace of $L^2(K\times K)$ spanned by $\{\phi(x)\colon x\in G\}$. Then it is clear that $f\in V_\mu$ and ϕ is a μ-spherical function. ϕ is called the μ-spherical function associated with f.

Notice that if τ is an irreducible unitary double representation of K on the finite dimensional Hilbert space V_τ, then τ can be regarded as an irreducible representation $\tau_1\otimes\tau_2^*$ of $K\times K$ on $V_1\otimes V_2^*$. Here τ_1 and τ_2 are irreducible representations of K on the spaces V_1 and V_2, and τ_2^* is the dual representation of τ_2 acting on V_2^*, the dual space of V_2. We write τ as (τ_1, τ_2) and $|\tau|$ as $|\tau_1|+|\tau_2|$. Let $\mathscr{E}_K^2$ be the set of equivalence classes of irreducible unitary double representations of K.

Suppose that $f(x)=(\Phi_1, \pi(x)\Phi_2)$, where π is a unitary representation of G on a Hilbert space $\mathscr{H}$, and for each $\alpha=1$ or 2, Φ_α is a unit vector in $\mathscr{H}$ that transforms under $\pi|K$ according to the irreducible unitary representation τ_α of K, acting on the Hilbert space V_α. Let $\tau=(\tau_1, \tau_2)\in\mathscr{E}_K^2$ act on the Hilbert space $V_\tau=V_1\otimes V_2^*$. We shall find a formula for the spherical function ϕ associated with f.

Let τ_α have dimension t_α and let $\{\xi_{\alpha 1}, \dots, \xi_{\alpha t_\alpha}\}$ be an orthonormal basis for V_α, for $\alpha=1$ or 2. Let V_α' be the subspace of $\mathscr{H}$ spanned by $\{\pi(k)\Phi\colon k\in K\}$. Choose an orthonormal basis $\{\Phi_{\alpha 1}, \dots, \Phi_{\alpha t_\alpha}\}$ of V_α' such that the correspondence

$$\xi_{\alpha i}\leftrightarrow\Phi_{\alpha i}, \qquad i=1, 2, \dots, t_\alpha,$$

gives an intertwining operator between τ_α and $\pi|K$ acting on the space V_α'. Define functions $e_{\alpha i}(i=1, 2, \dots, t_\alpha,\ \alpha=1, 2)$ on K as follows:

$$e_{1i}(k_1)=(\pi(k_1)\Phi_1, \Phi_{1i}), \qquad k_1\in K,$$

$$e_{2j}(k_2)=\overline{(\pi(k_2^{-1})\Phi_2, \Phi_{2j})}=(\pi(k_2)\Phi_{2j}, \Phi_2), \qquad k_2\in K,$$

where the bar denotes the complex conjugate. Then

$$\phi(x)(k_1, k_2)=f(k_1^{-1}xk_2^{-1})=(\pi(k_1)\Phi_1, \pi(x)\pi(k_2^{-1})\Phi_2).$$

This is equal to the expression

$$\textstyle\sum_{ij}e_{1i}(k_1)e_{2j}(k_2)(\Phi_{1i}, \pi(x)\Phi_{2j}).$$

V_μ is the subspace of $L^2(K\times K)$ spanned by the functions $e_{1i}(k_1)e_{2j}(k_2)$. Let $\{\xi_{21}^*, \dots, \xi_{2t_2}^*\}$ be the dual basis in V_2^* of $\{\xi_{21}, \dots, \xi_{2t_2}\}$. Then for $1\leqq i\leqq t_1$, $1\leqq j\leqq t_2$, identify $e_{1i}(k_1)\cdot e_{2j}(k_2)$ with $(t_1t_2)^{-1/2}\xi_{1i}\otimes\xi_{2j}^*$. This gives an intertwining operator between the double representations μ and τ. Therefore, we can regard ϕ as a τ-spherical function from G to V_τ. We have the formula

$$\phi(x)=(t_1t_2)^{-1/2}\sum_{i,j}\xi_{1i}\otimes\xi_{2j}^*(\Phi_{1i}, \pi(x)\Phi_{2j}), \qquad x\in G. \tag{6.1}$$

Now suppose $\pi=\pi_{\sigma,\lambda}(\sigma\in\mathscr{E}_M, \lambda\in\mathfrak{a}_\mathfrak{p}^*)$ and recall that $R(\tau_\alpha, \sigma)$ is the space of intertwining operators between τ_α and σ. Fix $T_\alpha\in R(\tau_\alpha, \sigma)$ such that

$$\|T_\alpha\|^2=\dim\tau_\alpha=t_\alpha, \qquad \alpha=1, 2.$$

Suppose that $\xi_1\in V_1$, $\xi_2\in V_2$ are unit vectors. Let

$$\Phi_\alpha(k)=T_\alpha(\tau_\alpha^*(k)\xi_\alpha), \qquad k\in K, \alpha=1, 2.$$

Then clearly, Φ_1 and Φ_2 are unit vectors in $\mathscr{H}_\sigma$. Define

$$\Phi_{\alpha i}(k)=T_\alpha(\tau_\alpha^*(k)\xi_{\alpha i}), \qquad k\in K, \qquad \alpha=1, 2, 1\leqq i\leqq t_\alpha.$$

Then $\{\Phi_{\alpha i}\}$ is an orthonormal basis of V_α'. Put

$$f_{\sigma,\lambda}(x)=(\Phi_1, \pi_{\sigma,\lambda}(x)\Phi_2)=(\pi_{\sigma,\lambda}(x^{-1})\Phi_1, \Phi_2).$$

This is equal to the expression

$$\int_K(T_1[\tau_1^*(\kappa(xk)\xi_1], T_2[\tau_2^*(k)\xi_2])_{V_\sigma}e^{(i\lambda-\varrho)(H(xk))}dx, \tag{6.2}$$

where $(\ ,\ )_{V_\sigma}$ denotes the inner product on V_σ, the space on which σ acts. Combining the formulae (6.1) and (6.2) we obtain the following formula:

$$\phi_{\sigma,\lambda}(x)=(t_1t_2)^{-1/2}\sum_{i,j}\xi_{1i}\otimes\xi_{2j}^*\int_K(T_1[\tau_1^*(\kappa(xk))\xi_{1i}], T_2[\tau_2^*(k)\xi_{2j}])_{V_\sigma}\cdot$$
$$\cdot e^{(i\lambda-\varrho)(H(xk))}dk. \tag{6.3}$$

Let $L=L^\tau$ be the following set of functions on M: $\{\psi: M\to V_\tau: \psi(m_1mm_2)=\tau(m_1)\psi(m)\tau(m_2),\ m,\ m_1,\ m_2\in M\}$.
Then L^τ is a Hilbert space with inner product

$$(\psi_1, \psi_2)_M=\int_M(\psi_1(m), \psi_2(m))dm=\int_M(\tau_1(m)\psi_1(1), \tau_1(m)\psi_2(1))dm$$
$$=(\psi_1(1), \psi_2(1)).$$

If $\psi\in L^\tau$, then $\psi(1)\in V_1\otimes V_2^*$ and it can be regarded as an intertwining operator from V_2 to V_1 for $\tau_2|M$ and $\tau_1|M$. Conversely, if S is such an intertwining operator, then

$$\psi(m)=\tau_1(m)S=S\tau_2(m)$$

is contained in L^{τ}.

If $\sigma\in\mathscr{E}_M$, let L^{τ}_{σ} be the set of functions $\psi\in L^{\tau}$ such that $\psi(m)$ transforms under left and right translates of M according to the representation σ of M. Then there exists a finite number of representations $\{\sigma_1, \dots, \sigma_r\}$ in $\mathscr{E}_M$ such that

$$L^{\tau}=L^{\tau}_{\sigma_1}\oplus\cdots\oplus L^{\tau}_{\sigma_r}.$$

For any $\psi\in L^{\tau}$ let us extend the domain of ψ to all of G by defining

$$\psi(kan)=\tau_1(k)\psi(1), \qquad k\in K,\ a\in A_{\mathfrak{p}},\ n\in N.$$

Let us return to our function $\phi(x)$ above. Define

$$T_1^*:\ V_{\sigma}\longrightarrow V_1$$

as the adjoint of T_1. Let

$$S=T_1^*T_2:\ V_2\longrightarrow V_1.$$

Then S is an intertwining operator for $\tau_2|M$ and $\tau_1|M$, and can be regarded canonically as an element in $V_1\otimes V_2^*$, and moreover,

$$\begin{aligned}S&=(t_1t_2)^{-1/2}\sum_{i,j}\xi_{1i}\otimes\xi_{2j}^*(\xi_{1i}, S\xi_{2j})_{V_1}\\&=(t_1t_2)^{-1/2}\sum_{i,j}\xi_{1i}\otimes\xi_{2j}^*(T_1\xi_{1i}, T_2\xi_{2j})_{V_{\sigma}},\end{aligned}$$

where subscripts V_1, V_{σ} indecate in what space the inner product is taken. If we define ψ by

$$\psi(m)=\tau_1(m)S=S\tau_2(m), \qquad m\in M,$$

then $\psi\in L^{\tau}_{\sigma}$ and

$$\begin{aligned}\|\psi\|_M^2&=(\psi(1), \psi(1))=(S, S)\\&=(t_1t_2)^{-1}\sum_{i,j}|(T_1\xi_{1i}, T_2\xi_{2j})|^2.\end{aligned}$$

From (5.5), the last expression is equal to $(\dim\sigma)^{-1}$.

From (6.3) and (6.4) we obtain the formula

$$\phi_{\sigma,\lambda}(x)=\int_K\psi(xk)\tau(k^{-1})e^{(i\lambda-\varrho)(H(xk))}dk. \tag{6.5}$$

For any $\psi \in L^{\tau}$ we write

$$E(\psi:\lambda:x)=\int_K \psi(xk)\tau(k^{-1})e^{(i\lambda-\varrho)(H(xk))}dk. \tag{6.6}$$

$E(\psi:\lambda:x)$ is called the *Eisenstein integral* of ψ and λ.

Suppose, conversely, that we were given $\psi \in L^{\tau}_{\sigma}$ such that $\|\psi\|_M^2=(\dim\sigma)^{-1}$. Then we could choose $T_\alpha \in R(\tau_\alpha, \sigma)$ with $\|T_\alpha\|^2=\dim\tau_\alpha$ for $\alpha=1$ or 2 such that

$$\psi(1)=T_1^* T_2.$$

Again we can define $\Phi_{\alpha i}$ by

$$\Phi_{\alpha i}(k)=T_\alpha(\tau_\alpha^*(k)\xi_{\alpha i}), \qquad k\in K,\ \alpha=1, 2.$$

Then $\Phi_{\alpha i}$ is a unit vector in $\mathscr{H}_\sigma$. Working backwards we can obtain the formula

$$E(\psi:\lambda:x)=(t_1t_2)^{-1/2}\sum_{i,j}\xi_{1i}\otimes\xi_{2j}^*(\Phi_{1i},\ \pi_{\sigma,\lambda}(x)\Phi_{2j}). \tag{6.7}$$

Now, if $\psi(1)=T_1^*T_2$ as above, and $\lambda\in\mathfrak{a}_{\mathfrak{p}}^{*\prime}$, then $n_\sigma^s(\lambda)T_\alpha\in R(\tau_\alpha, s\sigma)$ and $\|T_\alpha\|^2=t_\alpha$ for $\alpha=1$ or 2, $s\in W$. Define

$$(M_\sigma^s(\lambda)\psi)(1)=(n_\sigma^s(\lambda)T_1)^*(n_\sigma^s(\lambda)T_2). \tag{6.8}$$

Then $M_\sigma^s(\lambda)\psi$ can be regarded as a function in $L^{\tau}_{s\sigma}$. It has the same norm as ψ. Therefore, $M_\sigma^s(\lambda)$ is a unitary map of L^{τ}_{σ} onto $L^{\tau}_{s\sigma}$. We can then define a unitary linear transformation $M^s(\lambda)$ of L^{τ} by defining it to be $M_\sigma^s(\lambda)$ on each of the orthogonal subspaces L^{τ}_{σ} of L^{τ}.

If $\lambda\in\mathfrak{a}_{\mathfrak{p}}^{*\prime}$ we have the equation

$$(\Phi_1,\ \pi_{\sigma,\lambda}(x)\Phi_2)=(N_\sigma^s(\lambda)\Phi_1,\ \pi_{s\sigma,s\lambda}(x)N_\sigma^s(\lambda)\Phi_2).$$

Then from (6.7) we obtain the formula

$$E(\psi:\lambda:x)=E(M^s(\lambda)\psi:s\lambda:x), \qquad s\in W. \tag{6.9}$$

This is the functional equation for the Eisenstein integral with respect to the little Weyl group W.

7. Proof of the injectivity of the map $\mathscr{F}$

Let π be a unitary representation of G on a Hilbert space $\mathscr{F}$. If v is a vector in $\mathscr{H}$ such that the map from G to $\mathscr{H}$ given by

$$x\longrightarrow\pi(x)v, \qquad x\in G.$$

is infinitely differentiable, v is called a *differentiable vector*. Let $\mathscr{H}^\infty$ be the set of differentiable vectors in $\mathscr{H}$. If $v \in \mathscr{H}^\infty$ and $X \in \mathfrak{g}$, define

$$\pi(X)v = \lim_{t\to 0} \frac{1}{t}(\pi(\exp tX)v - v).$$

It can be checked that this gives a representation of the Lie algebra $\mathfrak{g}$ on the vector space $\mathscr{H}^\infty$. It extends to a representation, again denoted by π, of the universal enveloping algebra $\mathfrak{B}$ of $\mathfrak{g}^c$.

Let $\mathfrak{Z}$ be the center of $\mathfrak{B}$. If the restriction of π to $\mathfrak{Z}$ is one dimensional, we obtain a homomorphism

$$\chi: \mathfrak{Z} \longrightarrow \boldsymbol{C}.$$

In this case π is said to be *quasi-simple, and* χ is called the *infinitesimal character* of π. It is known that any irreducible unitary representation of G is quasi-simple.

Let η be the conjugation of $\mathfrak{g}^c$ with respect to the real form $\mathfrak{g}$. We define three involutions on $\mathfrak{g}^c$ by

$$X^* = -\eta X, \qquad X \in \mathfrak{g}^c,$$

$$X^+ = -X, \qquad X \in \mathfrak{g}^c,$$

$$\bar{X} = \eta X, \qquad X \in \mathfrak{g}^c.$$

If $X, Y \in \mathfrak{g}^c$ and $c \in \boldsymbol{C}$, it is easy to show that

$$[\bar{X}, \bar{Y}] = \overline{[X, Y]}, \qquad (\overline{cX}) = \bar{c}\bar{X},$$

$$[X^+, Y^+] = -[X, Y]^+, \quad (cX)^+ = cX_+,$$

$$[X^*, Y^*] = -[X, Y]^*, \quad (cX)^* = \bar{c}X^*.$$

All three involutions extend to involutions of $\mathfrak{B}$.

If π is a unitary representation of G, then for $g \in \mathfrak{B}$,

$$\pi(g^*) = \pi(\bar{g})^*,$$

where $\pi(g)^*$ denotes the adjoint operator of $\pi(g)$.

LEMMA 7.1. *Suppose that π is quasi-simple. Assume Φ_1 and Φ_2 are vectors in $\mathscr{H}$ such that the vector spaces*

$$\mathrm{sp}\{\pi(k)\Phi_\alpha : k \in K\}, \qquad \alpha = 1, 2,$$

are both finite dimensional. Then $\Phi_1, \Phi_2 \in \mathscr{H}^\infty$. Furthermore if $g_1, g_2 \in \mathfrak{B}$, and

$$h(x) = (\Phi_1, \pi(x)\Phi_2),$$

then

$$h(g_1; x; g_2) = (\pi(g_1^\dagger)\Phi_1, \pi(x)\pi(\bar{g}_2)\Phi_2).$$

For a proof see [1, Lemma 6].

Let $\mathfrak{M}$, $\mathfrak{A}_\mathfrak{p}$, $\mathfrak{A}_\mathfrak{k}$ and $\mathfrak{A}$ be the universal enveloping algebras of $\mathfrak{m}^c$, $\mathfrak{a}_\mathfrak{p}^c$, $\mathfrak{a}_\mathfrak{k}^c$ and $\mathfrak{a}^c$ respectively. Let $\mathfrak{Z}_M$ be the center of $\mathfrak{M}$. Then $\mathfrak{M}\mathfrak{A}_\mathfrak{p}$ is the universal enveloping algebra of $\mathfrak{m}^c + \mathfrak{a}_\mathfrak{p}^c$, and its center is $\mathfrak{Z}_M\mathfrak{A}_\mathfrak{p}$.

If $z \in \mathfrak{Z}$, there exists a unique element $\gamma_0'(z) \in \mathfrak{Z}_M\mathfrak{A}_\mathfrak{p}$ such that

$$z - \gamma_0'(z) \in \sum_{\alpha \in P_+} \mathfrak{B}X_\alpha$$

([6(e), Lemma 18]).

If $z_1 \in \mathfrak{Z}_M\mathfrak{A}_\mathfrak{p}$, there exists a unique element $\gamma_1'(z_1) \in \mathfrak{A}$ such that

$$z_1 - \gamma_1'(z_1) \in \sum_{\alpha \in P_-} \mathfrak{M}\mathfrak{A}_\mathfrak{p}X_\alpha$$

([6(e), Lemma 18]).

If $z \in \mathfrak{Z}$, there exists a unique element $\gamma'(z) \in \mathfrak{A}$ such that

$$z - \gamma'(z) \in \sum_{\alpha \in P} \mathfrak{B}X_\alpha.$$

Notice that if z is an element in $\mathfrak{Z}$,

$$z - \gamma_1'(\gamma_0'(z)) = (z - \gamma_0'(z)) + (\gamma_0'(z) - \gamma_1'(\gamma_0'(z))).$$

The right hand sum is an element in $\sum_{\alpha \in P} \mathfrak{B}X_\alpha$. Therefore

$$\gamma_1' \circ \gamma_0' = \gamma'. \tag{7.1}$$

Define automorphisms β and β_1 of $\mathfrak{A}$ by

$$\beta(H) = H + \rho(H), \qquad H \in \mathfrak{a}^c,$$

$$\beta_1(H) = H + \rho_\mathfrak{m}(H), \qquad H \in \mathfrak{a}^c.$$

Let

$$\gamma = \beta^{-1} \circ \gamma', \qquad \gamma_1 = \beta_1^{-1} \circ \gamma_1'.$$

It is known that the maps

$$\gamma: \mathfrak{Z} \longrightarrow \mathfrak{A}, \qquad \gamma_1: \mathfrak{Z}_M\mathfrak{A}_\mathfrak{p} \longrightarrow \mathfrak{A},$$

are algebraic isomorphisms onto the subalgabras consisting of the elements in $\mathfrak{A}$ which are invariant under $W_\mathfrak{g}$ and $W_\mathfrak{m}$ respectively ([6(e), Lemma 19]). $\mathfrak{A}$ can be regarded as the algebra of polynomial functions from the dual space $\mathfrak{a}^{c*}$

of $\mathfrak{a}^c$, into $\boldsymbol{C}$. If $\lambda \in \mathfrak{a}^{c*}$ denote the evaluation of $p \in S(\mathfrak{a}^c)$ at λ by $<p, \lambda>$. Then for any λ in $\mathfrak{a}^{c*}$ define a homomorphism χ_λ: $\mathfrak{Z} \to \boldsymbol{C}$ by

$$\chi_\lambda(z) = <\gamma(z), \lambda>, \qquad z \in \mathfrak{Z}.$$

Any homomorphism from $\mathfrak{Z}$ into $\boldsymbol{C}$ is of this form and $\chi_{\lambda_1} = \chi_{\lambda_2}$ if and only if $\lambda_1 = s\lambda_2$ for some $s \in W_{\mathfrak{g}}$. χ_λ is called the homomorphism corresponding to the linear function λ. Similarly, we can define χ_λ^M: $\mathfrak{Z}_M \mathfrak{A}_{\mathfrak{p}} \to \boldsymbol{C}$ by

$$\chi_\lambda^M(z_1) = <\gamma_1(z_1), \lambda>, \qquad z_1 \in \mathfrak{Z}_M \mathfrak{A}_{\mathfrak{p}}.$$

