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# $(p, q)$-Nuclear and $(p, q)$-Integral Operators

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Recently, investigations have been made on the various generalizations of nuclear operators on the basis of the theory of locally convex topological vector spaces and the classes of operators in them ([2], [12], [13], [11], [8]). Among other things, $p$-absolutely summing operators due to A. Pietsch ([13]) and $p$-nuclear, $p$-integral, $p$-quasi-nuclear and $p$-quasi-integral operators due to A. Persson and A. Pietsch ([11]), $1 \leq p \leq \infty$, have played an important role in the study of classes of operators in connection with the classes of nuclear and integral operators in Banach spaces. Not only these operators were defined by making use of the norms of spaces $L^p$ and $l^p$, but also their associated domains and ranges were closely related with spaces $L^p$ and $l^p$ ([13], [10], [11], [7], [8]). For instance, these operators were characterized with the aid of operators in $L^p$ and $l^p$ as follows ([11], [13]). A bounded linear operator T from a Banach space $E$ to a Banach space $F$ is $p$-nuclear (resp. $p$-integral, resp. $p$-absolutely summing) if and only if $T$ can be factorized in the form $T=Q_1DP_1$ where $P_1 \in \boldsymbol{L}(E, l^\infty)$ with $\|P_1\| \leq 1$, $Q_1 \in \boldsymbol{L}(l^p, F)$ with $\|Q_1\| \leq 1$ and $D$ is a multiplication operator by a sequence in $l^p$, (resp. if and only if $T$ can be factorized in the form $T=Q_2IP_2$ where $P_2 \in \boldsymbol{L}(E, L^\infty)$ with $\|P_2\| \leq 1$, $Q_2 \in \boldsymbol{L}(L^p, F)$ with $\|Q_2\| \leq 1$ and $I$ is the identity operator in $\boldsymbol{L}(L^\infty, L^p)$, resp. if and only if there exists a positive Radon measure $\mu$ on the weakly compact unit ball $U^\circ$ in $E'$ such that $\|Tu\| \leq \rho\{\int_{U^\circ} |<u, u'>|^p d\mu(u')\}^{1/p}$ for each $u \in E$ and with a positive constant $\rho$). With these in mind, by making use of Lorentz spaces $L^{p,q}$ and $l^{p,q}$ instead of $L^p$ and $l^p$, the definitions and investigations of new classes of operators will be expected to be made. In the present paper, using the Lorentz spaces we shall introduce the four distinct types of operators, namely, the $(p, q)$-nuclear, $(p, q)$-integral, $(p, q)$-quasi-nuclear and $(p, q)$-quasi-integral operators, $1 \leq p, q \leq \infty$, which, in case $p=q$, coincide with the $p$-nuclear, $p$-integral, $p$-quasi-nuclear and $p$-quasi-integral operators respectively. The main purpose of this paper is to investigate these operators and to obtain their properties, their characterizations and the relationships among them. We also study the properties of the spaces of these operators with adequate quasi-norms. In these processes we shall be often concerned with Lorentz spaces $L^{p,q}$, $l^{p,q}$, where the notion and general properties of rearrangements of functions and of sequences are frequently used. Such utilizations of Lorentz spaces are of interest in themselves.

Section 1 is devoted to the preliminary remarks. We shall recall the definitions and the fundamental properties related to Lorentz spaces in view of rearrangement in one way and by making use of the theory of interpolation between Banach spaces in another way. Especially, a result concerning the rearrangement of a sequence due to Hardy, Littlewood and Pólya (Lemma 1), and a generalized Hölder's inequality (Lemma 2) will be frequently used in the subsequent sections. In Sections 2, 3, 4 and 5, we deal with $(p, q)$-nuclear, $(p, q)$-integral, $(p, q)$-quasi-nuclear and $(p, q)$-quasi-integral operators respectively. In these sections we consider the spaces of these operators and introduce the quasi-norms into them. We also give there a characterization of the operator of each class by making use of a special operator of the same kind in $l^{p,q}$, $L^{p,q}$. In Section 6 we show that there are intimate connections among these classes of operators.

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## 1. Preliminaries

Let $E$ and $F$ be Banach spaces. We shall denote by $\boldsymbol{L}(E, F)$ the space of bounded linear operators $T$ from $E$ to $F$ with the usual operator norm

$$\|T\| = \sup_{\|u\|\leq 1} \|Tu\|.$$

We denote by $\boldsymbol{L}_0(E, F)$ and $\boldsymbol{K}(E, F)$ the subspaces of operators of finite rank and compact operators respectively.

A Banach space $F$ is said to have the extension property if each operator $T_0 \in \boldsymbol{L}(E_0, F)$, $E_0$ being any linear subspace of an arbitrary Banach space $E$, can be extended to a $T \in \boldsymbol{L}(E, F)$ preserving its norm ([9]). It is well known that the Banach space $L^\infty$ has the extension property ([14], [9]).

We next summarize the notations and properties concerning Lorentz spaces which will be repeatedly used in the following sections (cf. [1], [6], [4], [15]). The letter $X$ is used for a locally compact Hausdorff space and $\mu$ denotes a positive Radon measure on $X$. Let $L^p_\mu(X, E)$, $1 \leq p \leq \infty$, be the Banach space of (classes of)$E$-valued $\mu$-measurable functions $f$ on $X$ such that

$$\|f\|_{L^p_\mu(X,E)} = \begin{cases} \left(\int_X \|f(x)\|^p d\mu(x)\right)^{1/p} < \infty & \text{if } p < \infty, \\ \operatorname{ess\,sup} \|f(x)\| < \infty & \text{if } p = \infty. \end{cases}$$

The distribution function of $f$ is defined by

$$\lambda_f(y) = \mu(\{x \in X \mid \|f(x)\| > y\}), \qquad y > 0,$$

and the non-increasing rearrangement of $f$ onto $(0, \infty)$ is defined by

$$f^*(t)=\inf\{y>0 \mid \lambda_f(y)\leq t\}, \qquad 0<t<\infty.$$

The Lorentz space $L_\mu^{p,q}(X, E)$ or $L(p, q; X, \mu, E)$, $1\leq p, q\leq\infty$, is the collection of all $f$ such that $\|f\|_{L(p,q;X,\mu,E)}<\infty$, where

$$\|f\|_{L(p,q;X,\mu,E)}=\begin{cases}\left(\int_0^\infty t^{q/p-1}f^*(t)^q dt\right)^{1/q} & \text{if } 1\leq p<\infty, \quad 1\leq q<\infty,\\ \sup_{t>0} t^{1/p}f^*(t) & \text{if } 1\leq p\leq\infty, \quad q=\infty.\end{cases}$$

In particular, if $E$ is a scalar space $\boldsymbol{C}$ or if $E$ is understood, we write $L(p, q; X, \mu)$, $L(p, q; X)$ or $L(p, q)$ shortly instead of $L(p, q; X, \mu, E)$.

We notice some results concerning Lorentz spaces which will be utilized hereafter ([6]).

By a scalar valued simple function we mean a function $f(x)$ which can be written in the form $f(x)=\sum_{k=1}^n c_k\chi_{A_k}(x)$, where $c_1, c_2, \dots, c_n$ are complex numbers and $A_1, A_2, \dots, A_n$ are pairwise disjoint sets of finite measure and $\chi_A(x)$ denotes the characteristic function of the set $A$. For such a function, let $c_1^*\geq c_2^*\geq\cdots\geq c_n^*\geq 0$ be a non-increasing rearrangement of $|c_1|, |c_2|, \dots, |c_n|$, and let $A_k^*$ be the set $A_i$ corresponding to $c_i$ with $c_k^*=|c_i|$. Then we have

$$\|f\|_{L(p,q)}=(p/q)^{1/q}\{\sum_{k=1}^n c_k^{*q}(a_k^{q/p}-a_{k-1}^{q/p})\}^{1/q},$$

where $$a_j=\sum_{k=1}^j \mu(A_k^*), \quad j=1, 2, \dots, n \quad \text{and} \quad a_0=0.$$

For a fixed $r: 0<r\leq 1$, we put

$$f_r^{**}(t)=\begin{cases}\sup_{\mu(A)\geq t}\left\{(1/\mu(A))\int_A \|f(x)\|^r d\mu(x)\right\}^{1/r} & \text{if } 0<t\leq\mu(X),\\ \left\{(1/t)\int_X \|f(x)\|^r d\mu(x)\right\}^{1/r} & \text{if } t>\mu(X).\end{cases}$$

Then, by making use of this function, the norm of the Lorentz space $L(p, q; X, \mu, E)$ is seen to be equivalent to

$$\begin{cases}\left\{\int_0^\infty t^{q/p-1}(f_r^{**}(t))^q dt\right\}^{1/q} & \text{if } 1\leq p<\infty, \quad 1\leq q<\infty,\\ \sup_{t>0} t^{1/p}f_r^{**}(t) & \text{if } 1\leq p\leq\infty, \quad q=\infty.\end{cases}$$

In this case, it is to be noticed for our later purpose that $f_1^{**}(t)$ has the property

$$(f+g)_1^{**}(t)\leq f_1^{**}(t)+g_1^{**}(t).$$

The third equivalent norm of Lorentz spaces is given from the point of view of the interpolation theory ([1], [4]). We consider a couple $(E_0, E_1)$ of Banach spaces which are both continuously embedded in a Hausdorff topological vector space $\mathscr{E}$. For $0<\theta<1$, $1\leq q\leq\infty$, the space

$$(E_0, E_1)_{\theta,q}=\{a\in E_0+E_1 \mid \int_0^\infty (t^{-\theta}K(t, a))^q dt/t<\infty\}$$

with the norm

$$\|a\|_{(E_0,E_1)_{\theta,q}}=\left\{\int_0^\infty (t^{-\theta}K(t, a))^q dt/t\right\}^{1/q}$$

is called an interpolation space between $E_0$ and $E_1$, where

$$K(t, a)=\inf_{a=a_0+a_1,\, a_i\in E_i,\, i=0,1}(\|a_0\|_{E_0}+t\|a_1\|_{E_1}).$$

Then, in the particular case $E_0=L_\mu^1$, $E_1=L_\mu^\infty$, we get

$$K(t, f)=\int_0^t f^*(s)ds, \qquad 0<t<\infty,$$

for each $\qquad f\in L_\mu^1+L_\mu^\infty,$

and we have $\qquad L(p, q; X, \mu)\sim(L_\mu^1, L_\mu^\infty)_{1-1/p,\, q}$ ,

where $\sim$ means that both sides coincide algebraically and their norms are equivalent. Therefore, by the interpolation theory of spaces, it is noted that if $X$ is a compact Hausdorff space, the norms of $L(p, q; X)$ are monotonically increasing (resp. decreasing) with respect to $p$ (resp. $q$). Namely, if $1\leq p_0<p_1<\infty$, $1\leq q_0$, $q_1<\infty$, then we have

$$\|f\|_{L(p_0,q_0)}\leq\|f\|_{L(p_1,q_1)} \qquad \text{for each } f\in L(p_1, q_1; X, \mu),$$

and if $1\leq p<\infty$, $1\leq q_0\leq q_1<\infty$, then we have

$$\|f\|_{L(p,q_1)}\leq\|f\|_{L(p,q_0)} \qquad \text{for each } f\in L(p, q_0; X, \mu).$$

Since Lorentz spaces for the case of sequences are frequently used later, we notice some of their results. Let $l^p(E)$ be the collection of sequences $\{a_i\}\subset E$ such that

$$\|\{a_i\}\|_{l^p(E)}=\begin{cases}(\sum_i\|a_i\|^p)^{1/p}<\infty & \text{if } p<\infty,\\ \sup_i\|a_i\| \quad <\infty & \text{if } p=\infty.\end{cases}$$

The Lorentz space $l^{p,q}(E)$, $l(p, q; E)$ or shortly $l(p, q)$, $1\leq p, q\leq\infty$, is the collection of all sequences $\{a_i\}\in c_0(E)$ such that $\|\{a_i\}\|_{l(p,q;E)}<\infty$. Here, denoting by $\{\|a_i\|^*\}$ the non-increasing rearrangement of $\{\|a_i\|\}$, we put

$$\|\{a_i\}\|_{l(p,q;E)}=\begin{cases}(\sum_i i^{q/p-1}\|a_i\|^{*q})^{1/q} & \text{if } 1\le p<\infty,\quad 1\le q<\infty,\\ \sup_i i^{1/p}\|a_i\|^* & \text{if } 1\le p\le\infty,\quad q=\infty.\end{cases}$$