Define an automorphism β_0 as follows:

$$\beta_0(X) = X, \qquad X \in \mathfrak{m}^c,$$

$$\beta_0(H) = H + \rho(H), \qquad H \in \mathfrak{a}_{\mathfrak{p}}^c.$$

β_0 takes $\mathfrak{Z}_M \mathfrak{A}_{\mathfrak{p}}$ onto itself. Put

$$\gamma_0 = \beta_0^{-1} \circ \gamma_0'.$$

By (7.1)

$$\gamma = \gamma_1 \circ \gamma_0. \tag{7.2}$$

We can now find the infinitesimal character of $\pi_{\sigma,\lambda}$. If $\sigma \in \mathscr{E}_K$, let μ be a real linear function on $\sqrt{-1}\mathfrak{a}_{\mathfrak{k}}$ associated with σ. Regard μ as a linear function on $\sqrt{-1}\mathfrak{a}_{\mathfrak{k}} + \mathfrak{a}_{\mathfrak{p}}$ by making it equal zero on $\mathfrak{a}_{\mathfrak{p}}$. By looking at a highest weight vector for σ, we can easily check that for any $z_1 \in \mathfrak{Z}_M$

$$\sigma(z_1) = \chi_\mu^M(z_1) = <\gamma_1(z_1), \mu>. \tag{7.3}$$

LEMMA 7.2. *Fix $\sigma \in \mathscr{E}_M$ and $\lambda \in \mathfrak{a}_{\mathfrak{p}}^*$. Then for any $z \in \mathfrak{Z}$,*

$$\pi_{\sigma,\lambda}(z) = \chi_{-\mu - i\lambda}(z).$$

PROOF. It is known that the representation $\pi_{\sigma,\lambda}$ is quasi-simple ([6(a), pg. 243]). Therefore, there exists a complex linear function ν on $\mathfrak{a}^c$ such that

$$\pi_{\sigma,\lambda}(z) = \chi_\nu, \qquad z \in \mathfrak{Z}.$$

Choose a $\tau \in \mathscr{E}_K^2$ such that $L_\sigma^\tau \neq 0$. Fix ψ in L_σ^τ such that

$$\|\psi\|_M^2 = (\dim \sigma)^{-1}.$$

Then by (6.7) (using the notation in that formula) and Lemma 7.1, we obtain the formula

$$\begin{aligned}
&\overline{\chi_\nu(\bar{z})}E(\psi: \lambda: x)\\
&=\sum_{i,j}(t_1t_2)^{-1/2}\cdot\xi_{1i}\otimes\xi^*_{2j}(\Phi_{1i}, \pi_{\sigma,\lambda}(x)\pi_{\sigma,\lambda}(\bar{z})\Phi_{2j})\\
&=E(\psi: \lambda: x; z), \qquad z\in\mathfrak{Z}.
\end{aligned}$$

Put

$$F(x)=\psi(x)e^{(i\lambda-\varrho)(H(x))},$$

and define

$$F(x: k)=F(xk)\tau(k^{-1}), \qquad x\in G,\ k\in K.$$

Then by (6.6),

$$E(\psi: \lambda: x)=\int_K F(x: k)dk.$$

Let z be an arbitrary element in $\mathfrak{Z}$. It can be regarded as a left and right invariant differential operator, so

$$F(x; z: k)=F(xk; z)\tau(k^{-1}).$$

Therefore

$$\begin{aligned}
E(\psi: \lambda; x; z)&=\int_K F(x; z: k)dk\\
&=\int_K F(xk; z)\tau(k^{-1})dk.
\end{aligned} \tag{7.4}$$

Clearly $F(xn)=F(x)$ for any $n\in N$, so if $g\in\mathfrak{B}\mathfrak{n}$ then

$$F(x; g)=0.$$

Therefore

$$\begin{aligned}
E(\psi: \lambda: x; z)&=\int_K F(xk; \gamma_0'(z))\tau(k^{-1})dk\\
&=\int_K F(xk; \beta_0\gamma_0(z))\tau(k^{-1})dk.
\end{aligned}$$

Suppose that

$$\gamma_0(z)=\sum_i z_ih_i, \qquad z_i\in\mathfrak{Z}_M,\ h\in\mathfrak{a}_\mathfrak{p}. \tag{7.5}$$

Then

$$\beta_0\gamma_0(z)=\sum_i z_i\beta_0(h_i).$$

Now

$$F(y\exp H)=F(y)e^{(i\lambda-\varrho)(H)}, \qquad H\in\mathfrak{a}_{\mathfrak{p}},$$

so for any $h\in\mathfrak{A}_{\mathfrak{p}}$,

$$\begin{aligned} F(y;\beta_0(h)) &= F(y)<\beta_0(h), i\lambda-\rho> \\ &= F(y)<h, i\lambda>. \end{aligned}$$

On the other hand, if $m\in M$,

$$\begin{aligned} F(ym) &= \psi(ym)e^{(i\lambda-\varrho)(H(y))} \\ &= \tau_1(\kappa(y))\tau_1(m)\psi(1)e^{(i\lambda-\varrho)(H(y))} \\ &= \tau_1(\kappa(y))T_1^*\sigma(m)T_2e^{(i\lambda-\varrho)(H(y))}, \end{aligned}$$

where $\psi(1)=T_1^*T_2$ in the notation of Section 5. Therfore, if $z_M\in\mathfrak{Z}_M$,

$$\begin{aligned} F(y;z_M) &= \tau_1(\kappa(y))T_1^*\sigma(z_M)T_2e^{(i\lambda-\varrho)(H(y))} \\ &= \tau_1(\kappa(y))T_1^*<\gamma_1(z_M), \mu>T_2e^{(i\lambda-\varrho)(H(y))} \qquad (7.7) \\ &= <\gamma_1(z_M), \mu>F(y). \end{aligned}$$

From (7.4), (7.5), (7.6) and (7.7) we see that

$$\begin{aligned} F(y;\beta_0\gamma_0(z)) &= F(y)<\sum_i \gamma_1(z_i)h_i, \mu+i\lambda> \\ &= F(y)<\gamma_1\gamma_0(z), \mu+i\lambda> \\ &= F(y)<\gamma(z), \mu+i\lambda>. \end{aligned}$$

So, from (7.4) we have

$$E(\psi:\lambda:x;z)=<\gamma(z), \mu+i\lambda>E(\psi:\lambda:x), \qquad z\in\mathfrak{Z}.$$

It follows that

$$\overline{\chi_\nu(\bar z)}=<\gamma(z), \mu+i\lambda>.$$

It is easy to show that $\nu=-\mu-i\lambda$. This proves our Lemma 7.2. Q.E.D.

From the above proof we obtain the following corollary.

COROLLARY. *Let $\tau\in\mathcal{E}_K^2$ and $\sigma\in\mathcal{E}_M$ such that $L_\sigma^\tau\neq 0$. Let $\psi\in L_\sigma^\tau$ such* *that* $\|\psi\|_M^2=(\dim\sigma)^{-1}$. *If* $\lambda\in\mathfrak{a}_\mathfrak{p}^*$, *then for any* $z\in\mathfrak{Z}$,

$$E(\psi:\lambda:x;z)=\langle\gamma(z),\mu+i\lambda\rangle E(\psi:\lambda:x).$$

Let $X_1,\dots,X_n$ be a basis for $\mathfrak{g}^c$ and $g_{ij}=B(X_i,X_j)$ $(1\leqq i,j\leqq n)$. As is well known the matrix $(g_{ij})_{1\leqq i,j\leqq n}$ is non-singular. Let (g^{ij}) be the inverse matrix of the matrix (g_{ij}) and $\omega_\mathfrak{g}=\sum_{i,j}g^{ij}X_iX_j$. It is known that $\omega_\mathfrak{g}$ is independent of the choice of basis for $\mathfrak{g}^c$ and is contained in $\mathfrak{Z}$. $\omega_\mathfrak{g}$ is called the *Casimir element* of $\mathfrak{B}$. If ν is any linear function on $\mathfrak{a}^c$,

$$\chi_\nu(\omega_\mathfrak{g})=B(\nu,\nu)-B(\rho,\rho) \tag{7.8}$$

(see [1, (6.8)]).
Since

$$B(-\mu-i\lambda,\ -\mu-i\lambda)=B(\mu,\mu)-B(\lambda,\lambda),$$

from Lemma 7.2

$$\pi_{\sigma,\lambda}(\omega_\mathfrak{g})=|\sigma|^2-B(\lambda,\lambda)-B(\rho,\rho). \tag{7.9}$$

In order to prove the injectivity of the Fourier transform $\mathscr{F}$, we use the following

LEMMA 7.3. *Let* $\sup_0$ *denote the supremum over all* (σ,λ), (τ_1,i_1), (τ_2,i_2). *Then for non-negative integers* m, m_1, m_2 *and differential operators* $d\in\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^*)$, *the semi-norms*

$$\begin{aligned}&\|a\|_{(m,m_1,m_2,d)}\\ &\quad=\sup_0|d_\lambda[(\Phi_{\tau_1,i_1},\ a(\sigma,\lambda)\Phi_{\tau_2,i_2})(1+|\sigma|^2-|\lambda|^2)^m]|(1+|\tau_1|^2)^{m_1}\cdot\\ &\quad\quad\cdot(1+|\tau_2|^2)^{m_2},\qquad a\in\mathscr{C}(\hat{G}),\end{aligned}$$

form a basis for the topology of $\mathscr{C}(\hat{G})$.

PROOF. By Leibnitz' rule and induction on the degree of d, we can see that it is enough to prove the lemma for the semi-norms

$$\begin{aligned}&\sup_0|d_\lambda[(\Phi_{\tau_1,i_1},\ a(\sigma,\lambda)\Phi_{\tau_2,i_2})]\|(1+|\sigma|^2-|\lambda|^2)|^m\cdot\\ &\quad\cdot(1+|\tau_1|^2)^{m_1}(1+|\tau_2|^2)^{m_2},\qquad a\in\mathscr{C}(G).\end{aligned} \tag{7.10}$$

Fix $\sigma\in\mathscr{E}_M$ and $\tau\in\mathscr{E}_K$. There is a non-zero vector of the form $\Phi_{\tau,i}$ in our basis for $\mathscr{H}_\sigma$ if and only if the representation σ occurs in $\tau|M$. Suppose that this is the case. Then if τ acts on the finite dimensional vector space V_τ, σ acts on a subspace of V_τ. The Cartan subalgebras of $\mathfrak{m}$ and $\mathfrak{k}$ are both $\mathfrak{a}_\mathfrak{k}$, and we have already ordered its dual space. Let ν and μ be the highest weights of the representations τ and σ respectively. Then

$$|\sigma|^2 = B(\mu+\rho_M, \mu+\rho_M)$$

$$|\tau|^2 = B(\nu+\rho_K, \nu+\rho_K),$$

where ρ_M and ρ_K are the halves of the sum of positive roots of $(\mathfrak{m}^c, \mathfrak{a}_{\mathfrak{k}}^c)$ and $(\mathfrak{k}^c, \mathfrak{a}_{\mathfrak{k}}^c)$ respectively.

Let ξ be a highest weight vector in V_τ for σ. V_τ is a direct sum of weight spaces for τ. Examine the action of $\tau(\mathfrak{a}_{\mathfrak{k}})$ on ξ. Since $\mathfrak{a}_{\mathfrak{k}}$ is a Cartan subalgebra of $\mathfrak{k}$, we can regard μ as a weight for τ. However

$$B(\mu, \mu)^{1/2} \leqq B(\mu+\rho_K, \mu+\rho_K)^{1/2} + B(\rho_K, \rho_K)^{1/2}.$$

Since ν is the highest weight for τ, we see by [8, Lemma 3, pg. 248] that

$$B(\mu+\rho_K, \mu+\rho_K) \leqq B(\nu+\rho_K, \nu+\rho_K).$$

Therefore, we have

$$|\sigma| \leqq B(\nu+\rho_K, \nu+\rho_K)^{1/2} + B(\rho_K, \rho_K)^{1/2} + B(\rho_M, \rho_M)^{1/2}.$$

Hence, we can find a constant $c>0$, independent of τ and σ, such that

$$(7.11) \qquad |\sigma| \leqq |\tau| + c.$$

From this inequality we obtain the additional formula

$$(7.12) \qquad |\lambda|^2 \leqq |1+|\sigma|^2 - |\lambda|^2| + (1+(c+|\tau|)^2).$$

Formulae (7.11) and (7.12) show that any semi-norm of the form (5.6) is dominated by a semi-norm of the form (7.10). Since the semi-norms (5.6) form a basis for the topology of $\mathscr{C}(\hat{G})$, our Lemma 7.3 is proved. Q.E.D.

The Lie algebra $\mathfrak{k}$ is reductive, so $\mathfrak{k} = \mathfrak{k}_1 + \mathfrak{k}_2$, where $\mathfrak{k}_1$ is semisimple and $\mathfrak{k}_2$ is abelian. Let $\mathfrak{K}$ be the universal enveloping algebra of $\mathfrak{k}^c$, and let $\mathfrak{Z}_K$ be its center. $\mathfrak{a}_{\mathfrak{k}}$ is a Cartan subalgebra of $\mathfrak{k}$. For linear functions ν on $\mathfrak{a}_{\mathfrak{k}}^c$ we can define the homomorphisms

$$\chi_\nu^K : \mathfrak{Z}_K \longrightarrow \boldsymbol{C}.$$

Since the Killing form of $\mathfrak{g}^c$ when restricted to $\mathfrak{k}_1$ is K-invariant, this Killing form is a linear combination of the Killing forms of $\mathfrak{k}_{1,i}$ on $\mathfrak{k}_1$, where $\mathfrak{k}_{1,i}$ are simple ideals of $\mathfrak{k}_1$. Then it is clear that we can choose an element $\omega_{\mathfrak{k}} \in \mathfrak{Z}_K$ such that

$$\chi_{\nu}^{K}(\omega_{\mathfrak{k}})=B(\nu,\nu)-B(\rho_K,\rho_K) \tag{7.13}$$

for any linear function ν on $\mathfrak{a}_{\mathfrak{k}}^{c}$.

Notice that

$$\begin{aligned} &\omega_{\mathfrak{g}}^{*}=\omega_{\mathfrak{g}}, \quad \overline{\omega_{\mathfrak{g}}}=\omega_{\mathfrak{g}}, \quad \omega_{\mathfrak{g}}^{+}=\omega_{\mathfrak{g}}, \\ &\omega_{\mathfrak{k}}^{*}=\omega_{\mathfrak{k}}, \quad \overline{\omega_{\mathfrak{k}}}=\omega_{\mathfrak{k}}, \quad \omega_{\mathfrak{k}}^{+}=\omega_{\mathfrak{k}}. \end{aligned} \tag{7.14}$$

LEMMA 7.4. *The Fourier transform $\mathscr{F}$: $f\to\hat{f}$ is a continuous map from $\mathscr{C}(G)$ into $\mathscr{C}(\hat{G})$.*

PROOF. Let $\|\cdot\|_0$ be an arbitrary continuous semi-norm on $\mathscr{C}(\hat{G})$. Since $\dim\sigma$, $\dim\tau_1$ and $\dim\tau_2$ are bounded by polynomials in $|\sigma|$, $|\tau_1|$ and $|\tau_2|$ respectively, we can use the previous lemma to choose integers m, m_1, m_2 and a differential operator $d\in\boldsymbol{D}(\mathfrak{a}_{\mathfrak{p}}^{*})$ such that for any $a\in\mathscr{C}(\hat{G})$

$$\begin{aligned}\|a\|_0\leqq \sup\nolimits_0(\dim\sigma)^{1/2}(\dim\tau_1\cdot\dim\tau_2)^{-1/2}|d_\lambda[(\Phi_{\tau_1,i_1},a(\sigma,\lambda)\Phi_{\tau_2,i_2})\cdot\\ \cdot(1+|\sigma|^2-|\lambda|^2)^m]|(1+|\tau_1|^2)^{m_1}(1+|\tau_2|^2)^{m_2}.\end{aligned}$$

Define elements g_1 and g_2 in $\mathfrak{B}$ by

$$\begin{aligned} g_1&=(\omega_{\mathfrak{k}}+B(\rho_K,\rho_K)+1)^{m_1},\\ g_2&=(\omega_{\mathfrak{g}}+B(\rho,\rho)+1)^{m}\cdot(\omega_{\mathfrak{k}}+B(\rho_K,\rho_K)+1)^{m_2}.\end{aligned}$$

By (7.13), $g_1^{+}=g_1$ and $\bar{g}_2=g$. Since Φ_{τ_1,i_1} and Φ_{τ_2,i_2} transform under $\pi_{\sigma,\lambda}$ according to τ_1 and τ_2 respectively, we have by (7.9) that

$$\begin{aligned}\pi_{\sigma,\lambda}(g_1)\Phi_{\tau_1,i_1}&=(1+|\tau_1|^2)^{m_1}\Phi_{\tau_1,i_1},\\ \pi_{\sigma,\lambda}(g_2)\Phi_{\tau_2,i_2}&=(1+|\tau_2|^2)^{m_2}\cdot(1+|\sigma|^2-|\lambda|^2)^m\Phi_{\tau_2,i_2}.\end{aligned}$$

Therefore, for any $f\in C_c^\infty(G)$ we have from Lemma 7.1 that

$$\|\hat{f}\|_0\leqq\sup\nolimits_0(\dim\sigma)^{1/2}(t_1t_2)^{-1/2}\left|d_\lambda\int_G h(g_1;x;g_2)\overline{f(x)}dx\right|,$$

where

$$\begin{aligned} t_i&=\dim\tau_i, \qquad i=1,2,\\ h(x)&=(\Phi_{\tau_1,i_1},\pi_{\sigma,\lambda}(x)\Phi_{\tau_2,i_2}).\end{aligned}$$

Now in order to see that by means of the G-invariance of dx we can transfer the differentaition of $h(x)$ to the differentiation of $f(x)$ in the above inequality, we need a few lemmas.

LEMMA 7.5. *If $f \in \mathscr{C}(G)$, b_1, $b_2 \in \mathfrak{B}$ and Ω is any compact set in G, then the integral*

$$\int_G |h(xy;\, b_1)f(x;\, b_2)|dx$$

converges uniformly with respect to y on Ω.

PROOF. We have

$$h(x) = (\Phi_{\tau_1,i_1},\, \pi_{\sigma,\lambda}(x)\Phi_{\tau_2,i_2}).$$

By Lemma 7.1, for any $b \in \mathfrak{B}$ and $x \in G$

$$h(x;\, b) = (\Phi_{\tau_1,i_1},\, \pi_{\sigma,\lambda}(x)\pi_{\sigma,\lambda}(\bar{b})\Phi_{\tau_2,i_2}).$$

So, it follows that there exists a constant $c_1 > 0$ such that

$$|h(xy;\, b_1)| \leqq c_1 \int_K e^{-\varrho(H(y^{-1}x^{-1}k))}dk.$$

On the other hand, it is known that we can choose a constant $c_2 > 0$ such that

$$e^{-\varrho(H(y^{-1}x^{-1}k))} \leqq c_2 e^{-\varrho(H(x^{-1}k))}$$

for $y \in \Omega$ (see [6(h), the proof of Lemma 48]). Since $f \in \mathscr{C}(G)$, Lemma 7.5 follows. Q.E.D.

COROLLARY 1. *Let $f \in \mathscr{C}(G)$, b_1, $b_2 \in \mathfrak{B}$ and Ω be any compact set in $G \times G$. Then the integral*

$$\int_G |h(xy;\, b_1)f(xz;\, b_2)|dx$$

converges uniformly with respect to (y, z) on Ω.

PROOF. Without loss of generality we may assume that $\Omega = \omega_1 \times \omega_2$ where ω_1, ω_2 are compact subsets of G. Let ω be the image of Ω in G under the mapping $(y, z) \to z^{-1}y$. Then ω is also compact. For any $\varepsilon > 0$ we can, by the above lemma, select a compact set V in G such that

$$\int_{^cV} |h(xy;\, b_1)f(x;\, b_2)| < \varepsilon$$

for $y \in \omega$; where ${}^cV$ denotes the complement of V in G. Put $U = V\omega_2^{-1}$. Then U is also compact and ${}^cUz \subset {}^cV$ for $z \in \omega_2$. Hence if $y \in \omega_1$ and $z \in \omega_2$, it is clear that

$$\int_{^cU} |h(xy;\, b_1)f(xz;\, b_2)|dx = \int_{^cUz} |h(xz^{-1}y;\, b_1)f(x;\, b_2)|dx < \varepsilon$$

from the right-invariance of the measure dx. This proves Corollary 1.

Q.E.D.

Let $g \to g^*$ denote the anti-automorphism of $\mathfrak{B}$ over $\boldsymbol{R}$ defined by the involution $X \to X^*$ of $\mathfrak{g}^c$.

COLOLLARY 2. *For any* $g \in \mathfrak{B}, f \in \mathscr{C}(G)$

$$\int_G h(x;\, g)\overline{f(x)}dx = \int_G h(x)\overline{f(x;\, g^*)}dx.$$

PROOF. We put

$$F(y:\, z) = \int_G h(xy)\overline{f(xz)}dx, \qquad y,\, z \in G.$$

Then the above Corollary implies that F is a C^∞-function and

$$F(y;\, g_1:\, z;\, g_2) = \int_G h(xy;\, g_1)\overline{f(xz;\, g_2)}dx$$

for b_1, $b_2 \in \mathfrak{B}$. On the other hand

$$F(y;\, g_1:\, z;\, g_2) = F(z^{-1}y;\, g_1:\, 1;\, g_2)$$

by the right-invariance of dx. Hence if we put $y=1$ and $z=\exp tX$ ($X \in \mathfrak{g}$, $t \in \boldsymbol{R}$) and differentiate with respect to t at $t=0$, we get

$$F(1;\, g_1:\, 1;\, Xg_2) = -F(1;\, Xg_1:\, 1;\, g_2).$$

From this our Corollary 2 follows. Q.E.D.

By an argument similar to the above, we obtain the following

LEMMA 7.6. *For any* g_1, $g_2 \in \mathfrak{B}, f \in \mathscr{C}(G)$

$$\int_G h(g_1:\, x;\, g_2)\overline{f(x)}dx = \int_G h(x)\overline{f(g_1^*:\, x;\, g_2^*)}dx.$$

Now we return to the proof of Lemma 7.4. By Lemma 7.5 and (7.14), we have the inequality

$$\|\hat{f}\|_0 < \sup\nolimits_0 (\dim\sigma)^{1/2}(t_1t_2)^{-1/2}\left| d_\lambda \int_G h(x)\overline{f(g_1:\, x;\, g_2)}dx \right|.$$

Let τ be the double representation (τ_1, τ_2) of K. From (6.7) we obtain the following estimation.

$$\begin{aligned}|E(\psi: \lambda; d: x)| &= (t_1 t_2)^{-1/2} \Big|\sum_{i,j}(\xi_{1i}\otimes\xi^*_{2j})d_\lambda(\Phi_{1i}, \pi_{\sigma,\lambda}(x)\Phi_{2j})\\ &= (t_1 t_2)^{-1/2}\Big(\sum_{i,j}|d_\lambda(\Phi_{1i}, \pi_{\sigma,\lambda}(x)\Phi_{2j})|^2\Big)^{1/2}\\ &\geqq (t_1 t_2)^{-1/2}|d_\lambda(\Phi_{1i}, \pi_{\sigma,\lambda}(x)\Phi_{2j})|, \qquad 1\leqq i\leqq t_1,\ 1\leqq j\leqq t_2.\end{aligned}$$

So we can find a $\psi\in L^\tau_\sigma$, with $\|\psi\|^2_M=(\dim\sigma)^{-1}$, so that the last expression in the above inequality is bounded by

$$\sup_0(\dim\sigma)^{1/2}\int_G|f(g_1; x; g_2)||E(\psi: \lambda; d: x)|dx.$$

Now if $k\in K$,

$$|\psi(k)|=|\psi(1)|=\|\psi\|_M=(\dim\sigma)^{-1/2}.$$

So, by (6.6)

$$\begin{aligned}&(\dim\sigma)^{1/2}|E(\psi: \lambda; d: x)|\\ &=(\dim\sigma)^{1/2}\Big|\int_K\psi(xk)\tau(k^{-1})[i\lambda_d(H(xk))]e^{(i\lambda-\varrho)(H(xk))}dk\Big|\\ &\leqq\int_K|\lambda_d(H(xk))|e^{-\varrho(H(xk))}dk,\end{aligned}$$

where $\lambda_d\in S(\mathfrak{a}^*_\mathfrak{p})$ is the polynomial function corresponding to d. To complete the proof of our lemma we need the following

LEMMA 7.6. *For every $p\in S(\mathfrak{a}^*_\mathfrak{p})$ we can select a polynomial $\bar{p}$ such that for any $x\in G$*

$$|p(H(x))|\leqq\bar{p}(\sigma(x)),$$

where σ is the function defined in Section 5.

PROOF. Each $x\in G$ can be rwitten in the form

$$x=k'\cdot\exp tH\cdot k, \qquad k', k\in K, \exp H\in cl(A^+_\mathfrak{p}),\ t\geqq 0,$$

where $A^+_\mathfrak{p}$ denotes $\exp\mathfrak{a}^+_\mathfrak{p}$ ($\mathfrak{a}^+_\mathfrak{p}$ is the positive Weyl chamber of $\mathfrak{a}_\mathfrak{p}$). Then we can select a $t_k\in\boldsymbol{R}$ such that

$$H(x)=H(\exp tH\cdot k)=t_kH', \qquad H'\in cl(\mathfrak{a}^+_\mathfrak{p}).$$

Now [6(h), Lemma 35] and [6(h), Lemma 35, Corollary 2] establish precisely that

$$-t \leqq t_k \leqq t.$$

Therefore we can select a polynomial $\bar{p}$ such that

$$|p(H(x))| \leqq \bar{p}(|t_k|) \leqq \bar{p}(t) = \bar{p}(\sigma(x)).$$

Q.E.D.

Now let us return to the proof of Lemma 7.4. From Lemma 7.7 we can select a polynomial q which satisfies

$$\begin{aligned}
&(\dim \sigma)^{1/2} |E(\psi : \lambda;\ d : x)| \\
&\leqq \int_K q(\sigma(xk)) \cdot e^{-\varrho(H(xk))} dk \\
&= q(\sigma(x)) \int_K e^{-\varrho(H(xk))} dk = q(\sigma(x)) \Xi(x).
\end{aligned}$$

Therefore,

$$\|\hat{f}\|_0 \leqq \int_G |f(g_1 ;\ x;\ g_2)| q(\sigma(x)) \Xi(x)\, dx. \tag{7.15}$$

Clearly, there exist a positive integer n and $c \in \boldsymbol{R}$ ($c \geqq 0$) such that

$$q(\sigma(x)) \leqq c(1+\sigma(x))^n, \qquad x \in G.$$

By (5.3), the right hand side of (7.15) is bounded by

$$cN(r_0) \sup_{x \in G} \{\Xi(x)^{-1} \cdot (1+\sigma(x))^{n+r_0} |f(g_1 ;\ x;\ g_2)|\}.$$

We have dominated $\|\hat{f}\|_0$ by a continuous seminorm on f. Since $C_c^\infty(G)$ is dense in $\mathscr{C}(G)$, this is enough to prove Lemma 7.4. Q.E.D.

8. A sufficient condition for the map $\mathscr{F}$ to be surjective

In this Section we shall find the inversion formula and give a statement about a sufficient condition for $\mathscr{F}$ to be a surjective map from $\mathscr{C}(G)$ onto $\mathscr{C}(\hat{G})$.

Let a be an element in $\mathscr{F}(L^2(G))$, the image of $L^2(G)$ by $\mathscr{F}$. Then there exists a unique function $f \in L^2(G)$ such that $\hat{f} = a$. f is the unique function in $L^2(G)$ such that for every $g \in C_c^\infty(G)$

$$\int_G f(x) g(x) dx = (a,\ \hat{g}),$$

where the latter inner product is that of $L^2(\hat{G})$. We shall write Σ_0 to denote the summation over all σ, (τ_1, i_1) and (τ_2, i_2). Then

$$\begin{aligned}
&(a, \hat{g}) \\
&=\frac{1}{\omega}\sum\nolimits_0\int_{\mathfrak{a}_{\mathfrak{p}}^*}(a(\sigma, \lambda)\Phi_{\tau_2, i_2}, \Phi_{\tau_1, i_1})(\Phi_{\tau_1, i_1}, \hat{g}(\sigma, \lambda)\Phi_{\tau_2, i_2})\beta(\sigma: \lambda)d\lambda \\
&=\frac{1}{\omega}\sum\nolimits_0\int_{\mathfrak{a}_{\mathfrak{p}}^*}\int_G(a(\sigma, \lambda)\Phi_{\tau_2, i_2}, \Phi_{\tau_1, i_1})(\Phi_{\tau_1, i_1}, \pi_{\sigma, \lambda}(x)\Phi_{\tau_2, i_2})\beta(\sigma: \lambda)\overline{g(x)}dxd\lambda.
\end{aligned}$$

Let us assume that the integrals in the last espression are absolutely convergent. This is true for example if $a\in\mathscr{C}(\hat{G})$. Then we may take the integration of G outside. Define a function $\check{a}$ on G by

$$(8.1) \qquad \check{a}(x)=\frac{1}{\omega}\sum\nolimits_0\int_{\mathfrak{a}_{\mathfrak{p}}^*}(a(\sigma, \lambda)\Phi_{\tau_2, i_2}, \Phi_{\tau_1, i_1})(\Phi_{\tau_1, i_1}, \pi_{\sigma, \lambda}(x)\Phi_{\tau_2, i_2})\beta(\sigma: \lambda)d\lambda,$$

as is easily seen, which is equal to

$$(8.2) \qquad \frac{1}{\omega}\sum_{\sigma}\int_{\mathfrak{a}_{\mathfrak{p}}^*}\mathrm{tr}[\pi_{\sigma, \lambda}(x^{-1})a(\sigma, \lambda)]\beta(\sigma: \lambda)d\lambda.$$

Then we have the formula

$$\int_G f(x)\overline{g(x)}dx=\int_G\check{a}(x)\overline{g(x)}dx, \qquad g\in C_c^\infty(G).$$

Since the function g is arbitrary we have

$$f(x)=\check{a}(x) \qquad (a.e.).$$

We call the transform (8.2) the *Fourier inverse transform.*

To prove the surjectivity of the map $\mathscr{F}$ in Theorem 5.1, we have to show that $\check{a}$ is in $\mathscr{C}(G)$. Put

$$\begin{aligned}
h_{\tau_1, i_1, \tau_2, i_2}(\sigma: \lambda)&=(a(\sigma, \lambda)\Phi_{\tau_2, i_2}, \Phi_{\tau_1, i_1}), \\
\phi_{\tau_1, i_1, \tau_2, i_2}(\sigma: \lambda: x)&=(\Phi_{\tau_1, i_1}, \pi_{\sigma, \lambda}(x)\Phi_{\tau_2, i_2}).
\end{aligned}$$

Then, since the sum is absolutely convergent, for any g_1 and g_2 in $\mathfrak{B}$ we have

$$\begin{aligned}
&\omega\check{a}(g_1; x; g_2) \\
&=\sum\nolimits_0\int_{\mathfrak{a}_{\mathfrak{p}}^*}h_{\tau_1, i_1, \tau_2, i_2}(\sigma: \lambda)\phi_{\tau_1, i_1, \tau_2, i_2}(\sigma: \lambda: g_1; x; g_2)\beta(\sigma: \lambda)d\lambda \\
&=\sum_{\tau, \sigma}\sum_{i_1, i_2}\int_{\mathfrak{a}_{\mathfrak{p}}^*}h_{\tau_1, i_1, \tau_2, i_2}(\sigma: \lambda)\phi_{\tau_1, i_1, \tau_2, i_2}(\sigma: \lambda: g_1; x; g_2)\beta(\sigma; \lambda)d\lambda.
\end{aligned}$$

On the other hand it is clear that

$$(t_1t_2)^{-1/2}\left|\int_{\mathfrak{a}_\mathfrak{p}^*} h_{\tau_1,i_1,\tau_2,i_2}(\sigma:\lambda)\phi_{\tau_1,i_1,\tau_2,i_2}(\sigma:\lambda:g_1;x;g_2)\beta(\sigma:\lambda)d\lambda\right|$$

$$\leqq (t_1t_2)^{-1/2}\Big(\sum_{j_1=1}^{t_1}\sum_{j_2=1}^{t_2}\Big|\int_{\mathfrak{a}_\mathfrak{p}^*} h_{\tau_1,i_1,\tau_2,i_2}(\sigma:\lambda)\phi_{\tau_1,j_1,\tau_2,j_2}(\sigma:\lambda:g_1;x;g_2)\cdot$$

$$\cdot\beta(\sigma:\lambda)d\lambda\Big|^2\Big)^{1/2}$$

$$=\left|\int_{\mathfrak{a}_\mathfrak{p}^*} h_{\tau_1,i_1,\tau_2,i_2}(\sigma:\lambda)E(\psi:\lambda:g_1;x;g_2)\beta(\sigma:\lambda)d\lambda\right|.$$

So,

$$|\check{a}(g_1;x;g_2)|$$

$$\leqq (1/\omega)\sum_{\tau,\sigma}(t_1t_2)^{1/2}\sum_{i_1,i_2}\left|\int_{\mathfrak{a}_\mathfrak{p}^*} h_{\tau_1,i_1,\tau_2,i_2}(\sigma:\lambda)E(\psi:\lambda:g_1;x;g_2)\beta(\sigma:\lambda)d\lambda\right|.$$

From the dimension formula (Lemma 2.1), t_1 and t_2 are bounded by certain polynomials of $|\tau_1|$ and $|\tau_2|$ respectively, and $(t_1t_2)^{1/2}$ is also bounded by a polynomial of $|\tau_1|$ and $|\tau_2|$. Moreover, since $a\in\mathscr{C}(\hat{G})$, $h_{\tau_1,i_1,\tau_2,i_2}\in\mathscr{S}(\mathfrak{a}_\mathfrak{p}^*)$ for each $\sigma\in\mathscr{E}_M$.

Now it is easy to check that, to complete the proof of Theorem 5.1 it is enough to prove the following

THEOREM 8.1. *If $\sigma\in\mathscr{E}_M$ and $\psi\in L_\sigma^\tau$ with $\|\psi\|_M=1$, let $E(\psi:\lambda:x)$ be the Eisenstein integral as in (6.6). Then for each g_1, $g_2\in\mathfrak{B}$ and $s\in\boldsymbol{R}$, there exist a finite number of polynomials $p_1, p_{21},\dots,p_{2N}$, q and a finite number of differential operators $d_1,\dots,d_N\in\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^*)$ such that whenever $h\in\mathscr{S}(\mathfrak{a}_\mathfrak{p}^*)$*

$$\sup_{x\in G}\left|\int_{\mathfrak{a}_\mathfrak{p}^*} h(\lambda)E(\psi:\lambda:g_1;x;g_2)\beta(\sigma:\lambda)d\lambda\right|\Xi(x)^{-1}(1+\sigma(x))^s$$

$$\leqq p_1(|\sigma|)q(|\tau|)\sum_{i=1}^N\sup_{\lambda\in\mathfrak{a}_\mathfrak{p}^*}p_{2i}(|\lambda|)|h(\lambda;d_i)|.$$

9. Basic estimates for derivatives

We mention here some estimates for derivatives of matrix elements by Arthur [1].

Let η be the conjugation of $\mathfrak{g}^c$ with respect to the compact real form $\mathfrak{u}=\mathfrak{k}+\sqrt{-1}\mathfrak{p}$. As usual, we obtain a Hermitian scalar product on $\mathfrak{g}^c$ by

$$(X,Y)=-B(X,\eta Y),\quad X,Y\in\mathfrak{g}^c$$

and a Hermitian inner product on $\mathfrak{a}^c$ by restriction. This permits us to define an inner product on the dual space of $\mathfrak{a}^c$. If λ is a complex valued linear function on $\mathfrak{a}^c$, $|\lambda|$ denotes the norm of λ with respect to this inner product.

Suppose τ_1 and τ_2 are in $\mathscr{E}_K$. Write τ as the double representation (τ_1, τ_2) of K. Then

$$|\tau| = |\tau_1| + |\tau_2|,$$

$$\dim \tau = \dim \tau_1 \cdot \dim \tau_2.$$

LEMMA 9.1 ([1, Lemma 13]). *Let π be an irreducible unitary representation of G on the Hilbert space $\mathscr{H}$, with infinitesimal character χ_λ, for a linear function λ on $\mathfrak{a}^c$. Suppose $g_1, g_2 \in \mathfrak{B}$. Then there exist polynomials p and q, independent of λ, such that the following (some what complicated) property is satisfied:*

Whenever Φ_1 and Φ_2 are unit vectors in $\mathscr{H}$ that transform under $\pi|K$ according to the representations τ_1 and τ_2 in $\mathscr{E}_K$ set $f(x) = (\Phi_1, \pi(x)\Phi_2)$. Then there are two sets $\{\Psi_{1\alpha}: 1 \leqq \alpha \leqq t_1\}$, $\{\Psi_{2\beta}: 1 \leqq \beta \leqq t_2\}$ of orthogonal vectors in $\mathscr{H}$, and two sets $\{\tau_{1\alpha}: 1 \leqq \alpha \leqq t_1\}$, $\{\tau_{2\beta}, 1 \leqq \beta \leqq t_2\}$ of representations in $\mathscr{E}_K$ such that

(i) $(\|\Psi_{1\alpha}\| + \|\Psi_{2\beta}\|) \leqq p(|\lambda|) \cdot q(|\tau|)$,

(ii) *$\Psi_{1\alpha}$ and $\Psi_{2\beta}$ transform under $\pi|K$ according to the representations $\tau_{1\alpha}$ and $\tau_{2\beta}$ respectively,*

(iii) $(|\tau_{1\alpha}| + |\tau_{2\beta}|) \leqq q(|\tau|)$,

(iv) $t_1 + t_2 \leqq q(|\tau|)$,

(v) $f(g_1; x; g_2) = \sum_{\alpha=1}^{t_1} \sum_{\beta=1}^{t_2} (\Psi_{1\alpha}, \pi(x)\Psi_{2\beta})$.

For a proof see [1, Section 8].

If $\pi = \pi_{\sigma,\lambda}$ for $\sigma \in \mathscr{E}_M$, choose a real linear function μ on $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{t}}$ associated with σ. The infinitesimal character of $\pi_{\sigma,\lambda}$ is $\chi_{-\mu-i\lambda}$. $\eta(\mu) = -\mu$ and $\eta(i\lambda) = i\lambda$. Therefore

$$|-\mu - i\lambda|^2 = -B(\mu + i\lambda, \eta(\mu + i\lambda)) \tag{9.1}$$
$$= B(\mu, \mu) + B(\lambda, \lambda) = |\sigma|^2 + |\lambda|^2.$$

10. Spherical functions

Suppose $\{\phi_\nu^\tau: \nu \in E, \tau \in T\}$ is a collection of infinitely differentiable τ-spherical functions where τ indexes certain irreducible unitary double representations

$\tau=(\tau_1, \tau_2)$ of K on the finite dimensional Hilbert spaces $V_\tau=V_{\tau_1}\otimes V^*_{\tau_2}$ and ν indexes linear functions from $\mathfrak{a}^c$ to $\boldsymbol{C}$. We have the homomorphism $\chi_\nu\colon \mathfrak{Z}\to\boldsymbol{C}$ defined in § 7. $|\nu|$ denotes the real number $\{-B(\nu, \eta(\nu))\}^{1/2}$. We assume that ϕ^τ_ν satisfies the following conditions:

(i) $$z\phi^\tau_\nu=\chi_\nu(z)\phi^\tau_\nu, \qquad z\in\mathfrak{Z}.$$

(ii) *For any $g_1, g_2\in\mathfrak{B}$ there are polynomials p, q and a number $r\geqq 0$, independent of ν and τ, such that*

$$|\phi^\tau_\nu(x)|\leqq p(|\nu|)q(|\tau|)\Xi(x)(1+\sigma(x))^r, \qquad x\in G, \tag{10.1}$$

where the functions Ξ and σ are defined in § 5.

Now, $\mathfrak{p}$ can be regarded as a Hilbert space with respect to the norm $|X|=B(X, X)^{1/2}$ $(X\in\mathfrak{p})$. Consider the set S^+ of all points $H\in\mathfrak{a}^+_\mathfrak{p}$ with $|H|=1$. Fix $H_0\in S^+$ and let $\mathfrak{m}_1$ be the centralizer of H_0 in $\mathfrak{g}$ and M_1 be the analytic subgroup of G corresponding to $\mathfrak{m}_1$. Let $\mathfrak{l}$ be the centralizer of $\mathfrak{m}_1$ in $\mathfrak{a}_\mathfrak{p}$, Σ the set of all positive restricted roots of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$ and Σ_2 the subset of those $\alpha\in\Sigma$ which do not vanish identically on $\mathfrak{l}$. Put

$$\mathfrak{n}_2=\sum_{\alpha\in\Sigma_2}\mathfrak{g}^\alpha,$$

and

$$\rho_2=\frac{1}{2}\sum_{\alpha\in\Sigma_2}m_\alpha\alpha,$$

where m_α is the number of multiplicity of α.