The following inequalities show that this norm is defined. When we calculate $\|f\|_{L(p,q;I,\mu,E)}$ with $f(i)=a_i$, $i\in \boldsymbol{I}$, the set of positive integers, and with the counting measure $\mu$ on $\boldsymbol{I}$, we have

$$\begin{aligned}&\{\sum_i(i-1)^{q/p-1}\|a_i\|^{*q}\}^{1/q}\\ &\le(\text{resp. }\ge)\|f(i)\|_{L(p,q:I,\mu,E)}=(p/q)^{1/q}[\sum_i\|a_i\|^{*q}\{i^{q/p}-(i-1)^{q/p}\}]^{1/q}\\ &\le(\text{resp. }\ge)\{\sum_i i^{q/p-1}\|a_i\|^{*q}\}^{1/q}\end{aligned}$$

for $\qquad 1\le p\le q<\infty \qquad (\text{resp. } 1\le q\le p<\infty)$

and

$$\|f(i)\|_{L(p,\infty;I,\mu,E)}=\sup_i(i^{1/p}\|a_i\|^*)$$

for $\qquad 1\le p\le\infty,\ q=\infty.$

If it is necessary to specify the suffix of a sequence in the norm, we write $\|\{a_i\}\|_{i,l(p,q)}$. The interpolation theoretical description of Lorentz spaces $l(p, q; E)$ is

$$l(p,q;E)\sim(l^1(E),\ l^\infty(E))_{1-1/p,q}.$$

According to this, by the result of interpolation theory it is noted that the norms of $l(p, q; E)$ are monotonically decreasing with respect to both $p$ and $q$. Namely, if $1\le p_0<p_1<\infty$, $1\le q_0$, $q_1<\infty$, then we have

$$\|u\|_{l(p_1,q_1)}\le\|u\|_{l(p_0,q_0)}\qquad\text{for each } u\in l(p_0,q_0;E),$$

and if $1\le p<\infty$, $1\le q_0<q_1<\infty$, then we have

$$\|u\|_{l(p,q_1)}\le\|u\|_{l(p,q_0)}\qquad\text{for each } u\in l\ (p,q_0;E).$$

We shall notice the following two lemmas which are the fundamental tools of our subsequent discussions.

LEMMA 1. (*Hardy, Littlewood and Pólya* [3]). *Let* $\{c_i^*\}$ *and* $\{{}^*c_i\}$ *be respectively the non-increasing and non-decreasing rearrangements of a finite* *sequence* $\{c_i\}_{1\le i\le n}$ *of positive numbers. Then for two sequences* $\{a_i\}_{1\le i\le n}$ *and* $\{b_i\}_{1\le i\le n}$ *of positive numbers we have*

$$\sum_i a_i^* \cdot {}^*b_i \le \sum_i a_i b_i \le \sum_i a_i^* \cdot b_i^*.$$

Generalizing the usual Hölder's inequality, R. Hunt proved the next

LEMMA 2 (*Hunt* [6]). *Suppose* $1\le p, q, p', q', p'', q''\le\infty$, $1/p+1/p'=1/p''$ *and* $1/q+1/q'=1/q''$. *Then for any* $f\in L(p, q; X, \mu)$ *and* $g\in L(p', q'; X, \mu)$ *we have* $f\cdot g\in L(p'', q''; X, \mu)$ *and*

$$\|f\cdot g\|_{L(p'',q'';X,\mu)}\le C\|f\|_{L(p,q;X,\mu)}\cdot\|g\|_{L(p',q';X,\mu)}$$

*with a constant* $C\ge 1$. *In particular when* $p''=q''=1$, *we have*

$$\left|\int_X f(x)\cdot g(x)d\mu(x)\right|\le\|f\|_{L(p,q;X,\mu)}\cdot\|g\|_{L(p',q';X,\mu)}.$$

*The case of sequences is as follows: If* $\{a_i\}\in l(p, q)$ *and* $\{b_i\}\in l(p', q')$ *then* $\{a_ib_i\}\in l(p'', q'')$ *and*

$$\|\{a_ib_i\}\|_{l(p'',q'')}\le C\|\{a_i\}\|_{l(p,q)}\cdot\|\{b_i\}\|_{l(p',q')}.$$

Throughout this paper, unless otherwise stated, $E'$ stands for the dual space of $E$, and $U^0$ denotes the weakly compact unit ball in $E'$. We denote by $\|\cdot\|_E$ the norm in $E$, and briefly by $\|\cdot\|$ if there is no confusion.

Throughout this paper, for brevity, we only deal with $p, q: 1<p, q<\infty$ unless otherwise stated and we denote by $p', q'$ the conjugate exponents of $p, q: 1/p+1/p'=1$, $1/q+1/q'=1$ respectively. Concerning the general properties of interpolation spaces and Lorentz spaces we may refer to [1], [4], [6] and [15].

## 2. (*p*, *q*)-nuclear operators

We shall first define $(p, q)$-nuclear operators as follows.

DEFINITION 1. $T\in \boldsymbol{L}(E, F)$ is said to be a *left* $(p, q)$-*nuclear* or simply $(p, q)$-*nuclear* (resp. *right* $(p, q)$-*nuclear*) *operator*, $1\le p, q\le\infty$, if $T$ can be written in the form

(1) $$Tu=\sum_i <u, u_i'>v_i \quad \text{for each} \quad u\in E$$

with $$\{u_i'\}\subset E',\ \{v_i\}\subset F \quad \text{such that}$$

$$\|\{\|u_i'\|\}\|_{l(p,q)}<\infty$$

and

$$\sup_{\|v'\|\le 1}\|\{|<v_i, v'>|\}\|_{l(p',q')}<\infty$$

(resp. $$\sup_{\|u\|\leq 1}\|\{|<u, u_i'>|\}\|_{l(p',q')}<\infty$$

and $$\|\{\|v_i\|\}\|_{l(p,q)}<\infty).$$

The collection of $(p, q)$-nuclear (resp. right $(p, q)$-nuclear) operators is denoted by $N_{p,q}(E, F)$ (resp. $N^{p,q}(E, F)$). The quasi-norm (as proved later) is defined by

$$\nu_{p,q}(T)=\inf(\|\{\|u_i'\|\}\|_{l(p,q)}\cdot\sup_{\|v'\|\leq 1}\|\{|<v_i, v'>|\}\|_{l(p',q')})$$

(resp. $$\nu^{p,q}(T)=\inf(\sup_{\|u\|\leq 1}\|\{|<u, u_i'>|\}\|_{l(p',q')}\cdot\|\{\|v_i\|\}\|_{l(p,q)}),$$

where the infimum is taken over all representations (1) of $T$.

REMARK 1. In case of $p=q$, a $(p, q)$-nuclear (resp. right $(p, q)$-nuclear) operator coincides with a $p$-nuclear operator (resp. of type $N^p$) introduced in [11] (resp. [10]).

For $T\in N_{p,q}(E, F)$ and for each $u\in E$, the series (1) is convergent. In fact, for any finite set $J$ of natural numbers and for each $u\in E$ we have

$$\|\sum_{i\in J}<u, u_i'>v_i\|\leq\sup_{\|v'\|\leq 1}\sum_{i\in J}|<u, u_i'>|\cdot|<v_i, v'>|.$$

Applying Lemma 2 to the right hand side of this inequality, if $q>p$ (resp. $q\leq p$), with $\|u_i'\|=\|u_{m(i)}'\|^*$ (resp. $|<v_i, v'>|=|<v_{n(i)}, v'>|^*$) we have

$$\|\sum_{i\in J}<u, u_i'>v_i\|$$
$$\leq\|u\|\cdot(\sum_{i\in J}m(i)^{q/p-1}\|u_{m(i)}'\|^{*q})^{1/q}\cdot\sup_{\|v'\|\leq 1}\|\{<v_i, v'>\}\|_{l(p',q')}$$

(resp. $$\leq\|u\|\cdot\|\{\|u_i'\|\}\|_{l(p,q)}\cdot\sup_{\|v'\|\leq 1}(\sum_{i\in J}n(i)^{q'/p'-1}|<v_{n(i)}, v'>|^{*q'})^{1/q'}),$$

which shows the convergence of the series (1). A similar fact is valid for $T\in N^{p,q}(E, F)$.

From these considerations we obtain the following

PROPOSITION 1. *Let* $T\in N_{p,q}(E, F)$ *(resp.* $T\in N^{p,q}(E, F)$*). Then*

$$\|T\|\leq\nu_{p,q}(T)\qquad(\textit{resp. }\|T\|\leq\nu^{p,q}(T)).$$

Concerning the connection between $N_{p,q}$ and $N^{p,q}$ we have

PROPOSITION 2. *If* $T\in N_{p,q}(E, F)$*, then its adjoint* $T'$ *belongs to* $N^{p,q}(F', E')$ *and it satisfies*

$$\nu^{p,q}(T')\leq\nu_{p,q}(T).$$

*Furthermore assume E and F are reflexive. Then, if* $T'\in N^{p,q}(F', E')$

*we have*

$$T \in \boldsymbol{N}_{p,q}(E, F)$$

*and*

$$\boldsymbol{\nu}^{p,q}(T') = \boldsymbol{\nu}_{p,q}(T).$$

PROOF. If $T \in \boldsymbol{N}_{p,q}(E, F)$, then for any $\varepsilon > 0$ it can be written as

$$< Tu, v' > = \sum_i < u, u_i' > < v_i, v' >$$

for each $u \in E$ and $v' \in F'$, with

$$\|\{\|u_i'\|\}\|_{l(p,q)} \cdot \sup_{\|v'\| \leq 1} \|\{< v_i, v' >\}\|_{l(p',q')} \leq \boldsymbol{\nu}_{p,q}(T) + \varepsilon. \tag{2}$$

Hence we have

$$T'v' = \sum_i < v', v_i > u_i',$$

and (2) shows

$$\boldsymbol{\nu}^{p,q}(T') \leq \boldsymbol{\nu}_{p,q}(T).$$

When $E$, $F$ are reflexive, in the same way $T' \in \boldsymbol{N}^{p,q}(F', E')$ implies

$$T \in \boldsymbol{N}_{p,q}(E, F)$$

and

$$\boldsymbol{\nu}_{p,q}(T) \leq \boldsymbol{\nu}^{p,q}(T').$$

Thus

$$\boldsymbol{\nu}_{p,q}(T) = \boldsymbol{\nu}^{p,q}(T').$$

This completes the proof.

We shall next show that $\boldsymbol{N}_{p,q}(E, F)$ (resp. $\boldsymbol{N}^{p,q}(E, F)$) turns out to be a quasi-normed space ([12]) with respect to $\boldsymbol{\nu}_{p,q}(\cdot)$ (resp. $\boldsymbol{\nu}^{p,q}(\cdot)$).

THEOREM 1. *Let $T_k \in \boldsymbol{N}_{p,q}(E, F)$ for $k = 1, 2, \ldots, M$, $M$ being a positive integer. Then $\sum_{k=1}^{M} T_k \in \boldsymbol{N}_{p,q}(E, F)$ and*

$$\boldsymbol{\nu}_{p,q}\Big(\sum_{k=1}^{M} T_k\Big) \leq M^{|1/p - 1/q|}\Big(\sum_{k=1}^{M} \boldsymbol{\nu}_{p,q}(T_k)\Big).$$

*A similar statement holds for elements of $\boldsymbol{N}^{p,q}(E, F)$.*

PROOF. For any $\varepsilon > 0$ $T_k$ can be written in the form

$$T_k u = \sum_i \langle u, u'_{k,i} \rangle v_{k,i}, \qquad k=1, 2, \ldots, M$$

such that

$$\sum_i i^{q/p-1} \|u'_{k,i}\|^{*q} < \boldsymbol{\nu}_{p,q}(T_k) + \varepsilon/2^k$$

and

$$\sup_{\|v'\| \le 1} \sum_i i^{q'/p'-1} |\langle v_{k,i}, v' \rangle|^{*q'} < \boldsymbol{\nu}_{p,q}(T_k) + \varepsilon/2^k, \qquad k=1, 2, \ldots, M.$$

In order to calculate the $l(p, q)$-norm $\sigma$ of the countable set $\{\{\|u'_{1,i}\|^*\}_{1 \le i < \infty}, \{\|u'_{2,i}\|^*\}_{1 \le i < \infty}, \ldots, \{\|u'_{M,i}\|^*\}_{1 \le i < \infty}\}$, let $N$ be any positive integer and let $\sigma_N$ be the $l(p, q)$-norm of the first $N$ terms of the non-increasing rearrangement of $\{\{\|u'_{1,i}\|^*\}_{1 \le i < \infty}, \{\|u'_{2,i}\|^*\}_{1 \le i < \infty}, \ldots, \{\|u'_{M,i}\|^*\}_{1 \le i < \infty}\}$. Next, rearranging $\{\{\|u'_{1,i}\|^*\}_{1 \le i \le N}, \{\|u'_{2,i}\|^*\}_{1 \le i \le N}, \ldots, \{\|u'_{M,i}\|^*\}_{1 \le i \le N}\}$ in non-increasing order we denote by $n(k, i)$ the number corresponding to the term $\|u'_{k,i}\|^*$. Then, on account of Lemma 1 and the fact that $\{n(k, i) | 1 \le i \le N, 1 \le k \le M\}$ is a permutation of $\{1, 2, \ldots, MN\}$ we obtain

$$\begin{aligned}
&\sigma_N^q \\
&\le \sum_{k=1}^{M} \sum_{i=1}^{N} n(k, i)^{q/p-1} \|u'_{k,i}\|^{*q} \\
&\le \begin{cases} \displaystyle\sum_{k=1}^{M} \sum_{i=1}^{N} (M i - k + 1)^{q/p-1} \|u'_{k,i}\|^{*q} & \text{if } q \ge p, \\ \displaystyle\sum_{k=1}^{M} \sum_{i=1}^{N} i^{q/p-1} \|u'_{k,i}\|^{*q} & \text{if } q < p \end{cases} \\
&\le \max(1, M^{q/p-1}) \cdot \sum_{k=1}^{M} \sum_{i=1}^{N} i^{q/p-1} \|u'_{k,i}\|^{*q} \\
&< \max(1, M^{q/p-1}) \cdot \Big\{ \sum_{k=1}^{M} \boldsymbol{\nu}_{p,q}(T_k) + \varepsilon \Big\}.
\end{aligned}$$

Since $N$ is arbitrary, this shows

$$\sigma \le \max(1, M^{1/p-1/q}) \cdot \Big\{ \sum_{k=1}^{M} \boldsymbol{\nu}_{p,q}(T_k) + \varepsilon \Big\}^{1/q}.$$

In the same way, in order to estimate the $l(p', q')$-norm $\sigma'(v')$ of the sequence $\{\{|\langle v_{1,i}, v' \rangle|^*\}_{1 \le i < \infty}, \ldots, \{|\langle v_{M,i}, v' \rangle|^*\}_{1 \le i < \infty}\}$, for each fixed $v'$: $\|v'\| \le 1$, let $\sigma'_N(v')$ be the $l(p', q')$-norm of the first $N$ terms of the non-increasing rearrangement of $\{\{|\langle v_{1,i}, v' \rangle|^*\}_{1 \le i < \infty}, \ldots, \{|\langle v_{M,i}, v' \rangle|^*\}_{1 \le i < \infty}\}$. When we rearrange the sequence $\{\{|\langle v_{1,i}, v' \rangle|^*\}_{1 \le i \le N}, \ldots, \{|\langle v_{M,i}, v' \rangle|^*\}_{1 \le i \le N}\}$ in non-increasing order, we denote by $m(k, i)$ the number corresponding to the term $|\langle v_{k,i}, v' \rangle|^*$.

Then we have

$$\begin{aligned}
&(\sigma'_N(v'))^{q'}\\
&\leq \sum_{k=1}^{M}\sum_{i=1}^{N} m(k, i)^{q'/p'-1}|<v_{k,i}, v'>|^{*q'}\\
&\leq \max(1, M^{q'/p'-1})\cdot\sum_{k=1}^{M}\sum_{i=1}^{N} i^{q'/p'-1}|<v_{k,i}, v'>|^{*q'}\\
&\leq \max(1, M^{q'/p'-1})\{\sum_{k=1}^{M}\boldsymbol{\nu}_{p,q}(T_k)+\varepsilon\},
\end{aligned}$$

which shows

$$\sigma'(v')\leq \max(1, M^{1/p'-1/q'})\{\sum_{k=1}^{M}\boldsymbol{\nu}_{p,q}(T_k)+\varepsilon\}^{1/q'}.$$

Therefore

$$(\sum_{k=1}^{M}T_k)u=\sum_{k=1}^{M}\sum_{i=1}^{\infty}<u, u'_{k,i}>v_{k,i}$$

and this satisfies

$$\boldsymbol{\nu}_{p,q}(\sum_{k=1}^{M}T_k)\leq \sigma\cdot\sup_{\|v'\|\leq 1}\sigma'(v')<M^{|1/p-1/q|}\cdot(\sum_{k=1}^{M}\boldsymbol{\nu}_{p,q}(T_k)+\varepsilon).$$

Since $\varepsilon$ is arbitrary, this completes the proof.

For some special parameters $p$, $q$, we have the following inclusion relations between the spaces $\boldsymbol{N}_{p,q}$.

PROPOSITION 3. (i) *If* $1\leq p<p_1$, $1\leq q<q_1$, $p_1q<pq_1$ *and* $1/q-1/q_1<1/p-1/p_1$, *then*

$$\boldsymbol{N}_{p,q}(E, F)\subset \boldsymbol{N}_{p_1,q_1}(E, F)$$

*and* $\boldsymbol{\nu}_{p_1,q_1}(T)\leq C\boldsymbol{\nu}_{p,q}(T)$ *for each* $T\in\boldsymbol{N}_{p,q}(E, F)$, *with some positive constant* $C$.

(ii) *If* $1\leq p\leq p_1\leq q$, *then*

$$\boldsymbol{N}_{p,q}(E, F)\subset \boldsymbol{N}_{p_1,q}(E, F)$$

*and*

$$\boldsymbol{\nu}_{p_1,q}(T)\leq C\boldsymbol{\nu}_{p,q}(T) \qquad \textit{for each} \quad T\in\boldsymbol{N}_{p,q}(E, F).$$

PROOF. (i) For $T\in\boldsymbol{N}_{p,q}(E, F)$ and for any $\varepsilon>0$ we may write

$$Tu=\sum_i<u, u'_i>v_i \qquad \text{for each} \quad u\in E,$$

with

$$\|\{\|u_i'\|\}\|_{l(p,q)} \leq 1$$

and

$$\sup_{\|v'\| \leq 1} \|\{<v_i, v'>\}\|_{l(p',q')} < \nu_{p,q}(T) + \varepsilon,$$

where

$$1/p + 1/p' = 1 \quad \text{and} \quad 1/q + 1/q' = 1.$$

Here we may assume $\|u_i'\|^* = \|u_i'\|$, $u_i' \neq 0$. We put

$$\hat{u}_i' = i^{q/(pq_1) - 1/p_1} \|u_i'\|^{-q/s} u_i'$$

and

$$\hat{v}_i = i^{1/p_1 - q/(pq_1)} \|u_i'\|^{q/s} v_i$$

with $1/q_1 + 1/s = 1/q$. Then, since by the assumptions we have $q/(pq_1) - 1/p_1 < 0$ and

$$\|\hat{u}_i'\|^* = i^{q/(pq_1) - 1/p_1} \|u_i'\|^{q/q_1},$$

we have

$$\sum_i i^{q_1/p_1 - 1} \|\hat{u}_i'\|^{*q_1} = \sum_i i^{q/p - 1} \|u_i'\|^{*q} \leq 1.$$

On the other hand, by Lemma 2 we have

$$\|\{<\hat{v}_i, v'>\}\|_{l(p_1', q_1')} \leq C \|\{i^{1/p_1 - q/(pq_1)} \|u_i'\|^{q/s}\}\|_{l(r,s)} \cdot \|\{<v_i, v'>\}\|_{l(p',q')}$$

with $1/p_1 + 1/r = 1/p$, $1/p_1 + 1/p_1' = 1$, $1/q_1 + 1/q_1' = 1$ and with a constant $C > 0$. Furthermore, since $s/r - 1 > 0$, by Lemma 1 we have

$$\|\{i^{1/p_1 - q/(pq_1)} \|u_i'\|^{q/s}\}\|_{l(r,s)} = \Big(\sum_i i^{s/r - 1} |i^{1/p_1 - q/(pq_1)} \|u_i'\|^{q/s}|^{*s}\Big)^{1/s}$$

$$\leq \Big(\sum_i i^{s/r - 1 + s\{1/p_1 - q/(pq_1)\}} \|u_i'\|^{q}\Big)^{1/s} = \Big(\sum_i i^{q/p - 1} \|u_i'\|^{*q}\Big)^{1/s} \leq 1.$$

Hence we have

$$\|\{<\hat{v}_i, v'>\}\|_{l(p_1', q_1')} < C(\nu_{p,q}(T) + \varepsilon)$$

and

$$Tu = \sum_i <u, \hat{u}_i'> \hat{v}_i,$$

which shows

$$T \in N_{p_1, q_1}(E, F)$$

and

$$\nu_{p_1, q_1}(T) \leq C \nu_{p,q}(T).$$

(ii) Let $1\leq p\leq p_1\leq q$, $1/p_1+1/r=1/p$, $1/p+1/p'=1$, $1/p_1+1/p_1'=1$, $1/q+1/q'=1$ and let $T\in \boldsymbol{N}_{p,q}(E, F)$. Then for any $\varepsilon>0$ $Tu$ may be written as

$$Tu=\sum_i <u, u_i'>v_i \qquad \text{for each} \quad u\in E,$$

with
$$\|\{\|u_i'\|\}\|_{l(p,q)}\leq 1$$

and
$$\sup_{\|v'\|\leq 1}\|\{<v_i, v'>\}\|_{l(p',q')}<\boldsymbol{\nu}_{p,q}(T)+\varepsilon.$$

Here we may assume
$$\|u_i'\|^*=\|u_i'\|.$$

Putting
$$\hat{u}_i'=i^{1/p-1/p_1}u_i'$$

and
$$\hat{v}_i=i^{1/p_1-1/p}v_i,$$

as in (i) we have

$$Tu=\sum_i <u, \hat{u}_i'>\hat{v}_i,$$

$$\sum_i i^{q/p_1-1}\|\hat{u}_i'\|^{*q}$$
$$\leq \sum_i i^{q/p_1-1}i^{q/p-q/p_1}\|u_i'\|^q \qquad \text{by Lemma 1}$$
$$=\sum_i i^{q/p-1}\|u_i'\|^{*q}\leq 1$$

and

$$\|\{<\hat{v}_i, v'>\}\|_{l(p_1',q')}$$
$$\leq C\|\{i^{1/p_1-1/p}\}\|_{l(r,\infty)}\|\{<v_i, v'>\}\|_{l(p',q')} \qquad \text{by Lemma 2}$$
$$=C\|\{<v_i, v'>\}\|_{l(p',q')}$$

with some positive constant $C$. This completes the proof.

PROPOSITION 4. *Let $E$, $F$ and $G$ be Banach spaces. If $T\in \boldsymbol{N}_{p,q}(E, F)$ and $S\in \boldsymbol{L}(F, G)$, then $ST\in \boldsymbol{N}_{p,q}(E, G)$ and*

$$\boldsymbol{\nu}_{p,q}(ST)\leq \|S\|\cdot\boldsymbol{\nu}_{p,q}(T).$$

*If $T\in \boldsymbol{L}(E, F)$ and $S\in \boldsymbol{N}_{p,q}(F, G)$, then $ST\in \boldsymbol{N}_{p,q}(E, G)$ and*

$$\boldsymbol{\nu}_{p,q}(ST)\leq \boldsymbol{\nu}_{p,q}(S)\cdot\|T\|.$$

*The analogues for the operators of $\boldsymbol{N}^{p,q}$ are valid.*

PROOF. For each $u \in E$, by the assumption that $T \in \boldsymbol{N}_{p,q}(E, F)$ and $S \in \boldsymbol{L}(F, G)$ we have

$$STu = \sum_i \langle u, u'_i \rangle Sv_i \qquad \text{for each} \quad u \in E,$$

with

$$\|\{\|u'_i\|\}\|_{l(p,q)} < \infty$$

and

$$\begin{aligned}
&\sup_{\|w'\|_{G'} \le 1} \|\{\langle Sv_i, w' \rangle\}\|_{l(p',q')} \\
&\le \|S\| \cdot \sup_{\|w'\|_{G'} \le 1} \|\{\langle v_i, \|S\|^{-1} S' w' \rangle\}\|_{l(p',q')} \\
&\le \|S\| \cdot \sup_{\|v'\|_{F'} \le 1} \|\{\langle v_i, v' \rangle\}\|_{l(p',q')},
\end{aligned}$$

where $S'$ denotes the adjoint of $S$. This implies $ST \in \boldsymbol{N}_{p,q}(E, G)$ and

$$\boldsymbol{\nu}_{p,q}(ST) \le \|S\| \cdot \boldsymbol{\nu}_{p,q}(T).$$

If $T \in \boldsymbol{L}(E, F)$ and $S \in \boldsymbol{N}_{p,q}(F, G)$, then we observe

$$\begin{aligned}
STu &= \sum_i \langle Tu, v'_i \rangle w_i \\
&= \|T\| \cdot \sum_i \langle u, \|T\|^{-1} T' v'_i \rangle w_i \qquad \text{for each} \quad u \in E,
\end{aligned}$$

$$\|\{\|T\|^{-1} T' v'_i\}\|_{l(p,q;E')} \le \|\{\|v'_i\|_{F'}\}\|_{l(p,q)} < \infty$$

and

$$\sup_{\|w'\|_{G'} \le 1} \|\{\langle w_i, w' \rangle\}\|_{l(p',q')} < \infty.$$

Hence

$$ST \in \boldsymbol{N}_{p,q}(E, G)$$

and

$$\boldsymbol{\nu}_{p,q}(ST) \le \boldsymbol{\nu}_{p,q}(S) \cdot \|T\|.$$

This completes the proof.