In [6(m), § 27–§ 30] Harish-Chandra defined for each ϕ^τ_ν a C^∞ function θ which maps M_1 into V_τ. θ is $\bar\tau$-spherical, where $\bar\tau=\tau|M_1$. We shall make two assumptions on the collection of linear functions $\{\nu\colon \nu\in E\}$. It then turns out that there exist polynomials p and q, an open neighborhood U of H_0 in S_+ and positive numbers ε, $r>0$, independent of (ν, τ), such that for each $t\geqq 0$ and $H\in U$,

$$|e^{t\rho_2(H)}\phi^\tau_\nu(\exp tH)-\theta^\tau_\nu(\exp tH)|<p(|\nu|)q(|\tau|)e^{-\varepsilon t}(1+t)^r. \tag{10.2}$$

We shall review Harish-Chandra's work and prove the estimate (10.2) in § 10–§ 13.

LFMMA 10.1. *For any two elements $g, g'\in\mathfrak{B}$, we can choose a finite number of $g_i\in\mathfrak{B}$ $(1\leqq i\leqq p)$ with the following property. If ϕ is a C^∞ τ-spherical function, then*

$$|\phi(g; x; g')|\leqq\sum_{1\leqq j\leqq p}|\phi(x; g_j)|, \qquad x\in G,$$

$$|\phi(g;x;g')|\leqq\sum_{1\leqq j\leqq p}|\phi(g_j;x)|,\qquad x\in G.$$

(See [6(m), Lemma 17].)

Define the convolution $f*g$ ($f\in C_c(G)$, $g\in C(G)$), as usual, by

$$(f*g)(x)=\int_G f(y)g(y^{-1}x)dy,\qquad x\in G.$$

For $f\in C_c(G)$ we denote the support of f by Supp f. Let $\mathfrak{K}$ be the subalgebra of $\mathfrak{B}$ generated by $(1, \mathfrak{k}^c)$. The following lemma is proved by Harish-Chandra [6(m), Theorem 1].

LEMMA 10.2. *Let V be a complex vector space of finite dimension and f a C^∞ function from G to V such that the functions zf ($z\in\mathfrak{Z}\mathfrak{K}$) span a finite dimensional space. Fix a neighborhood U of* 1 *in G and let J be the space of all $\alpha\in C_c^\infty(G)$ such that* Supp $\alpha\subset U$ *and $\alpha(kxk^{-1})=\alpha(x)$($k\in K$, $x\in G$). Then there exists an element $\alpha\in J$ such that $f*\alpha=f$.*

LEMMA 10.3. *For any g_1, $g_2\in\mathfrak{B}$ there exist polynomials p, q and a number $r\geqq 0$, independent of (ν,τ), such that*

$$|\phi_\nu^\tau(g_1;x;g_2)|\leqq p(|\nu|)q(|\tau|)\Xi(x)(1+\sigma(x))^r,\qquad x\in G. \tag{10.3}$$

PROOF. To prove this, by Lemma 10.1, it is sufficient to consider the case $g_1=1$. By Lemma 10.2 we can find a function $\alpha\in C_c^\infty(G)$ such that

$$\phi_\nu^\tau*\alpha=\phi_\nu^\tau,$$

and

$$g\phi_\nu^\tau=\phi_\nu^\tau*(g\alpha),\qquad g\in\mathfrak{B}.$$

Then

$$|\phi_\nu^\tau(x;g)|=|\phi_\nu^\tau*(g\alpha)(x)|=\left|\int_G\phi_\nu^\tau(y)\alpha(y^{-1}x;g)dy\right|.$$

Since there exist polynomials p, q and a number $r'>0$ such that

$$|\phi_\nu^\tau(y)|\leqq p(|\nu|)q(|\tau|)\Xi(y)(1+\sigma(y))^{r'},$$

the last expression is bounded by

$$p(|\nu|)q(|\tau|)\int_G\Xi(y)(1+\sigma(y))^{r'}|\alpha(y^{-1}x;g)|dy$$

$$=p(|\nu|)q(|\tau|)\int_G\Xi(xy^{-1})(1+\sigma(xy^{-1}))^{r'}|(g\alpha)(y)|dy,$$

where we use that g is left invariant. It is known that

$$\sigma(xy^{-1}) \leqq \sigma(x) + \sigma(y^{-1}) = \sigma(x) + \sigma(y).$$

(See [6(m), Lemma 10].)
So, we can select a number $r > 0$ such that

$$(1 + \sigma(xy^{-1}))^{r'} \leqq (1 + \sigma(x))^r (1 + \sigma(y))^r.$$

On the other hand there exists a number $c' > 0$ such that

$$\Xi(xy^{-1}) \leqq c' \Xi(x), \qquad y \in \operatorname{Supp} \alpha.$$

(See [6(1), Lemma 32].) Therefore

$$|\phi_\nu^\tau(x; g)| \leqq c' p(|\nu|) q(|\tau|) \Xi(x) (1 + \sigma(x))^r \int_G (1 + \sigma(y))^r |\alpha(y; g)| dy.$$

If we put

$$c(g) = c' \int_G (1 + \sigma(y))^r |\alpha(y; g)| dy,$$

we have the inequality

$$|\phi_\nu^\tau(x; g)| \leqq c(g) p(|\nu|) q(|\tau|) \Xi(x) (1 + \sigma(x))^r.$$

This proves our assertion. Q.E.D.

11. The functions Φ and Ψ.

Let $\mathfrak{M}_1$ be the universal enveloping algebra of $\mathfrak{m}_1^c$ and $\mathfrak{Z}_1$ the center of $\mathfrak{M}_1$. $\mathfrak{a} = \mathfrak{a}_\mathfrak{k} + \mathfrak{a}_\mathfrak{p}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{m}_1$. Let $W_\mathfrak{g}$, $W_{\mathfrak{m}_1}$ be the Weyl groups of $(\mathfrak{g}, \mathfrak{a})$ and $(\mathfrak{m}_1, \mathfrak{a})$ respectively. Then $W_{\mathfrak{m}_1}$ can be regarded as the subgroup of $W_\mathfrak{g}$, consisting of the elements of $W_\mathfrak{g}$ which are generated by reflections with respect to the roots of $(\mathfrak{g}, \mathfrak{a})$ vanishing at H_0. Let $\mathfrak{A}_\mathfrak{p}$ and $\mathfrak{A}$ be the universal enveloping algebras of $\mathfrak{a}_\mathfrak{p}^c$ and $\mathfrak{a}^c$ respectively. Let $S(\mathfrak{a}^c)$ be the symmetric algebra over $\mathfrak{a}^c$. Let J and J_1 be the subalgebras of $S(\mathfrak{a}^c)$ consisting of the elements in $S(\mathfrak{a}^c)$ which are invariant under $W_\mathfrak{g}$ and $W_{\mathfrak{m}_1}$ respectively. J is contained in J_1. Let P_0 be the set of positive roots $\alpha \in P$ which vanish at H_0 and P_1 the complement of P_0 in P. In § 7 we define the isomorphism

$$\gamma : \mathfrak{Z} \longrightarrow J.$$

On the other hand, for each $z_1 \in \mathfrak{Z}_1$ there exists a unique element $\gamma'_{\mathfrak{m}_1}(z_1) \in \mathfrak{A}$ such that

$$z_1-\gamma'_{\mathfrak{m}_1}(z_1)\in\sum_{\alpha\in P_0}\mathfrak{M}_1X_\alpha.$$

Put

$$\rho_{\mathfrak{m}_1}=\frac{1}{2}\sum_{\alpha\in P_1}\alpha,$$

and define an automorphism $\beta_{\mathfrak{m}_1}$ of $\mathfrak{A}$ by

$$\beta_{\mathfrak{m}_1}(H)=H+\rho_{\mathfrak{m}_1}(H),\qquad H\in\mathfrak{a}^c.$$

Let $\gamma_1=\beta_{\mathfrak{m}_1}^{-1}\circ\gamma'_{\mathfrak{m}_1}$. Then

$$\gamma_1:\mathfrak{Z}_1\longrightarrow J_1$$

is an algebraic isomorphism (surjective) ([6(e), Lemma 19]), and $\mu_0=\gamma_1^{-1}\circ\gamma$ is a homomorphism (injective) of $\mathfrak{Z}$ into $\mathfrak{Z}_1$.

Now we fix $\tau\in\mathscr{E}_K^2$. Let $\mathfrak{U}_\nu$ be the annihilator of ϕ_ν^τ in $\mathfrak{Z}$, and let $\mathfrak{U}_{1\nu}=\mathfrak{Z}_1\cdot\mu_0(\mathfrak{U}_\nu)$. Let $\mathfrak{Z}_1^*$ be the quotient algebra $\mathfrak{Z}_1/\mathfrak{U}_{1\nu}$. We can regard $\mathfrak{Z}_1^*$ as a complex vector space on which there is a natural $\mathfrak{Z}_1$ action.

If $\zeta\in\mathfrak{Z}_1$, let ζ^* be the projection of ζ onto $\mathfrak{Z}_1^*$. Let $\mathfrak{Z}_1^{**}$ be the dual vector space of $\mathfrak{Z}_1^*$. Let $\mathscr{V}_\tau=V_\tau\otimes\mathfrak{Z}_1^{**}$. Make $\mathscr{V}_\tau$ a double K-module by letting K act trivially on $\mathfrak{Z}_1^{**}$. Make it a $\mathfrak{Z}_1$-module by defining

$$\Gamma(z)(v\otimes\zeta^{**})=v\otimes z\zeta^{**},\qquad z\in\mathfrak{Z}_1,\ v\in V_\tau,\ \zeta^{**}\in\mathfrak{Z}_1^{**}.$$

(Since $\mathfrak{Z}_1^*$ is a $\mathfrak{Z}_1$-module, there is a natural action of $\mathfrak{Z}_1$ on $\mathfrak{Z}_1^{**}$ obtained by taking transposes.)

To obtain a basis of $\mathfrak{Z}_1^*$, we examine the algebra J and J_1 more closely. Such results appear in [6(h), § 3]. We identify $S=S(\mathfrak{a}^c)$ with the algebra of polynomial functions on $\mathfrak{a}^{c*}$, the dual space of $\mathfrak{a}^c$. Let $C(S)$, $C(J_1)$ and $C(J)$ be the quotient fields of S, J_1 and J respectively.

LEMMA 11.1. *If $[W_{\mathfrak{g}}: W_{\mathfrak{m}_1}]=r$, then there are homogeneous elements $v_1=1, v_2, \ldots, v_r\in J_1$ such that $J_1=\sum_{1\leq j\leq r}Jv_j$. Moreover, the elements $v_1, \ldots, v_r$ are linearly independent over $C(J)$.*

For a proof see [6(h), Lemma 8].

Now suppose that $\nu\in\mathfrak{a}^{c*}$. Let S_ν be the ideal of polynomial functions in S that vanish at ν. Let $J_\nu=J\cap S_\nu$ and let $J_{1\nu}=J_1\cap S_\nu$. $J=\boldsymbol{C}+J_\nu$ is a vector space decomposition of J and the projection from J onto $J/J_\nu=\boldsymbol{C}$ is given by

$$w\longrightarrow w(\nu),\qquad w\in J.$$

J_1J_ν is an ideal in J_1 and it is clear that $\gamma_{\mathfrak{m}_1}$ defines an isomorphism from $\mathfrak{Z}_1^*$ onto

J_1/J_1J_ν. We shall obtain a basis of J_1/J_1J_ν over the field $\boldsymbol{C}$ of complex numbers.
We have the formula

$$J_1 = \sum_{1\leqq j\leqq r} Jv_j = \sum_j \boldsymbol{C}v_j + \sum_j J_\nu v_j.$$

But $\Sigma_j J_\nu v_j \in J_1J_\nu$, so $\{v_j: 1\leqq j\leqq r\}$ spans J_1/J_1J_ν. On the other hand, suppose $\{c_j\}$ is a set of complex numbers such that the vector $\Sigma_j c_j v_j \in J_1J_\nu$. Now

$$J_1J_\nu = (\sum_j Jv_j)J_\nu = \sum_j J_\nu v_j.$$

However, $\{v_j: 1\leqq j\leqq r\}$ is linearly independent over $C(J)$, so each $c_j \in J_\nu$. This implies that each c_j equals zero. Therefore, $\{v_j: 1\leqq j\leqq r\}$ is a basis for the space J_1/J_1J_ν. Let us regard J_1/J_1J_ν as a Hilbert space with orthonormal basis $\{v_1, \dots, v_r\}$.

Define elements $\eta_1 = 1, \eta_2, \dots, \eta_r \in \mathfrak{Z}_1$ by

$$\gamma_1(\eta_j) = v_j, \qquad 1\leqq j\leqq r.$$

Then $\{\eta_1^*, \dots, \eta_r^*\}$ is a basis for $\mathfrak{Z}_1^*$. Let $\{\eta_1^{**}, \dots, \eta_r^{**}\}$ be the dual basis of $\mathfrak{Z}_1^{**}$. If we make $\mathfrak{Z}_1^{**}$ into a Hilbert space with orthonormal basis $\{\eta_1^{**}, \dots, \eta_r^{**}\}$, we can regard $\mathscr{V}_\tau$ as a Hilbert space.

We defined Σ, Σ_2 and $\mathfrak{l}$ in the previous section. Let Σ_1 be the complement of Σ_2 in Σ, that is, the subset of those $\alpha\in\Sigma$ which vanish identically on $\mathfrak{l}=\mathfrak{a}_2$. Put

$$\mathfrak{n}_j = \sum_{\alpha\in\Sigma_j} \mathfrak{g}^\alpha, \qquad (j=1, 2),$$

where $\mathfrak{g}^\alpha$ is defined by

$$\mathfrak{g}^\alpha = \{X\in\mathfrak{g}: [H, X] = \alpha(H)X \qquad \textit{for all } H\in\mathfrak{a}_\mathfrak{p}\},$$

then $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$. Now M_1 normalizes $\mathfrak{n}_2$. Define a function d on M_1 by

$$d(m) = |\det(Ad(m))_{\mathfrak{n}_2}|^{1/2} \qquad m\in M_1,$$

and put $\zeta' = d^{-1}\zeta\circ d$ for $\zeta\in\mathfrak{Z}_1$. It is easy to verify (see [6(m), Appendix, § 45]) that $\zeta\to\zeta'$ is an automorphism of $\mathfrak{Z}_1$. Now define

$$\Phi(m) = \sum_{1\leqq j\leqq r} \phi_j(m)\otimes\eta_j^{**}, \qquad m\in M_1,$$

where

$$\phi_j(m) = d(m)\phi(m;\, \eta_j').$$

For any $\zeta\in\mathfrak{Z}_1$, there exist unique complex numbers c_{ij} such that

$$u_l(\zeta)=\zeta\eta_l-\sum_{1\leqq j\leqq r} c_{lj}\eta_j\in\mathfrak{U}_{1\nu}, \qquad 1\leqq l\leqq r.$$

Then define

$$\Psi_\zeta(m)=\sum_j d(m)\cdot\phi_\nu^\tau(m;u_j(\zeta)')\otimes\eta_j^{**}, \qquad m\in M_1.$$

Φ and Ψ are functions from M_1 to the vector space $\mathscr{V}_\tau$. Also, Φ and Ψ_ζ are both $\bar\tau$(the restriction of τ to $K_1=M_1\cap K$)-spherical functions on M_1, since elements in $\mathfrak{Z}_1$ act on $C^\infty(M_1)$ as left and right invariant differential operators.

Lemma 11.2. *Let* $\zeta\in\mathfrak{Z}_1$. *Then*

$$\Phi(m;\zeta)=\Gamma(\zeta)\Phi(m)+\Psi_\zeta(m), \qquad m\in M_1.$$

Proof. We have the equation

$$\Phi(m;\zeta)=\sum_l d(m)\phi_\nu^\tau(m;\zeta'\eta_l')\otimes\eta_l^{**}.$$

Also

$$\zeta'\eta_l'=(\zeta\eta_l)'=\sum_j c_{lj}\eta_j'+u_l(\xi)'.$$

Therefore, $\Phi(m;\zeta)$ is equal to

$$\sum_l\sum_j c_{lj}d(m)\phi_\nu^\tau(m;\eta_j')\otimes\eta_l^{**}+\sum_l d(m)\phi_\nu^\tau(m;u_l(\zeta)')\eta_l^{**},$$

which in turn equals

$$\sum_{l,j} c_{lj}\phi_j(m)\otimes\eta_l^{**}+\Psi_\zeta(m).$$

Since $u_l(\zeta)^*=0$, we have the formula

$$(\zeta\eta_l)^*=\sum_j c_{lj}\eta_j^*.$$

Since $\{\eta_l^{**}\}$ is the dual basis of $\{\eta_l^*\}$, the matrix of the linear transformation ζ acting on $\mathfrak{Z}_1^{**}$ is the transposed matrix of its action on $\mathfrak{Z}_1^*$, with respect to these bases. Therefore,

$$\begin{aligned}\Gamma(\zeta)\Phi(m)&=\sum_{l,j} c_{jl}\phi_l(m)\otimes\eta_j^{**}\\&=\sum_{l,j} c_{lj}\phi_j(m)\otimes\eta_j^{**}.\end{aligned}$$

This proves Lemma 11.2. Q.E.D.

Now $\mathfrak{l}$ lies in the center of $\mathfrak{m}_1$. Hence if $H\in\mathfrak{l}$, we conclude from the above lemma that

$$d(e^{-t\Gamma(H)}(m\cdot\exp tH))/dt = e^{-t\Gamma(H)}\Psi_H(m\cdot\exp tH), \qquad t\in \boldsymbol{R},\ m\in M_1.$$

Therefore the following result is now obvious.

COROLLARY. *For any $m\in M_1$, $H\in\mathfrak{l}$ and $T\in\boldsymbol{R}$, we have the integral equation*

$$\Phi(m\cdot\exp TH) = e^{T(H)}\Phi(m) + \int_0^T e^{(T-t)\Gamma(H)}\Psi_H(m\cdot\exp tH)dt.$$

12. Some estimates for Φ and Ψ

We shall obtain some estimates for Φ and Ψ.

Let $K_1 = M_1\cap K$. Let Ξ_1 be the function on M_1 which corresponds to Ξ when (G, K) is replaced by (M_1, K_1). Put $M_1^+ = K_1 A_{\mathfrak{p}}^+ K_1$.

LEMMA 12.1. *For a given $v\in\mathfrak{M}_1$, we can choose polynomials p, q and a real number $r\geqq 0$ such that*

$$|\Phi(m; v)| \leqq p(|\nu|)q(|\tau|)\Xi_1(m)(1+\sigma(m))^r, \qquad m\in M_1^+.$$

PROOF. This lemma follows easily from (10.1) and the definition of Φ. Q.E.D.

Now fix $\zeta\in\mathfrak{Z}_1$. From Lemma 11.1, it follows that there exist elements $w_{lj}\in J$ such that for $1\leqq l\leqq r$,

$$\gamma_1(\zeta)v_l = \sum_j w_{lj}v_j.$$

Now, the coordinates of $\gamma_1(\zeta)v_l$ relative to the basis $\{v_j\}$ of J_1/J_1J_ν are clearly $\{w_{lj}(\nu)\colon 1\leqq j\leqq r\}$. Then the element

$$\gamma_1(u_l(\zeta)) = \gamma_1(\zeta)v_l - \sum_j w_{lj}(\nu)v_j$$

is contained in J_1J_ν. $\gamma_1(u_l(\zeta))$ is equal to

$$\sum_j (w_{lj} - w_{lj}(\nu))v_j.$$

For each l and j, $(w_{lj} - w_{lj}(\nu))\in J_\nu$. Let

$$z_{lj}^\nu = \gamma^{-1}(w_{lj} - w_{lj}(\nu)).$$

Then

$$u_l(\zeta) = \sum_j \mu_0(z_{lj}^\nu)\gamma_1^{-1}(v_j) = \sum_j \mu_0(z_{lj})\eta_j.$$

Let u_{lj}^{ν} be the differential operator given by

$$u_{lj}^{\nu} = -z_{lj}^{\nu} + \mu_0(z_{lj}^{\nu})'.$$

Then u_{lj}^{ν} is an element of $\mathfrak{B}$, and it is independent of ν. So, we write u_{lj}^{ν} simply as u_{lj}. Recall that θ is the Cartan involution of $\mathfrak{g}^c$. In the appendix of [6(m)] it is known that there exist elements $N_{lj} \in \mathfrak{n}_2$, and $g_{lj} \in \mathfrak{B}$, both independent of ν, such that

$$u_{lj} = \theta(N_{lj})g_{lj}, \qquad 1 \leqq l, j \leqq r.$$

Also it is known ([6(m), Appendix]) that $v \to v' = d^{-1} v \circ d (v \in \mathfrak{M}_1)$ gives an automorphism of $\mathfrak{M}_1$ which preserves $\mathfrak{Z}_1$.