PROPOSITION 5. *$\boldsymbol{L}_0(E, F)$ is dense in $\boldsymbol{N}_{p,q}(E, F)$ and in $\boldsymbol{N}^{p,q}(E, F)$.*

PROOF. Let $T \in \boldsymbol{N}_{p,q}(E, F)$. Then $Tu = \sum_i \langle u, u'_i \rangle v_i$ for each $u \in E$,

with

$$\sum_{i=1}^{\infty} i^{q/p-1}\|u_i'\|^{*q}<\infty$$

and

$$\sup_{\|v'\|\leq 1}\sum_{i=1}^{\infty} i^{q'/p'-1}|<v_i, v'>|^{*q'}<\infty,$$

where * stands for the non-increasing rearrangement of sequences with respect to $i$. Putting

$$T_k u=\sum_{i=1}^{k}<u, u_i'>v_i,$$

we obtain

$$T_k\in \boldsymbol{L}_0(E, F)$$

and

$$(T-T_k)u=\sum_{i=1}^{\infty}<u, u_{k+i}'>v_{k+i}.$$

Assume first $q\geq p$. In order to estimate the $l(p, q)$-norm (resp. $l(p', q')$-norm) of the sequence $\{\|u_{k+i}'\|\}_{1\leq i<\infty}$ (resp. $\{|<v_{k+i}, v'>|\}_{1\leq i<\infty}$), rearranging $\{\|u_{k+i}'\|\}_{1\leq i<\infty}$ (resp. $\{|<v_{k+i}, v'>|\}_{1\leq i<\infty}$) in non-increasing order we denote the rearrangement by $\{\|u_{k,i}'\|^{**}\}_{1\leq i<\infty}$ (resp. $\{|<v_{k,i}, v'>|^{**}\}_{1\leq i<\infty}$). Let $m(k, i)$ (resp. $n(k, i)$) be the number corresponding to the term $\|u_{k,i}'\|^{**}$ (resp. $|<v_{k,i}, v'>|^{**}$) in the non-increasing rearrangement $\{\|u_i'\|^{*}\}_{1\leq i<\infty}$ (resp. $\{|<v_i, v'>|^{*}\}_{1\leq i<\infty}$) of $\{\|u_i'\|\}_{1\leq i<\infty}$ (resp. $\{|<v_i, v'>|\}_{1\leq i<\infty}$). Since $m(k, i)\geq i$, we have

$$\sum_{i=1}^{\infty} i^{q/p-1}\|u_{k,i}'\|^{**q}\leq\sum_{i=1}^{\infty} m(k, i)^{q/p-1}\|u_{k,i}'\|^{**q},$$

where the right hand side tends to 0 as $k\to\infty$ because

$$\sum_{i=1}^{\infty} i^{q/p-1}\|u_i'\|^{*q}<\infty \quad \text{and } m(k, i)\to\infty \text{ as } k\to\infty.$$

On the other hand, taking in mind that $|<v_{k,i}, v'>|^{**}\leq|<v_i, v'>|^{*}$, $i=1, 2, \ldots$, we have

$$\begin{aligned}&\sup_{\|v'\|\leq 1}\sum_{i=1}^{\infty} i^{q'/p'-1}|<v_{k,i}, v'>|^{**q'}\\ &\leq\sup_{\|v'\|\leq 1}\sum_{i=1}^{\infty} i^{q'/p'-1}|<v_i, v'>|^{*q'}<\infty.\end{aligned}$$

Next, in case $q<p$ $(q'>p')$, we similarly have

$$\sum_{i=1}^{\infty} i^{q/p-1} \|u'_{k,i}\|^{**q} \leq \sum_{i=1}^{\infty} i^{q/p-1} \|u'_i\|^{*q} < \infty,$$

and

$$\begin{aligned} &\sup_{\|v'\|\leq 1} \sum_{i=1}^{\infty} i^{q'/p'-1} |<v_{k,i}, v'>|^{**q'} \\ &\leq \sup_{\|v'\|\leq 1} \sum_{i=1}^{\infty} n(k, i)^{q'/p'-1} |<v_{k,i}, v'>|^{**q'} \\ &\to 0 \quad \text{as} \quad k\to\infty. \end{aligned}$$

Hence we have

$$\begin{aligned} &\nu_{p,q}(T-T_k) \\ &\leq \Big(\sum_{i=1}^{\infty} i^{q/p-1} \ \|u'_{k,i}\|^{**q}\Big)^{1/q} \sup_{\|v'\|\leq 1} \Big(\sum_{i=1}^{\infty} i^{q'/p'-1} |<v_{k,i}, v'>|^{**q'}\Big)^{1/q'} \\ &\to 0 (\text{as } k\to\infty), \end{aligned}$$

as desired.

In the same way we can show that $\boldsymbol{L}_0(E, F)$ is dense in $\boldsymbol{N}^{p,q}(E, F)$ and the proof is complete.

By making use of Propositions 1 and 5 the following corollary is readily shown.

Corollary 1. *$\boldsymbol{N}_{p,q}(E, F)\subset \boldsymbol{K}(E, F)$ and $\boldsymbol{N}^{p,q}(E, F)\subset \boldsymbol{K}(E, F)$.*

Example 1. *Let $\{\delta_i\}\in l(p, q)$ and $D_1$ be the operator from $l^\infty$ into $l(p, q)$ defined by*

$$D_1(\{a_i\})=\{\delta_i a_i\} \qquad \textit{for each } \{a_i\}\in l^\infty.$$

*Then*

$$D_1\in \boldsymbol{N}_{p,q}(l^\infty, l(p, q))$$

*and*

$$\nu_{p,q}(D_1)=\|\{\delta_i\}\|_{l(p,q)}.$$

In fact, let $e_i=\{\overbrace{0,\dots,0, 1}^{i}, 0, \dots\}$ in $l(p, q)$, and define $u'_i\in(l^\infty)'$ by $<u, u'_i>=\delta_i a_i$ for each $u=\{a_i\}\in l^\infty$. Then

$$D_1 u=\sum_i <u, u'_i> e_i$$

and

$$\|\{\|u_i'\|\}\|_{l(p,q)} = \|\{\delta_i\}\|_{l(p,q)},$$

$$\sup_{\|v'\|_{l(p',q')}\leq 1} \|\{<e_i, v'>\}\|_{l(p',q')} = 1,$$

which shows

$$D_1 \in \boldsymbol{N}_{p,q}(l^\infty, l(p, q)) \quad \text{and} \quad \boldsymbol{\nu}_{p,q}(D_1) \leq \|\{\delta_i\}\|_{l(p,q)}.$$

On the other hand for $e=\{1, 1, \dots\} \in l^\infty$ we get

$$\|\{\delta_i\}\|_{l(p,q)} = \|D_1 e\| \leq \|D_1\| \leq \boldsymbol{\nu}_{p,q}(D_1).$$

Hence

$$\boldsymbol{\nu}_{p,q}(D_1) = \|\{\delta_i\}\|_{l(p,q)}.$$

In a similar way we obtain

EXAMPLE 1′. *Let $\{\delta_i\}$ be the same one as in Example* 1, *and $D_2$: $l(p', q') \to l^1$ be defined as*

$$D_2(\{b_i\}) = \{\delta_i b_i\} \qquad \textit{for each} \quad \{b_i\} \in l(p', q').$$

*Then*

$$D_2 \in \boldsymbol{N}^{p,q}(l(p', q'), l^1) \quad \textit{and} \quad \boldsymbol{\nu}^{p,q}(D_2) = \|\{\delta_i\}\|_{l(p,q)}.$$

These examples illustrate the fact stated in Proposition 2.

By making use of these examples, a $(p, q)$-nuclear (or right $(p, q)$-nuclear) operator is characterized in the following decomposition theorems.

THEOREM 2. *$T \in \boldsymbol{L}(E, F)$ is $(p, q)$-nuclear if and only if $T$ can be factorized in the form $T = Q_1 D_1 P_1$:*

$$E \xrightarrow{P_1} l^\infty \xrightarrow{D_1} l(p, q) \xrightarrow{Q_1} F$$

*where $P_1 \in \boldsymbol{L}(E, l^\infty)$ with $\|P_1\| \leq 1$, $Q_1 \in \boldsymbol{L}(l(p, q), F)$ with $\|Q_1\| \leq 1$ and $D_1$ is the operator in Example* 1.

PROOF. The sufficiency is evident by Proposition 4 and Example 1. The necessity is proved by virtue of the definition of $T \in \boldsymbol{N}_{p,q}(E, F)$ and the following natural decomposition of $T$. Since

$$Tu = \sum_i <u, u_i'> v_i$$

with

$$\|\{\|u_i'\|\}\|_{l(p,q)} < \boldsymbol{\nu}_{p,q}(T)+\varepsilon,\ u_i' \neq 0$$

and

$$\sup_{\|v'\|\leq 1} \|\{<v_i,\ v'>\}\|_{l(p',q')} \leq 1,$$

we can get the decomposition of $T$:

$$E \xrightarrow{P_1} l^\infty \xrightarrow{D_1} l(p,\ q) \xrightarrow{Q_1} F$$

defined by

$$P_1 u = \{<u,\ u_i'/\|u_i'\|>\} \in l^\infty \qquad \text{for each} \quad u \in E,$$

$$D_1(\{a_i\}) = \{\|u_i'\| a_i\} \in l(p,\ q) \qquad \text{for each} \quad \{a_i\} \in l^\infty$$

and

$$Q_1(\{<u,\ u_i'>\}) = \sum_i <u,\ u_i'> v_i \in F.$$

It is easy to verify that $\|P_1\| \leq 1$ and $\|Q_1\| \leq 1$.

Repeating the same procedure with the aid of Proposition 4 and Example 1′ we get

THEOREM 2′. *$T \in \boldsymbol{L}(E, F)$ is right $(p, q)$-nuclear if and only if $T$ can be written in the form $T = Q_2 D_2 P_2$:*

$$E \xrightarrow{P_2} l(p',\ q') \xrightarrow{D_2} l^1 \xrightarrow{Q_2} F,$$

*where*

$$P_2 \in \boldsymbol{L}(E,\ l(p',\ q'))\ \text{with}\ \|P_2\| \leq 1, \quad Q_2 \in \boldsymbol{L}(l^1,\ F)\ \text{with}\ \|Q_2\| \leq 1$$

*and $D_2$ is the operator in Example 1′.*

## 3. $(p, q)$-integral operators

Before introducing $(p, q)$-integral operators, we shall begin with the definition of $(p, q)$-majorizable measures.

DEFINITION 2. Let $X$ be a compact Hausdorff space and denote by $C(X)$ the set of continuous functions on $X$. If $M \in \boldsymbol{L}(C(X), F)$ satisfies the following condition, $M$ is said to be a *$(p, q)$-majorizable measure on* $X$, $1 \leq p, q \leq \infty$: There exists a positive Radon measure $\mu$ on $X$ such that

(3) $$\|M(\varphi)\| \leq \|\varphi\|_{L(p,q;X,\mu)} \quad \text{for each} \quad \varphi \in C(X).$$

$M(\varphi)$ is described symbolically as $\int_X \varphi(x) dM(x)$.

The collection of $(p, q)$-majorizable measures is denoted by $\boldsymbol{M}_{p,q}(X; F)$ and $\boldsymbol{m}_{p,q}(M)$ is defined as

$$\boldsymbol{m}_{p,q}(M) = (p/q)^{1/q} \inf\{\mu(X)\}^{1/p},$$

where the infimum is taken over all $\mu$ satisfying (3).

PROPOSITION 6. *If* $1 < p_0 < p_1 < \infty$, $1 < q_0$, $q_1 < \infty$, *then* $\boldsymbol{M}_{p_0,q_0}(X; F) \subset \boldsymbol{M}_{p_1,q_1}(X; F)$.

*If* $1 < p < \infty$, $1 < q_0 \leq q_1 < \infty$, *then*

$$\boldsymbol{M}_{p,q_0}(X; F) \supset \boldsymbol{M}_{p,q_1}(X; F).$$

PROOF. This is easily shown by the definition of $\boldsymbol{M}_{p,q}$ and the monotonicities of norms with respect to $p$ and $q$ in $L(p, q; X, \mu, F)$ mentioned in Section 1.

PROPOSITION 7. *Let* $M \in \boldsymbol{M}_{p,q}(X; F)$. *Then for any scalar* $\alpha$

$$\alpha M \in \boldsymbol{M}_{p,q}(X; F)$$

*and*

$$\boldsymbol{m}_{p,q}(\alpha M) = |\alpha| \cdot \boldsymbol{m}_{p,q}(M).$$

PROOF. By the assumption we have $M \in \boldsymbol{L}(C(X), F)$ and there is a positive Radon measure $\mu$ on $X$ such that

$$\|M(\varphi)\| \leq \|\varphi\|_{L(p,q;X,\mu)} \quad \text{for each} \quad \varphi \in C(X).$$

Therefore

$$\|\alpha M(\varphi)\| \leq |\alpha| \cdot \|\varphi\|_{L(p,q;X,\mu)} = \|\varphi\|_{L(p,q;X,|\alpha|^p \mu)},$$

where the last equation is obtained by the definition of the norm of $L(p, q; X, \mu)$ in Section 1. This shows $\alpha M \in \boldsymbol{M}_{p,q}(X; F)$ and

$$\boldsymbol{m}_{p,q}(\alpha M) = |\alpha| \cdot \boldsymbol{m}_{p,q}(M).$$

This completes the proof.

PROPOSITION 8. *Let* $M_1$ *and* $M_2$ *be in* $\boldsymbol{M}_{p,q}(X; F)$. *Then*

$$M_1 + M_2 \in \boldsymbol{M}_{p,q}(X; F)$$

*and*

$$\boldsymbol{m}_{p,q}(M_1+M_2)\le C(\boldsymbol{m}_{p,q}(M_1)+\boldsymbol{m}_{p,q}(M_2))$$

*with a positive constant C.*

PROOF. By the assumption, for every $\varepsilon>0$ and $k=1, 2$ there exists a positive Radon measure $\mu_k$ on $X$ such that

$$\|M_k(\varphi)\|\le\|\varphi\|_{L(p,q;X,\mu_k)}\qquad\text{for each}\quad \varphi\in C(X)$$

and

$$(p/q)^{1/q}\{\mu_k(X)\}^{1/p}\le\boldsymbol{m}_{p,q}(M_k)+\varepsilon/2.$$

If we put $\nu_k=\{\boldsymbol{m}_{p,q}(M_k)+\varepsilon/2\}^{1-p}\mu_k$, $\mu=\nu_1+\nu_2$ and $M=M_1+M_2$, then we have

$$\mu(X)\le(q/p)^{p/q}\Big(\sum_{k=1}^{2}\boldsymbol{m}_{p,q}(M_k)+\varepsilon\Big),$$

whence

$$(p/q)^{1/q}\{\mu(X)\}^{1/p}\le\Big(\sum_{k=1}^{2}\boldsymbol{m}_{p,q}(M_k)+\varepsilon\Big)^{1/p}.$$

On the other hand, for each $\varphi\in C(X)$ we have

$$\begin{aligned}&\|M(\varphi)\|\\ &\le\|M_1(\varphi)\|+\|M_2(\varphi)\|\\ &\le\sum_{k=1}^{2}\{\boldsymbol{m}_{p,q}(M_k)+\varepsilon/2\}^{1-1/p}\|\varphi\|_{L(p,q;X,\nu_k)},\end{aligned}$$

where the last inequality is obtained by the same calculation as in the proof of Proposition 7. Furthermore, by Hölder's inequality the right hand side of the above inequality is majorized by

$$\Big(\sum_{k=1}^{2}\boldsymbol{m}_{p,q}(M_k)+\varepsilon\Big)^{1/p'}\Big(\sum_{k=1}^{2}\|\varphi\|^p_{L(p,q;X,\nu_k)}\Big)^{1/p}$$

with $1/p+1/p'=1$. We here denote by $\varphi_k^*(t)$ and $\varphi^*(t)$ the non-increasing rearrangements of $\varphi$ with respect to the measures $\nu_k$ and $\mu$ respectively. Then, by the definitions of the non-increasing rearrangement and the norm of Lorentz spaces, taking into account of the fact $\varphi_k^*(t)\le\varphi^*(t)$ we obtain

$$\sum_{k=1}^{2}\|\varphi\|^p_{L(p,q;X,\nu_k)}\le 2\|\varphi\|^p_{L(p,q;X,\mu)}.$$

From the above discussions we have

$$\|M(\varphi)\| \leq 2^{1/p}(\sum_{k=1}^{2} \boldsymbol{m}_{p,q}(M_k)+\varepsilon)^{1/p'} \|\varphi\|_{L(p,q;X,\mu)}.$$

Since $\varepsilon>0$ is arbitrary it follows that

$$M=M_1+M_2 \in \boldsymbol{M}_{p,q}(X; F)$$

and

$$\boldsymbol{m}_{p,q}(M_1+M_2) \leq 2^{1/p}(\boldsymbol{m}_{p,q}(M_1)+\boldsymbol{m}_{p,q}(M_2)).$$

This completes the proof.

By the aid of $(p, q)$-majorizable measures we shall define $(p, q)$-integral operators.

DEFINITION 3. $T \in \boldsymbol{L}(E, F)$ is called a *$(p, q)$-integral operator*, $1 \leq p$, $q \leq \infty$, if there exists an $M \in \boldsymbol{M}_{p,q}(U^0; F)$ such that

(4) $$Tu=\int_{U^0} \langle u, u' \rangle dM(u') \qquad \text{for each } u \in E,$$

where $U^0$ is the weakly compact unit ball in $E'$.

Let $\boldsymbol{\iota}_{p,q}(T)$ be $\inf\{\boldsymbol{m}_{p,q}(M)\}$, where the infimum is taken over all $M$ satisfying (4). The collection of $(p, q)$-integral operators is denoted by $\boldsymbol{I}_{p,q}(E, F)$.

By the definition and by Proposition 6 respectively, the following two propositions are immediately obtained.

PROPOSITION 9. *For any* $T \in \boldsymbol{I}_{p,q}(E, F)$

$$\|T\| \leq \boldsymbol{\iota}_{p,q}(T).$$

PROPOSITION 10. *If* $1<p_0<p_1<\infty$, $1<q_0$, $q_1<\infty$, *then* $\boldsymbol{I}_{p_0,q_0}(E, F) \subset \boldsymbol{I}_{p_1,q_1}(E, F)$.

*If* $1<p<\infty$, $1<q_0 \leq q_1<\infty$, *then* $\boldsymbol{I}_{p,q_0}(E, F) \supset \boldsymbol{I}_{p,q_1}(E, F)$.

By making use of Propositions 7, 8 and by Definition 3 we also have

PROPOSITION 11. (i) *Let* $T \in \boldsymbol{I}_{p,q}(E, F)$ *and* $\alpha$ *be any scalar. Then* $\alpha T \in \boldsymbol{I}_{p,q}(E, F)$ *and*

$$\boldsymbol{\iota}_{p,q}(\alpha T)=|\alpha| \cdot \boldsymbol{\iota}_{p,q}(T).$$

(ii) *Let* $T_1$ *and* $T_2 \in \boldsymbol{I}_{p,q}(E, F)$. *Then* $T_1+T_2 \in \boldsymbol{I}_{p,q}(E, F)$ *and*

$$\boldsymbol{\iota}_{p,q}(T_1+T_2) \leq C(\boldsymbol{\iota}_{p,q}(T_1)+\boldsymbol{\iota}_{p,q}(T_2))$$

*with the constant* $C$ *in Proposition* 8.

PROPOSITION 12. *For* $T \in \boldsymbol{I}_{p,q}(E, F)$ *and* $S \in \boldsymbol{L}(F, G)$, *we have* $ST \in \boldsymbol{I}_{p,q}$

*$(E, G)$ and $\iota_{p,q}(ST) \leq \|S\| \cdot \iota_{p,q}(T)$.*

*When $T \in \boldsymbol{L}(E, F)$ and $S \in \boldsymbol{I}_{p,q}(F, G)$, we have $ST \in \boldsymbol{I}_{p,q}(E, G)$ and $\iota_{p,q}(ST) \leq \iota_{p,q}(S) \cdot \|T\|$.*

PROOF. Let $T \in \boldsymbol{I}_{p,q}(E, F)$. Then for any $\varepsilon > 0$ there exists an $M \in \boldsymbol{M}_{p,q}(U^0; F)$, $U^0$ being the weakly compact unit ball in $E'$, such that

$$Tu = \int_{U^0} \langle u, u' \rangle dM(u')$$

and

$$\boldsymbol{m}_{p,q}(M) < \iota_{p,q}(T) + \varepsilon.$$

Putting $N = SM$ with $S \in \boldsymbol{L}(F, G)$, we have $N \in \boldsymbol{M}_{p,q}(U^0; G)$,

$$STu = \int_{U^0} \langle u, u' \rangle dN(u')$$

and $\boldsymbol{m}_{p,q}(N) \leq \|S\| \cdot \boldsymbol{m}_{p,q}(M)$ as in the proof of Proposition 7. This shows $ST \in \boldsymbol{I}_{p,q}(E, G)$ and

$$\iota_{p,q}(ST) \leq \|S\| \cdot \iota_{p,q}(T).$$

Next, we assume that $T \in \boldsymbol{L}(E, F)$ and $S \in \boldsymbol{I}_{p,q}(F, G)$. Then for any $\varepsilon > 0$ there exists an $N \in \boldsymbol{M}_{p,q}(V^0; G)$, $V^0$ being the weakly compact unit ball in $F'$, such that

$$Sv = \int_{V^0} \langle v, v' \rangle dN(v')$$

and

$$\boldsymbol{m}_{p,q}(N) < \iota_{p,q}(S) + \varepsilon/2.$$

$N \in \boldsymbol{M}_{p,q}(V^0; G)$ means that there exists a positive Radon measure $\nu$ on $V^0$ such that

$$\|N(\psi)\| \leq \|\psi\|_{L(p,q;V^0,\nu)} \qquad \text{for each } \psi \in C(V^0)$$

and

$$(p/q)^{1/q}\{\nu(V^0)\}^{1/p} < \boldsymbol{m}_{p,q}(N) + \varepsilon/2.$$

We now define $M(\varphi)$, that is, $\int_{U^0} \varphi(u') dM(u')$ by

$$(5) \qquad M(\varphi) = \|T\| \int_{V^0} \varphi(T'v'/\|T\|) dN(v') \qquad \text{for each } \varphi(u') \in C(U^0),$$

where $U^0$ is the weakly compact unit ball in $E'$ and $T': F' \to E'$ is the adjoint of $T$. Then we have

$$\|M(\varphi)\| \leq \|T\| \cdot \|\varphi(T'v'/\|T\|)\|_{L(p,q;V^0,\nu)}$$

and

$$M \in \boldsymbol{M}_{p,q}(U^0; G),$$

that is,

$$\boldsymbol{m}_{p,q}(M) \leq \|T\| \cdot (p/q)^{1/q}\{\nu(V^0)\}^{1/p}. \tag{6}$$

Consequently we obtain by virtue of (5)

$$\begin{aligned} STu &= \int_{V^0} <Tu, v'> dN(v') \\ &= \|T\| \cdot \int_{V^0} <u, T'v'/\|T\|> dN(v') \\ &= \int_{U^0} <u, u'> dM(u'), \end{aligned}$$

and by virtue of (6)

$$\boldsymbol{\iota}_{p,q}(ST) < (\boldsymbol{\iota}_{p,q}(S) + \varepsilon) \cdot \|T\|,$$

which finishes the proof.