It is known ([6(m), Lemma 47 and its corollary]) that there exist numbers c_0, $r_0 > 0$ such that

$$d(\exp H)\Xi(\exp H) \leqq c_0 \Xi_1(\exp H)(1+|H|)^{r_0}, \qquad H \in \mathfrak{a}_{\mathfrak{p}}^+, \tag{12.1}$$

and

$$d(m)\Xi(m) \leqq c_0 \Xi_1(m)(1+\sigma(m))^{r_0}, \qquad m \in M_1^+. \tag{12.2}$$

LEMMA 12.2. *For fixed $\zeta \in \mathfrak{Z}_1$ and $v \in \mathfrak{M}_1$, there are polynomials p, q and an integer $n \geqq 0$, independent of (ν, τ), such that*

$$|\Psi_\zeta(m \exp tH; v)| \leqq p(|\nu|)q(|\tau|)e^{-t\beta(H)}(1+t)^n \Xi_1(m)(1+\sigma(m))^n$$

for every $t \geqq 0$, $H \in S^+ \cap \mathfrak{l}$ and $m \in M_1^+$. Here

$$\beta(H) = \min_{\alpha \in \Sigma_2} \alpha(H).$$

PROOF. We have the equation

$$\begin{aligned}\Psi_\zeta(m \exp tH; v) &= \sum_l d(m \exp tH)\phi_\nu^\tau(m \exp tH; v' u_l(\zeta)') \otimes \eta_l^{**} \\ &= \sum_{l,j} d(m \exp tH)\phi_\nu^\tau(m \exp tH; v' \eta_j' \mu_0(z_{lj}^\nu)) \otimes \eta_l^{**}.\end{aligned}$$

Since $z_{lj}^\nu \in \mathfrak{U}_\nu$, the annihilator of ϕ_ν^τ, this last expression is equal to

$$\begin{aligned}&\sum_{l,j} d(m \exp tH)\phi_\nu^\tau(m \exp tH; u_{lj} v' \eta_j') \otimes \eta_l^{**} \\ &= \sum_{l,j} d(m \exp tH)\phi_\nu^\tau(m \exp tH; \theta(N_{lj}) g_{lj} \eta_j' v') \otimes \eta_l^{**},\end{aligned}$$

here we used that u_{lj}, $\eta_j \in \mathfrak{Z}_1$.

If $N \in \mathfrak{n}_2$, N is written in the form $\sum_{\alpha \in \Sigma_2} X_\alpha$, where $X_\alpha \in \mathfrak{g}^\alpha$. But if $\alpha \in \Sigma_2$,

$$\phi_v^\tau(m\exp tH;\ \theta(X_\alpha)g_{lj}\eta'_j v') = e^{-t\alpha(H)}\phi_v^\tau(\theta(X_\alpha);\ m\exp tH;\ g_{lj}\eta'_j v')$$

for $H \in S^+ \cap \mathfrak{l}$, $t \geqq 0$, $m \in M_1^+$. Our lemma 12.2 now follows from (5.3), (10.1) and (12.2). Q.E.D.

13. The functions Θ and θ

We shall obtain a $\bar{\tau}$-spherical function Θ mapping M_1 into $\mathscr{V}_\tau$, and a $\bar{\tau}$-spherical V_τ-valued function θ on M_1, which is the η_1^{**}-coefficient of Θ. The function θ plays an essential role in the discussion of the induction to prove Theorem 8.1.

Recall $\varpi = \prod_{\alpha \in P} H_\alpha$, which is an element in S. In this section we make the following assumption.

Assumption 1: *For each* $v \in E$, $\varpi(v) \neq 0$.

We would like to find the eigenvalues of the linear transformation H_0 acting on the vector space $J_1/J_1 J_v$. We also want to find the norms of the projections of $J_1/J_1 J_v$ into these eigenspaces. (These are norms as operators on the Hilbert space $J_1/J_1 J_v$; these projections are not necessarily self-adjoint).

The field $C(J_1)$ is an extension of degree r of the field $C(J)$. Therefore the trace $tr_{C(J_1)/C(J)}$ is a function from $C(J_1)$ into $C(J)$. Define an element v^i in $C(J_1)$ by

$$\mathrm{tr}_{C(J_1)/C(J)}(v^i v_j) = \delta^i_j, \qquad 1 \leqq i,\ j \leqq r,$$

($\delta^i_j = 0$ if $i \neq j$, $\delta^i_i = 1$).

Recall that P_0 is the set of all positive roots $\alpha \in P$ which vanish at H_0 and that $\varpi^{\mathfrak{m}_1} = \prod_{\alpha \in P_0} H_\alpha$. ϖ and $\varpi^{\mathfrak{m}_1}$ are both in S.

Define

$$D = \varpi / \varpi^{\mathfrak{m}_1}, \qquad \tau^i = D v^i.$$

Then D and τ^i are both in $C(S)$. In [6(h), Lemma 12] Harish-Chandra shows that τ^i is actually in S.

Let $\{s_1 = 1, s_2, \dots, s_r\}$ be a set of representatives of right cosets of $W_{\mathfrak{m}_1}$ in $W_{\mathfrak{g}}$. If $v \in C(J_1)$, then

$$\mathrm{tr}_{C(J_1)/C(J)}(v) = \sum_{1 \leqq i \leqq r} s_i^{-1}(v). \tag{13.1}$$

However,

$$\sum_{1 \leqq k \leqq r} \tau^i(s_k \nu) v_j(s_k \nu) / D(s_k \nu) = \delta^i_j. \tag{13.2}$$

Difine

$$A_{ik}=\tau^i(s_k\nu),\quad B_{kj}=v_j(s_k\nu)/D(s_k\nu).$$

Then $A=(A_{ik})$ and $B=(B_{kj})$ are both $r\times r$ matrices, and $AB=I$ (identity matrix). Define

$$f_{s_i\nu}=\sum_{1\leqq k\leqq r}\tau^k(s_i\nu)v_k. \tag{13.3}$$

Then $\{f_{s_i\nu}: 1\leqq i\leqq r\}$ is a basis for J_1/J_1J_ν. Also,

$$v_j=\sum_k (v_j(s_k\nu)/D(s_k\nu))f_{s_k\nu}. \tag{13.4}$$

LEMMA 13.1. *If $p\in J_1$, then we have the equation*

$$pf_{s_i\nu}\equiv p(s_i\nu)f_{s_i\nu} \qquad (\mathrm{mod}\, J_1J_\nu).$$

For a proof see [6(h), Lemma 15].

In particular, the operator p on J_1/J_1J_ν is semisimple. Since $H_0\in J_1$, the lemma tells us that the set of eigenvalues of H_0 is

$$Q=\{\nu(H_0),\ \nu(s_2^{-1}H_0),\ldots,\ \nu(s_r^{-1}H_0)\}. \tag{13.5}$$

We can lift J_1 to $\mathfrak{Z}_1$ by γ_1^{-1}. Then if $\zeta\in\mathfrak{Z}_1$, we have an analogous statement to Lemma 17 in [1] for the eigenvalues of the operator $\Gamma(\zeta)$ on $\mathscr{V}_\tau$. In particular, the eigenvalues of $\Gamma(H_0)$ are also given by (13.5).

Let $\{E'_{s_1},\ldots,E'_{s_r}\}$ be the projections in J_1/J_1J_ν relative to the direct decomposition

$$J_1/J_1J_\nu=\boldsymbol{C}f_{s_1\nu}+\cdots+\boldsymbol{C}f_{s_r\nu}.$$

Difine

$$E_{s_l}=(\mathrm{Id})\otimes(\gamma_1^{-1}E'_{s_l}\gamma_1)^*,\qquad 1\leqq l\leqq r,$$

where Id denotes the identity operator on V_τ and the star * denotes the vector space transpose operator. Then the operators $\{E_{s_l}\}$ are the projections of $\mathscr{V}_\tau$ onto the eigenspaces of $\Gamma(\mathfrak{Z}_1)$, and $\Gamma(\zeta)$ ($\zeta\in\mathfrak{Z}_1$) are commutative with the projections E_{s_l}.

If $u\in\mathscr{V}_\tau$, there are elements $u_1,\ldots,u_r\in V_\tau$ such that

$$u=(u_1\otimes\eta_1^{**})+\cdots+(u_r\otimes\eta_r^{**}).$$

Denote u_i by $t_i(u)$.

LEMMA 13.2. *There is a fixed set of elements $\{p_{ij}^l: 1\leqq i, j, l\leqq r\}$ in S,* *independent of* τ *and* ν, *such that for any* $u \in \mathscr{V}_\tau$,

$$t_i(E_{s_l}u) = \sum_{1 \leqq j \leqq r} (p_{ij}^l(\nu)/D(\nu))t_j(u), \qquad u \in \mathscr{V}_\tau.$$

PROOF. (13.4) implies the formula

$$E'_{s_l}v_i = E'_{s_l}\Big(\sum_{1 \leqq j \leqq r} (v_i(s_j\nu)/D(s_j\nu))f_{s_j\nu}\Big)$$

$$=(v_i(s_l\nu)/D(s_l\nu))f_{s_l\nu}.$$

From the definition of $f_{s_l\nu}$ it follows that the last expression is equal to

$$(v_i(s_l\nu)/D(s_l\nu)) \sum_{1 \leqq j \leqq r} \tau^j(s_l\nu)v_j$$

where $D = \varpi/\varpi^{m_1}$ and $\tau^i = Dv^i$. As we mentioned above, it is known that $\tau^i \in S$ $(1 \leqq i \leqq r)$. $D(s_l\nu)$ equals $(\varpi(\nu)/\varpi^{m_1}(\nu))\varepsilon$, where ε equals either 1 or -1. Therefore, our assertion follows. Q.E.D.

Either the set of eigenvalues of the linear transformation H_0 on J_1/J_1J_ν or that of $\Gamma(H_0)$ on $\mathscr{V}_\tau$ is given by (13.5). Let Q^+, Q^- and Q^0 be the subsets of these eigenvalues with real parts greater than, less than and equal to zero respectively. Let E'^+, E'^-, E'^0, and E^+, E^-, E^0 be the corresponding projections in J_1/J_1J_ν and $\mathscr{V}_\tau$ respectively. Put $Q' = Q^+ \cup Q^-$ and let

$$4\varepsilon_0 = \min_{\lambda \in Q'} |\lambda_R|.$$

Set $\varepsilon_0 = 1$ if $Q^+ \cup Q^-$ is empty. Then $\varepsilon_0 > 0$. Put

$$\beta(H) = \min_{\alpha \in \Sigma_2} \alpha(H), \qquad H \in \mathfrak{l}.$$

Then $\beta(H_0) > 0$. Fix a number $\varepsilon (0 < \varepsilon \leqq \varepsilon_0)$ and an open and relatively compact neighborhood Ω of H_0 in $\mathfrak{l}$. By selecting them sufficiently small, we can arrange that

$$\beta(H_0) \geqq 5\varepsilon, \quad 0 < \varepsilon < 1, \qquad (1-\varepsilon)H_0 \in \Omega$$

and

$$\beta(H) \geqq 4\varepsilon, \quad |\Gamma(H) - \Gamma(H_0)| < \varepsilon/2$$

for $H \in \Omega$.

For the remainder of this section and for the following section we make the following assumption.

Assumption 2. *The real parts of all eigenvalues* (13.5), *as* ν *ranges in* E, *are the sum of a finite number of lattices in* $\boldsymbol{R}$.

Lemma 13.3. *Choose a complex number ξ in the resolvent set of the operator H_0. Let d be the distance from ξ to the spectrum of H_0. Then there are polynomials p_1 and p_2, independent of ξ and ν, such that the norm of the operator $(\xi-H_0)^{-1}$ is bounded by $d^{-r}p_1(|\xi|)p_2(|\nu|)$.*

For a proof see [1, Lemma 20].

Lemma 13.4. *There exists a polynomial p, independent of ν, such that*

$$|e^{-t\Gamma(H)}E^+|+|e^{t\Gamma(H)}E^-|\leqq p(|\nu|)e^{-2\varepsilon t}, \qquad t\geqq 0,$$

and

$$|e^{t\Gamma(H)}E^0|\leqq p(|\nu|)e^{\varepsilon t}, \qquad t\in \boldsymbol{R},$$

for $H\in\Omega$.

Proof. First we can prove the lemma for $H=H_0$ as follows, by an argument similar to the one in the proof of [1, Lemma 21]. To do this, it is clearly enough to prove the same statements for the linear transformations $e^{-tH_0}E'^+$, $e^{tH_0}E'^-$ and $e^{tH_0}E'^0$ on the Hilbert space J_1/J_1J_ν.

Let Γ^+, Γ^- and Γ^0 be closed curves in the complex plane that wind around the corresponding sets of eigenvalues Q^+, Q^- and Q^0 in a positive sense, but which contain no other eigenvalues. By looking at Assumption 2 and the eigenvalues (13.5) we see that the curves can be chosen to satisfy the following conditions:

(i) $|\xi|$ is bounded by a polynomial in $|\nu|$ for any ξ on one of the curves.

(ii) The arc length of each of the curves is bounded by a polynomial in $|\nu|$.

(iii) If ξ is on one of the curves, the distance from ξ to the spectrum of H_0 is not less than ε.

(iv) If ξ is a complex number, let $\mathscr{R}\xi$ be its real part. Then

$$\mathscr{R}\xi\geqq 2\varepsilon \qquad \text{for } \xi \text{ on } \Gamma^+,$$

$$\mathscr{R}\xi\leqq -2\varepsilon \qquad \text{for } \xi \text{ on } \Gamma^-,$$

$$|\mathscr{R}\xi|\leqq \varepsilon \qquad \text{for } \xi \text{ on } \Gamma^0.$$

From the spectral theory for a linear transformation on a finite dimensional vector space we have the following formulae.

$$e^{-tH_0}E'^+=\int_{\Gamma^+}e^{-t\xi}(\xi-H_0)^{-1}d\xi, \qquad t\geqq 0,$$

$$e^{tH_0}E'^- =\int_{\Gamma^-}e^{t\xi}(\xi-H_0)^{-1}d\xi, \qquad t\geqq 0,$$

$$e^{tH_0}E'^0 = \int_{\Gamma^0} e^{t\xi}(\xi - H_0)^{-1}d\xi, \qquad t \in \boldsymbol{R}.$$

Therefore if $t \geqq 0$

$$|e^{-tH_0}E'^+| \leqq \int_{\Gamma^+} |e^{-t\xi}| \cdot |(\xi - H_0)^{-1}| d\xi.$$

By Lemma 13.3 and conditions (iii) and (iv), this last expression is bounded by

$$e^{-2\varepsilon t} d^{-r} p_2(|\nu|) \int_{\Gamma^+} p_1(|\xi|) d\xi.$$

Therefore by conditions (i) and (ii) there exists a polynomial p such that

$$|e^{-tH_0}E'^+| \leqq e^{-2\varepsilon t} p(|\nu|), \qquad t \geqq 0.$$

The inequalities for $|e^{tH_0}E'^-|$ and $|e^{tH_0}E'^0|$ follow from the same way.

Next, from the definition it follows that

$$\begin{aligned} |e^{-t\Gamma(H)}E^+| &\leqq |e^{-t(\Gamma(H) - \Gamma(H_0))}| \cdot |e^{-t\Gamma(H_0)}E^+| \\ &\leqq p(|\nu|) e^{-\varepsilon t/2} e^{-2\varepsilon t} \leqq p(|\nu|) e^{-2\varepsilon t}, \qquad t \geqq 0,\ H \in \Omega. \end{aligned}$$

The inequalities for $|e^{t\Gamma(H)}E^-|$ and $|e^{t\Gamma(H)}E^0|$ follow from the same way.

Q.E.D.

Put $\quad \Phi^{\pm}(m) = E^{\pm}\Phi(m), \quad \Phi^0(m) = E^0\Phi(m), \qquad m \in M_1.$

LEMMA 13.5. *For any fixed $\nu \in \mathfrak{M}_1$ there are polynomials p and q, and a real number $d > 0$, independent of (ν, τ), such that*

$$|\Phi^-(m \cdot \exp TH; \upsilon)| \leqq p(|\nu|) q(|\tau|) e^{-\varepsilon T} \Xi_1(m)(1 + \sigma(m))^d$$

for $H \in \Omega$, $T \geqq 0$ and $m \in M_1^+$.

PROOF. $\Gamma(H)$ and E^- commute. Therefore by the corollary to Lemma 11.2, for any $H \in \Omega$ and $T \geqq 0$ we have the equation

$$\Phi^-(m \cdot \exp TH; \upsilon) = e^{T\Gamma(H)} \Phi^-(m; \upsilon) + \int_0^T e^{(T-t)\Gamma(H)} \cdot E^- \Psi_H(m \cdot \exp tH; \upsilon) dt.$$

The first term of this expression is easily handled with the help of Lemma 12.1 and Lemma 13.4. On the other hand, Lemma 13.4 tells us that there exists a polynomial p such that

$$\left| \int_0^T e^{(T-t)\Gamma(H)} \cdot E^- \Psi_H(m \cdot \exp tH; \upsilon) dt \right|$$

$$\leqq p(|v|)\int_0^T e^{-2\varepsilon(T-t)}|\Psi_H(m\cdot\exp tH;\, v)|dt$$

$$\leqq p(|v|)e^{-\varepsilon T}\int_0^{T/2}|\Psi_H(m\cdot\exp tH;\, v)|dt+p(|v|)\int_{T/2}^T|\Psi_H\,(m\cdot\exp tH;\, v)|dt.$$

Our Lemma 13.5 then follows from Lemma 12.2. Q.E.D.

LEMMA 13.6. *For any fixed $v\in\mathfrak{M}_1$ there are polynomials p and q, and a real number $d>0$, independent of (v, τ), such that*

$$|\Phi^+(m\cdot\exp tH;\, v)|\leqq p(|v|)q(|\tau|)e^{-\varepsilon T}\Xi_1(m)(1+\sigma(m))^d$$

for $H\in\Omega$, $T\geqq 0$ and $m\in M_1^+$.

PROOF. By means of a change of variables we can rewrite the integral equation of the corollary to Lemma 11.2. Then for $h\in A_{\mathfrak{p}}$, $H\in\Omega$, $T\geqq 0$ and $m\in M_1$

$$\Phi(mh;\, v)=e^{-T\Gamma(H)}\Phi(mh\cdot\exp TH;\, v)-\int_0^T e^{-t\Gamma(H)}\Psi_H\,(mh\cdot\exp tH;\, v)dt.$$

Operate E^+ on both sides of this equation, and let T approach $+\infty$. Now $|e^{-T\Gamma(H)}E^+|$ decreases exponentially in T. However, by Lemma 12.1 and (5.1) $|\Phi(mh\cdot\exp TH;\, v)|$ is bounded by a polynomial in T. Therefore, the first term of the right hand side of the above equation approaches zero. We have the formula

$$\Phi^+(mh;\, v)=-\int_0^\infty e^{-t\Gamma(H)}E^+\Psi_H\,(mh\cdot\exp tH;\, v)dt.$$

Let h equal $\exp TH$. We obtain the equation

$$\Phi^+(m\cdot\exp tH;\, v)=-\int_T^\infty e^{-(t-T)\Gamma(H)}E^+\Psi_H\,(m\cdot\exp tH;\, v)dt.$$

Now $|e^{-(t-T)\Gamma(H)}E^+|$ is bounded by a polynomial in $|v|$ if $t\geqq T$. Our Lemma 13.6 then follows from Lemma 12.2. Q.E.D.

For $m\in M_1$ and $H\in\Omega$, define

$$\Theta_H(m)=\Phi^0(m)+\int_0^\infty e^{-t\Gamma(H)}E^0\Psi_H(m\cdot\exp tH)dt.$$

(The integral converges absolutely by Lemma 12.2 and Lemma 13.4). It is clear that Θ_H is a C^∞ function on M_1 and

$$\Theta_H(m;\, v)=\Phi^0(m;\, v)+\int_0^\infty e^{-t\Gamma(H)}E^0\Psi_H\,(m\cdot\exp tH;\, v)dt.$$

Moreover, it follows from the corollary to Lemma 11.2 that

$$\Theta_H(m;v)=\lim_{T\to\infty} e^{-T\Gamma(H)}\Phi^0(m\cdot\exp TH;v).$$

Therefore

$$\Theta_H(m\cdot\exp tH)=e^{t\Gamma(H)}\Theta_H(m),\qquad m\in M_1,\ H\in\Omega,\ t\in\boldsymbol{R}. \tag{13.6}$$

We now claim that Θ_H is actually independent of H. Fix $H_1, H_2\in\Omega$ and $m\in M_1$. We can choose ([6(m), Lemma 54]) $T_0\geqq 0$ such that $m\cdot\exp tH\in M_1^+$ for $t\geqq T_0$ and $H\in\Omega$. Put $m_2=m\cdot\exp T_2H_2$ $(T_2\geqq T_0)$. Then by Corollary to Lemma 11.2

$$e^{-\Gamma(T_1H_1)}\Phi^0(m_2\cdot\exp T_1H_1)=\Phi^0(m_2)+\int_0^{T_1} e^{-t\Gamma(H_1)}E^0\Psi_{H_1}(m_2\cdot\exp tH_1)dt,$$

and therefore,

$$e^{-\Gamma(T_1H_1+T_2H_2)}\Phi^0(m\cdot\exp(T_1H_1+T_2H_2))-e^{-\Gamma(T_2H_2)}\Phi^0(m\cdot\exp T_2H_2)$$

$$=\int_0^{T_1} e^{-\Gamma(t_1H_1+T_2H_2)}E^0\Psi_{H_1}(m\cdot\exp(t_1H_1+T_2H_2))dt_1,\qquad T_1\geqq 0.$$

It follows from Lemma 12.2 and Lemma 13.4 that there exist polynomials p, q and a number $d\geqq 0$ such that for $t_1\geqq 0$ and $T_2\geqq T_0$

$$e^{-\Gamma(t_1H_1+T_2H_2)}E^0\Psi_{H_1}(m\cdot\exp(t_1H_1+T_2H_2))|<p(|\nu|)q(|\tau|)e^{-\varepsilon(t_1+T_2-T_0)}.$$

Therefore by making T_1, T_2 tend to $+\infty$, we get

$$\Theta_{H_2}(m)=\lim_{T_1,T_2\to\infty} e^{-\Gamma(T_1H_1+T_2H_2)}\Phi^0(m\cdot\exp(T_1H_1+T_2H_2)).$$

Since the right hand side is symmetrical in H_1, H_2, we conclude that $\Theta_{H_1}(m)=\Theta_{H_2}(m)$, $m\in M_1$.

Hence we may now write Θ instead of Θ_H.

Since Ω is open in $\mathfrak{l}$, every $H\in\mathfrak{l}$ can be written in the form $H=\sum_{1\leqq i\leqq q} t_iH_i(t_i\in\boldsymbol{R}$, $H_i\in\Omega)$. So, from (13.6), we have

$$\Theta(m\cdot\exp H)=e^{\Gamma(H)}\Theta(m),\qquad m\in M_1,\ H\in\mathfrak{l}. \tag{13.7}$$

Since Ψ is $\bar{\tau}$-spherical, and since both the left and right actions of $\bar{\tau}(m)$ on $\mathscr{V}_\tau$ commute with E^0, we have the formula

$$\Theta(k_1mk_2)=\tau(k_1)\Theta(m)\tau(k_2),\qquad k_1, k_2\in K_1,\ m\in M_1. \tag{13.8}$$

If $\zeta\in\mathfrak{Z}_1$

$$\Theta(m;\zeta)=\lim_{t\to\infty} e^{-t\Gamma(H)}\Phi^0(m\cdot\exp tH;\zeta).$$

By Lemma 11.2 this expression is equal to

$$\Gamma(\zeta)\Theta(m)+\lim_{t\to\infty} e^{-t\Gamma(H)}E^0\Psi_\zeta(m\cdot\exp tH).$$

Now, the last term in this formula approaches 0 as t approaches $+\infty$ by Lemma 12.2 and Lemma 13.4. Therefore

$$\Theta(m;\zeta)=\Gamma(\zeta)\Theta(m),\qquad \zeta\in\mathfrak{Z}_1,\ m\in M_1. \tag{13.9}$$

LEMMA 13.7. *For any fixed $v\in\mathfrak{M}_1$ there are polynomials p, q and a number $d>0$, independent of (v,τ), such that*

$$|e^{-t\Gamma(H)}\Phi^0(m\exp tH;v)-\Theta(m;v)|<p(|v|)q(|\tau|)e^{-2\varepsilon t}\Xi_1(m)(1+\sigma(m))^d$$

for $H\in\Omega$, $t\geqq 0$ and $m\in M_1^+$.

PROOF. Using the definition of Θ and the formula (13.7) we see that

$$\Theta(m;v)=e^{-T\Gamma(H)}\Theta(m\exp TH;v)$$

$$=e^{-T\Gamma(H)}\Phi^0(m\exp TH;v)+\int_T^\infty e^{-t\Gamma(H)}E^0\Psi(m\exp tH;v)dt.$$

Lemma 13.7 then follows from Lemma 12.2 and Lemma 13.4. Q.E.D.

COROLLARY. *For any fixed $v\in\mathfrak{M}_1$ there are polynomials p, q and a number $d>0$, independent of (v,τ), such that*

$$|\Phi(m\exp tH;v)-\Theta(m\exp tH;v)|<p(|v|)q(|\tau|)e^{-\varepsilon t}\Xi_1(m)(1+\sigma(m))^d$$

for $H\in\Omega$, $t\geqq 0$ and $m\in M_1$.

PROOF. We see that

$$|\Phi(m\cdot\exp tH;v)-\Theta(m\cdot\exp tH;v)|$$

$$\leqq|\Phi^+(m\cdot\exp tH;v)|+|\Phi^-(m\cdot\exp tH;v)|$$

$$+|e^{t\Gamma(H)}E^0|\cdot|e^{-t\Gamma(H)}\Phi^0(m\cdot\exp tH;v)-\Theta(m;v)|.$$

Our assertion follows from Lemmas 13.4, 13.5, 13.6 and 13.7. Q.E.D.

LEMMA 13.8. *For any fixed $v\in\mathfrak{M}_1$ there are a neighborhood U of H_0 in $cl(\mathfrak{a}_\mathfrak{p}^+)$, polynomials p, q and numbers ε_1, $d_1>0$, independent of (v,τ), such that*

$$|\Phi(\exp tH;v)-\Theta(\exp tH;v)|<p(|v|)q(|\tau|)e^{-\varepsilon_1 t}(1+t)^{d_1}$$

for any $H\in U$ *and* $t\geqq 0$.

PROOF. Recall that ε satisfies $0<\varepsilon<1$. Let U be the set of all $H\in cl(\mathfrak{a}_\mathfrak{p}^+)$ which satisfy $\alpha(H)\geqq(1-\varepsilon)\alpha(H_0)$ for any $\alpha\in P$ and $|H|\leqq(1+\varepsilon)||H_0|$. Then U is a neighborhood of H_0 in $cl(\mathfrak{a}_\mathfrak{p}^+)$. Put $H_\varepsilon=H-(1-\varepsilon)H_0$ for $H\in U$. Then $H_\varepsilon\in cl(\mathfrak{a}_\mathfrak{p}^+)$ and

$$\exp tH=\exp t(1-\varepsilon)H_0\cdot\exp tH_\varepsilon(t\in\boldsymbol{R}).$$

Hence, it follows from the corollary to Lemma 13.7 that there exist polynomials p, q and a number $d\geqq 0$, independent of (ν,τ), such that

$$\begin{aligned}&|\Phi(\exp tH;\,v)-\Theta(\exp tH;\,v)|\\&\quad=|\Phi(\exp tH_\varepsilon\cdot\exp t(1-\varepsilon)H_0;\,v)-\Theta(\exp tH_\varepsilon\cdot\exp t(1-\varepsilon)H_0;\,v)\\&\quad<p(|\nu|)q(|\tau|)e^{-\varepsilon t(1-\varepsilon)}\Xi_1(\exp tH_\varepsilon)(1+t|H_\varepsilon|)^d.\end{aligned}$$

Now put $\varepsilon(1-\varepsilon)=\varepsilon_1$ and remark that

$$\begin{aligned}|H_\varepsilon|&\leqq|H|+(1-\varepsilon)|H_0|\\&\leqq(1+\varepsilon)|H_0|+(1-\varepsilon)|H_0|=2.\end{aligned}$$

Then it follows from the inequality (5.2) that we can select a number $d_1\geqq 0$ so that the last expression of the above inequality is bounded by

$$p(|\nu|)q(|\tau|)e^{-\varepsilon_1 t}(1+t)^{d_1}.$$

Q.E.D.

For $a\in A_\mathfrak{p}$ and $1\leqq j\leqq r$, $t_j(\Theta(a))$ is the vector in V_τ such that

$$\Theta(a)=\sum_{1\leqq j\leqq r}t_j(\Theta(a))\otimes\eta_j^{**}.$$

We put $t_1(\Theta)=\theta$. Recall that $\rho_2=\dfrac{1}{2}\sum_{\alpha\in\Sigma_2}m_\alpha\alpha$.

COROLLARY. *There exist a neighborhood* U *of* H_0 *in* $cl(\mathfrak{a}_\mathfrak{p}^+)$, *polynomials* p, q *and numbers* ε_1, $d_1>0$ *such that*

$$|e^{t\rho_2(H)}\phi_\nu^\tau(\exp tH)-\theta_\nu^\tau(\exp tH)|<p(|\nu|)q(|\tau|)e^{-\varepsilon_1 t}(1+t)^{d_1}$$

for any $H\in U$ *and* $t\geqq 0$.

PROOF. From the definition we have

$$e^{t\rho_2(H)}\phi_\nu^\tau(\exp tH)-\theta_\nu^\tau(\exp tH)$$

$$=t_1(\Phi(\exp tH)-\Theta(\exp tH)).$$

Since for any $u\in\mathscr{V}_\tau$, $|t_1(u)|\leqq|u|$, our Corollary follows from Lemma 13.8.

Q.E.D.

Now put $v=1$ and $t=0$ in Lemma 13.7. Then we conclude from Lemma 12.1 and Lemma 12.2 that there exist polynomials p, q and a number $d_1\geqq 0$, independent of (ν, τ), such that

$$|\Theta(m)|<p(|\nu|)q(|\tau|)\Xi_1(m)(1+\sigma(m))^{d_1},\ m\in M_1^+.$$

On the other hand we can obviously choose a number $\delta_0\geqq 0$ such that

$$|\alpha(\log h)|\leqq\delta_0\sigma(h),\qquad \alpha\in\Sigma_2,\ h\in A_{\mathfrak{p}}.$$

Put $\delta=\max(1, \delta_0/4\varepsilon)$. Then it is clear that $m\exp tH\in M_1^+$ $(m\in M_1,\ H\in\Omega)$ provided $t\geqq\delta\sigma(m)$. Now fix $m_0\in M_1$ and put $t_0=\delta\sigma(m)$ and $m_0=m\exp t_0H_0$. Then $m_0\in M_1^+$ and by (13.6)

$$\Theta(m)=e^{-t_0\Gamma(H_0)}\,\Theta(m_0).$$

Therefore,

$$|\Theta(m)|<p(|\nu|)q(|\tau|)|e^{-t_0\Gamma(H_0)}E^0|\Xi_1(m_0)(1+\sigma(m_0))^{d_1}.$$

But $\Xi_1(m_0)=\Xi_1(m)$, $\sigma(m_0)\leqq\sigma(m)+t_0=(\delta+1)\sigma(m)$ and $\Gamma(H_0)E^0$ has only pure imaginary eigenvalues. Therefore, for a sufficiently large $d\geqq 0$, we have

$$|\Theta(m)|<p(|\nu|)q(|\tau|)\Xi_1(m)(1+\sigma(m))^{d},\quad m\in M_1. \tag{13.10}$$

Form (13.8), (13.9) and (13.10) we obtain the following lemma.

LEMMA 13.9. *Let k_1, $k_2\in K_1$, $m\in M_1$ and $\zeta\in Z_1$. Then*

(i) $\Theta(k_1mk_2)=\tau(k_1)\Theta(m)\tau(k_2)$,

(ii) $\Theta(m;\zeta)=\Gamma(\zeta)\Theta(m)$,

(iii) *there are polynomials p, q and a number $d\geqq 0$, independent of (ν, τ), such that*

$$|\Theta(m)|<p(|\nu|)q(|\tau|)\Xi_1(m)(1+\sigma(m))^{d},\ m\in M_1.$$

For any linear function λ on $\mathfrak{l}^c$, let $\mathscr{V}_\tau(\lambda)$ denote the subspace of all $v\in\mathscr{V}_\tau$ such that

$$(\Gamma(H)-\lambda(H))^r v=0,\qquad H\in\mathfrak{l}^c,$$

where $r=\dim J_1/J_1J_\nu$. Let ${}^0\mathscr{V}_\tau$ denote the sum

$$\sum_{\mathscr{R}\lambda=0} \mathscr{V}_\tau(\lambda),$$

where λ runs over those linear functions which take only pure imaginary values on $\mathfrak{l}$.

LEMMA 13.10. $\qquad \Theta(m)\in {}^0\mathscr{V}_\tau, \qquad m\in M_1.$

PROOF. Since $\Theta(m\exp tH)=e^{t\Gamma(H)}\Theta(m)$ $(H\in\Omega,\ t\in\boldsymbol{R})$, this is obvious from the last statement of Lemma 13.9. Q.E.D.

14. Some estimates for θ

Let τ be an element in $\mathscr{E}_K^2$. In § 10 we put two assumptions (i), (ii) on the spherical function ϕ_ν^τ on G. We put another two assumptions (iii), (iv) on ϕ_ν^τ, which are satisfied for the Eisenstein integrals (see following Section 15), and then we show that the spherical function θ on M_1 is a direct sum of functions $D(\nu)\theta_j$, and that $D(\nu)\theta_j$ are the functions on M_1 corresponding to ϕ_ν^τ on G and satisfy these assumptions. The functions $D(\nu)\theta_j$ are important tools when we use induction to prove the main theorem.

Now we assume that ν is parametrized by a real linear function μ on $\sqrt{-1}\,\mathfrak{a}_\mathfrak{t}$ and a λ in $\mathfrak{a}_\mathfrak{p}^*$ with $\nu=\mu+i\lambda$ and ϕ_ν^τ satisfies the following two conditions:

(iii) *For any fixed real linear function μ on $\sqrt{-1}\,\mathfrak{a}_\mathfrak{t}$, $g_1, g_2\in\mathfrak{B}$ and $x\in G$, the function*

$$\phi_\nu^\tau(g_1;x;g_2)=\phi^\tau(\mu:\lambda:g_1;x;g_2)$$

in λ can be regarded as an entire holomorphic function on $\mathfrak{a}_\mathfrak{p}^{*c}$. We write $\lambda\in\mathfrak{a}_\mathfrak{p}^{*c}$ as $\lambda=\lambda_R+i\lambda_I$, $\lambda_R, \lambda_I\in\mathfrak{a}_\mathfrak{p}^*$.

(iv) *For any $g_1, g_2\in\mathfrak{B}$ and $d\in\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^{*c})$ we can choose polynomials p_1, p_2, q and an integer $n\geqq 0$, independent of (μ, λ, τ), such that*

$$|\phi^\tau(\mu:\lambda;d:g_1;a;g_2)|$$

$$<p_1(|\mu|)p_2(|\lambda|)q(|\tau|)e^{|\lambda_I||\log a|}\Xi(a)\,(1+|\log a|)^n$$

for each $a\in cl(A_\mathfrak{p}^+)$ and $\lambda\in\mathfrak{a}_\mathfrak{p}^{*c}$.

For $u\in\mathscr{V}_\tau$ there are elements $u_1, \dots, u_r\in V_\tau$ such that

$$u=(u_1\otimes\eta_1^{**})+\cdots+(u_r\otimes\eta_r^{**}).$$

Then t_l is defined by $t_l(u)=u_l$. Recall the function Ψ,

$$\Psi(m)=\Psi(\mu:\lambda:m)=\sum_j d(m)\phi^\tau(\mu:\lambda:m;u_j(H_0)')\otimes\eta_j^{**}, \qquad m\in M_1.$$

Then we have

$$t_l(\Psi(\mu:\lambda:m))=d(m)\phi^\tau(\mu:\lambda:m;u_l(H_0)').$$

Let ε and U be the positive number and the neighborhood of H_0 given in § 13, respectively. Recall that ρ_1 is the half of the sum of positive restricted roots which vanish identically on $\mathfrak{l}$.

LEMMA 14.1. *Fix any element $m\in M_1$ and a real linear function μ on $\sqrt{-1}\mathfrak{a}_{\mathfrak{t}}$. Then for each $l(1\leqq l\leqq r)$ the function $t_l(\Psi(\mu:\lambda:m))$ in λ can be regarded as an entire holomorphic function on $\mathfrak{a}_{\mathfrak{p}}^{*c}$. Moreover, for each $d\in D(\mathfrak{a}_{\mathfrak{p}}^{*c})$ there exist polynomials p_1, p_2, q and an integer $n\geqq 0$, independent of (μ, λ, τ), such that*

$$|t_l(\Psi(\mu:\lambda;d:a\exp tH)|$$
$$<p_1(|\mu|)p_2(|\lambda|)q(|\tau|)e^{-(4\varepsilon-|\lambda_I|)t}(1+t)^n e^{|\lambda_I||\log a|}\Xi_1(a)(1+|\log a|)^n$$

for each $a\in cl(A_{\mathfrak{p}}^+)$, $H\in U$, $t\geqq 0$ and $\lambda\in\mathfrak{a}_{\mathfrak{p}}^{*c}$.

PROOF. The first statement of our assertion follows clearly from the assumption (iii) and the formula of $t_l(\Psi(\mu:\lambda:m))$.

Now using the notation of the proof of Lemma 12.2, we see that for every $a\in cl(A_{\mathfrak{p}}^+)$, $H\in U$, $t\geqq 0$ and $1\leqq l\leqq r$,

$$|t_l(\Psi(\mu:\lambda;d:a\exp tH))|$$
$$=|\sum_j e^{t\rho_2(H)+\rho_2(\log a)}\phi^\tau(\mu:\lambda;d:a\exp tH;\theta(N_{lj})g_{lj}\eta'_j)|$$
$$<e^{-t\beta(H)-\beta(\log a)+\rho_2(tH+\log a)}|\sum_j \phi^\tau(\mu:\lambda;d:\theta(N_{lj});a\exp tH;g_{lj}\eta'_j)|.$$

By the assumption (iv) we can choose polynomials p_1, p_2, q and an integer $n'\geqq 0$, independent of (μ, λ, τ), so that

$$|\sum_j \phi^\tau(\mu:\lambda;d:\theta(N_{lj});a\exp tH;g_{lj}\eta'_j)|$$

is bounded by

$$p_1(|\mu|)p_2(|\lambda|)q(|\tau|)e^{|\lambda_I|(|\log a|+t)}\Xi(a\exp tH)(1+|\log a+tH|)^{n'}.$$

Therefore, by (5.2) we can choose an integer $n\geqq 0$ such that

$$|t_l(\Psi(\mu:\lambda;d:a\exp tH))|$$
$$\leqq p_1(|\mu|)p_2(|\lambda|)q(|\tau|)e^{-t(\beta(H)-|\lambda_I|)+|\lambda_I||\log a|}(1+t)^n\cdot e^{-\rho_1(\log a)}(1+|\log a|)^n.$$

From the definition of ε and the relation for M_1 corresponding to (5.2) (see [6(m), Appendix]) it follows that the last expression is bounded by

$$p_1(|\mu|)p_2(|\lambda|)q(|\tau|)\, e^{-(4\varepsilon-|\lambda_I|)t}(1+t)^n e^{|\lambda_1||\log a|}\, \Xi_1(a)(1+|\log a|)^n.$$

This proves our Lemma 14.1. Q.E.D.