Example 2. *Let $X$ be a compact Hausdorff space. Then the identity operator $I$: $C(X) \to L(p, q; X, \mu, C)$ is $(p, q)$-integral and*

$$\boldsymbol{\iota}_{p,q}(I) = (p/q)^{1/q}\{\mu(X)\}^{1/p}.$$

In fact, by making use of the mapping $x \to \delta_x$ (Dirac measure at $x$) from $X$ into $C(X)'$, $X$ can be embedded into the weakly compact unit ball $U^0$ in $C(X)'$. On account of this fact, with any $\varphi \in C(U^0)$ we can associate a function $\varphi_X \in C(X)$ by $\varphi_X(x) = \varphi(\delta_x)$. Hence, defining a positive Radon measure $\hat{\mu}$ on $U^0$ by means of $<\varphi, \hat{\mu}> = <\varphi_X, \mu>$, we get an $M \in \boldsymbol{M}_{p,q}(U^0; L(p, q; X, \mu))$ and

$$\|\varphi_X\|_{L(p,q;X,\mu)} = \|\varphi\|_{L(p,q;U^0,\hat{\mu})}.$$

Therefore

$$\boldsymbol{m}_{p,q}(M) \leq (p/q)^{1/q}\{\mu(X)\}^{1/p} = (p/q)^{1/q}\{\hat{\mu}(U^0)\}^{1/p}.$$

On the other hand, we have

$$(p/q)^{1/q}\{\mu(X)\}^{1/p} \leq \|I\| \leq \boldsymbol{\iota}_{p,q}(I) \leq \boldsymbol{m}_{p,q}(M),$$

and therefore $\boldsymbol{\iota}_{p,q}(I) = (p/q)^{1/q}\{\mu(X)\}^{1/p}$, as desired.

Owing to this example we can state the decomposition theorem of $(p, q)$-integral operators.

THEOREM 3. *$T \in L(E, F)$ is $(p, q)$-integral if and only if $T$ can be decomposed in the form $T=QIP$, where $P \in \boldsymbol{L}(E, C(X))$, $Q \in \boldsymbol{L}(L(p, q; X, \mu), F)$ with $\|P\| \leq 1$ and $\|Q\| \leq 1$ and $I$ is the operator defined in Example 2.*

PROOF. Since the sufficiency is obvious by Example 2 and Proposition 12, we need only to prove that any $T \in \boldsymbol{I}_{p,q}(E, F)$ can be factorized in the above mentioned form. To begin with, for any $\varepsilon>0$ $T$ may be written as

$$Tu=\int_{U^0} <u, u'> dM(u') \qquad \text{for each } u \in E,$$

with $M \in \boldsymbol{M}_{p,q}(U^0; F)$ such that $\boldsymbol{m}_{p,q}(M) < \boldsymbol{\iota}_{p,q}(T)+\varepsilon$. That is, there exists a positive Radon measure $\mu$ on the weakly compact unit ball $U^0$ in $E'$ such that

$$\|M(\varphi)\| \leq \|\varphi\|_{L(p, q; U^0, \mu)} \qquad \text{for each } \varphi \in C(U^0).$$

According to this, $M$ can be extended to a bounded linear operator $Q$: $L(p, q; U^0, \mu, \boldsymbol{C}) \to F$ satisfying $\|Q\| \leq 1$. Thus, when we put $Pu=<u, \cdot>$ and let $I$: $C(U^0) \to L(p, q; U^0, \mu, \boldsymbol{C})$ be the identity operator, we obtain the decomposition $Tu=QIPu$, $\|P\| \leq 1$. This proves the theorem.

We now notice the well known theorem that for the Banach space $L_\mu^\infty(X)$ there exists a compact Stonian space $\hat{X}$ such that $L_\mu^\infty(X)$ becomes isometrically isomorphic with $C(\hat{X})$ ([14]). We denote by $\hat{\mu}$ the measure on $\hat{X}$ which is obtained by transforming $\mu$ on $X$ by this isomorphism. Then $L(p, q; X, \mu)$ and $L(p, q; \hat{X}, \hat{\mu})$ are isometrically isomorphic. According to this fact the following example and theorem are easily obtained.

EXAMPLE 2′. *Let $X$ be a compact Hausdorff space. Then the identity operator $I$: $L_\mu^\infty(X) \to L(p, q; X, \mu)$ is $(p, q)$-integral and $\boldsymbol{\iota}_{p,q}(I)=(p/q)^{1/q} \{\mu(X)\}^{1/p}$.*

THEOREM 3′. *$T \in \boldsymbol{L}(E, F)$ is $(p, q)$-integral if and only if $T$ can be decomposed in the form $T=QIP$ where $P \in \boldsymbol{L}(E, L_\mu^\infty(X))$ with $\|P\| \leq 1$, $Q \in \boldsymbol{L}(L(p, q; X, \mu), F)$ with $\|Q\| \leq 1$ and $I$ is the operator defined in Example 2′.*

REMARK 2. Example 2′ and Theorem 3′ with $C(X)$ replaced by $L_\mu^\infty(X)$ are useful, as later seen, because $L_\mu^\infty(X)$ has the extension property.

## 4. $(p, q)$-quasi-nuclear operators

Similarly to the definition of $p$-quasi-nuclear operators [11] we shall give

DEFINITION 4. $T\in \boldsymbol{L}(E, F)$ is said to be a *$(p, q)$-quasi-nuclear operator*, $1\leq p, q\leq\infty$, if there exists a sequence $\{u_i'\}\in l(p, q; E')$ such that

$$\|Tu\|\leq\|\{<u, u_i'>\}\|_{l(p,q)}\qquad\text{for each } u\in E. \tag{7}$$

The $\inf\|\{\|u_i'\|\}\|_{l(p,q)}$ which is taken over all $\{u_i'\}$ satisfying the above condition is denoted by $\boldsymbol{\nu}_{p,q}^{Q}(T)$. The collection of all $(p, q)$-quasi-nuclear operators is denoted by $\boldsymbol{N}_{p,q}^{Q}(E, F)$.

In view of (7) the following proposition is clear.

PROPOSITION 13. *For any $T\in\boldsymbol{N}_{p,q}^{Q}(E, F)$ we have*

$$\|T\|\leq\boldsymbol{\nu}_{p,q}^{Q}(T).$$

We next show the relation between $(p, q)$-nuclear operators and $(p, q)$-quasi-nuclear operators.

PROPOSITION 14. *We have*

$$\boldsymbol{N}_{p,q}(E, F)\subset\boldsymbol{N}_{p,q}^{Q}(E, F)\ (\text{resp. } \boldsymbol{N}^{p,q}(E, F)\subset\boldsymbol{N}_{p,q}^{Q}(E, F))$$

and

$$\boldsymbol{\nu}_{p,q}^{Q}(T)\leq\boldsymbol{\nu}_{p,q}(T)\ (\text{resp. } \boldsymbol{\nu}_{p,q}^{Q}(T)\leq\boldsymbol{\nu}^{p,q}(T))$$

*for each $T\in\boldsymbol{N}_{p,q}(E, F)$* (resp. $\boldsymbol{N}^{p,q}(E, F)$).

PROOF. $T\in\boldsymbol{N}_{p,q}(E, F)$ can be expressed as follows. For any $\varepsilon>0$ there exist sequences $\{u_i'\}\subset E'$ and $\{v_i\}\subset F$ such that

$$Tu=\sum_i<u, u_i'>v_i\qquad\text{for each } u\in E,$$

and

$$\|\{\|u_i'\|\}\|_{l(p,q)}<\boldsymbol{\nu}_{p,q}(T)+\varepsilon,$$

$$\sup_{\|v'\|\leq 1}\|\{<v_i, v'>\}\|_{l(p',q')}\leq 1$$

with $1/p+1/p'=1$ and $1/q+1/q'=1$.
Therefore we have

$$\begin{aligned}&\|Tu\|\\ &\leq\sup_{\|v'\|\leq 1}\sum_i|<u, u_i'>|\cdot|<v_i, v'>|\\ &\leq\|\{<u, u_i'>\}\|_{l(p,q)}\cdot\sup_{\|v'\|\leq 1}\|\{<v_i, v'>\}\|_{l(p',q')}\end{aligned}$$

$$\leq \|\{<u, u'_i>\}\|_{l(p,q)},$$

which shows $\qquad T \in \boldsymbol{N}^Q_{p,q}(E, F)$

and $\qquad \boldsymbol{\nu}^Q_{p,q}(T) \leq \boldsymbol{\nu}_{p,q}(T).$

In a similar way it can be easily seen that

$$\boldsymbol{N}^{p,q}(E, F) \subset \boldsymbol{N}^Q_{p,q}(E, F)$$

and $\qquad \boldsymbol{\nu}^Q_{p,q}(T) \leq \boldsymbol{\nu}^{p,q}(T) \qquad$ for each $T \in \boldsymbol{N}^{p,q}(E, F)$.

This completes the proof.

We next show that $\boldsymbol{N}^Q_{p,q}(E, F)$ is a quasi-normed space with respect to $\boldsymbol{\nu}^Q_{p,q}(\cdot)$.

THEOREM 4. *Let* $T_k \in \boldsymbol{N}^Q_{p,q}(E, F)$, $k=1, 2, \ldots, M$. *Then* $\sum_{k=1}^{M} T_k \in \boldsymbol{N}^Q_{p,q}(E, F)$ *and*

$$\boldsymbol{\nu}^Q_{p,q}\Big(\sum_{k=1}^{M} T_k\Big) \leq M^{|1/p-1/q|}\Big(\sum_{k=1}^{M} \boldsymbol{\nu}^Q_{p,q}(T_k)\Big).$$

PROOF. For any $\varepsilon > 0$ there exist sequences

$$\{u'_{k,i}\}_{1 \leq i < \infty} \in l(p, q; E'), \qquad k=1, 2, \ldots, M$$

such that

$$\|\{\|u'_{k,i}\|\}\|_{i,l(p,q)} < \boldsymbol{\nu}^Q_{p,q}(T_k) + \varepsilon/2^k$$

and $\qquad \|T_k u\| \leq \|\{<u, u'_{k,i}>\}\|_{i,l(p,q)} \qquad$ for each $u \in E$, $k=1, 2, \ldots, M$.

Therefore, if we put

$$\hat{u}'_{k,i} = (\boldsymbol{\nu}^Q_{p,q}(T_k) + \varepsilon/2^k)^{-1/q'} u'_{k,i},\ k=1, 2, \ldots, M,$$

with $1/q + 1/q' = 1$, then $(\sum_{k=1}^{M} T_k)u = \sum_{k=1}^{M} T_k u$ satisfies

$$\begin{aligned} &\Big\|\Big(\sum_{k=1}^{M} T_k\Big)u\Big\| \\ &\leq \sum_{k=1}^{M}(\boldsymbol{\nu}^Q_{p,q}(T_k) + \varepsilon/2^k)^{1/q'}\Big(\sum_i i^{q/p-1}|<u, \hat{u}'_{k,i}>|^{*q}\Big)^{1/q} \qquad (8) \\ &\leq \Big(\sum_{k=1}^{M} \boldsymbol{\nu}^Q_{p,q}(T_k) + \varepsilon\Big)^{1/q'}\Big(\sum_{k=1}^{M}\sum_i i^{q/p-1}|<u, \hat{u}'_{k,i}>|^{*q}\Big)^{1/q}, \end{aligned}$$

where $*$ stands for the non-increasing rearrangement of the sequence with respect to $i$. Now, in order to estimate the summation

$$\sigma(u)=\Big(\sum_{k=1}^{M}\sum_{i=1}^{\infty}i^{q/p-1}|<u,\ \hat{u}'_{k,i}>|^{*q}\Big)^{1/q}$$

for each $u\in E$, let $N$ be any positive integer and let $\sigma_N(u)$ be the summation of any first $N$ terms in the summation $\sigma(u)$. Putting

$$\varepsilon_1=\min_{1\leq k\leq M,\ 1\leq i\leq N,\ |<u,\hat{u}'_{k,i}>|^*\neq 0}|<u,\ \hat{u}'_{k,i}>|^*,$$

we denote by $\sum_1$ the summation taken over all $i$ such that $|<u,\ \hat{u}'_{k,i}>|^*\geq\varepsilon_1$, and we write $n(k, i)$ the index number of the term corresponding to $|<u,\ \hat{u}'_{k,i}>|^*$ in the non-increasing rearrangement of $\{|<u,\ \hat{u}'_{k,i}>|^*\ |\ |<u,\ \hat{u}'_{k,i}>|^*\geq\varepsilon_1\}$. Then, in case $q\geq p$, since $n(k, i)\geq i$ we have

$$\begin{aligned}
&\sigma_N(u)\\
&\leq\Big(\sum_{k=1}^{M}\sum_{i=1}^{N}i^{q/p-1}|<u,\ \hat{u}'_{k,i}>|^{*q}\Big)^{1/q}\\
&\leq\Big(\sum_{k=1}^{M}\sum\nolimits_1 i^{q/p-1}|<u,\ \hat{u}'_{k,i}>|^{*q}\Big)^{1/q}\\
&\leq\Big(\sum_{k=1}^{M}\sum\nolimits_1 n(k,\ i)^{q/p-1}|<u,\ \hat{u}'_{k,i}>|^{*q}\Big)^{1/q}\\
&\leq\|\{<u,\ \hat{u}'_{k,i}>\}_{1\leq k\leq M,\ 1\leq i<\infty}\|_{l(p,q)}.
\end{aligned}\tag{9}$$