Recall that E_{s_j} is the projection of $\mathscr{V}_\tau$ into the eigenspace of $\Gamma(\mathfrak{Z}_1)$ and that $D=\varpi/\varpi^{\mathfrak{m}_1}$. By definition

$$\Theta(\mu\colon \lambda\colon a)=E^0\Phi(\mu\colon \lambda\colon a)+\int_0^\infty e^{-t\Gamma(H_0)}E^0\Psi(\mu\colon \lambda\colon a\exp tH_0))dt.$$

We write $E_{s_j}\Theta$ as Θ_j.

LEMMA 14.2. *Fix any element $m\in M_1$ and a real linear function μ on $\sqrt{-1}\mathfrak{a}_{\mathfrak{t}}$. Then for each $l(1\leqq l\leqq r)$ the function $D(\mu+i\lambda)t_l(\Theta_j(\mu\colon \lambda\colon m))$ in λ is entire holomorphic on $\mathfrak{a}_{\mathfrak{p}}^{*c}$. Moreover, for each $d\in \boldsymbol{D}(\mathfrak{a}_{\mathfrak{p}}^{*c})$ there exist polynomials p_1, p_2, q and an integer $n\geqq 0$, independent of (μ, λ, τ), such that*

$$|d_\lambda[D(\mu+i\lambda)t_l\Theta_j(\mu\colon \lambda\colon a)]|$$
$$<p_1(|\mu|)p_2(|\lambda|)q(|\tau|)\, e^{|\lambda_I||\log a|}\, \Xi_1(a)(1+|\log a|)^n$$

for each $a\in cl(A_{\mathfrak{p}}^+)$ and $\lambda\in\mathfrak{a}_{\mathfrak{p}}^{*c}$.

PROOF. E_{s_j} is the projection of $\mathscr{V}_\tau$ onto the eigenspace of $\Gamma(H_0)$ corresponding to the eigenvalue $<s_j(\mu+i\lambda), H_0>$. If the eigenvalue $<s_j(\mu+i\lambda), H_0>$ is not pure imaginary for $\lambda\in\mathfrak{a}_{\mathfrak{p}}^*$ then by definition $E_{s_j}E^0=0$, and if pure imaginary then $E_{s_j}E^0=E_{s_j}$. So, to prove our lemma, it is sufficient to consider the latter case. Now if $1\leqq l\leqq r$ then

$$t_l(\Phi(\mu\colon \lambda;\, d\colon a))=d(a)\phi^\tau(\mu\colon \lambda;\, d\colon a;\, \eta_l').$$

By Lemma 13.2 and the assumption (iv), $D(\mu+i\lambda)t_l(E_{s_j}\Phi(\mu\colon \lambda\colon a))$ can be clearly regarded as an entire holomorphic function on $\mathfrak{a}_{\mathfrak{p}}^{*c}$ and for any $d\in\boldsymbol{D}(\mathfrak{a}_{\mathfrak{p}}^{*c})$ there are polynomials p_1, p_2, q and an integer $n'\geqq 0$, independent of (μ, λ, τ), such that for every $a\in cl(A_{\mathfrak{p}}^+)$ and $\lambda\in\mathfrak{a}_{\mathfrak{p}}^{*c}$

$$|d_\lambda[D(\mu+i\lambda)t_l(E_{s_j}\Phi(\mu\colon \lambda\colon a))]|$$
$$<p_1(|\mu|)p_2(|\lambda|)q(|\tau|)\, e^{|\lambda_I||\log a|}\Xi(a)(1+|\log a|)^{n'}d(a).$$

From the formula (5.2), it follows immediately that there exist numbers c_0, $r_0>0$ such that

$$d(a)\Xi(a)\leqq c_0\Xi_1(a)(1+|\log a|)^{r_0},\qquad a\in cl(A_p^+).$$

Hence we can choose polynomials p_1, p_2, q and an integer $n \geqq 0$, independent of (μ, λ, τ), such that for every $a \in cl(A_{\mathfrak{p}}^+)$ and $\lambda \in \mathfrak{a}_{\mathfrak{p}}^{*c}$

$$|d_\lambda[D(\mu+i\lambda)t_l(E_{s_j}\Phi(\mu: \lambda: a))]|$$

$$< p_1(|\mu|)p_2(|\lambda|)q(|\tau|)\, e^{|\lambda_I||\log a|}\, \Xi_1(a)(1++|\log a|)^n.$$

On the other hand, by the last lemma for any $d' \in \boldsymbol{D}(\mathfrak{a}_{\mathfrak{p}}^{*c})$ we can also choose polynomials p_1, p_2, q and an integer $n \geqq 0$, independent of (μ, λ, τ), such that

$$|e^{<s_j(\mu+i\lambda), H_0>}\, t_l(\Psi(\mu: \lambda;\, d': a\exp tH_0))|$$

$$< p_1(|\mu|)p_2(|\lambda|)q(|\tau|)\, e^{-(4\varepsilon-|\lambda_I|)t}(1+t)^n e^{|\lambda_I||\log a|-\rho_1(\log a)}(1+|\log a|)^n$$

for $a \in cl(A_{\mathfrak{p}}^+)$ and $\lambda \in \mathfrak{a}_{\mathfrak{p}}^{*c}$. Therefore, the integral

$$\int_0^\infty e^{<s_j(\mu+i\lambda), H_0>}\, t_1(\Psi(\mu: \lambda;\, d': a\exp tH_0))dt$$

converges uniformly for λ in compact sets in $\mathfrak{a}_{\mathfrak{p}}^{*c}$, and so is an entire holomorphic function on $\mathfrak{a}_{\mathfrak{p}}^{*c}$. By Lemma 13.2 we can choose polynomials p_1, p_2, q and an integer n, independent of (μ, λ, τ), such that

$$|d\lambda[D(\mu \pm i\lambda)E_{s_j}\int_0^\infty e^{-t\Gamma(H_0)}\, t_l(E^0\Psi(\mu: \lambda: a\exp tH_0))dt]|$$

$$< p_1(|\mu|)p_2(|\lambda|)q(|\tau|)\, e^{|\lambda_I||\log a|}\, \Xi_1(a)(1+|\log a|)^n$$

for each $a \in cl(A_{\mathfrak{p}}^+)$, $\lambda \in \mathfrak{a}_{\mathfrak{p}}^{*c}$ and $t \geqq 0$. Now our Lemma 14.2 follows immediately. Q.E.D.

Now we put

$$t_1\Theta(\mu: \lambda :m) = \theta(m),$$

$$t_1\Theta_j(\mu: \lambda: m) = \theta_j(m),\ m \in M_1.$$

Then

$$\theta = \theta_1 + \cdots + \theta_r \tag{14.1}$$

and the following corollary is obvious.

COROLLARY 1. *Fix any element* $m \in M_1$ *and a real linear function* μ *on* $\sqrt{-1}\,\mathfrak{a}_t$. *Then the function* $D(\mu+i\lambda)\theta_j(\mu: \lambda: m)$ *in* λ *can be regarded as an entire holomorphic function on* $\mathfrak{a}_{\mathfrak{p}}^{*c}$. *Moreover, for each* $d \in \boldsymbol{D}(\mathfrak{a}_{\mathfrak{p}}^{*c})$ *there exist polynomials* p_1, p_2, q *and an integer* $n \geqq 0$, *independent of* (μ, λ, τ), *such that*

$$|d_\lambda[D(\mu+i\lambda)\theta_j(\mu:\lambda:a)]|$$
$$<p_1(|\mu|)p_2(|\lambda|)q(|\tau|)e^{|\lambda_I||\log a|}\Xi_1(a)(1+|\log a|)^n$$

for each $a\in cl(A_\mathfrak{p}^+)$ *and* $\lambda\in\mathfrak{a}_\mathfrak{p}^{*c}$.

In particular, if we put

$$\Gamma_j(\mu:\lambda)=D(\mu+i\lambda)\theta_j(\mu:\lambda:1)$$

we have the following

COROLLARY 2. *Fix a real linear function* μ *on* $\sqrt{-1}\mathfrak{a}_\mathfrak{k}$. *Then the function* $\Gamma_j(\mu:\lambda)$ *in* λ *can be regarded as an entire holomorphic function on* $\mathfrak{a}_\mathfrak{p}^{*c}$. *Moreover, for any* $d\in\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^{*c})$ *there exist polynomials* p_1, p_2 *and* q, *independent of* (μ,λ,τ), *such that*

$$|\Gamma_j(\mu:\lambda;d)|<p_1(|\mu|)p_2(|\lambda|)q(|\tau|)$$

for any $\lambda\in\mathfrak{a}_\mathfrak{p}^{*c}$.

Recall that $\bar\tau$ is the restriction of τ to K_1. By Lemma 13.9, Θ is $\bar\tau$-spherical. Since the action of $\bar\tau(K_1)$ and $\Gamma(\mathfrak{Z}_1)$ on the vector space $\mathscr{V}_\tau$ commute and the Θ_j are eigenvectors of $\Gamma(\zeta)$, each Θ_j is a $\bar\tau$-spherical function. By definition $\mathscr{V}_\tau=V_\tau\otimes\mathfrak{Z}_1^{**}$ is a double K-module so that K acts trivially on $\mathfrak{Z}_1^{**}$. So the maps $t_l(\mathscr{V}_\tau\to V_\tau(1\leqq l\leqq r))$ and the action of $\bar\tau(K_1)$ also commute. Therefore the function $D(\mu+i\lambda)\theta_j(\mu:\lambda:m)$ on M_1 is $\bar\tau$-spherical and it is clear that for any $d\in\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^{*c})$ the function $d_\lambda[D(\mu+i\lambda)\theta_j(\mu:\lambda:m)]$ on M_1 is also $\bar\tau$-spherical. This being so, from Corollary 1 it follows that for any $d\in\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^{*c})$ we can choose polynomials p_1, p_2, q and an integer $n\geqq 0$, independent of (μ,λ,τ), such that for any $m\in M_1$ and $\lambda\in\mathfrak{a}_\mathfrak{p}^{*c}$

$$|d_\lambda[D(\mu+i\lambda)\theta_j(\mu:\lambda:m)]|<p_1(|\mu|)p_2(|\lambda|)q(|\tau|)e^{|\lambda_I||\log a|}\Xi_1(a)(1+|\log a|)^n,$$

where $m=k_1ak_2$, k_1, $k_2\in K_1$ and $a\in cl(A_\mathfrak{p}^+)$. Clearly, for fixed v_1, $v_2\in M_1$ and $m\in M_1$ the function $D(\mu+i\lambda)\theta_j(\mu:\lambda:v_1;m;v_2)$ is entire holomorphic on $\mathfrak{a}_\mathfrak{p}^{*c}$. By an argument similar to the proof of Lemma 13.2 it can be proved easily that for any $d\in\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^{*c})$ and v_1, $v_2\in\mathfrak{M}_1$ we can choose polynomials p_1, p_2, q and an integer $n\geqq 0$, independent of (μ,λ,τ), such that

$$|d_\lambda[D(\mu+i\lambda)\theta_j(\mu:\lambda:v_1;a;v_2)]|$$
$$<p_1(|\mu|)p_2(|\lambda|)q(|\tau|)e^{|\lambda_I||\log a|}\Xi_1(a)(1+|\log a|)^n$$

for every $a\in cl(A_\mathfrak{p}^+)$ and $\lambda\in\mathfrak{a}_\mathfrak{p}^{*c}$. Thus we obtain the following

LEMMA 14.3. *Fix a real linear function* μ *on* $\sqrt{-1}\mathfrak{a}_{\mathfrak{t}}$. *Then the function* $D(\mu+i\lambda)\theta_j(\mu: \lambda: m)$ *on* M_1 *is* $\bar{\tau}$-spherical and satisfies the following conditions:

(i) $D(\mu+i\lambda)\theta_j(\mu: \lambda: m; \zeta)=\langle\gamma_1(\zeta), s_j(\mu+i\lambda)\rangle D(\mu+i\lambda)\theta_j(\mu: \lambda: m)$ *for* $m\in M_1$, $\lambda\in\mathfrak{a}_{\mathfrak{p}}^{*c}$ *and* $\zeta\in\mathfrak{Z}_1$.

(ii) *For any fixed* $v_1, v_2\in\mathfrak{M}_1$ *and* $m\in M_1$ *the function* $D(\mu+i\lambda)\theta_j(\mu: \lambda: v_1; m; v_2)$ *in* λ *is entire holomorphic on* $\mathfrak{a}_{\mathfrak{p}}^{*c}$. *Moreover for any* v_1, $v_2\in\mathfrak{M}_1$ *and* $d\in D(\mathfrak{a}_{\mathfrak{p}}^{*c})$ *we can choose polynomials* p_1, p_2, q *and an integer* $n\geqq 0$, *independent of* (μ, λ, τ), *such that*

$$|d_\lambda[D(\mu+i\lambda)\theta_j(\mu: \lambda: v_1; a; v_2)]| < p_1(|\mu|)p_2(|\lambda|)q(|\tau|)e^{|\lambda_I||\log a|}\Xi_1(a)(1+|\log a|)^n$$

for any $a\in cl(A_{\mathfrak{p}}^+)$ *and* $\lambda\in\mathfrak{a}_{\mathfrak{p}}^{*c}$.

15. Application to the continuous series

Let τ be an irreducible unitary double representation of K on the Hilbert space V_τ. Recall from § 6 that

$$L^\tau=\bigoplus_{\sigma\in\mathscr{E}_M} L_\sigma^\tau$$

is an orthogonal direct sum. For any fixed τ, $L_\sigma^\tau=0$ for all but a finite number of σ.

We would like to apply the results of the previous sections to the collection $\{E(\psi_\sigma^\tau: \lambda: x)\}$ of Eisenstein integrals. λ is to range over $\mathfrak{a}_{\mathfrak{p}}^{*\prime}$, and τ and σ will range over $\mathscr{E}_K^2$ and $\mathscr{E}_M$ respectively. ψ_σ^τ will be any unit vector in L_σ^τ.

We will have to check the estimate (10.1), Assumptions 1 and 2 of § 13 and the conditions (iii) and (iv) in § 14 for our collection $\{E(\psi_\sigma^\tau: \lambda: x)\}$.

LEMMA 15.1. *Fix* $g_1, g_2\in\mathfrak{B}$. *Then for fixed* $x\in G$, $E(\psi_\sigma^\tau: \lambda: g_1; x; g_2)$ *can be regarded as an entire holomorphic function of* λ. *Also, there are polynomials* p_1, p_2 *and* q, *dependent only on* g_1 *and* g_2, *and for any* $d\in\boldsymbol{D}(\mathfrak{a}_{\mathfrak{p}}^{*c})$ *there exists a polynomial* p, *dependent only on* d, *such that*

$$|E(\psi_\sigma^\tau: \lambda; d: g_1; a; g_2)| < p_1(|\sigma|)p_2(|\lambda|)e^{|\lambda_I||\log a|}\Xi(a)p(|\log a|)$$

for any $a\in cl(A_{\mathfrak{p}}^+)$.

PROOF. We have the formula (6.6):

$$E(\psi_\sigma^\tau: \lambda: x)=\int_K \psi_\sigma^\tau(xk)\tau(k^{-1})e^{(i\lambda-\rho)(H(xk))}dk.$$

For a fixed x, this is clearly an entire function of λ. Also, derivatives of $E(\psi_\sigma^\tau: \lambda: x)$ with respect to left or right invariant differential operators are entire functions in λ.

From (6.7) we obtain the formula

$$E(\psi_\sigma^\tau: \lambda; d: x)=(t_1t_2)^{1/2}\sum_{i,j}\xi_{1i}\otimes\xi_{2j}^*[d_\lambda(\Phi_{1i}, \pi_{\sigma,\lambda}(x)\Phi_{2j})],$$

where t_1t_2 is the dimension of the representation τ. We apply Lemma 9.1 to each of the functions $(\Phi_{1i}, \pi_{\sigma,\lambda}(x)\Phi_{2j})$. As a result we obtain polynomials p_1, q, orthogonal sets of vectors

$$\{\Psi_{1\alpha}^i: 1\leqq\alpha\leqq t_1\}, \{\Psi_{2\beta}^j: 1\leqq\beta\leqq t_2\}, \qquad (1\leqq i\leqq t_1, 1\leqq j\leqq t_2)$$

in $\mathscr{H}_{\sigma,\lambda}$, and the representations

$$\{\tau_{1\alpha}^i: 1\leqq\alpha\leqq t_1\}, \{\tau_{2\beta}^j: 1\leqq\beta\leqq t_2\}, \qquad (1\leqq i\leqq t_1, 1\leqq j\leqq t_2),$$

in $\mathscr{E}_K$ that satisfy the conditions of Lemma 9.1. In addition

$$|E(\psi_\sigma^\tau: \lambda; d: g_1: x; g_2)|^2=\sum_{i,\alpha}\sum_{j,\beta}|d_\lambda(\Psi_{1\alpha}^i, \pi_{\sigma,\lambda}(x)\Psi_{2\beta}^j)|^2. \tag{15.1}$$

For any (i, j, α, β), the vectors $\Psi_{1\alpha}^i$ and $\Psi_{2\beta}^j$ transform under $\pi_{\sigma,\lambda}|K$ according to the representations $\tau_{1\alpha}^i$ and $\tau_{2\beta}^j$ respectively. Let $\tau(i, j, \alpha, \beta)$ be the double representation $(\tau_{1\alpha}^i, \tau_{2\beta}^j)$ of K. Fix a vector $\psi_{\alpha\beta}^{ij}$ in $L_\sigma^{\tau(i,j,\alpha,\beta)}$ such that $E(\psi_{\alpha\beta}^{ij}: \lambda: x)$ is the $\tau(i, j, \alpha, \beta)$-spherical function associated with the function $(\Psi_{1\alpha}^i, \pi_{\sigma,\lambda}(x)\Psi_{2\beta}^j)$. By (6.6) we have that for $H\in S^+$ and $t\geqq 0$

$$\begin{aligned}
&|d_\lambda(\Psi_{1\alpha}^i, \pi_{\sigma,\lambda}(\exp tH)\Psi_{2\beta}^j)|\\
&\quad\leqq(\dim\tau(i, j, \alpha, \beta))^{1/2}|E(\psi_{\alpha\beta}^{ij}: \lambda; d: \exp tH)|\\
&\quad=(\dim\tau(i, j, \alpha, \beta))^{1/2}\left|d_\lambda\int_K\psi_{\alpha\beta}^{ij}(\exp tH\cdot k)\tau(i, j, \alpha, \beta: k^{-1})\cdot\right.\\
&\qquad\left.\cdot e^{(i\lambda-\rho)(H(\exp tH\cdot k))}dk\right|,
\end{aligned}$$

and this expression equals

$$\begin{aligned}
&(\dim\tau(i, j, \alpha, \beta))^{1/2}\left|\int_K\psi_{\alpha\beta}^{ij}(\exp tH\cdot k)\tau(i, j, \alpha, \beta: k^{-1})p'(\lambda: H(\exp tH\cdot k))\cdot\right.\\
&\qquad\left.\cdot e^{(i\lambda-\rho)(H(\exp tH\cdot k)}dk\right|,
\end{aligned}$$

where p' is a function of λ and $H(\exp tH\cdot k)$, and dependent only on d. Now there exist a number $t_k\geqq 0$, $s\in W(W$ is the little Weyl group) and $H_1\in S^+$ such that

$$H(\exp tH\cdot k)=t_k(sH_1).$$

Since

$$\sigma(a)\leqq\sigma(an), \qquad a\in A_{\mathfrak{p}},\ n\in N.$$

we have

$$|t_k|\leqq t, \qquad t\geqq 0,\ k\in K.$$

Therefore we can choose polynomials p_2 and p, dependent only on d, such that

$$|p'(\lambda\colon H(\exp tH\cdot k)|)<p_2(|\lambda|)p(t).$$

On the other hand we have that

$$|e^{(i\lambda-\rho)(H(\exp tH\cdot k))}|=e^{|\lambda_I|t-\rho(H(\exp tH\cdot k))}$$

and that

$$|\psi^{ij}_{\alpha\beta}(\exp tH\cdot k)\tau(i,\ j,\ \alpha,\ \beta\colon k^{-1})|=|\psi^{ij}_{\alpha\beta}(1)|.$$

Therefore, for each $t\geqq 0$, we have the inequality

$$\begin{aligned}&|d_\lambda(\Psi^i_{1\alpha},\ \pi_{\sigma,\lambda}(\exp tH)\Psi^j_{2\beta})|\\ &\leqq(\dim\tau(i,\ j,\ \alpha,\ \beta))^{1/2}|\psi^{ij}_{\alpha\beta}(1)|e^{|\lambda_I|t}p(t)\int_K e^{-\rho(H(\exp tH\cdot k))}dk\\ &=(\dim\tau(i,\ j,\ \alpha,\ \beta))^{1/2}|\psi^{ij}_{\alpha\beta}(1)|e^{|\lambda_I|t}p(t)\,\Xi(\exp tH).\end{aligned}$$

But $|\psi^{ij}_{\alpha\beta}(1)|=\|\psi^{ij}_{\alpha\beta}\|_M$. From the remarks in §6 we see that

$$\|\psi^{ij}_{\alpha\beta}\|_M=(\dim\sigma)^{-1/2}\|\Psi^i_{1\alpha}\|\|\Psi^j_{2\beta}\|.$$

However, by Lemma 9.1 and formula (9.1),

$$\|\Psi^i_{1\alpha}\|^2\|\Psi^j_{2\beta}\|^2\leqq p'_1(((|\sigma|^2+|\lambda|^2)^{1/2})^2\cdot q'(|\tau|)^2.$$

Now $\dim\tau(i, j, \alpha, \beta)$ and $\dim\sigma$ are bounded by polynomials in $|\tau(i, j, \alpha, \beta)|$ and $|\sigma|$ respectively, by the Weyl dimension formula. But Lemma 9.1 $|\tau(i, j, \alpha, \beta)|$, t_1 and t_2 are all bounded by polynomials in $|\tau|$. The inequality in our Lemma 15.1 then follows from (15.1). Q.E.D.