In case $q<p$, we have

$$\begin{aligned}
&\sigma_N(u)\\
&\leq\Big(\sum_{k=1}^{M}\sum_{i=1}^{N}i^{q/p-1}|<u,\ \hat{u}'_{k,i}>|^{*q}\Big)^{1/q}\\
&\leq\left\{\sum_{k=1}^{M}\sum_{i=1}^{N}\left(\frac{(i-1)\ M+k}{M}\right)^{q/p-1}|<u,\ \hat{u}'_{k,i}>|^{*q}\right\}^{1/q}\\
&=\{M^{1-q/p}\sum_{k=1}^{M}\sum_{i=1}^{N}((i-1)M+k)^{q/p-1}|<u,\ \hat{u}'_{k,i}>|^{*q}\}^{1/q}.
\end{aligned}\tag{10}$$

Here we denote by $n'(k, i)$ the number of the term corresponding to $|<u,\hat{u}'_{k,i}>|^*$ in the non-increasing rearrangement of the sequence $\{|<u,\hat{u}'_{k,i}>|^*|\ 1\leq k\leq M,\ 1\leq i\leq N\}$. Then, noting that the set $\{(i-1)M+k\ \ |1\leq k\leq M,\ 1\leq i\leq N\}$ is a permutation of the sequence $\{1, 2,\ldots, MN\}$ and $q<p$, by the help of Lemma 1 the right hand side of (10) is majorized by

$$M^{1/q-1/p}\Big(\sum_{k=1}^{M}\sum_{i=1}^{N}n'(k,\ i)^{q/p-1}|<u,\ \hat{u}'_{k,i}>|^{*q}\Big)^{1/q}$$

$$(11) \qquad \le M^{1/q-1/p}\Big(\sum_{k=1}^{M}\sum{}_1 n(k, i)^{q/p-1}|\langle u, \hat{u}'_{k,i}\rangle|^{*q}\Big)^{1/q}$$

$$\le M^{1/q-1/p}\|\{\langle u, \hat{u}'_{k,i}\rangle\}_{1\le k\le M,\,1\le i<\infty}\|_{l(p,q)}.$$

Since $N$ is arbitrary, by (9), (10) and (11) we get

$$\sigma(u)\le \max(1, M^{1/q-1/p})\cdot\|\{\langle u, \hat{u}'_{k,i}\rangle\}_{1\le k\le M,\,1\le i<\infty}\|_{l(p,q)}.$$

With the aid of this inequality, (8) yields

(12)

$$\|(\sum_{k=1}^{M} T_k)u\|$$

$$\le\Big(\sum_{k=1}^{M}\nu^{Q}_{p,q}(T_k)+\varepsilon\Big)^{1/q'}\cdot\max(1, M^{1/q-1/p})\cdot\|\{\langle u, \hat{u}'_{k,i}\rangle\}_{1\le k\le M,\,1\le i<\infty}\|_{l(p,q)}.$$

On the other hand, for any positive integer $N$ we denote by $\sigma'_N$ the $l(p, q)$-norm of the first $N$ terms of the non-increasing rearrangement of $\{\{\|\hat{u}'_{1,i}\|^*\}_{1\le i<\infty}, \dots, \{\|\hat{u}'_{M,i}\|^*\}_{1\le i<\infty}\}$. Let $m(k, i)$ be the index number of the term corresponding to $\|\hat{u}'_{k,i}\|^*$ in the non-increasing rearrangement of the sequence $\{\{\|\hat{u}'_{1,i}\|^*\}_{1\le i\le N}, \dots, \{\|\hat{u}'_{M,i}\|^*\}_{1\le i\le N}\}$. Then, as in the proof of Theorem 1, we can apply Lemma 1 to obtain the following inequalities:

$$\sigma'^{q}_N$$

$$\le\sum_{k=1}^{M}\sum_{i=1}^{N} m(k, i)^{q/p-1}\|\hat{u}'_{k,i}\|^{*q}$$

$$=\sum_{k=1}^{M}(\nu^{Q}_{p,q}(T_k)+\varepsilon/2^k)^{-q/q'}\sum_{i=1}^{N} m(k, i)^{q/p-1}\|u'_{k,i}\|^{*q}$$

$$\le\max(1, M^{q/p-1})\sum_{k=1}^{M}\{(\nu^{Q}_{p,q}(T_k)+\varepsilon/2^k)^{-q/q'}\sum_{i=1}^{N} i^{q/p-1}\|u'_{k,i}\|^{*q}\}$$

$$<\max(1, M^{q/p-1})\cdot\Big(\sum_{k=1}^{M}\nu^{Q}_{p,q}(T_k)+\varepsilon\Big).$$

Since $N$ is arbitrary, we have

$$\|\{\|\hat{u}'_{k,i}\|\}_{1\le k\le M,\,1\le i<\infty}\|_{l(p,q)}$$

$$\le\max(1, M^{1/p-1/q})\cdot\Big(\sum_{k=1}^{M}\nu^{Q}_{p,q}(T_k)+\varepsilon\Big)^{1/q}.$$

Therefore, combining this with (12) and on account of the fact that $\varepsilon$ is arbitrary we have

$$\sum_{k=1}^{M} T_k \in \boldsymbol{N}_{p,q}^{Q}(E, F)$$

and

$$\boldsymbol{\nu}_{p,q}^{Q}\left(\sum_{k=1}^{M} T_k\right) \leq M^{|1/p-1/q|}\left(\sum_{k=1}^{M} \boldsymbol{\nu}_{p,q}^{Q}(T_k)\right),$$

which completes the proof.

Since the next proposition can be shown by a reasoning similar to the proof of Proposition 4, we omit the proof.

PROPOSITION 15. *If $T \in \boldsymbol{N}_{p,q}^{Q}(E, F)$ and $S \in \boldsymbol{L}(F, G)$, then $ST \in \boldsymbol{N}_{p,q}^{Q}(E, G)$* and

$$\boldsymbol{\nu}_{p,q}^{Q}(ST) \leq \|S\| \cdot \boldsymbol{\nu}_{p,q}^{Q}(T).$$

*If $T \in \boldsymbol{L}(E, F)$ and $S \in \boldsymbol{N}_{p,q}^{Q}(F, G)$, then $ST \in \boldsymbol{N}_{p,q}^{Q}(E, G)$ and*

$$\boldsymbol{\nu}_{p,q}^{Q}(ST) \leq \boldsymbol{\nu}_{p,q}^{Q}(S) \cdot \|T\|.$$

## 5. $(p, q)$-quasi-integral operators

DEFINITION 5. *$T \in \boldsymbol{L}(E, F)$ is said to be a $(p, q)$-quasi-integral operator* if there exists a positive Radon measure $\mu$ on the weakly compact unit ball $U^0$ in $E'$ and the following inequality is satisfied

$$\|Tu\| \leq \|\langle u, u'\rangle\|_{L(p,q;U^0,\mu,\boldsymbol{C})} \qquad \text{for each } u \in E. \tag{13}$$

Let $\boldsymbol{\iota}_{p,q}^{Q}(T) = (p/q)^{1/q} \inf\{\mu(U^0)\}^{1/p}$, where the infimum is taken over all $\mu$ satisfying (13). The collection of $(p, q)$-quasi-integral operators is denoted by $\boldsymbol{I}_{p,q}^{Q}(E, F)$.

REMARK 3. In view of the result established in [13] to the effect that the notions of $p$-quasi-integral and $p$-absolutely summing operators coincide, it might be well for us to say that a $(p, q)$-quasi-integral operator is a generalization of $p$-absolutely summing operator.

From Definitions 3 and 5 the following proposition is clear.

PROPOSITION 16. *$\boldsymbol{I}_{p,q}(E, F) \subset \boldsymbol{I}_{p,q}^{Q}(E, F)$, and for each $T \in \boldsymbol{I}_{p,q}(E, F)$ we have*

$$\boldsymbol{\iota}_{p,q}^{Q}(T) \leq \boldsymbol{\iota}_{p,q}(T).$$

It is an easy matter to obtain the following propositions corresponding to Propositions 9, 10, 11 and 12.

PROPOSITION 17. *For each* $T \in \boldsymbol{I}^Q_{p,q}(E, F)$ *we have*

$$\|T\| \leq \boldsymbol{\iota}^Q_{p,q}(T).$$

PROPOSITION 18. *If* $1 < p_0 < p_1 < \infty$ *and* $1 < q_0, q_1 < \infty$, *then* $\boldsymbol{I}^Q_{p_0,q_0}(E, F) \subset \boldsymbol{I}^Q_{p_1,q_1}(E, F)$.

*If* $1 < p < \infty$ *and* $1 < q_0 \leq q_1 < \infty$, *then* $\boldsymbol{I}^Q_{p,q_0}(E, F) \supset \boldsymbol{I}^Q_{p,q_1}(E, F)$.

PROPOSITION 19. (i) *Let* $T \in \boldsymbol{I}^Q_{p,q}(E, F)$ *and* $\alpha$ *be any scalar. Then* $\alpha T \in \boldsymbol{I}^Q_{p,q}(E, F)$ *and*

$$\boldsymbol{\iota}^Q_{p,q}(\alpha T) = |\alpha| \cdot \boldsymbol{\iota}^Q_{p,q}(T).$$

(ii) *Let* $T_1$ *and* $T_2 \in \boldsymbol{I}^Q_{p,q}(E, F)$. *Then* $T_1 + T_2 \in \boldsymbol{I}^Q_{p,q}(E, F)$ *and*

$$\boldsymbol{\iota}^Q_{p,q}(T_1 + T_2) \leq C(\boldsymbol{\iota}^Q_{p,q}(T_1) + \boldsymbol{\iota}^Q_{p,q}(T_2))$$

*with a constant* $C \geq 1$.

PROPOSITION 20. *For* $T \in \boldsymbol{I}^Q_{p,q}(E, F)$ *and* $S \in \boldsymbol{L}(F, G)$ *we have* $ST \in \boldsymbol{I}^Q_{p,q}(E, G)$ *and*

$$\boldsymbol{\iota}^Q_{p,q}(ST) \leq \|S\| \cdot \boldsymbol{\iota}^Q_{p,q}(T).$$

*If* $T \in \boldsymbol{L}(E, F)$ *and* $S \in \boldsymbol{I}^Q_{p,q}(F, G)$, *then* $ST \in \boldsymbol{I}^Q_{p,q}(E, G)$ *and*

$$\boldsymbol{\iota}^Q_{p,q}(ST) \leq \boldsymbol{\iota}^Q_{p,q}(S) \cdot \|T\|.$$

## 6. Interrelations among $\boldsymbol{N}_{p,q}$, $\boldsymbol{I}_{p,q}$, $\boldsymbol{N}^Q_{p,q}$ and $\boldsymbol{I}^Q_{p,q}$

We have already seen that

$$\boldsymbol{N}_{p,q}(E, F) \subset \boldsymbol{N}^Q_{p,q}(E, F)$$

and

$$\boldsymbol{I}_{p,q}(E, F) \subset \boldsymbol{I}^Q_{p,q}(E, F)$$

in Propositions 14 and 16 respectively. In this section we shall furthermore investigate the relations between $\boldsymbol{N}_{p,q}(E, F)$ and $\boldsymbol{I}_{p,q}(E, F)$ and between $\boldsymbol{N}^Q_{p,q}(E, F)$ and $\boldsymbol{I}^Q_{p,q}(E, F)$.