Recall the definitions of $\varpi^{\mathfrak{m}_1}$, Π and Π' in §2. Let $\tilde{\Pi}$ be the lattice of real linear functions on $\sqrt{-1}\mathfrak{a}_{\mathfrak{k}}$ which is generated by the roots of $(\mathfrak{m}_1, \mathfrak{a}_{\mathfrak{k}})$. Then $\tilde{\Pi}\subset\Pi$ and $\tilde{\Pi}$ is a lattice of finite index in Π. We put

$$\varpi_1=\prod_{\alpha\in P_+}H_\alpha=\varpi/\varpi^{\mathfrak{m}_1}.$$

LEMMA 15.2. *There is a number $\delta_1 > 0$ such that for any $\mu \in \Pi'$ the function*

$$\varpi_1(\lambda)\varpi(-\mu - i\lambda)^{-1}$$

is holomorphic in the region $|\lambda_I| < \delta_1$. In addition, there exists a polynomial p, independent of μ, such that for $\mu \in \Pi'$ and $|\lambda_I| < \delta_1$

$$|\varpi_1(\lambda)\varpi(-\mu - i\lambda)^{-1}| < p(|\lambda|).$$

PROOF. We have the formula

$$\varpi(-\mu - i\lambda)^{-1} = \varpi^{\mathfrak{m}_1}(-\mu)^{-1} \prod_{\alpha \in P_+} <-\mu - i\lambda, H_\alpha>^{-1}.$$

$\varpi_1(\lambda)\varpi(-\mu - i\lambda)$ is a meromorphic function in λ.

Since $\mu \in \Pi'$, $\varpi^{\mathfrak{m}_1}(\mu)$ is not equal to zero. The numbers $\{|\varpi^{\mathfrak{m}_1}(\mu)| : \mu \in \Pi'\}$ are actually bounded away from zero. In fact, it is known that $\Pi / \tilde{\Pi}$ is isomorphic to the center of M, which is finite. For any $\alpha \in P_M \{\tilde{\mu}(H_\alpha) : \mu \in \Pi\}$ is a sum of a finite number of lattices in $\boldsymbol{R}$, and hence $\{\mu(H_\alpha) : \mu \in \Pi\}$ is also so. Since, for $\mu \in \Pi'$, $\mu(H_\alpha) \neq 0$, $\{\varpi^{\mathfrak{m}_1}(-\mu) : \mu \in \Pi'\}$ is bounded away from zero.

If $\alpha \in P_+$, then the number $<-\mu - i\lambda, H_\alpha>$ equals zero only if

$$\lambda(H_\alpha) = -i\mu(H_\alpha).$$

Since $\{\tilde{\mu}(H_\alpha) : \tilde{\mu} \in \tilde{\Pi}\}$ is a sum of a finite number of lattices in $\boldsymbol{R}$ and $\tilde{\Pi}$ is of finite index in Π, $\{\mu(H_\alpha) : \mu \in \Pi\}$ is also a sum of a finite number of lattices in $\boldsymbol{R}$. Now let ε_α be the positive generators of them. Put

$$\delta_1' = \inf_{\alpha \in P_+} \left\{ \frac{1}{2} \varepsilon_\alpha \right\}.$$

Then for any $\alpha \in P_+$ and $\mu \in \Pi'$,

$$\text{either} \qquad \mu(H_\alpha) = 0$$

$$\text{or} \qquad |\mu(H_\alpha)| \geqq 2\delta_1'.$$

In either case, we can choose a sufficiently small $\delta_1 > 0$ such that

$$\alpha(\lambda) <-\mu - i\lambda, H_\alpha>^{-1}$$

is holomorphic in the region $|\lambda_I| < \delta_1$, and it is bounded by a polynomial in $|\lambda|$ and its bound is independent of μ. Our Lemma 15.2 follows immediately.

Q.E.D.

COROLLARY. *If $\mu \in \Pi'$ and $\lambda \in \mathfrak{a}_\mathfrak{p}^{*\prime}$ then $\varpi(-\mu - i\lambda)$ is not equal to zero.*

LEMMA 15.3. *The set of real parts of the set*

$$\{<-\mu-i\lambda, sH_0> : \mu\in\Pi,\ \lambda\in\mathfrak{a}_{\mathfrak{p}}^*,\ s\in W_{\mathfrak{g}}\}$$

of complex numbers is a sum of a finite number of lattices in $\boldsymbol{R}$.

PROOF. Since each element $s\in W_{\mathfrak{g}}$ maps $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}+\mathfrak{a}_{\mathfrak{p}}$ onto itself and since $\mathfrak{a}_{\mathfrak{k}}$ and $\mathfrak{a}_{\mathfrak{p}}$ are mutually orthogonal, the real part of $<-\mu-i\lambda, sH_0>$ is equal to $-<s^{-1}\mu, H_0>$. If $\mu\in\tilde{\Pi}$ we can regard μ as an integral sum of roots of $(\mathfrak{g}, \mathfrak{a})$. Then $s^{-1}\mu$ is also an integral sum of roots of $(\mathfrak{g}, \mathfrak{a})$. Therefore $\{\mathscr{R}<-\mu-i\lambda, sH_0> : \mu\in\tilde{\Pi}\}$ is a sum of a finite number of lattices in $\boldsymbol{R}$. Since $\tilde{\Pi}$ is of finite index in Π, $\{\mathscr{R}<-\mu-i\lambda, sH_0> : \mu\in\Pi\}$ is a sum of a finite number of lattices in $\boldsymbol{R}$. Our Lemma 15.3 is now obvious. Q.E.D.

16. Completion of the proof of Theorem 8.1

We shall prove Theorem 8.1.

Let $\phi^{\tau}(\mu\colon\lambda\colon\cdot)$ be the τ-spherical function which satisfies the conditions (i), (ii) in § 10 and (iii), (iv) in § 14. We shall prove the following lemma instead of Theorem 8.1.

LEMMA 16.1. *Fix a real linear function* μ_σ *on* $\sqrt{-1}\,\mathfrak{a}_{\mathfrak{k}}$ *which is associated with a* $\sigma\in\mathscr{E}_M$. *Then for every non-negative integer s we can choose a finite number of elements* $d_1, \dots, d_N$ *in* $\boldsymbol{D}(\mathfrak{a}_{\mathfrak{p}}^*)$ *and polynomials* $p_1, q, p_{21}, \dots, p_{2N}$ *such that for every* $a\in\mathscr{S}(\mathfrak{a}_{\mathfrak{p}}^*)$

$$\sup_{x\in G}\left|\int_{\mathfrak{a}_{\mathfrak{p}}^*} a(\lambda)\phi^{\tau}(\mu_\sigma\colon\lambda\colon x)\beta(\sigma\colon\lambda)d\lambda\cdot\Xi(x)^{-1}(1+\sigma(x))^s\right|$$

$$\leqq p_1(|\sigma|)q(|\tau|)\sum_j \sup_{\lambda\in\mathfrak{a}_{\mathfrak{p}}^*} p_{2j}(|\lambda|)|a(\lambda; d_j)|.$$

PROOF. Since ϕ^{τ} is τ-spherical, it is sufficient to prove for $h\in cl(A_{\mathfrak{p}}^+)$ instead of x. We shall use induction on $\dim\mathfrak{a}_{\mathfrak{p}}$. The case $\mathfrak{a}_{\mathfrak{p}}=\{0\}$ is obviously trivial. So let us assume that $\dim\mathfrak{a}_{\mathfrak{p}}\geqq 1$. Define S^+ as in § 10. Since S^+ is compact, if we attend to (5.2) it is enough to prove the following

LEMMA 16.2. *Fix* μ_σ *and s as above. Then for a given* $H_0\in S^+$, *we can choose a neighborhood V of* H_0 *in* S^+ *and a finite number of elements* $d_1, \dots, d_N$ *in* $\boldsymbol{D}(\mathfrak{a}_{\mathfrak{p}}^*)$ *and polynomials* $p_1, q, p_{21}, \dots, p_{2N}$ *such that for every* $a\in\mathscr{S}(\mathfrak{a}_{\mathfrak{p}}^*)$, $H\in V$ *and* $t\geqq 0$

$$(1+t)^s e^{t\rho(H)}\left|\int_{\mathfrak{a}_{\mathfrak{p}}^*} a(\lambda)\phi^{\tau}(\mu_\sigma\colon\lambda\colon x)\beta(\rho\colon\lambda)d\lambda\right|$$

$$\leqq p_1(|\sigma|)q(|\tau|)\sum_j \sup_{\lambda\in\mathfrak{a}_{\mathfrak{p}}^*} p_{2j}(|\lambda|)|a(\lambda; d_j)|.$$

Proof. If $t \geqq 0$ and $H \in V$, by (14.1) the expression

$$(1+t)^s e^{t\rho(H)} \left| \int_{\mathfrak{a}_\mathfrak{p}^*} a(\lambda) \phi^\tau(\mu_\sigma : \lambda : x) \beta(\sigma : \lambda) d\lambda \right|$$

is bounded by the sum of the following two expressions:

(i) $$I_1(t) = (1+t)^s e^{t\rho_1(H)} \left| \int_{\mathfrak{a}_\mathfrak{p}^*} a(\lambda) (e^{t\rho_2(H)} \phi^\tau(\mu_\sigma : \lambda : \exp tH) - \theta^\tau(\mu_\sigma : \lambda : \exp tH)) \beta(\sigma : \lambda) d\lambda \right|,$$

(ii) $$I_2(t) = (1+t)^s e^{t\rho_1(H)} \sum_j \left| \int_{\mathfrak{a}_\mathfrak{p}^*} a(\lambda) (\beta(\sigma : \lambda)/\beta_1(\sigma : \lambda)) \cdot \theta_j^\tau(\mu_\sigma : \lambda : \exp tH) \beta_1(\sigma : \lambda) d\lambda \right|.$$

By the corollary to Lemma 13.8 and the continuity of the function on $\mathfrak{a}_\mathfrak{p}^*$, we can choose a neighborhood V of H_0 in S^+, polynomials p_1', p_2' and q, and numbers ε_1, $d_1 > 0$, independent of (σ, λ, τ), such that

$$e^{t\rho_1(H)} | e^{t\rho_2(H)} \phi^\tau(\mu_\sigma : \lambda : \exp tH) - \theta^\tau(\mu_\sigma : \lambda : \exp tH)|$$
$$< p_1'(|\sigma|) p_2(|\lambda|) q(|\tau|) e^{-\varepsilon_1 t} (1+t)^{d_1},$$

for every $H \in V$, $t \geqq 0$ and $\lambda \in \mathfrak{a}_\mathfrak{p}^*$. This being so,

$$I_1(t) \leqq p_1'(|\sigma|) q(|\tau|) e^{-\varepsilon_1 t} (1+t)^{s+d_1} \int_{\mathfrak{a}_\mathfrak{p}^*} p_2'(|\lambda|) |a(\lambda)| |\beta(\sigma : \lambda)| d\lambda.$$

But by Lemma 4.1, there exist polynomials p_1'' and p_2'' such that

$$|\beta(\sigma : \lambda)| \leqq p_1(|\sigma|) p_2(|\lambda|).$$

Therefore

$$\sup_{t \geqq 0} I_1(t)$$
$$\leqq c \cdot p_1'(|\sigma|) p_1''(|\sigma|) q(|\tau|) \cdot \sup_{\lambda \in \mathfrak{a}_\mathfrak{p}^*} [p_2'(|\lambda|) p_2''(|\lambda|) (1+|\lambda|^2)^r |a(\lambda)|],$$

where $r = \dim \mathfrak{a}_\mathfrak{p}$ and

$$c = \int_{\mathfrak{a}_\mathfrak{p}^*} (1+|\lambda|^2)^{-r} d\lambda \cdot \sup_{t \geqq 0} [e^{-\varepsilon_1 t} (1+t)^{t+d_1}].$$

This deals with $I_1(t)$.

We now obtain a bound for $I_2(t)$. From the definitions of the Plancherel measures $\beta(\sigma:\lambda)$, $\beta_1(\sigma:\lambda)$ and the function $D(\mu+i\lambda)$, there exists a constant $c'\neq 0$ such that

$$\beta(\sigma:\lambda)/\beta_1(\sigma:\lambda)=c'\cdot D(\mu_\sigma+i\lambda).$$

Then by induction hypothesis, Lemma 16.1 is applicable to $\mathfrak{m}_1'=[\mathfrak{m}_1,\mathfrak{m}_1]$. Recall that $\mathfrak{a}_2=\mathfrak{l}$, the centralizer of $\mathfrak{m}_1$ in $\mathfrak{a}_\mathfrak{p}$, and that $\mathfrak{a}_1$ is the orthogonal complement of $\mathfrak{a}_2$ in $\mathfrak{a}_\mathfrak{p}$. Let E_j denote the orthogonal projections of $\mathfrak{a}_\mathfrak{p}$ on $\mathfrak{a}_j$ $(j=1, 2)$. For $H\in V$ we can write H as $H=H_1+H_2(H_j=E_jH, j=1, 2)$. Then for $t\in\boldsymbol{R}$

$$\theta_j^\tau(\mu:\lambda:\exp tH)=e^{t<s_j(\mu+i\lambda),H_2>}\theta_j^\tau(\mu:\lambda:\exp tH_1),$$

where $<s_j(\mu+i\lambda), H_2>$ is pure imaginary by Lemma 13.10. Attending to (5.2) and Corollary 2 to Lemma 14.2, for each $j(1\leqq j\leqq r)$ we can choose a finite number of elements $d_1',\dots, d_k'$ in $\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^*)$ and polynomials $p_1', q', p_{21}',\dots, p_{2k}'$ such that for every $a\in\mathscr{S}(\mathfrak{a}_\mathfrak{p}^*)$, $H\in V$ and $t\geqq 0$

$$(1+t)^s e^{t\rho_1(H)}\left|\int_{\mathfrak{a}_\mathfrak{p}^*}a(\lambda)(\beta(\sigma:\lambda)/\beta_1(\sigma:\lambda))\theta_j^\tau(\mu_\sigma:\lambda:\exp tH)\beta_1(\sigma:\lambda)d\lambda\right|$$
$$<p_1'(|\sigma|)q'(|\tau|)\sum_{1\leqq l\leqq k}\sup_{\lambda\in\mathfrak{a}_\mathfrak{p}^*}p_{2l}'(|\lambda|)|a(\lambda;d_l')|.$$

Now our assertion follows immediately. Q.E.D.

In particular, applying Lemma 16.1 for Eisenstein integrals, we obtain the following corollary.

COROLLARY 1. *Fix $\psi\in L_\sigma^\tau$ with $\|\psi\|_M=1$. Then for every non-negative integer s we can choose a finite number of elements $d_1,\dots, d_N$ in $\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^*)$ and polynomials p_1, q, $p_{21},\dots, p_{2N}$ such that for every $a\in\mathscr{S}(\mathfrak{a}_\mathfrak{p}^*)$,*

$$\sup_{x\in G}\left|\int_{\mathfrak{a}_\mathfrak{p}^*}a(\lambda)E(\psi_\sigma^\tau:\lambda:x)\beta(\sigma:\lambda)d\lambda\cdot\Xi(x)^{-1}(1+\sigma(x))^s\right|$$
$$\leqq p_1(|\sigma|)q(|\tau|)\sum_j\sup_{\lambda\in\mathfrak{a}_\mathfrak{p}^*}p_{2j}(|\lambda|)|a(\lambda;d_j)|.$$

COROLLARY 2. *Fix $\psi\in L_\sigma^\tau$ with $\|\psi\|_M=1$. Then for every g_1, g_2 in $\mathfrak{B}$ and non-negative integer s, we can choose a finite number of elements $d_1,\dots, d_N$ in $\boldsymbol{D}(\mathfrak{a}_\mathfrak{p}^*)$ and polynomials p_1, q $p_{21},\dots, p_{2N}$ such that for every $a\in\mathscr{S}(\mathfrak{a}_\mathfrak{p}^*)$,*

$$\sup_{x\in G}\left|\int_{\mathfrak{a}_\mathfrak{p}^*}a(\lambda)E(\psi_\sigma^\tau:\lambda:g_1;x;g_2)\beta(\sigma:\lambda)d\lambda\cdot\Xi(x)^{-1}(1+\sigma(x))^s\right|$$

$$\leqq p_1(|\sigma|)q(|\tau|)\sum_j \sup_{\lambda\in\mathfrak{a}_\mathfrak{p}^*} p_{2j}(|\lambda|)|a(\lambda; d_j)|.$$

PROOF. By (15.1)

$$|E(\psi_\sigma^\tau: \lambda: g_1; x; g_2)|^2 = \sum_{i,\alpha=1}^{t_1} \sum_{j,\beta=1}^{t_2} |\Psi_{1\alpha}^i, \pi_{\sigma,\lambda}(x)\Psi_{2\beta}^j)|^2,$$

where $\Psi_{1\alpha}^i$ and $\Psi_{2\beta}^j$ are the same as in (15.1). Now let $E(\psi_{\alpha\beta}^{ij}: \lambda: x)$ be the $(\tau_{1\alpha}^i, \tau_{2\beta}^j)$-spherical function corresponding to $(\Psi_{1\alpha}^i, \pi_{\sigma,\lambda}(x)\Psi_{2\beta}^j)$. Apply Lemma 16.1 to $E(\psi_{\alpha\beta}^{ij}: \lambda: x)$. The corollary follows from the conditions on $\{\Psi_{1\alpha}^i\}$, $\{\Psi_{2\beta}^j\}$ and $\{\tau_{1\alpha}^i\}$, $\{\tau_{2\beta}^j\}$ given in Lemma 9.1. Q.E.D.

This Corollary gives Theorem 8.1. Therefore the proof of Theorem 5.1 is now complete.

17. The Fourier transform of tempered distributions

Having proved Theorem 5.1, we can now extend the definition of the Fourier transform to tempered distributions on G.

A distribution on G is said to be *tempered* if it extends to a continuous linear functional from $\mathscr{C}(G)$ to $\boldsymbol{C}$. Since $C_c^\infty(G)$ is dense in $\mathscr{C}(G)$, and since the inclusion map

$$C_c^\infty(G) \subset \mathscr{C}(G)$$

is continuous, we can regard the space of tempered distributions as the dual space of $\mathscr{C}(G)$; that is, the space of continuous linear functionals from $\mathscr{C}(G)$ into $\boldsymbol{C}$.

Let $\mathscr{C}'(G)$ be the set of tempered distributions on G. It becomes a locally convex topological vector space when endowed with the strong topology.

Let $\mathscr{C}'(\hat{G})$ be the strong dual space of $\mathscr{C}(\hat{G})$ and let $(\mathscr{F}^{-1})^*$ denote the transposed inverse of the Fourier transform $\mathscr{F}: \mathscr{C}(G) \to \mathscr{C}(\hat{G})$.

THEOREM 17.1. *The mapping $(\mathscr{F}^{-1})^*$ is a linear topological isomorphism of $\mathscr{C}'(G)$ onto $\mathscr{C}'(\hat{G})$.*

PROOF. The theorem follows directly from the fact that $\mathscr{F}$ is a topological isomorphism of $\mathscr{C}(G)$ onto $\mathscr{C}(\hat{G})$. Q.E.D.

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