We shall first show

THEOREM 5. *If* $q \geq p$, *then every* $(p, q)$*-nuclear operator* $T$ *is* $q$*-integral and satisfies*

$$\boldsymbol{\nu}_{p,q}(T) \geq \boldsymbol{\iota}_q(T).$$

PROOF. Let $q \geq p$ and let $T$ be a $(p, q)$-nuclear operator from $E$ into $F$, that is, for any $\varepsilon > 0$ there exist two sequences $\{u_i'\} \subset E'$ and $\{v_i\} \subset F$ such that

$$Tu = \sum_i < u, u_i' > v_i \qquad \text{for each } u \in E,$$

with

$$\|\{\|u_i'\|\}\|_{l(p,q)} < \boldsymbol{\nu}_{p,q}(T) + \varepsilon$$

and

$$\sup_{\|v'\| \leq 1} \|\{< v_i, v' >\}\|_{l(p',q')} \leq 1.$$

Here we may assume

$$\|u_i'\|^* = \|u_i'\| \quad \text{and} \quad \|u_i'\| \neq 0.$$

We put

$$\hat{u}_i' = u_i' / \|u_i'\|, \quad \hat{v}_i = \|u_i'\| v_i$$

and

$$\mu_i = i^{q/p-1} \|u_i'\|^q.$$

Let $U^0$ be the weakly compact unit ball in $E'$ and define the mapping $M$ from $C(U^0)$ into $F$ dy $M(\varphi) = \sum_i \varphi(\hat{u}_i') \hat{v}_i$. Then, for each $\varphi \in C(U^0)$ we have

$$Tu = \sum_i < u, \hat{u}_i' > \hat{v}_i = M(< u, \cdot >)$$

and

$$\begin{aligned} \|M(\varphi)\| &= \sup_{\|v'\| \leq 1} \left| \sum_i \varphi(\hat{u}_i') \|u_i'\| < v_i, v' > \right| \\ &\leq \|\{\varphi(\hat{u}_i') \|u_i'\|\}\|_{l(p,q)} && \text{by Lemma 2} \\ &\leq \left(\sum_i i^{q/p-1} |\varphi(\hat{u}_i')|^q \|u_i'\|^q\right)^{1/q} && \text{by Lemma 1.} \end{aligned}$$

On the other hand, by using the counting measure $\mu$ defined by $\mu(\hat{u}_i') = \mu_i$, we obtain

$$\|\varphi\|_{L(q,q;U^0,\mu)} = \left(\sum_i |\varphi(\hat{u}_i')|^q i^{q/p-1} \|u_i'\|^q\right)^{1/q}.$$

Therefore we have

$$\|M(\varphi)\| \leq \|\varphi\|_{L(q,q;U^0,\mu)} \quad \text{and} \quad \mu(U^0) = \sum_i i^{q/p-1}\|u_i'\|^q.$$

Hence we get

$$\iota_{q,q}(T) \leq m_{q,q}(M) \leq \{\mu(U^0)\}^{1/q} < \nu_{p,q}(T) + \varepsilon,$$

which completes the proof.

We here notice that this result can also be shown by making use of Proposition 3 and the theorem proved in [11] that $N_q \subset I_q$ and $\nu_q(T) \geq \iota_q(T)$ for each $T \in N_q$.

Concerning the relation between $N_{p,q}$ and $N^Q_{p,q}$, if $F$ has the extension property, Proposition 14 can be precised as follows.

PROPOSITION 21. *Assume that (A): $F$ has the extension property. Then any $(p, q)$-quasi-nuclear operator $T: E \to F$ is $(p, q)$-nuclear, and*

$$\nu_{p,q}(T) = \nu^Q_{p,q}(T).$$

PROOF. From the definition, for any $\varepsilon > 0$ there exists a sequence $\{u_i'\} \subset E'$ such that

$$\|Tu\| \leq \|\{<u, u_i'>\}\|_{l(p,q)} \text{ for each } u \in E \tag{14}$$

and

$$\|\{\|u_i'\|\}\|_{l(p,q)} < \nu^Q_{p,q}(T) + \varepsilon.$$

Let us denote by $Q_0$ the operator from the subspace $\{\{<u, u_i'>\} | u \in E\}$ of $l(p, q)$ into $F$ defined by

$$Q_0(\{<u, u_i'>\}) = Tu.$$

Then, taking into account of (14) we have $\|Q_0\| \leq 1$. Thus, by the help of the assumption (A), there exists an extension $Q$: $l(p, q) \to F$ of $Q_0$ with $\|Q\| \leq 1$. If we put

$$e_i = (\overbrace{0, \ldots, 0, 1}^{i}, 0, \ldots) \qquad \text{and } Qe_i = v_i,$$

then we have

$$Tu = \sum_i <u, u_i'> v_i$$

and

$$\begin{aligned} &\|\{<v_i, v'>\}\|_{l(p',q')} \\ &\quad = \|\{<e_i, Q'v'>\}\|_{l(p',q')} \end{aligned}$$

$$\leq \|Q'v'\|_{l(p',q')} \leq \|v'\| \qquad \text{for each } v' \in F'.$$

Therefore

$$\sup_{\|v'\| \leq 1} \|\{<v_i, v'>\}\|_{l(p',q')} \leq 1$$

and

$$\boldsymbol{\nu}_{p,q}(T) \leq \boldsymbol{\nu}^{Q}_{p,q}(T) + \varepsilon.$$

Owing to Proposition 14 we now obtain the conclusion of the proposition.

We now consider the mapping $P: F \rightarrow B(V^0)$ defined by $Pv = <v, \cdot>$ where $V^0$ denotes the weakly compact unit ball in $F'$ and $B(V^0)$ denotes the space of bounded functions on $V^0$. Since P is isometric, $F$ may be considered as a subspace of $B(V^0)$.

Taking into account of the fact that $B(V^0)$ has the extension property, by virtue of Proposition 21 we have

COROLLARY 2. *$T \in \boldsymbol{L}(E, F)$ is $(p, q)$-quasi-nuclear if and only if $T$ is $(p, q)$-nuclear when we regard $T$ as an operator from $E$ into $B(V^0)$.*

Concerning the inclusion relation between $\boldsymbol{I}_{p,q}$ and $\boldsymbol{I}^{Q}_{p,q}$, Proposition 16 can be precised as follows.

PROPOSITION 22. *Under the assumption* (A) *in Proposition* 21, *any $(p, q)$-quasi-integral operator $T$ is $(p, q)$-integral and we have*

$$\boldsymbol{\iota}_{p,q}(T) = \boldsymbol{\iota}^{Q}_{p,q}(T).$$

PROOF. By the definition there exists a positive Radon measure $\mu$ on the weakly compact unit ball $U^0$ in $E'$ such that

$$\|Tu\| \leq \|<u, u'>\|_{L(p,q;U^0,\mu)} \qquad \text{for each } u \in E, \tag{15}$$

and

$$\boldsymbol{\iota}^{Q}_{p,q}(T) = (p/q)^{1/q} \inf\{\mu(U^0)\}^{1/p}.$$

Let $Q_0$ be the operator from the subspace $\{<u, \cdot> | u \in E\}$ of $L(p, q; U^0, \mu)$ into $F$, defined by $Q_0(<u, \cdot>) = Tu$. Then, in view of (15) and the assumption (A), $Q_0$ may be extended to an operator $Q: L(p, q; U^0, \mu) \rightarrow F$ with $\|Q\| \leq 1$. Therefore there exists an $F$-valued measure $M: C(U^0) \rightarrow F$ such that $M(<u, \cdot>) = Tu$ and $\|Tu\| \leq \|<u, u'>\|_{L(p,q;U^0,\mu)}$. This shows that $T \in \boldsymbol{I}_{p\,q}(E, F)$ and $\boldsymbol{\iota}_{p,q}(T) \leq \boldsymbol{\iota}^{Q}_{p,q}(T)$. By virtue of this fact and Proposition 16 we get

$$\boldsymbol{\iota}_{p,q}(T) = \boldsymbol{\iota}^{Q}_{p,q}(T).$$

This completes the proof.

REMARK 4. By this proposition we obtain an example of a (2, 1)-integral but not 1-integral operator. In fact, as shown in [11], the identity operator **I**: $l^1 \to l^2$ is 1-quasi-integral. Hence, by Proposition 18 $I$ is (2, 1)-quasi-integral and furthermore by Proposition 22 $I$ is (2, 1)-integral. But $I$ is not 1-integral as shown in [11].

In the same way as Corollary of Proposition 21 we obtain

COROLLARY 3. *$T \in \boldsymbol{L}(E, F)$ is $(p, q)$-quasi-integral if and only if $T$ is $(p, q)$-integral when we regard it as an operator from $E$ into $B(V^0)$.*

In the last of this section we shall observe the relation between $(p, q)$-quasi-nuclear operators and $(p, q)$-quasi-integral operators.

THEOREM 6. *If $q \geq p$, then we have $\boldsymbol{N}^Q_{p,q}(E, F) \subset \boldsymbol{I}^Q_q(E, F)$ and for each $T \in \boldsymbol{N}^Q_{p,q}(E, F)$*

$$\boldsymbol{\nu}^Q_{p,q}(T) \geq \boldsymbol{\iota}^Q_q(T).$$

PROOF. As in Corollary of Proposition 21, let $P$ be the isometric embedding from $F$ into $B(V^0)$. On account of the fact that $B(V^0)$ has the property $(A)$ ([9]), we have

$$\boldsymbol{\nu}^Q_{p,q}(T) = \boldsymbol{\nu}^Q_{p,q}(PT) = \boldsymbol{\nu}_{p,q}(PT) \quad \text{by Proposition 21.}$$

On the other hand, by Theorem 5 it holds that

$$\boldsymbol{\nu}_{p,q}(PT) \geq \boldsymbol{\iota}_q(PT).$$

Furthermore, by Proposition 22 we have

$$\boldsymbol{\iota}_q(PT) = \boldsymbol{\iota}^Q_q(PT) = \boldsymbol{\iota}^Q_q(T).$$

Thus we obtain

$$\boldsymbol{\nu}^Q_{p,q}(T) \geq \boldsymbol{\iota}^Q_q(T).$$

This completes the proof.

## References

[1] P. L. Butzer and H. Berens, *Semi-groups of operators and approximation*, Grundlehren der Mathematische Wissenschaften Bd. 145, Springer, Berlin/Heidelberg/New York, 1967.

[2] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16, 1955.

[3] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1967.

[4] T. Holmstedt, *Interpolation of quasi-normed spaces*, Math. Scand. **26** (1970), 177–199.
[5] J. R. Holub, *A characterization of subspaces of $L^p(\mu)$*, Studia Math. **42** (1972), 265–270.
[6] R. Hunt, *On $L(p, q)$ spaces*, Enseignement Math. **12** (1966), 249–276.
[7] S. Kwapień, *Some remarks on $(p, q)$-absolutely summing operators in $l_p$-spaces*, Studia Math. **29** (1968), 327–337.
[8] K. Miyazaki, *$(p, q; r)$-absolutely summing operators*, J. Math. Soc. Japan **24** (1972), 341–354.
[9] L. Nachbin, *Some problems in extending and lifting continuous linear transformations*, Proc. Internat. Sympos. Linear Spaces, Jerusalem Academic Press, Jerusalen (1961), 340–350.
[10] A. Persson, *On some properties of p-nuclear and p-integral operators*, Studia Math. **33** (1969), 213–222.
[11] ——— and A. Pietsch, *p-nukleare und p-integrale Abbildungen in Banachräumen*, Studia Math. **33** (1969), 19–62.
[12] A. Pietsch, *Nuclear locally convex spaces*, Ergebnisse der Mathematik Bd. 66, Springer, Berlin/Heidelberg/New York, 1972.
[13] ———, *Absolut p-summierende Abbildungen in normierten Räumen*, Studia Math. **28** (1967), 333–353.
[14] L. Schwartz, *Produits tensoriels topologiques d'espaces vectoriels topologiques. Espaces vectoriels topologiques nucléaires. Applications*, Séminaire Schwartz de la Faculté des Sciences de Paris, 1953/1954.
[15] K. Yoshinaga, *Lorentz spaces and the Calderón-Zygmund theorem*, Bull. Kyushu Inst. Tech. **16** (1969), 1–38.

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